CS 106 | Homework 1 | 8 April 2004
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Problem 1.1

Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1

1.1(a)

$$\begin{array}{|cccc|}\hline x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \\ x_9 & x_{10} & x_{11} & x_{12} \\ x_{13} & x_{14} & X_{15} & x_{16} \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline\end{array}$$

(B) (C) (D) (E)

$$\begin{array}{|cccc|}\hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ \hline\end{array} \quad \begin{array}{|cccc|}\hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ \hline\end{array} \quad \begin{array}{|ccc|}\hline 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline\end{array}$$

(F) (G) (H) (I)

To apply the requested operations, we mutliply the matrices listed above. Given a matrix X, we perform the operation YX when we want to work on X's rows with the matrix Y, and XY when we want to work on X's columns. And so, we perform the following steps:

1. BC
2. $D(BC)$
3. $E(D(BC))$
4. $E(D(BC))F$
5. $G(E(D(BC))F)$
6. $G(E(D(BC))F)H$
7. $G(E(D(BC))F)HI$

1.1 (b)

To achieve the same result as a product of three matrices, we need to group together the rows operations and the column operations. That is, perform all the operations that work on the rows in a matrix A, compute AB, and then multiply ABC for the remaining operations.

Problem 1.3

A square or rectangular matrix R is upper triangular if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a non-singular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular.
Let $Z = A^{-1}$. Equation (1.8) gives us

$$e_j = \sum_{i=1}^{m} z_{ij} a_i.$$

Furthermore, we know that $e_j = Az_j$. In other words, each e_i is a column of the identity matrix. Therefore, considering each Z_i as a vector, Z_i is some linear combination of A in the first i places and 0 everywhere else. Z_1 takes the form $(x_1, 0, \ldots, 0)$ for some x_1, Z_2 takes the form $(x_2, x_3, 0, \ldots, 0)$ for some x_2, x_3, and so on. Therefore the matrix Z must have 0's below the diagonal; it is upper-triangular.

Problem 1.4

Let $f_1, \ldots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, i = 1, \ldots, 8.$$

1.4(a)

Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.
If A is the 8×8 matrix representing the linear mapping from $d_1, \ldots, d_8$ to $c_1, \ldots, c_8$, then consider the basis matrices e_j for $j = 1, 2, \ldots, 8$. For any e_j, the operation Be_j gives you a unique c_i. Furthermore, the matrix formed by combining all the e_js must have $rank = m$. Since the e_j's are bases for the d_i's, we can use theorem 1.2 directly to see that A maps no two distinct vectors to the same vector.

1.4(b)

Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?
The i, j entry of A^{-1} is $f_j(i)$.

Problem 2.1

Show that if a matrix A is both triangular and unitary, then it is diagonal.
Assume that A is upper-triangular. By the definition of unitary, $A* = A^{-1}$. Since A is triangular, we know from exercise 1.3 that A^{-1} is also upper-triangular. Moreover, since A is unitary, we know that A^* is upper-triangular as well. In order for A and its transpose to be upper-triangular, A must be diagonal. The same follows if A is lower-triangular.

Problem 2.3

Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.

2.3(a) (first solution)

Prove that all eigenvalues of A are real.
We have that $Ax = \lambda x$. Multplying both sides by x^* we get

$$\begin{aligned} Ax &= \lambda x \\ x^*Ax &= x^*\lambda x \\ &= \lambda ||x||^2 \end{aligned}$$

In other words, $\lambda = x^*Ax/||x||^2$. First we need to show that x^*Ax is real. We'll start by showing that x^*Ax is hermitian:

$$\begin{aligned} (x^*Ax)^* &= x^*A^*(x^*)^* \\ &= x^*Ax \end{aligned}$$

And so, x^*Ax is hermitian. Therefore, it must have reals on the main diagonal. We know $||x||^2$ is real by the definition of norm. Therefore, the eigenvalue λ must also be real.

2.3(b) (second solution)

We have $Ax = \lambda x$. We can rewrite as follows:

$$\begin{aligned} Ax &= \lambda x \\ A^*x &= \lambda \\ x^*A^*x = \lambda x^*x \end{aligned}$$

Now we can transpose both sides of the first equality:

$$\begin{aligned} Ax &= \lambda x \\ x^*A^* = \overline{\lambda}x^* \\ x^*A^*x = x\overline{\lambda}x^* \end{aligned}$$

And so, we have that $\lambda = \overline{\lambda}$, and so λ must be real.

2.3(b)

Prove that if x and y are eigenvectors corresponding to different eigenvalues, then x and y are orthogonal.
By definition, x and y are orthogonal if $x \cdot y = 0$. We have $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ with $\lambda_1 \neq \lambda_2$. Therefore we get

$$\begin{aligned} Ax \cdot y &= x \cdot A^T y \\ &= x \cdot Ay \\ \lambda_1 x \cdot y &= x \cdot \lambda_2 y \\ (\lambda_1 - \lambda_2)(x \cdot y) &= 0 \end{aligned}$$

And since $\lambda_2 \neq \lambda_2$, it must be the case that $x \cdot y = 0$.

Problem 2.4

What can be said about the eigenvalues of a unitary matrix?
All eigenvalues of a unitary matrix lie on the unit circle, and so must have length 1. We can show this with the property that unitary matrices don't stretch or dilate the matrix, that is: $||Ax|| = ||x||$. And so we get:

$$\begin{aligned} ||Ax|| &= ||x|| \\ ||Ax|| = ||\lambda x|| &= |\lambda|||x|| \end{aligned}$$

And so we get that $|\lambda|||x|| = ||x||$, so λ must be on the unit circle.

Problem 2.5

Let $S \in \mathbb{C}^{m \times m}$ be skew-hermitian, i.e., $S* = -S$.

2.5(a)

Show by using exercise 2.1 that the eigenvalues of S are pure imaginary.
We have $Ax = \lambda x$. Since $A^* = -A$ we get

$$\begin{aligned} xA^*x &= x - Ax \\ &= x - \lambda x \\ = x - \overline{\lambda} x, & \end{aligned}$$

telling us that $\lambda = -\overline{\lambda}$ and so λ must be imaginary.

2.5(c)

Show that $Q = (I - S)^{-1}(I + S)$, known as the Cayley transform of S, is unitary.
We have

$$QQ^* = (I - S)^{-1}(I + S)(I - S)(I + S)^{-1}.$$

Now we need to show that $(I+S)(I-S) = (I-S)(I+S)$, which we can do by applying distribuve laws of matrix arithmetic:

$$\begin{aligned}
(I+S)(I-S) &= (I+S)I - (I+S)S \\
&= (I+S)I - (IS + SS) \\
&= I + S - S - SS \\
&= I + S + S^* + S^*S
\end{aligned}$$

and

$$\begin{aligned}
(I-S)(I+S) &= I(I+S) - S(I+S) \\
&= I(I+S) - (SI + SS) \\
&= I + S - S - SS \\
&= I + S + S^* + S^*S \\
&= (I+S)(I-S)
\end{aligned}$$

Now that we have $(I+S)(I-S) = (I-S)(I+S)$, we can amend the original equation:

$$\begin{aligned}
QQ^* &= (I-S)^{-1}(I+S)(I-S)(I+S)^{-1} \\
&= (I-S)^{-1}(I-S)(I+S)(I+S)^{-1} \\
&= I
\end{aligned}$$

And so we conclude that Q is unitary.