CS 106: Homework 2

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Suppose W is an arbitrary nonsingular matrix and define $||x||_W$ as $||Wx||$ for some norm $||\cdot||$. Prove that $||\cdot||_W$ is a norm.

To prove that $||\cdot||_W$ is a norm, we need to show that it satisfies three properties.

(1) $||x||_W \geq 0$ and $||x||_W$ if and only if $x = 0$.

Let $b = Wx$. Since W is nonsingular, $b = 0$ if and only if $x = 0$. Since $||\cdot||$ is a norm, $||b||$ = 0 if and only if $b = 0$. Therefore, $||b|| = 0$ if and only if $x = 0$. For $b \neq 0$, $||b|| > 0$. We have just shown that $||Wx|| \geq 0$ and $||Wx|| = 0$ if and only if $x = 0$.

(2) $||x+y||_W \leq ||x||_W + ||y||_W$

We use the linearity of the norm $||\cdot||$ and the linearity of W, as follows.

$$||x+y||_W = ||W(x+y)|| = ||Wx + Wy|| \leq ||Wx|| + ||Wy|| = ||x||_W + ||y||_W \; .$$

(3) $||\alpha x||_W \leq |\alpha| \; ||x||_W$

Again, we use the linearity of the norm $||\cdot||$ and the linearity of W.

$$||\alpha x||_W = ||W(\alpha x)|| = ||\alpha W x|| \leq |\alpha| \; ||Wx|| = |\alpha| \; ||x||_W \; .$$

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Let $||\cdot||$ be any norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$.

Suppose $A \in \mathbb{C}^{m\times m}$ and λ is the largest eigenvalue of A. Let $x \in \mathbb{C}^m$ be the corresponding eigenvector such that $Ax = \lambda x$. Then taking the norm and using linearity, $||Ax|| = ||\lambda x|| = |\lambda| \; ||x||$, so

$$|\lambda| = \frac{||Ax||}{||x||} \; .$$

The induced matrix norm $||A||$ is the supremum, or least upper bound, of the set

$$\left\{ \frac{||Ax||}{||x||} \;\middle|\; x \in \mathbb{C}^m \text{ and } x \neq 0 \right\} \; ,$$

so it is greater than or equal to every element in the set. Since $|\lambda|$ is in the set, $|\lambda| \leq ||A||$ and $|\lambda| = \rho(A)$ because λ is the largest eigenvalue of A. Therefore,

$$\rho(A) \leq ||A|| \; .$$

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The approach to finding the SVD of a matrix A follows four steps: (1) find the eigenvectors and eigenvalues of AA^T and A^TA, (2) form matrix Σ from the square roots of the eigenvalues, (3) form matrices U and V from the eigenvectors, and (4) adjust the signs of the eigenvectors as necessary. In the solutions below, the SVD shown is the full SVD.

(a) Let A = $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$. Then $AA^T = A^TA = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$. The eigenvalues of AA^T are 9 and 4 so the diagonal matrix of singular values is $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. The eigenvectors of AA^T are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the SVD of A is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

after adjusting the sign.

(b) Let A = $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $AA^T = A^TA = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$. The eigenvalues of AA^T are 9 and 4 so the diagonal matrix of singular values is $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. The eigenvectors of AA^T are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the SVD of A is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that the arrangement of the eigenvectors of AA^T in U corresponds with the arrangement of their associated singular values in Σ.

(c) Let A = $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AA^T = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$. The nonzero eigenvalue of AA^T and A^TA is 4, so the diagonal matrix of singular values is $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. The nonzero eigenvector of AA^T is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and the nonzero eigenvector of A^TA is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so the SVD of A is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ was added to V instead of the zero vector to make V orthogonal.

(d) Let A = $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $A^TA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The nonzero eigenvalue of AA^T and A^TA is 2, so the diagonal matrix of singular values is $\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$. The nonzero eigenvector of AA^T is $\begin{bmatrix}1\\0\end{bmatrix}$ and the nonzero eigenvector of A^TA is $\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}$, so the SVD of A is

$$A = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}.$$

Note that the vectors corresponding to the zero singular value were added to U and V to make them orthogonal.

(e) Let A = $\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$. Then $AA^T = A^TA = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$. The nonzero eigenvalue of AA^T and A^TA is 4, so the diagonal matrix of singular values is $\Sigma = \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}$. The nonzero eigenvector of AA^T and A^TA is $\begin{bmatrix}\sqrt{2}/2\\\sqrt{2}/2\end{bmatrix}$, so the SVD of A is

$$A = \begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\sqrt{2}/2 & \sqrt{2}/2\\\sqrt{2}/2 & -\sqrt{2}/2\end{bmatrix}.$$

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Both directions are not true. Clearly, if $A = QBQ^*$ and the SVD for B is $B = U\Sigma V^*$, then

$$A = QBQ^* = QU\Sigma V^*Q^* = (QU)\Sigma(QV)^*$$

is a factorization of A using the singular values of B. The factorization is an SVD for A because the matrices QU and QV are unitary. So we have shown one direction: unitarily equivalent implies same singular values.

The other direction is not true. As a counterexample, let B be a non-square matrix, such as the matrix from problem 1 part c. In the reduced SVD of B, the singular values are in a square diagonal matrix $\hat{\Sigma}$. We can construct a square matrix A with the same singular values as B by multiplying $\hat{\Sigma}$ by unitary U and V. Clearly, no unitary Q exists such that $A = QBQ^*$ because A and Q are square and B is not square.