5.1 According to Theorem 5.4, we try to find the nonzero eigenvalues of A^*A

$$A^*A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

Assume $A^*AX = \lambda X$ $\quad X = (x_1, x_2)^T \quad x \neq 0$

Then

$$x_1 + 2x_2 = \lambda x_1 \quad ①$$

$$2x_1 + 8x_2 = \lambda x_2 \quad ②$$

With ① and ②, we get:

$$(\lambda - 1)(\lambda - 8) = 4$$

$$\lambda = \frac{1}{2}(9 \pm \sqrt{65})$$

So $\sigma_{max}(A) = \sqrt{\frac{1}{2}(9 + \sqrt{65})}$

$\sigma_{min}(A) = \sqrt{\frac{1}{2}(9 - \sqrt{65})}$

5.3 (a) Using the solution for 5.1, we try to find the nonzero eigenvalues of A^*A

$$A^*A = \begin{bmatrix} 104 & -72 \\ -72 & 146 \end{bmatrix}$$

The eigenvalues of A^*A are 200, 50

Because $A = U\Sigma V^*$, $A^*A = V\Sigma^*\Sigma V^*$

$$(A^*A)V = V\begin{bmatrix} 200 & 0 \\ 0 & 50 \end{bmatrix}$$

So the columns of V are eigenvectors of A^*A

Assume

$$A^*AX = \lambda X \quad X = (x_1, x_2)^T \quad \lambda = 200, 50$$

We get

$$V_1 = (-0.6 \quad 0.8)^* \qquad V_2 = (0.8 \quad 0.6)^*$$

Because $A = U\Sigma V^*$, $\quad U = AV\Sigma^{-1}$

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix} \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

Here in U, V, there is one minus sign. Because U, V are orthogonal, we cannot get less minus sign.

5.3 (b) From the solution of (a), we know that

$$\Sigma = \begin{bmatrix} 10\sqrt{2} & 0 \\ 0 & 5\sqrt{2} \end{bmatrix}, \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad V = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

Please refer to figures for the graph.

5.3 (c) (I) $\|A\|_1 = 16$

$$\|AX\|_1 = |11x_2 - 2x_1| + |5x_2 - 10x_1| \leq 16|x_2| + 12|x_1|$$

So $\|AX\|_1 / \|X\|_1 \leq 16$

when $x_1 = 0$ $\quad \|AX\|_1 / \|X\|_1 = 16$

(II) $\|A\|_2 = 10\sqrt{2}$ (From solution of (a))

(III) $\|A\|_\infty = 15$

$$\|AX\|_\infty = \max(|11x_2 - 2x_1|, |5x_2 - 10x_1|)$$

$$\|X\|_\infty = \max(|x_1|, |x_2|)$$

So $\|AX\|_\infty / \|X\|_\infty \leq 15$

when $x_2 = -x_1 \neq 0$ $\quad \|AX\|_\infty / \|X\|_\infty = 15$

(IV) $\|A\|_F = \sqrt{250}$

5.3 (d) From solution of (a), we know that

$A = U \Sigma V^*$

So $A^{-1} = (V^*)^{-1} \Sigma^{-1} U^{-1} = V \Sigma^{-1} U^*$

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sqrt{200}} & 0 \\ 0 & \frac{1}{\sqrt{50}} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{200}} & 0 \\ 0 & \frac{1}{\sqrt{50}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{20} & \frac{-11}{100} \\ \frac{1}{10} & \frac{-1}{50} \end{bmatrix}$$

5.3 (e) Assume $AX = \lambda X$ $\quad X = (x_1, x_2)^T \quad X \neq 0$

Then
$-2x_1 + 11x_2 = \lambda x_1$ ①
$-10x_1 + 5x_2 = \lambda x_2$ ②

From ① and ②, we get
$\lambda = \frac{3}{2} \pm i \frac{\sqrt{391}}{2}$

5.3 (f) $\lambda_1 \lambda_2 = 100$

$\sigma_1 \sigma_2 = 100$

$\det(A) = 11$

5.3 (g) the area of the ellipsoid is $\pi \sigma_1 \sigma_2 = 100\pi$

5.4

$$A = U \Sigma V^*$$

$$A^* = V \Sigma^* U^*$$

$$A v_i = \sigma_i u_i$$

$$A^* u_i = \sigma_i v_i$$

So $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ u_i \end{bmatrix} = \sigma_i \begin{bmatrix} v_i \\ u_i \end{bmatrix}$

$$\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \begin{bmatrix} v_i \\ -u_i \end{bmatrix} = -\sigma_i \begin{bmatrix} v_i \\ -u_i \end{bmatrix}$$

It is obvious that

$$\begin{bmatrix} v_i \\ u_i \end{bmatrix}^* \begin{bmatrix} v_j \\ u_j \end{bmatrix} = 0 \qquad i \neq j$$

$$\begin{bmatrix} v_i \\ -u_i \end{bmatrix}^* \begin{bmatrix} v_j \\ -u_j \end{bmatrix} = 0 \qquad i \neq j$$

$$\begin{bmatrix} v_i \\ -u_i \end{bmatrix}^* \begin{bmatrix} v_j \\ u_j \end{bmatrix} = 0 \qquad 1 \leq i, j \leq m$$

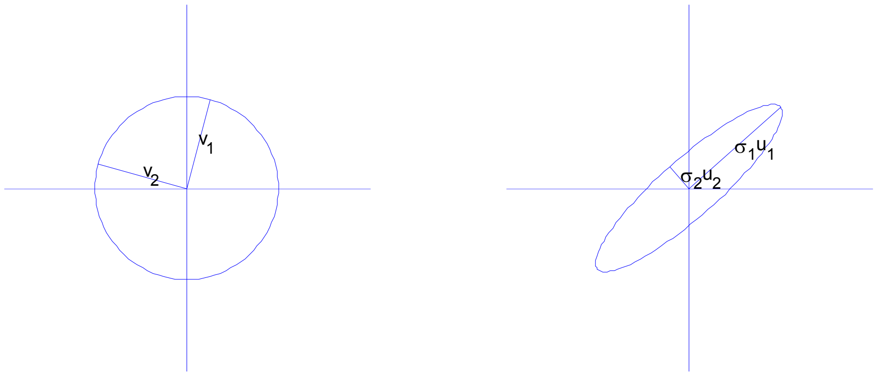
Let $t_j = \begin{bmatrix} v_i \\ u_i \end{bmatrix} \qquad 1 \leq i \leq m \qquad j = i$

$$t_j = \begin{bmatrix} v_i \\ -u_i \end{bmatrix} \qquad 1 \leq i \leq m \qquad j = m+i$$

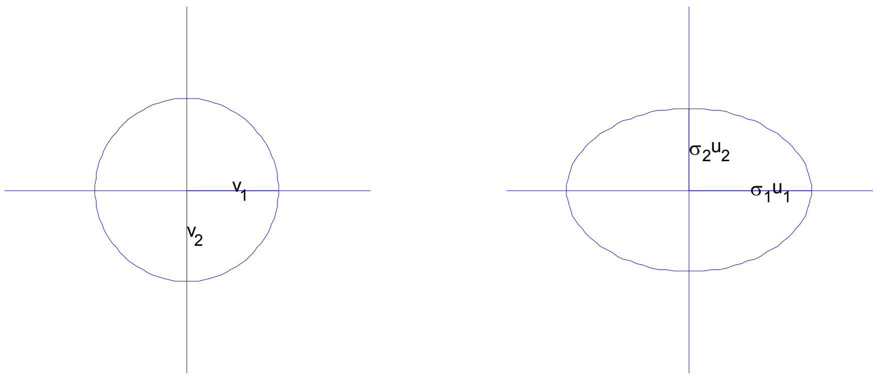
It is obvious that $T = [t_1 | t_2 | \cdots | t_{2m}]$ is nonsingular and all the columns in T are eigenvectors of $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$

so $\begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} = T \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} T^{-1}$

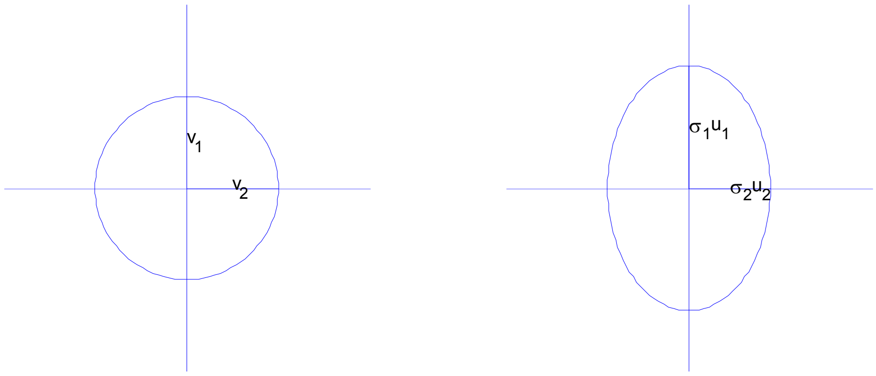
$$A = \begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix}$$



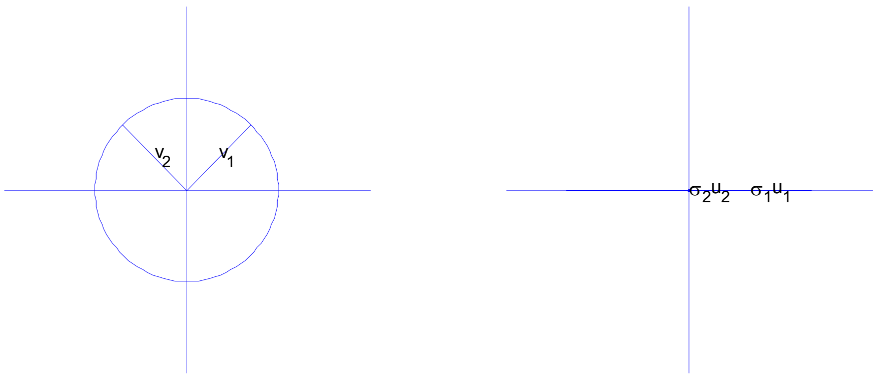
$$A = \begin{matrix} 3 & 0 \\ 0 & -2 \end{matrix}$$



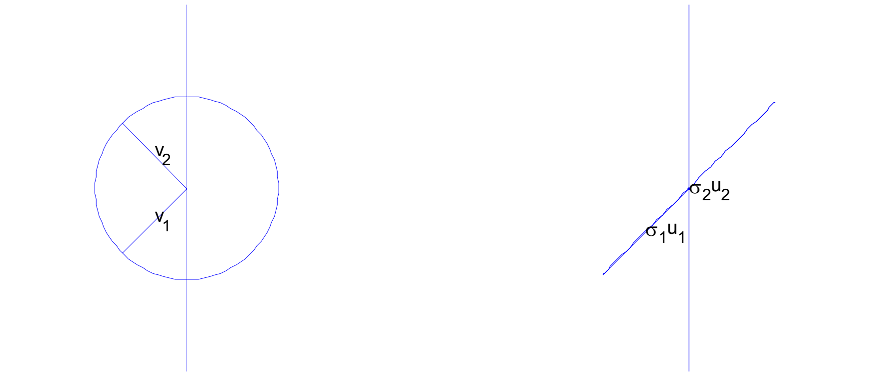
$$A = \begin{matrix} 2 & 0 \\ 0 & 3 \end{matrix}$$



$$A = \begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}$$



$$A = \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$



$$A = \begin{matrix} -2 & 11 \\ -10 & 5 \end{matrix}$$