CS 106 Homework 5

April 15,2004

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Algorithm 8.1

for $i=1$ to n
  $v_i = a_i$] only assignment $\therefore 0$

for $i=1$ to n
  $r_{ii} = \|v_i\|$ to get $\|v_i\|$ we need m multiplication and $m-1$ addition. all together $n \cdot m$ multi. $(m-1)n$ add

  $q_i = v_i / r_{ii}$ m division

  for $j = i+1$ to n

    $r_{ij} = q_i^* v_j$ $\rightarrow$ m multi. $m-1$ add

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=i+1}^{n} m \text{ multi} = \frac{n(n-1)}{2} m$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} (m-1) \text{ add} = \frac{n(n-1)}{2} (m-1)$$

    $v_j = v_j - r_{ij} q_i$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} m \text{ multiplication} = \frac{n(n-1)}{2} m$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} m \text{ subtraction} = \frac{n(n-1)}{2} m$$

$\therefore$ Addition: $(m-1)n + \frac{n(n-1)}{2}(m-1)$

Subtraction $\frac{n(n-1)}{2} m$

multiplication. $n \cdot m + \frac{n(n-1)}{2} m + \frac{n(n-1)}{2} m = n^2 m$

Division: m *

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```
function [Q,R]=mgs(A);
V = A;
[m,n] = size(A);
Q = zeros(m,n);
R = zeros(n,n);
for i = 1:n
    R(i,i) = norm(V(:,i));
    Q(:,i) = V(:,i)/R(i,i);
    for j = i+1 : n
        R(i,j) = conj(Q(:,i))' *V(:,j);
        V(:,j) = V(:,j) - R(i,j)* Q(:,i);
    end
end

%sample output
%A = [ 1 2 ; 3 4];
%[Q, R] = mgs(A)
%Q =
%
%     0.3162     0.4472
%     0.9487     0.8944
%
%
%R =
%
%     3.1623          0
%          0     4.4721
```

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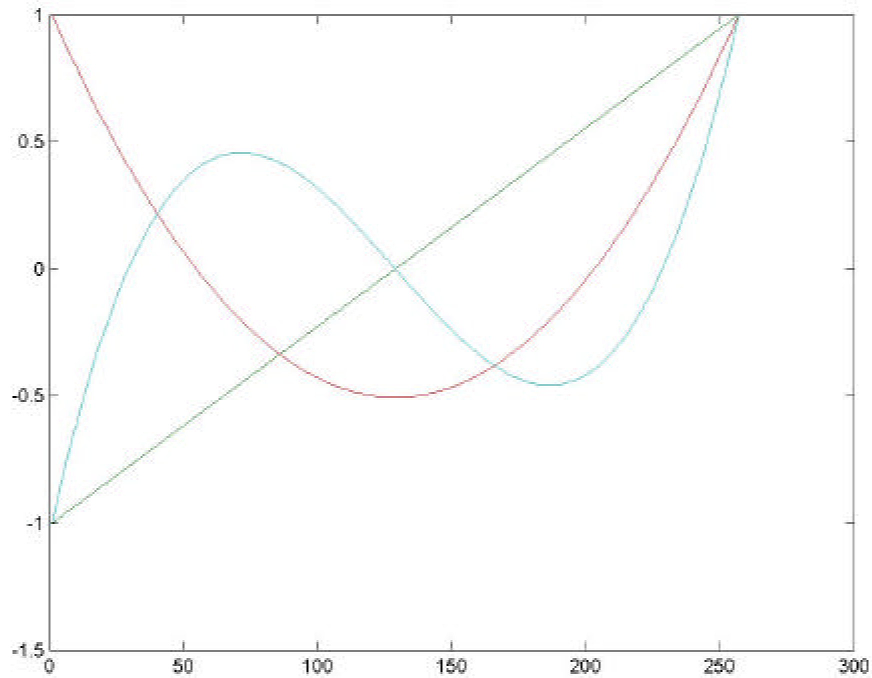
a)

```
x = (-128:128)'/128;
A = [x.^0 x.^1 x.^2 x.^3];
[Q,R]= qr(A,0);
scale = Q(257,:);
Q = Q * diag(1 ./scale);
plot(Q)
```

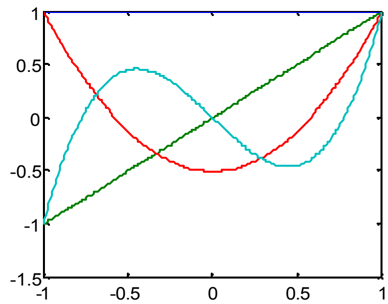
Plot:

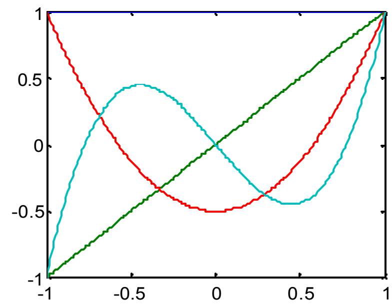


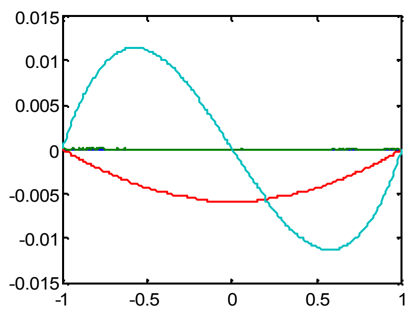
b)

```
x = (-128:128)'/128;
A = [x.^0 x.^1 x.^2 x.^3];
[Q,R]= qr(A,0);
scale = Q(257,:);
Q = Q * diag(1 ./scale);
P = [x.^0 x.^1 1.5*x.^2 - 0.5 2.5*x.^3-1.5*x.^1];
E = Q - P;
subplot(2,2,1); plot(x,Q);
subplot(2,2,2); plot(x,P);
subplot(2,2,3); plot(x,E);
maxerror = max(abs(E));
sumerror =[norm(E(:,1),1) norm(E(:,2),1) norm(E(:,3),1) norm(E(:,4),1)];

maxerror =

    0.0000    0.0000    0.0059    0.0114

sumerror =

    0.0000    0.0000    1.0039    1.8921
```

Plot:







c)

```
maxerror = zeros(1, 12);
for i = 7: 18;
      range = 2^i;
      x = (-range: range)'/range;
      A = [x.^0 x.^1 x.^2 x.^3];
      [Q,R]= qr(A,0);
      scale = Q(2*range + 1,:);
      Q = Q * diag(1 ./scale);
      P = [x.^0 x.^1 1.5*x.^2 - 0.5 2.5*x.^3-1.5*x.^1];
      E = Q - P;
      plot(x,E);
      maxerror(1,i-6) = max(abs(E(:,4)));
end


maxerror =

  Columns 1 through 6

      0.0114       0.0057       0.0028       0.0014       0.0007       0.0004

  Columns 7 through 12
```

0.0002	0.0001	0.0000	0.0000	0.0000	0.0000

From the maxerror matrix, we can see that when v= 14, the error is already controlled to 0.0001.

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9.1.

$$A = \begin{bmatrix} 1 & 2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 2 \\ 0 & & & 1 \end{bmatrix}_{m \times m}$$

a) obviously the eigenvalues of A is all 1.

the determinant of A $\det(A) = 1$

$\text{rank}(A) = m$

b) $[A \ E] \xrightarrow[\text{operation}]{\text{using only row}} [A^{-1} \ E]$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 & 4 & \cdots & (-2)^{m-1} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & 4 \\ & & & \ddots & -2 \\ 0 & & & & 1 \end{bmatrix}$$

c) σ_m. the m^{th} singular value of A is the smallest singular value.

$\because A = U \Sigma V^*$

$A^{-1} = V \Sigma^{-1} U^*$

$\therefore$ singular value of $A^{-1} = \frac{1}{\text{singular value of } A}$

$\therefore \frac{1}{\sigma_m}$ is the largest singular value of A^{-1}

$\frac{1}{\sigma_m} = \|A^{-1}\|_2 \geq \|A^{-1}x\|_2$ $\quad \|x\|=1$, $x \in \mathbb{C}^m$

$\therefore$ by carefully choose x we can get a good upper bound on σ_m.

say choose $x = [0 \ 0 \ 0 \ \cdots \ 1]$

$\|A^{-1}x\|_2 = \|a_m\|_2 = |(-2)^0 + (-2)^2 + (-2)^4 \cdots (-2)^{2m-2}|$

$= \left| \frac{1(1-(-2)^{2m})}{1+2} \right| = \frac{4^m - 1}{3}$

$\therefore \|\sigma_m\| \leq \frac{3}{4^m - 1}$

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11.1. Any $b \in \mathbb{C}^m$. $Ax = Pb$. P is an orthogonal projector of range(A)

$x = A^+ b$

$\because \|A^+\|_2 = \sup \frac{\|A^+b\|_2}{\|b\|_2}$

and for P. (6.5) shows that

$\frac{\|Pb\|}{\|b\|} \le \|P\| = 1$

$\therefore \|A^+\|_2 \le \sup \frac{\|A^+b\|_2}{\|Pb\|_2} = \sup \frac{\|x\|_2}{\|Ax\|_2}$

$\because A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ $\quad \|Ax\|_2^2 = \|A_1x\|_2^2 + \|A_2x\|_2^2$ (by definition of Norms)

$\le \|A_1x\|_2^2$

$\therefore \|A^+\|_2 \le \sup \frac{\|x\|_2}{\|A_1x\|_2^2}$

$\because A_1$ is non singular

$\therefore$ we can find a unique x'

$A_1 x = x'$

$x = A_1^{-1} x'$

$\therefore \|A^+\|_2 \le \sup \frac{\|A_1^{-1}x'\|_2}{\|x'\|_2} = \|A_1^{-1}\|_2$

$\therefore \|A^+\|_2 \le \|A_1^{-1}\|_2$

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format long e

a)

```
m = 50;
n = 12;
t = linspace(0,1,m);
V = vander(t);
V = fliplr(V);
A = zeros(m,n);
for i = 1:  n
      A(:,i) = V(:,i);
end
b = cos(4*t);
AC = conj(A)';
C = AC * A;
B = AC * b';
x = C \ B;

x =

   1.000000014651697e+000
  -4.497145736268313e-006
  -7.999827321532813e+000
  -2.597307341739053e-003
   1.068694453795228e+001
  -9.313280176145686e-002
  -5.421513572594567e+000
  -4.895234796470432e-001
   2.184201984933989e+000
  -3.558858838528528e-001
  -2.230054458592560e-001
   6.070014261594592e-002
```

b)

```
[Q R] = mgs(A);
QC = conj(Q)';
B = QC * b';
x = R \B;

x =

   1.000000003504095e+000
  -1.174057613479715e-006
  -7.999952810401643e+000
  -7.400164107536113e-004
   1.067267129913343e+001
  -2.850450420330584e-002
  -5.605292573075273e+000
```

```
-1.520769432115129e-001
  1.784557541196063e+000
-6.108461047808881e-002
-3.461875432229727e-001
  8.296770979626271e-002
```

c)

```
[Q,R] = house(A);
QC = conj(Q)';
B = QC * b';
x = R \B;
```

house.m

```
function [Q,R] = house(A)
[m,n] = size(A);
V = zeros(m,n);
R = A;
for k = 1:n
    x = R(k:m,k);
    if (x(1) == 0)
        s =1;
    else
        s = sign(x(1));
    end
    x(1)= s * norm(x) + x(1);
    v =x;
    v = v/norm(v);
    V(k:m, k) = v;
    R(k:m, k:n) = R (k:m, k:n) - 2 *v*(v' * R(k:m,k:n));
end
R = R(1:n,1:n);
Q = eye(m,n);
for j = 1:n
    for k = n:-1:1
        Q(k:m, j) = Q(k:m, j) - 2 * V(k:m, k) * (V(k:m, k)' * Q(k:m,j));
    end
end
```

```
x =

  1.000000000996591e+000
 -4.227421753859120e-007
 -7.999981235703829e+000
 -3.187630270533617e-004
```

```
   1.066943079455400e+001
  -1.382028254843984e-002
  -5.647075641507811e+000
  -7.531600017639149e-002
   1.693606936612471e+000
   6.032127571537506e-003
  -3.742417109750871e-001
   8.804057738442990e-002
```

d)

```
[Q,R] = qr(A);
QC = conj(Q)';
B = QC * b';
x = R\B;

x =

   1.000000000996609e+000
  -4.227433097715201e-007
  -7.999981235679233e+000
  -3.187633039879644e-004
   1.066943079635676e+001
  -1.382028980041875e-002
  -5.647075622701471e+000
  -7.531603222675481e-002
   1.693606972325646e+000
   6.032102504262157e-003
  -3.742417009122402e-001
   8.804057562237955e-002
```

e)

```
x = A\b';

x =

   1.000000000996610e+000
  -4.227436678520266e-007
  -7.999981235667154e+000
  -3.187634590862581e-004
   1.066943079740723e+001
  -1.382029406980476e-002
  -5.647075611636319e+000
  -7.531605097010199e-002
   1.693606993001497e+000
   6.032088188582607e-003
```

-3.742416952621144e-001
8.804057465256371e-002

f)

```
[U S V] = svd(A,0);
T = conj(U)' * b';
W = S \ T;
x = V * W;
```

x =

```
 1.000000000996611e+000
-4.227432961907494e-007
-7.999981235679583e+000
-3.187632989027636e-004
 1.066943079631425e+001
-1.382028959570143e-002
-5.647075623303711e+000
-7.531603110511279e-002
 1.693606970995698e+000
 6.032103478108175e-003
-3.742417013139172e-001
 8.804057569378899e-002
```

g)

The normal equations exhibit instability. And the red digit above shows the potential rounding error.