

# PHOTON SURFACES FOR ROBUST, UNBIASED VOLUMETRIC DENSITY ESTIMATION

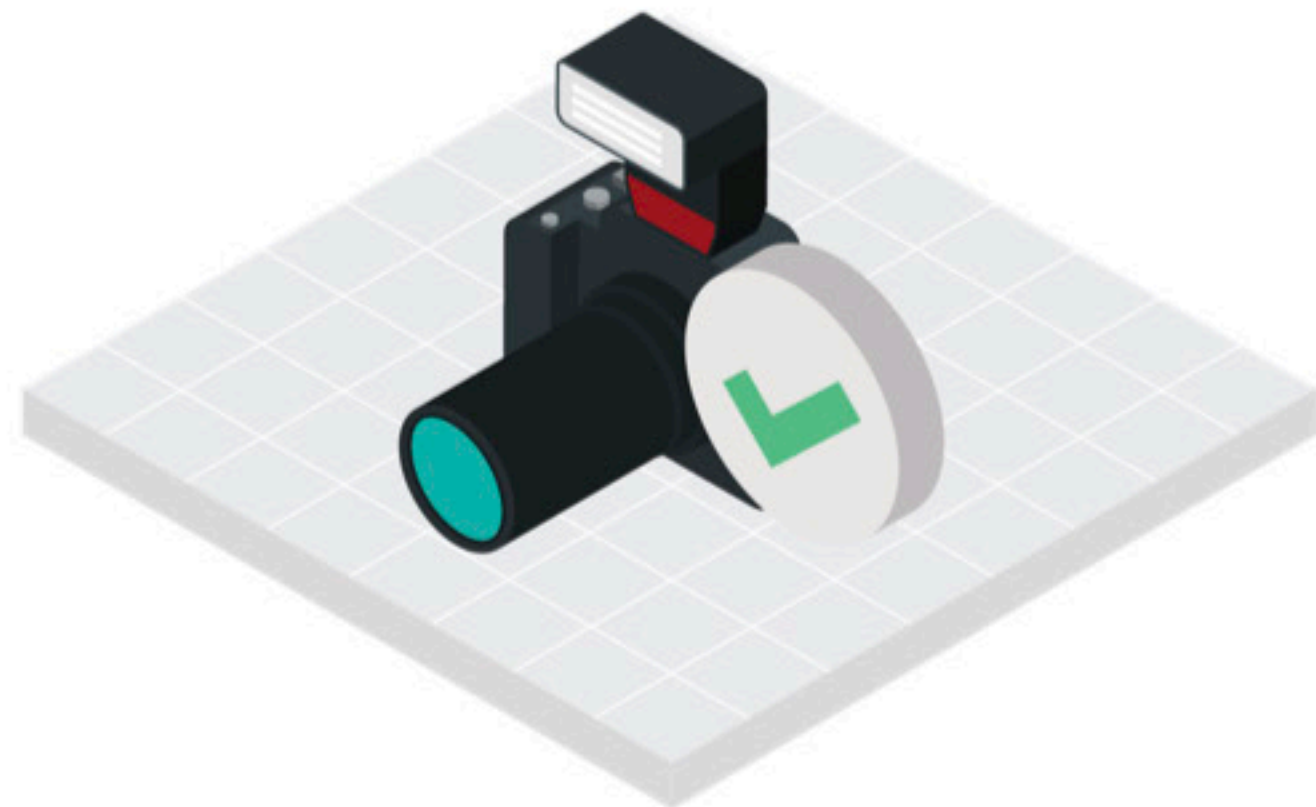
Xi Deng\*

Shaojie Jiao\*

Benedikt Bitterli

Wojciech Jarosz





# PHOTOGRAPHY & RECORDING ENCOURAGED

# PHOTON SURFACES FOR ROBUST, UNBIASED VOLUMETRIC DENSITY ESTIMATION



Xi Deng   Shaojie Jiao   Benedikt Bitterli   Wojciech Jarosz

# Objective



# Problem setup

# Problem setup

Light Source



Camera

# Problem setup

Light Source



Camera



Geometry

# Problem setup

Light Source



Medium  
(Homogeneous)



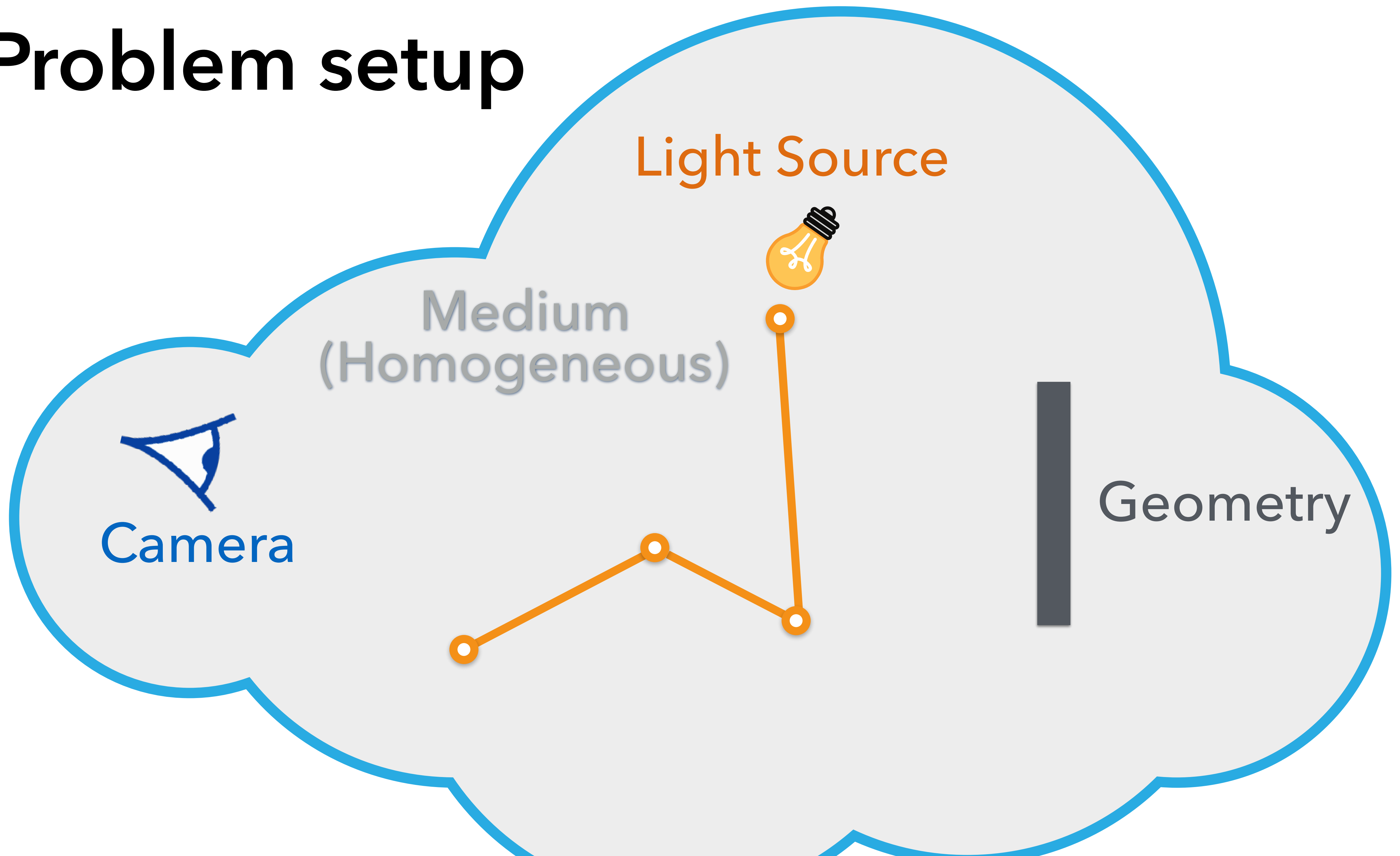
Camera

Geometry

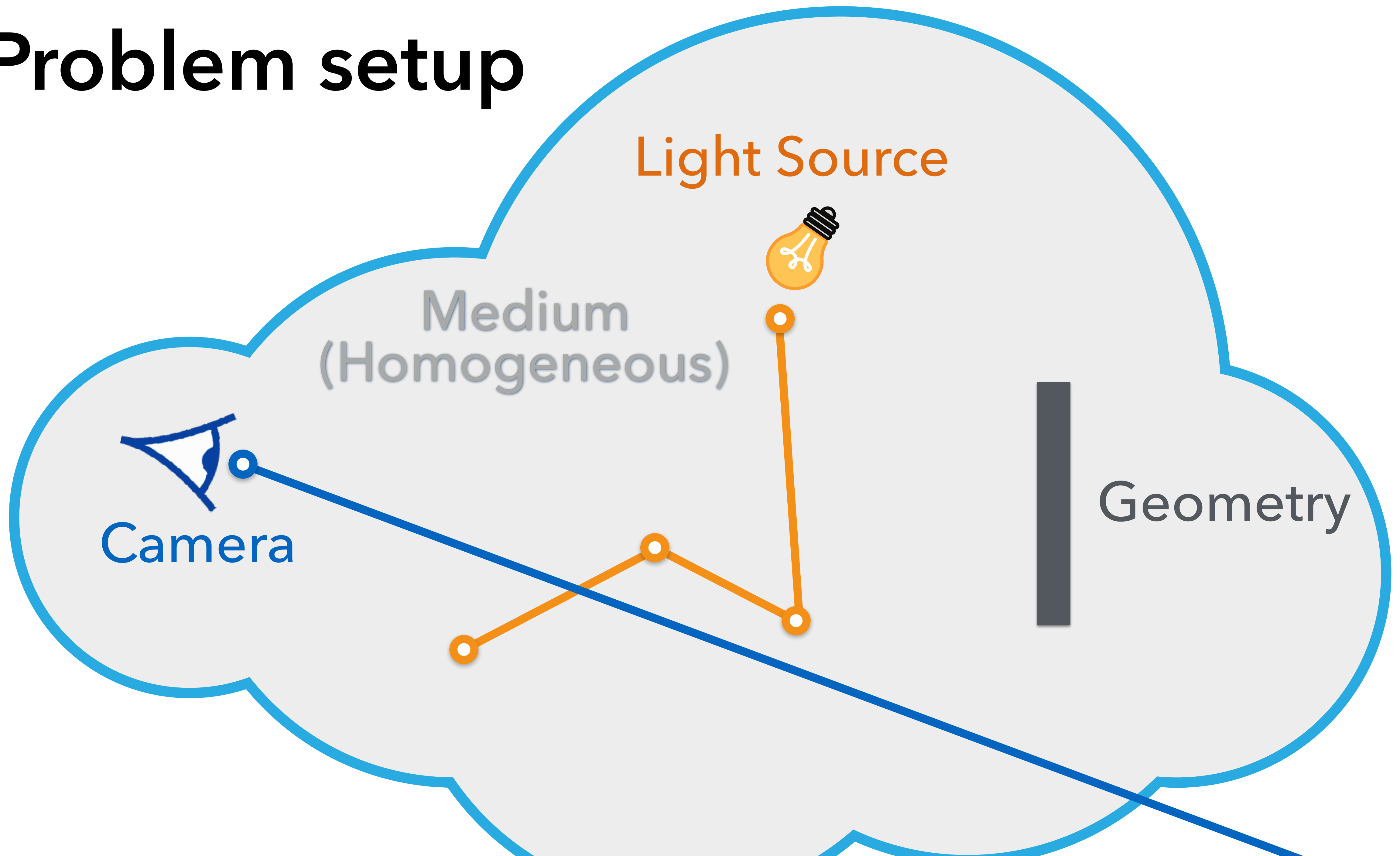




# Problem setup



# Problem setup



# Full Light Transport



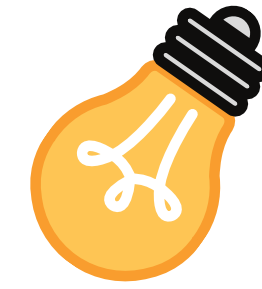


# PREVIOUS WORK

# Volumetric photon mapping

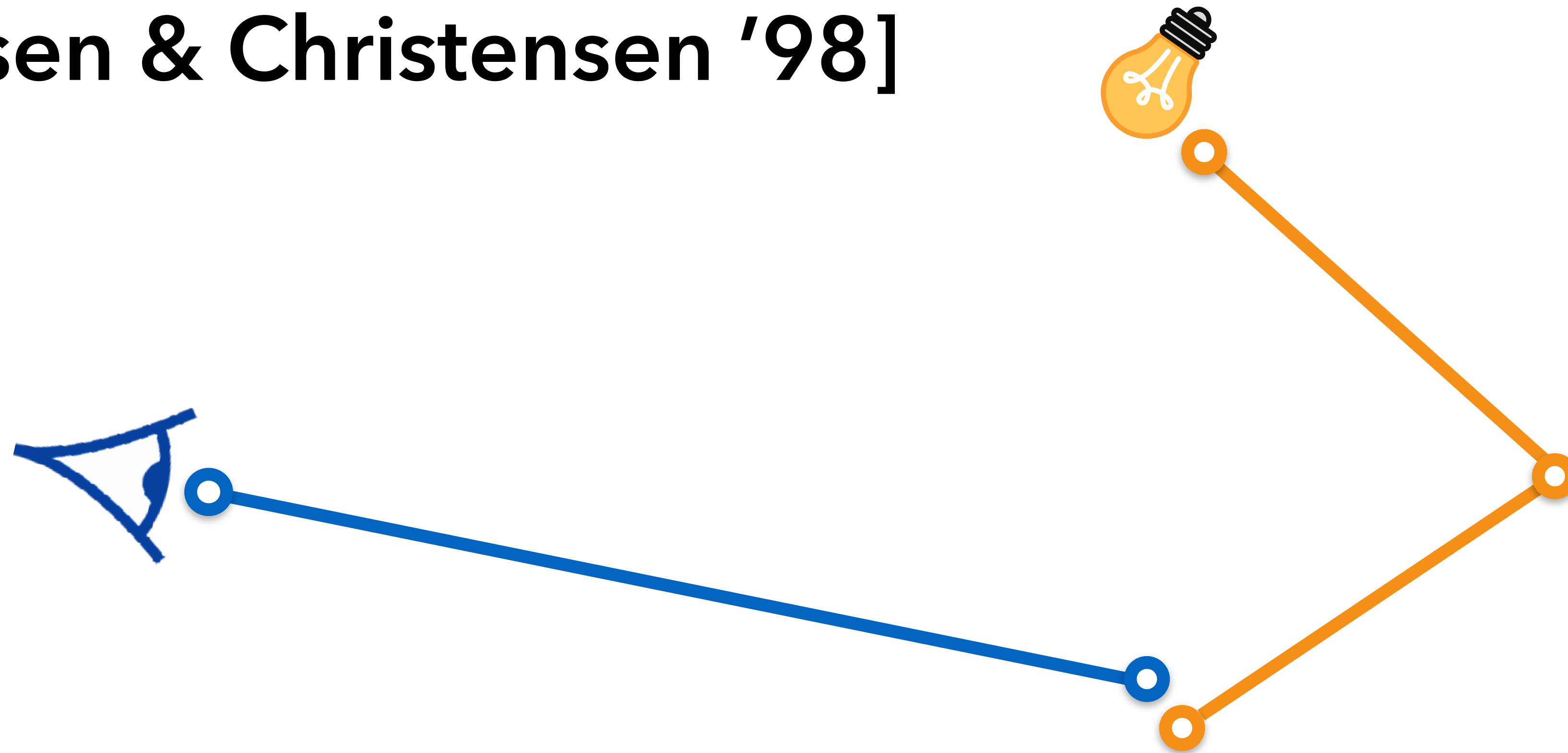
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[Jensen & Christensen '98]



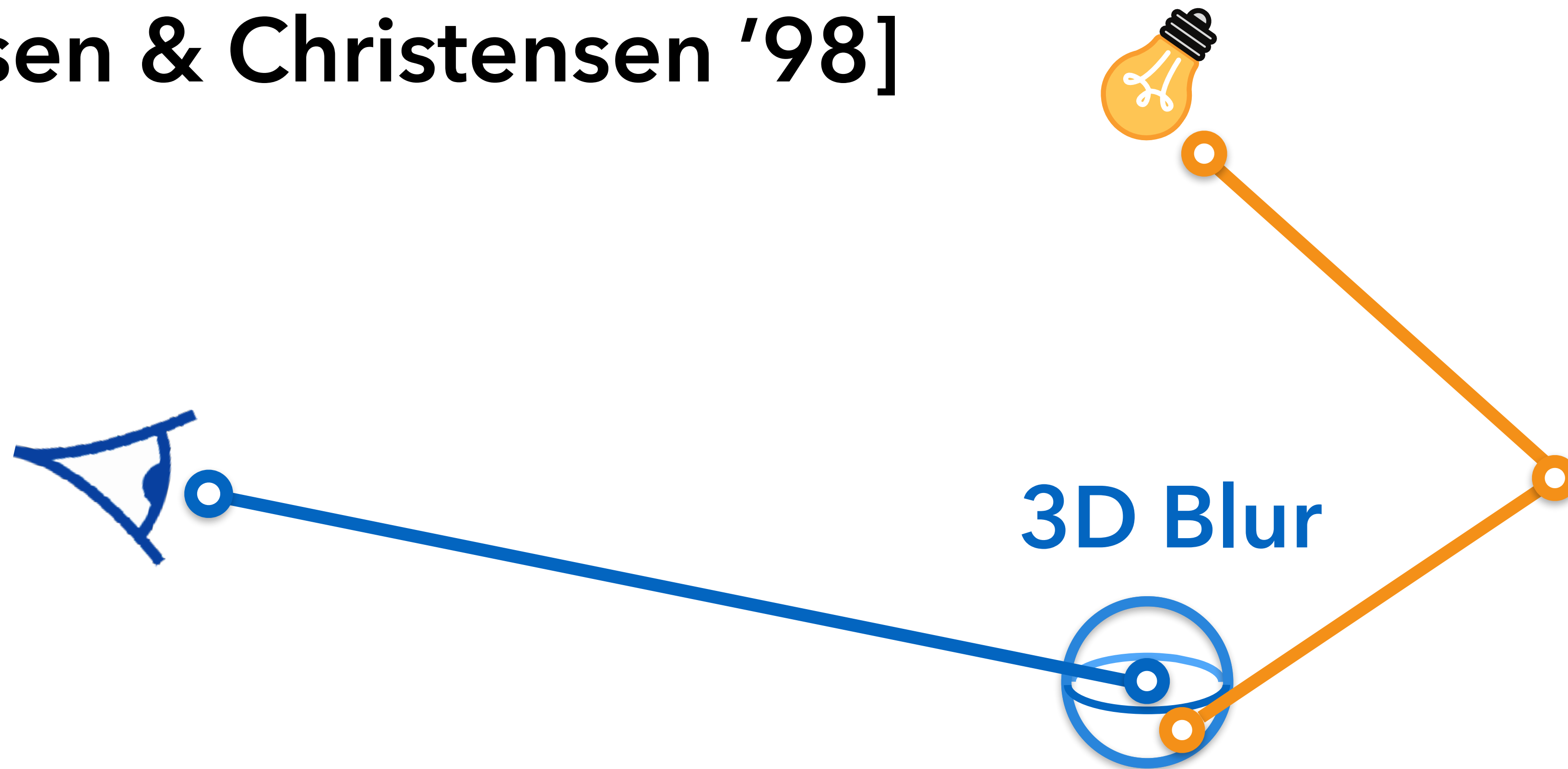
# Volumetric photon mapping

[Jensen & Christensen '98]



# Volumetric photon mapping

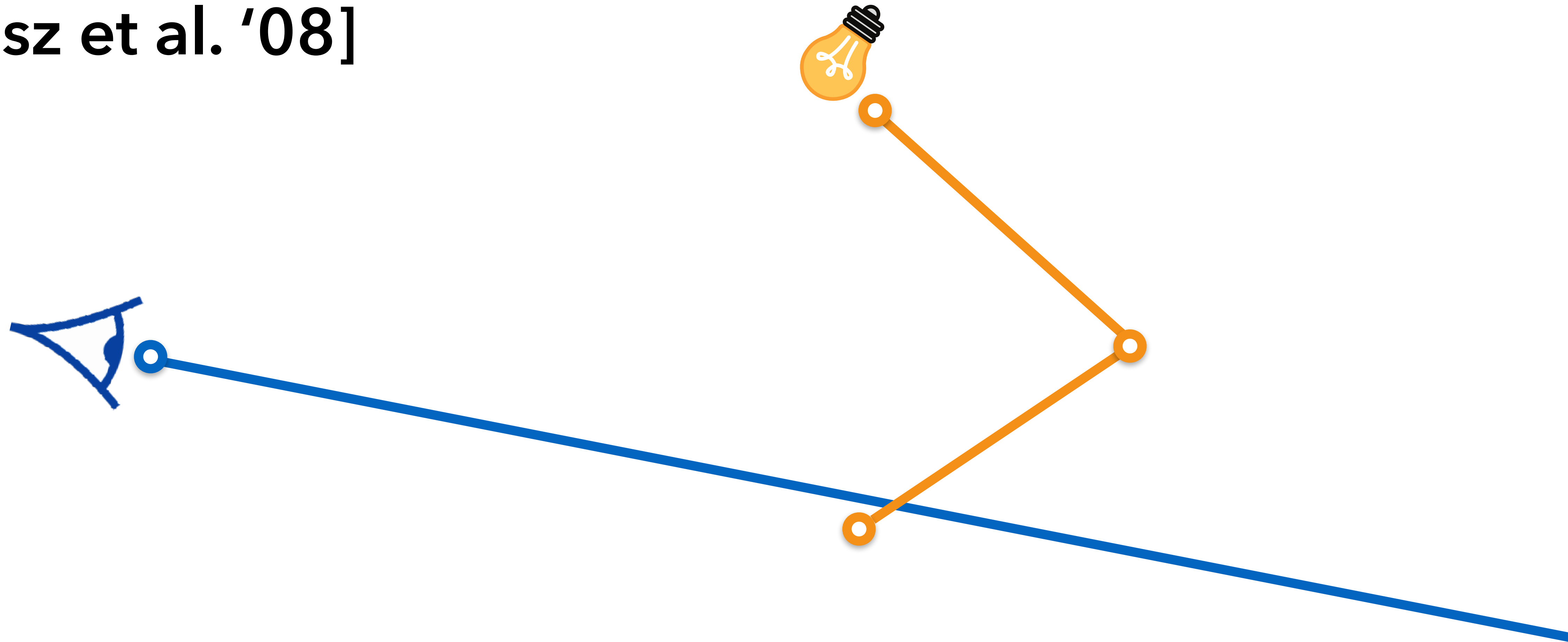
[Jensen & Christensen '98]





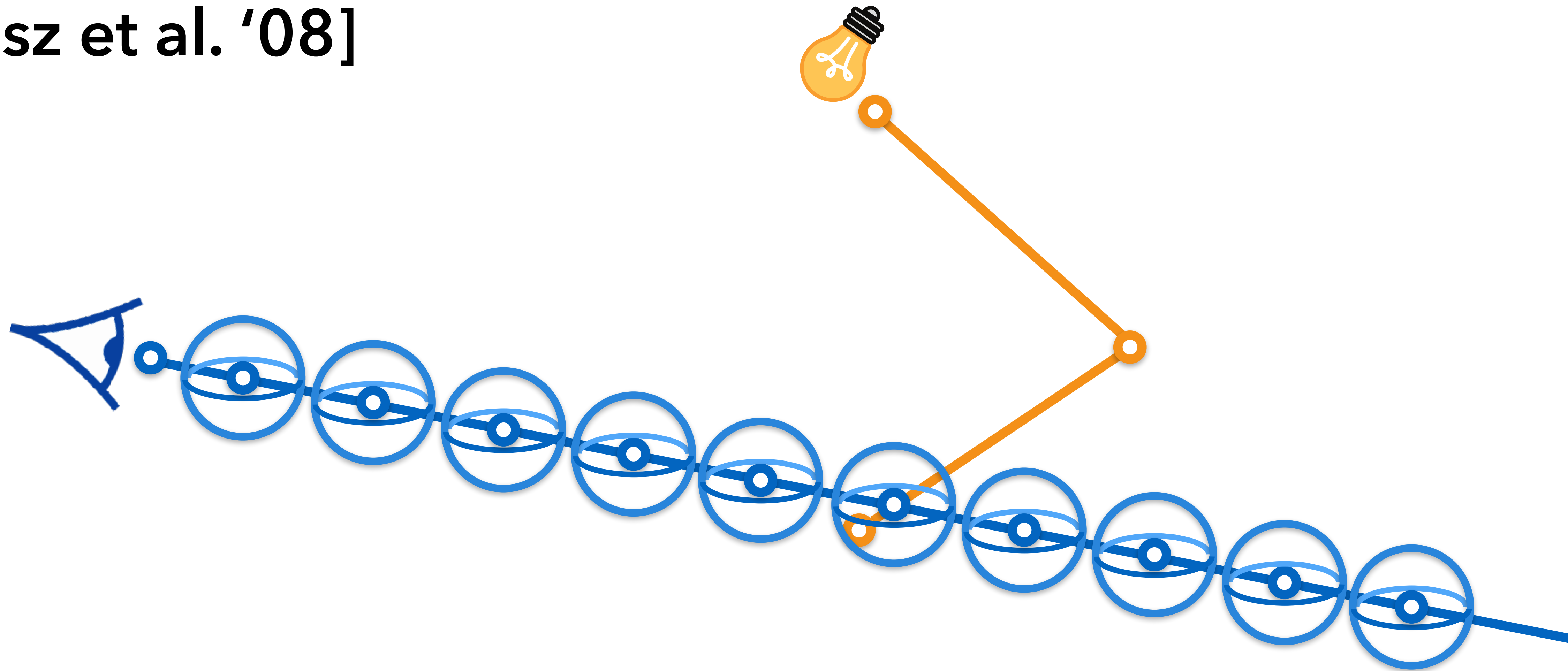
# Sensor beam

[Jarosz et al. '08]



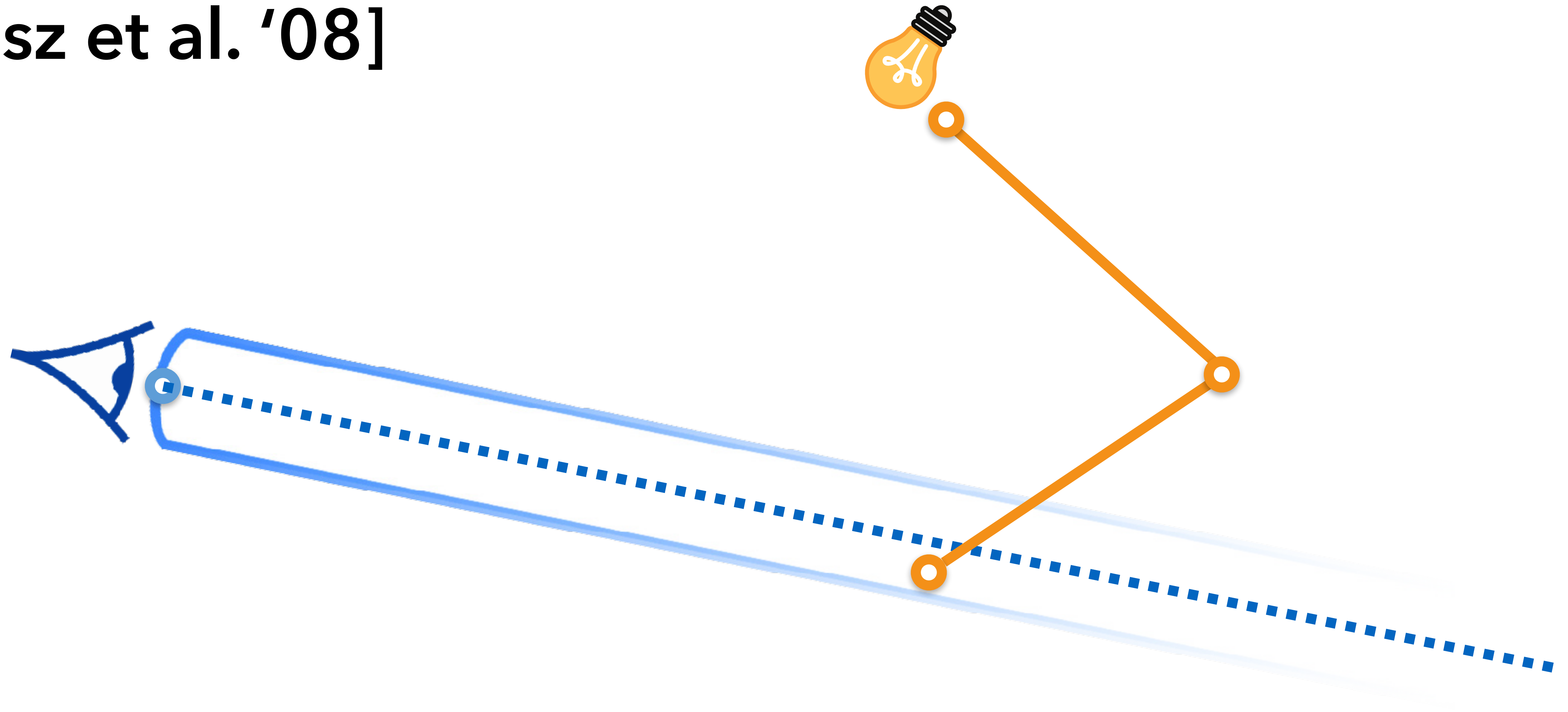
# Sensor beam

[Jarosz et al. '08]



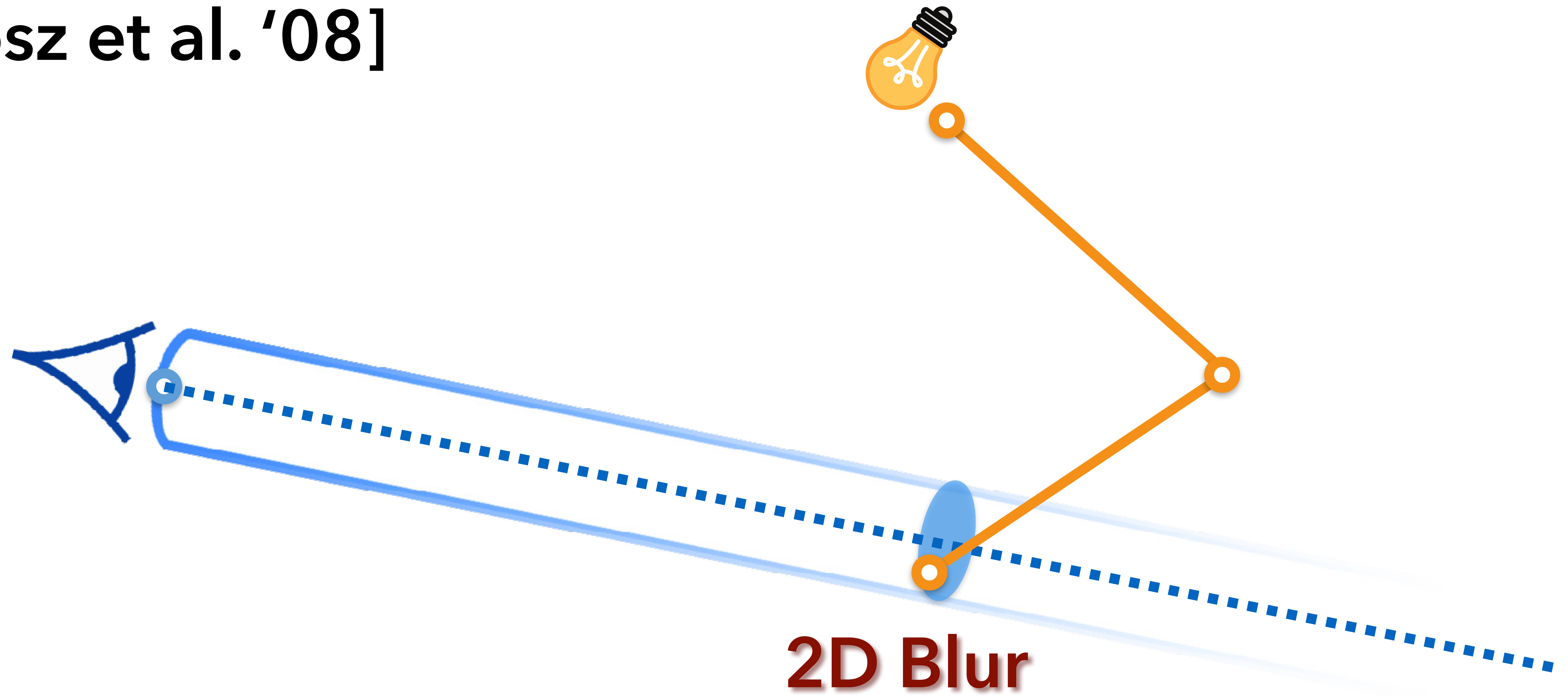
# Sensor beam

[Jarosz et al. '08]



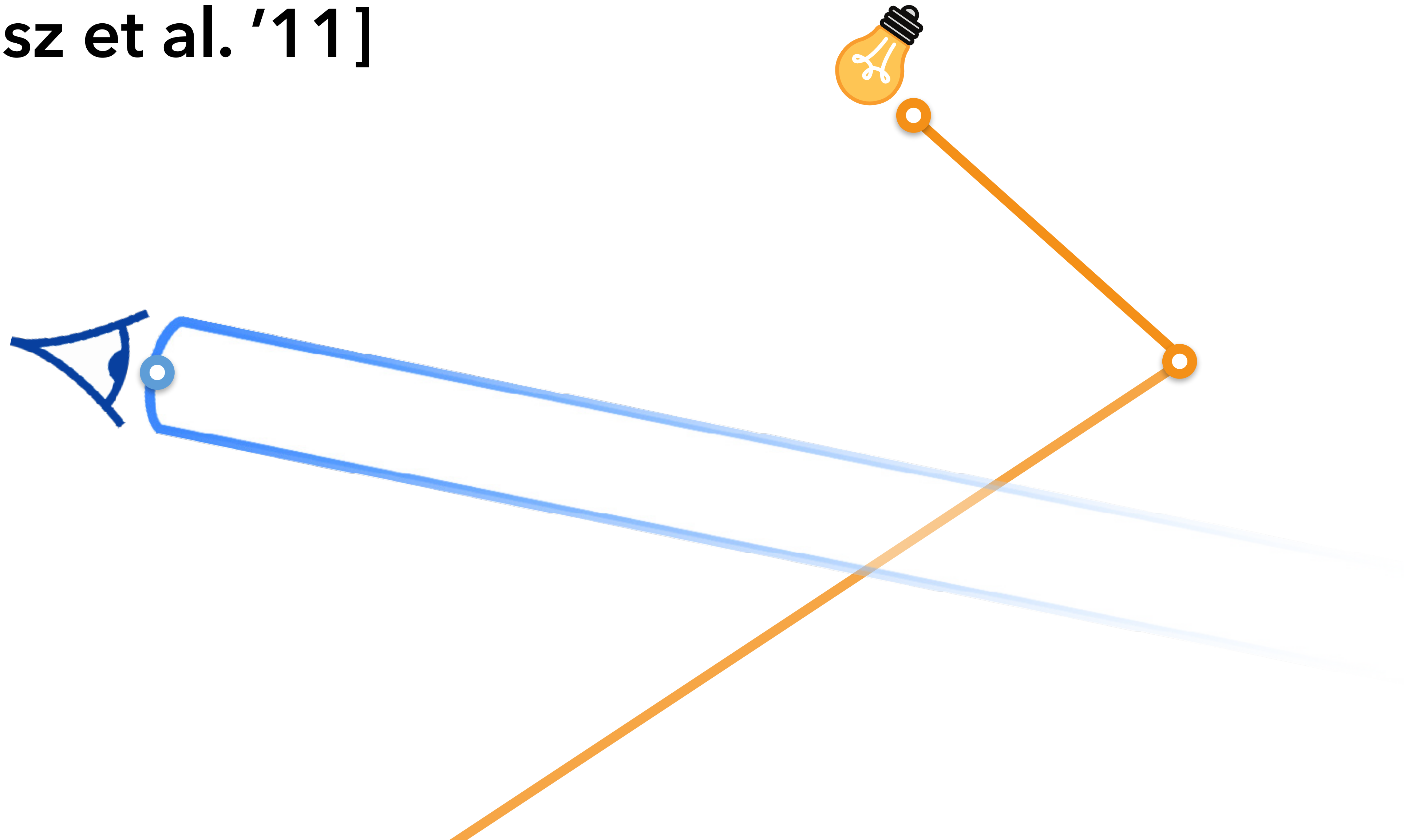
# Sensor beam

[Jarosz et al. '08]



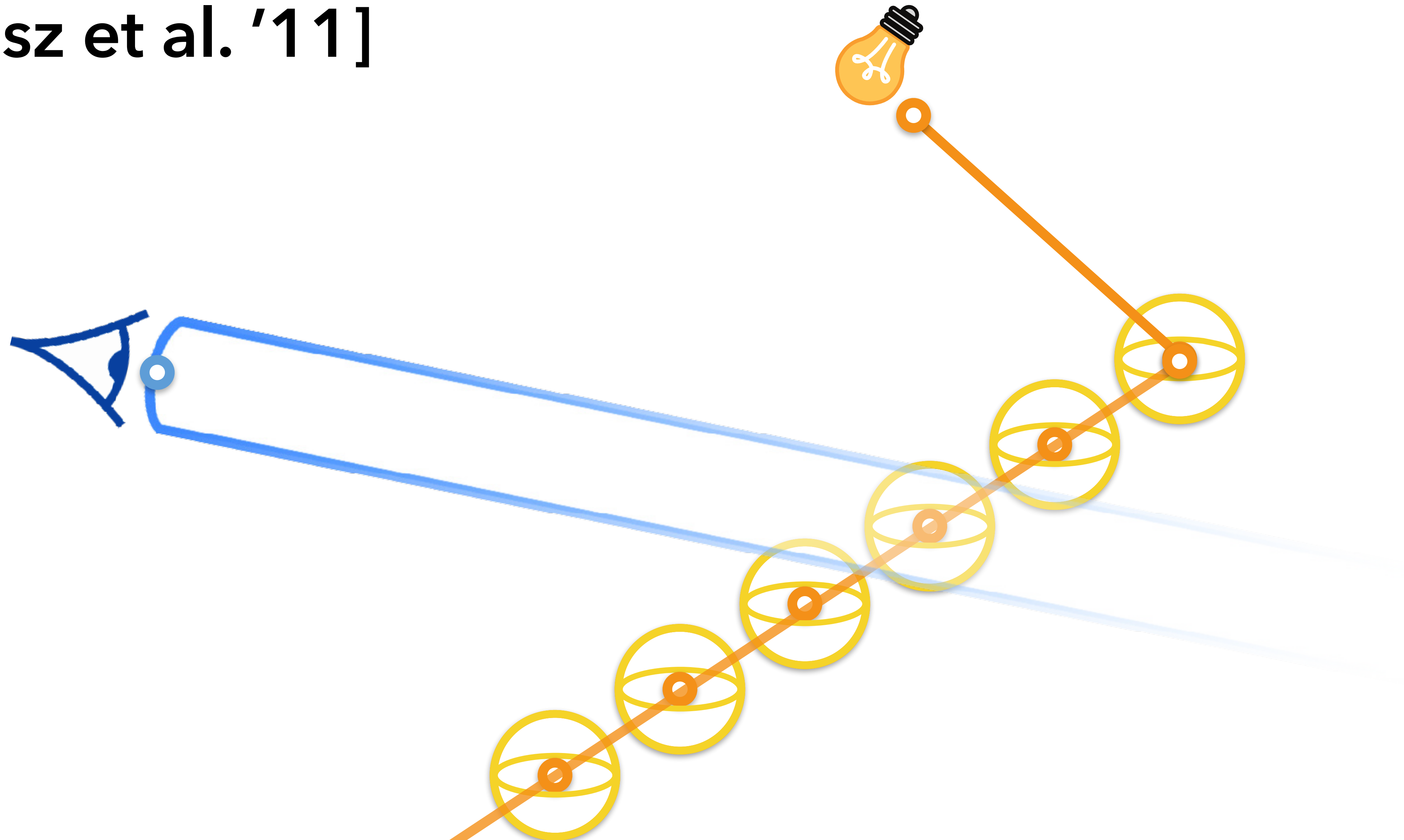
# Photon beam

[Jarosz et al. '11]



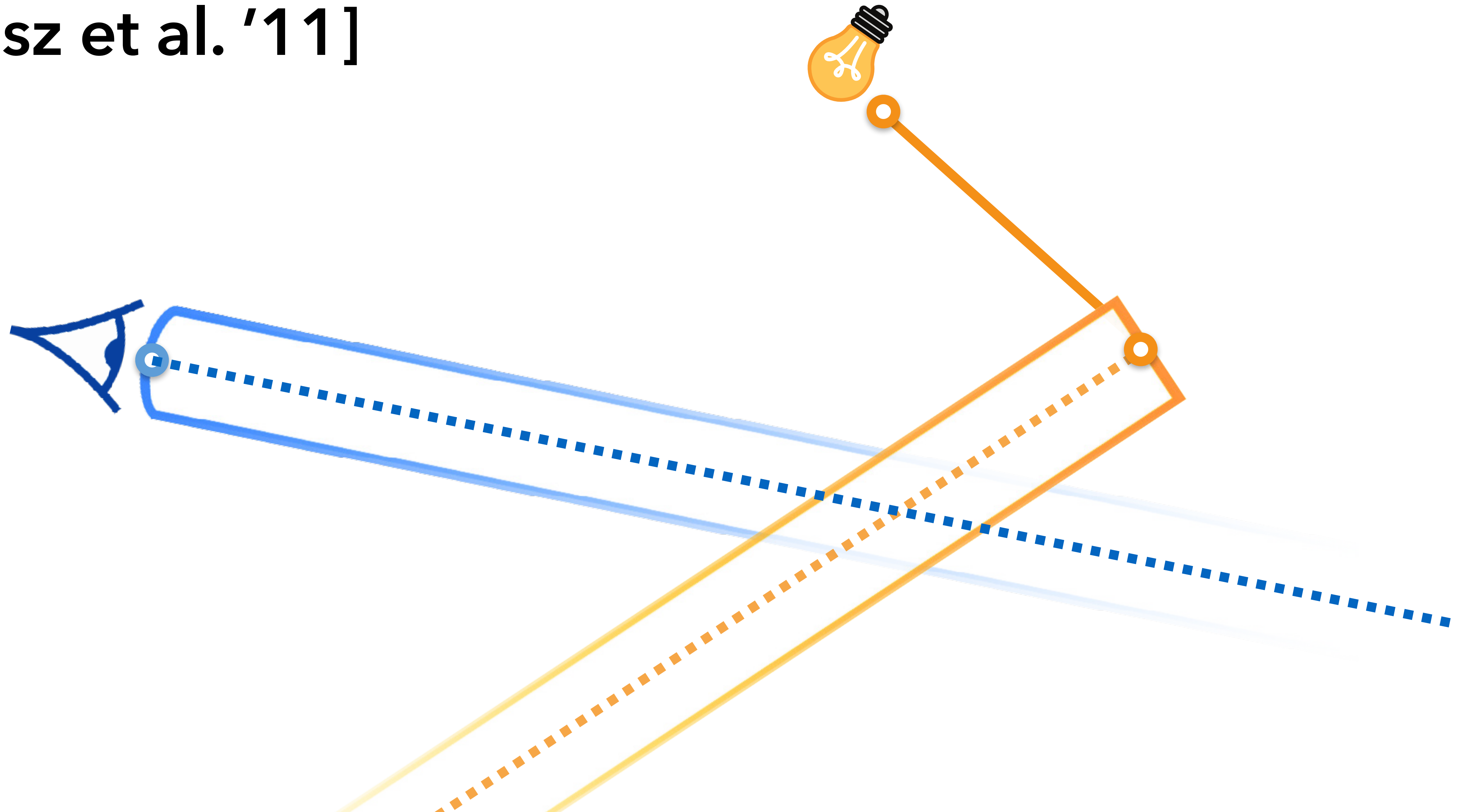
# Photon beam

[Jarosz et al. '11]



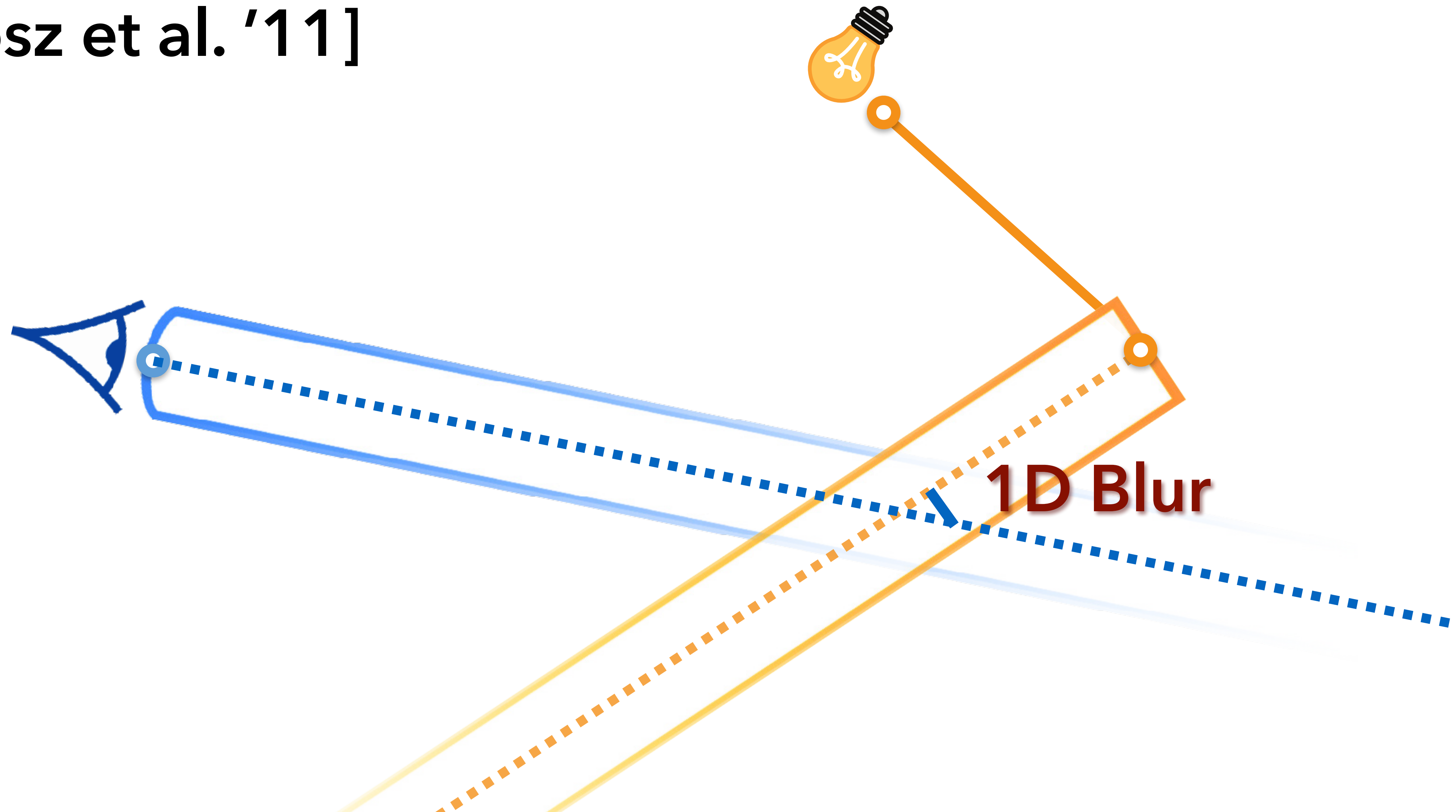
# Photon beam

[Jarosz et al. '11]



# Photon beam

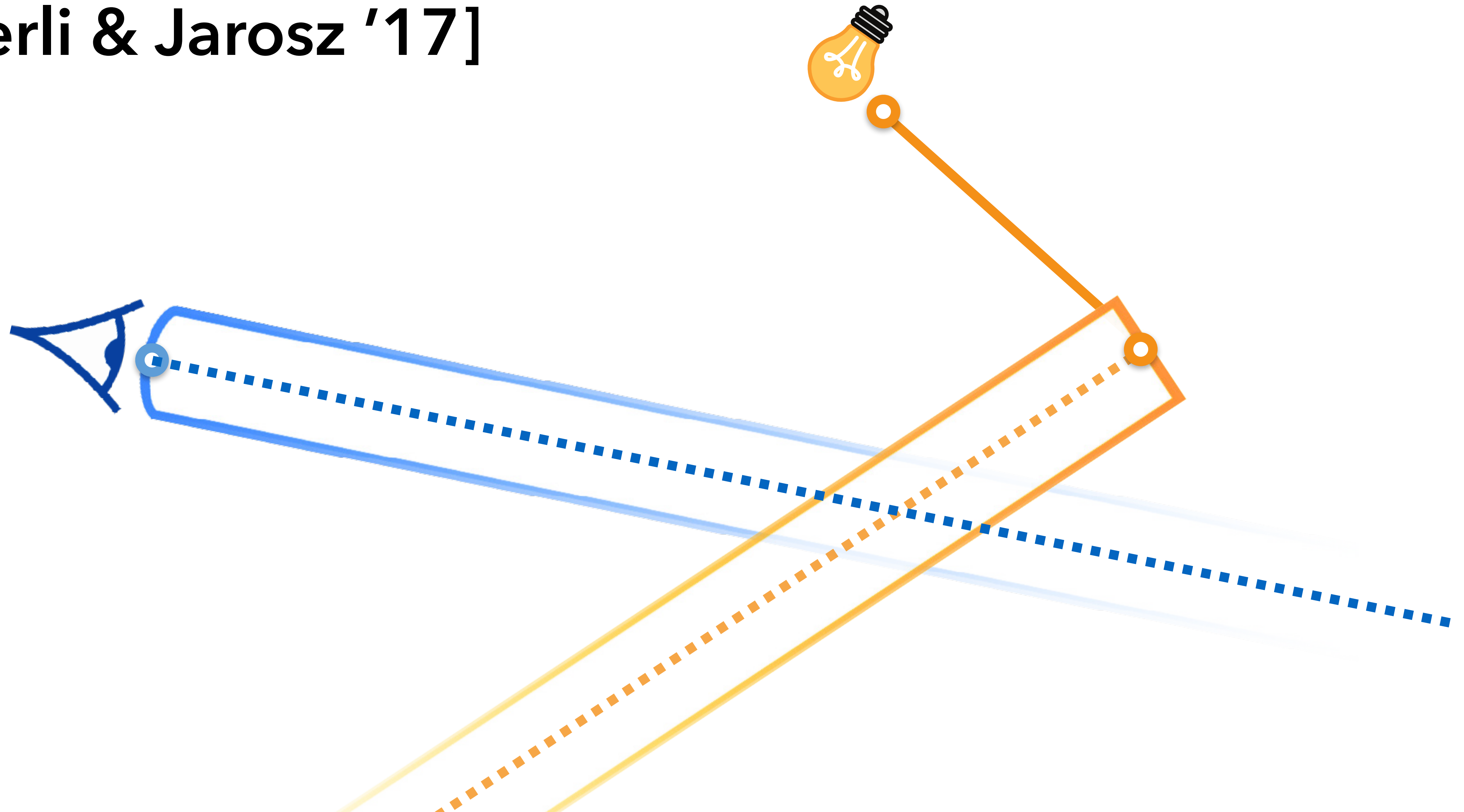
[Jarosz et al. '11]





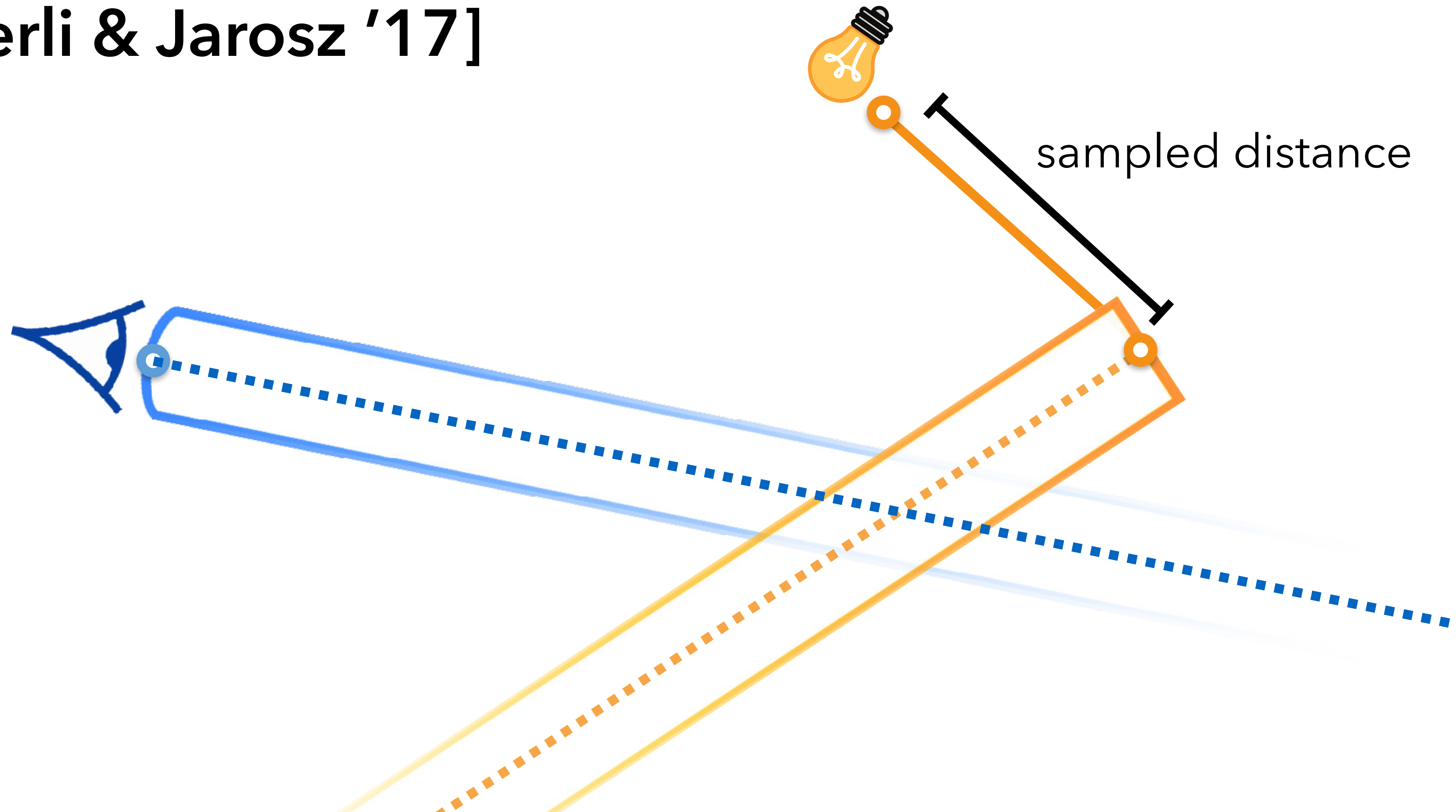
# Photon beam

[Bitterli & Jarosz '17]



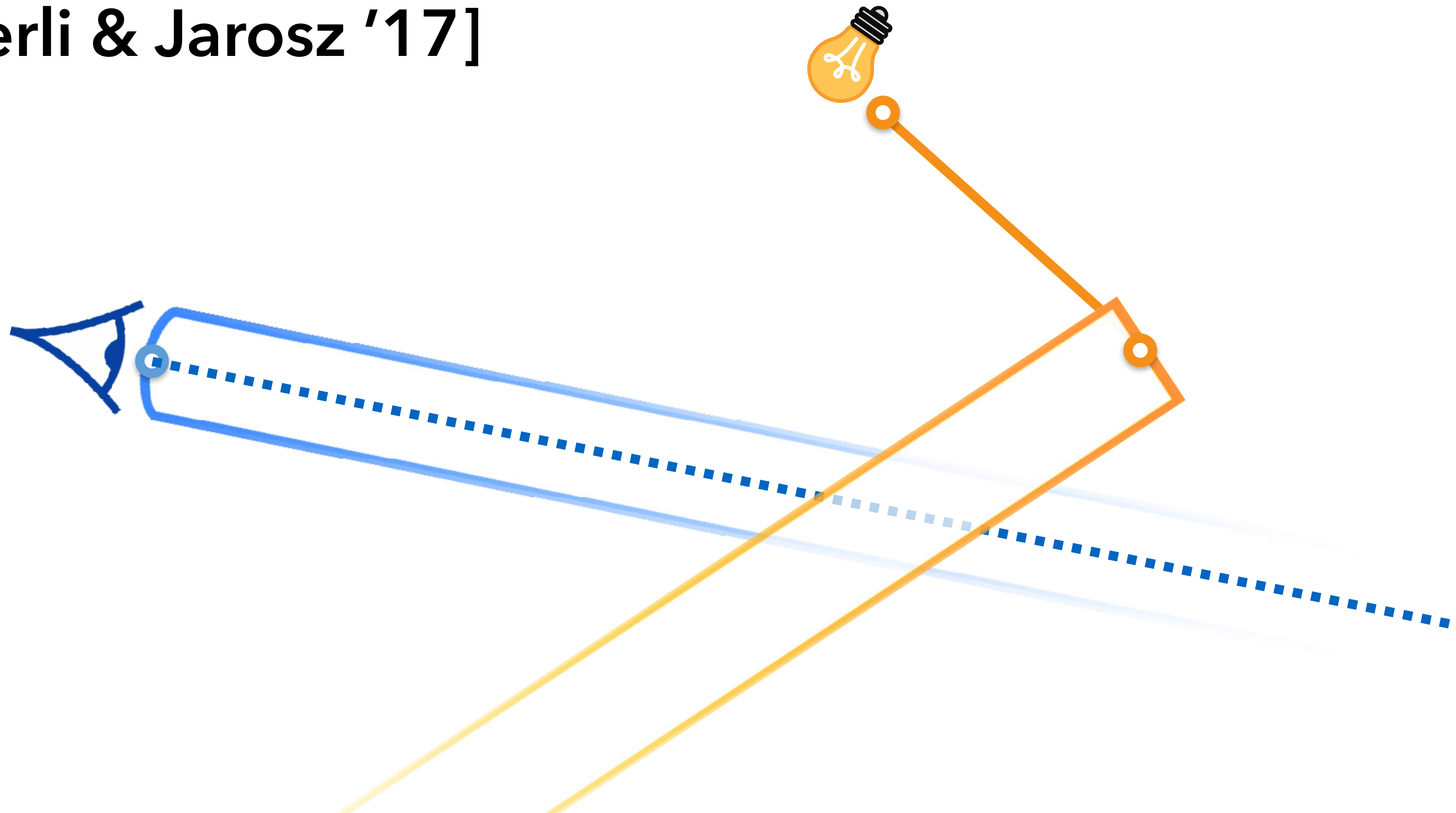
# Photon beam

[Bitterli & Jarosz '17]



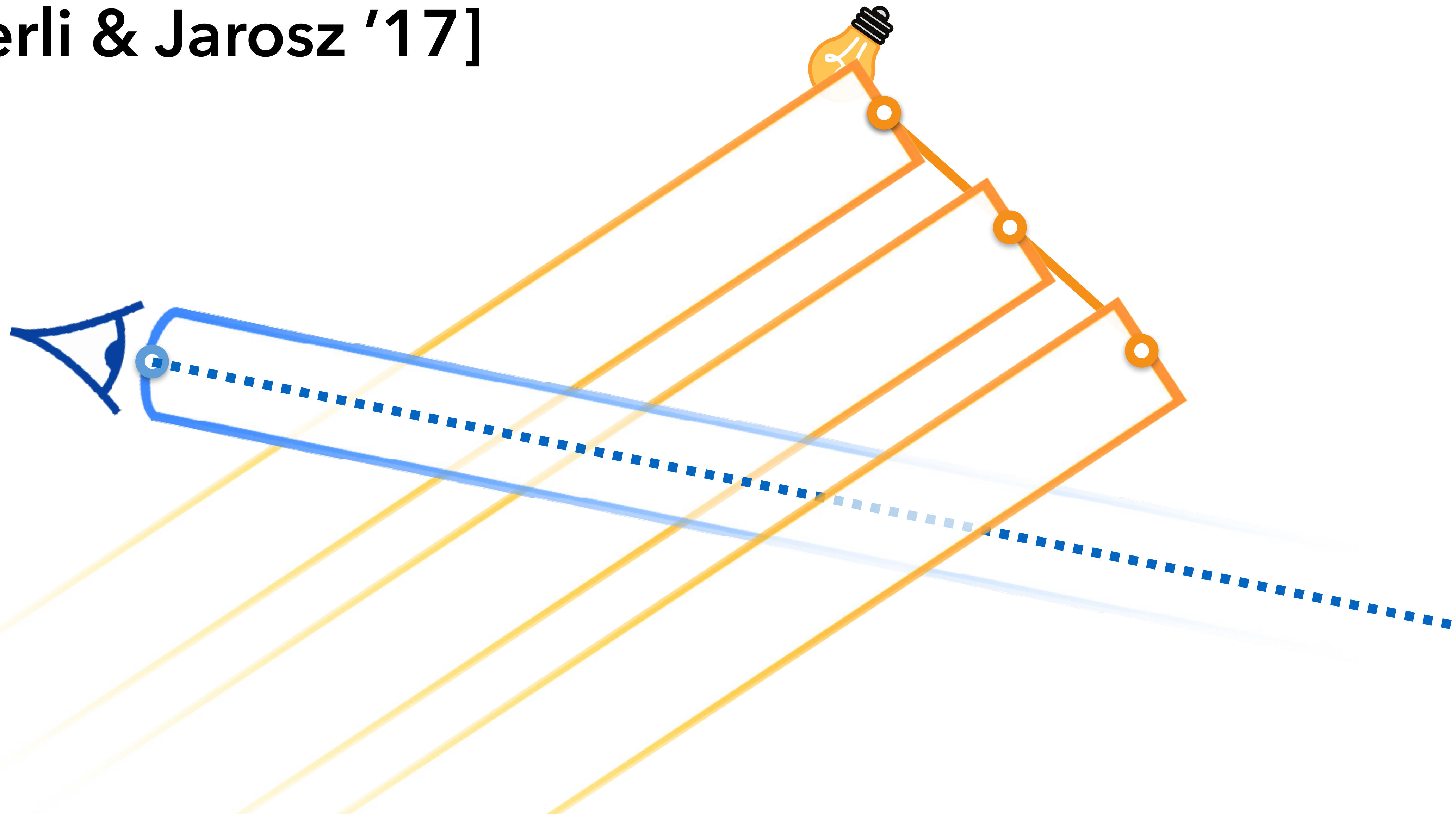
# Photon plane

[Bitterli & Jarosz '17]



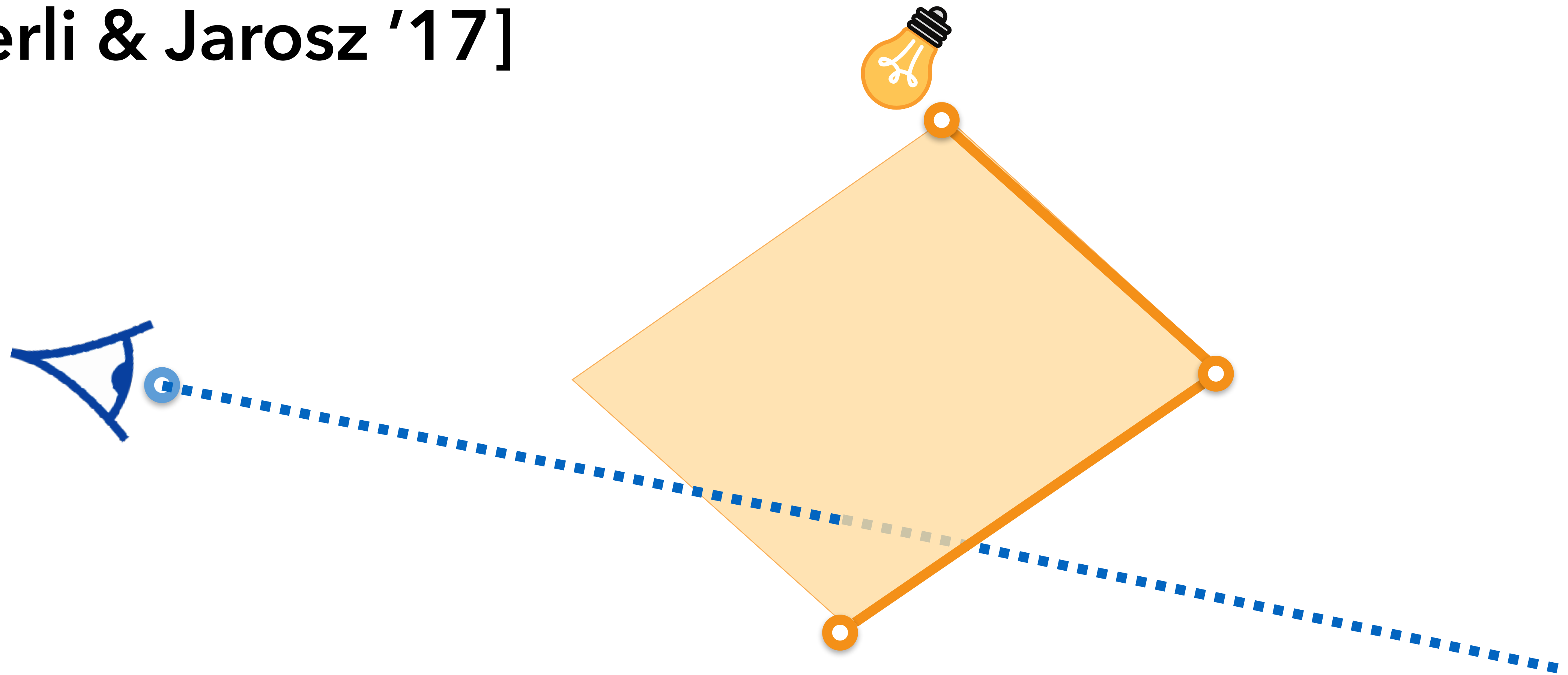
# Photon plane

[Bitterli & Jarosz '17]



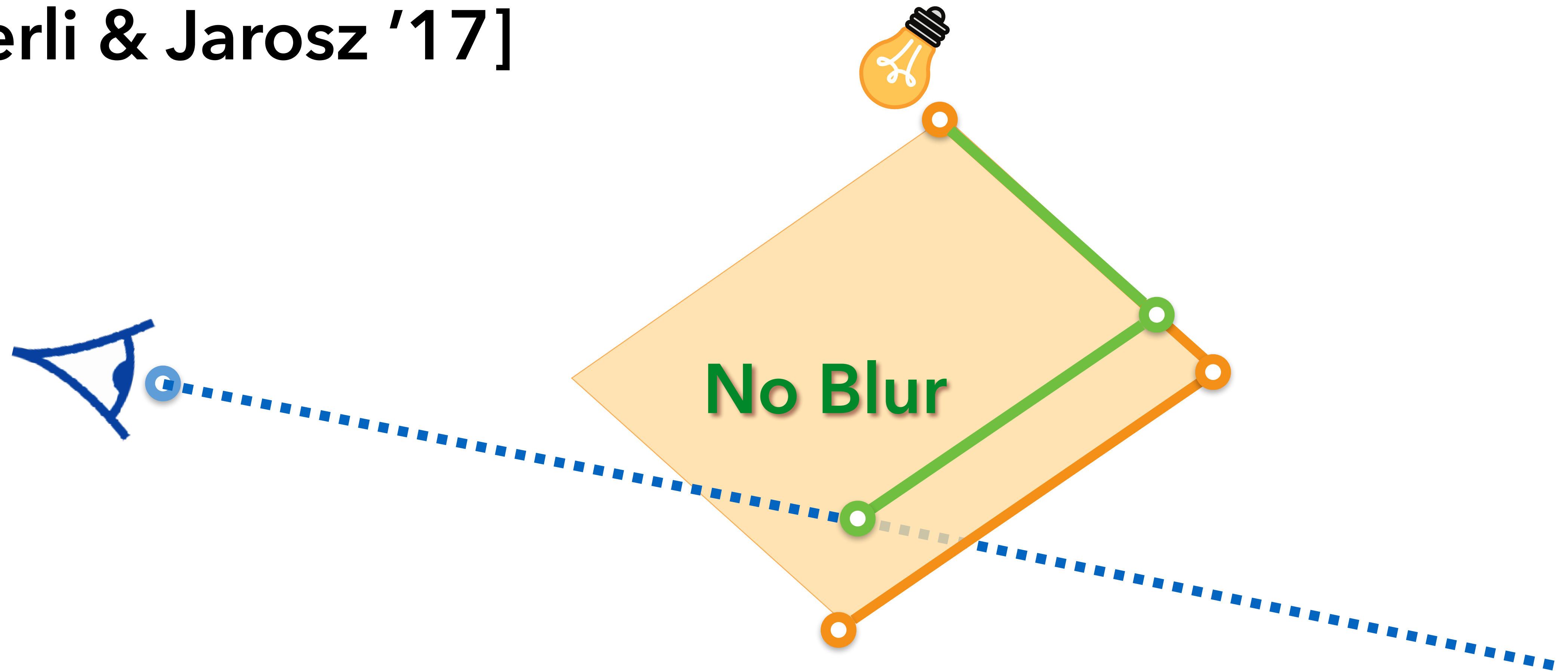
# Photon plane

[Bitterli & Jarosz '17]



# Photon plane

[Bitterli & Jarosz '17]



**Photon Points**  
(biased)



**Photon Beams**  
(biased)



**Photon Planes**  
(unbiased)



**Photon Points**  
(biased)



**Photon Beams**  
(biased)

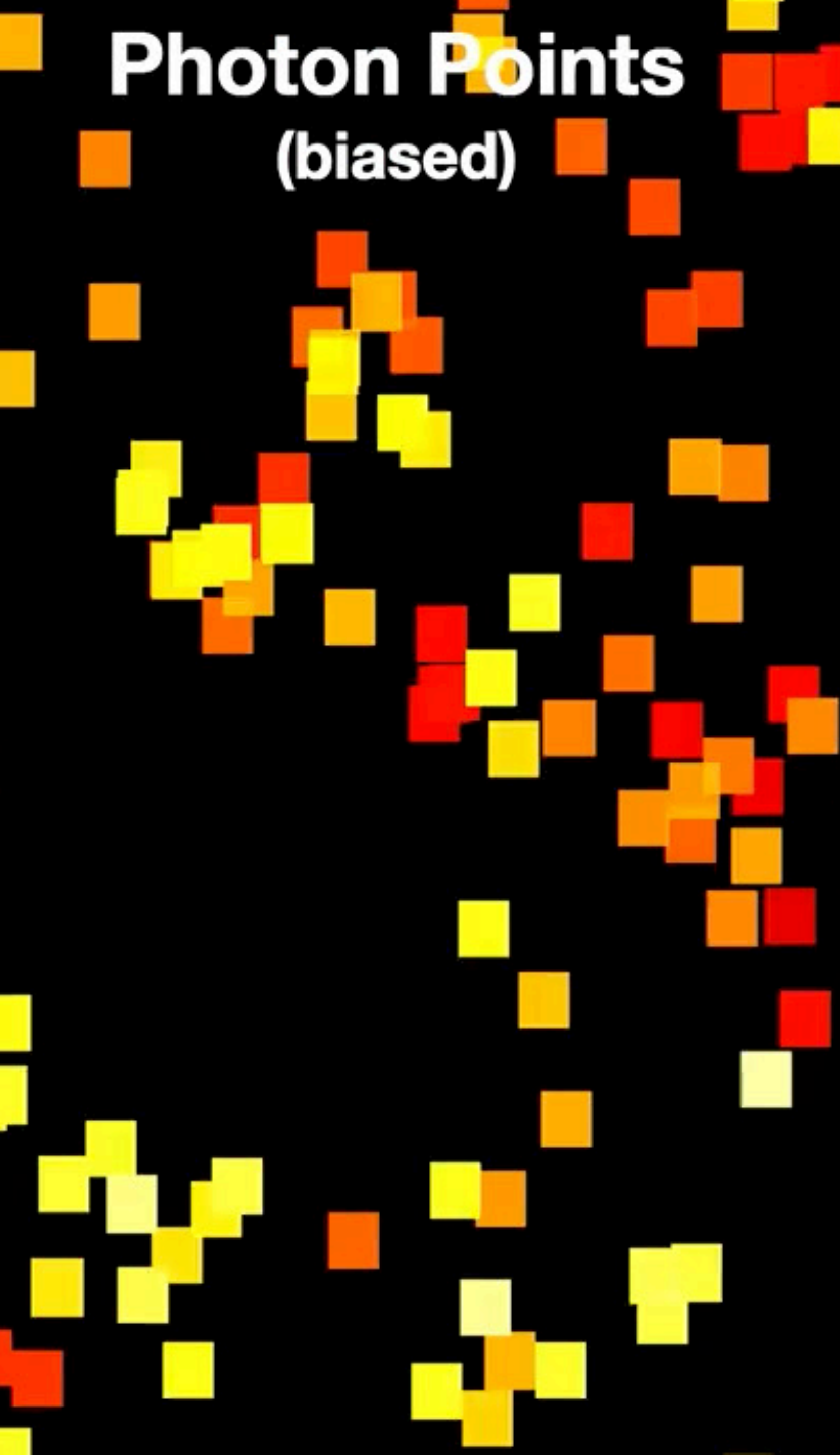


**Photon Planes**  
(unbiased)

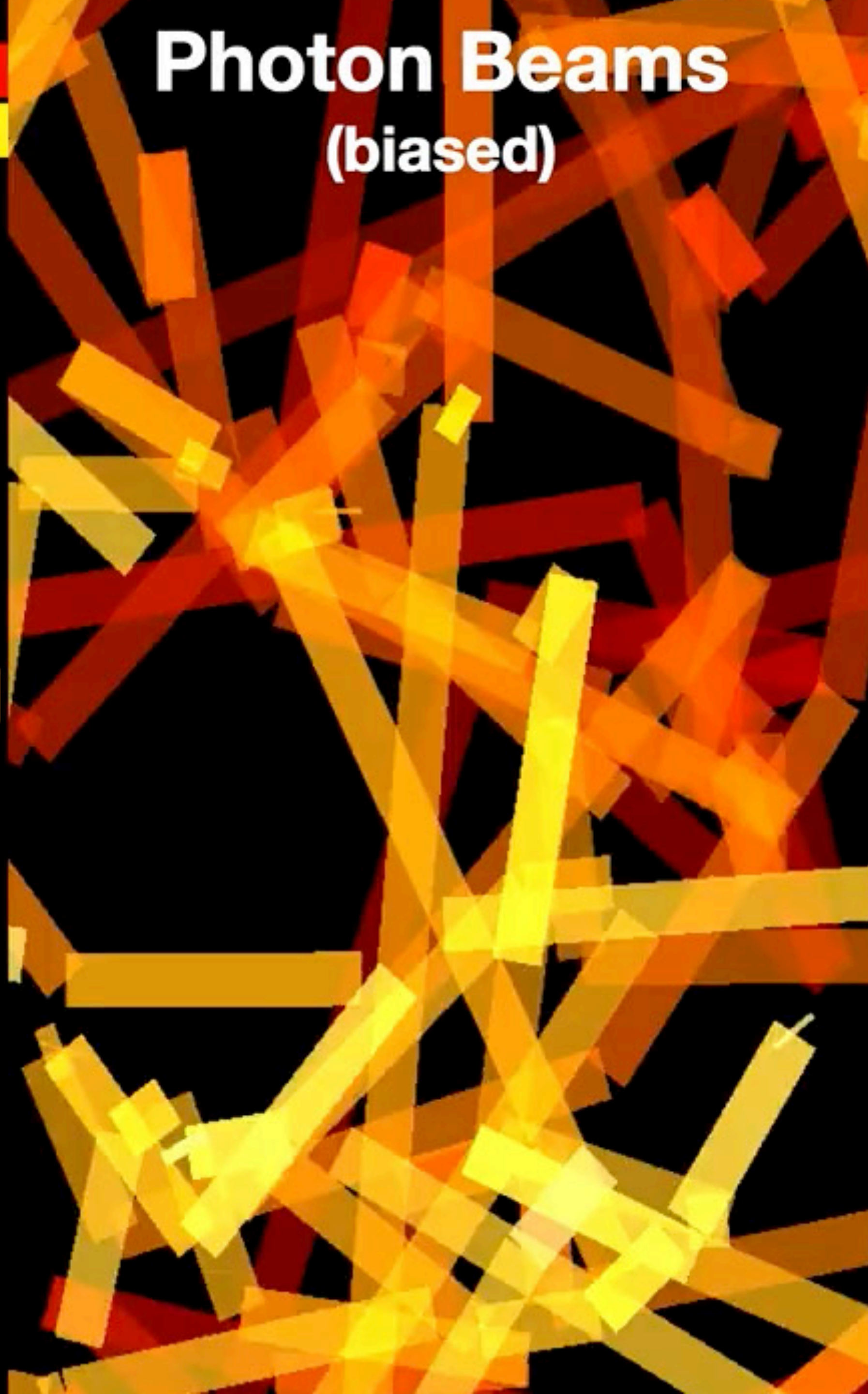




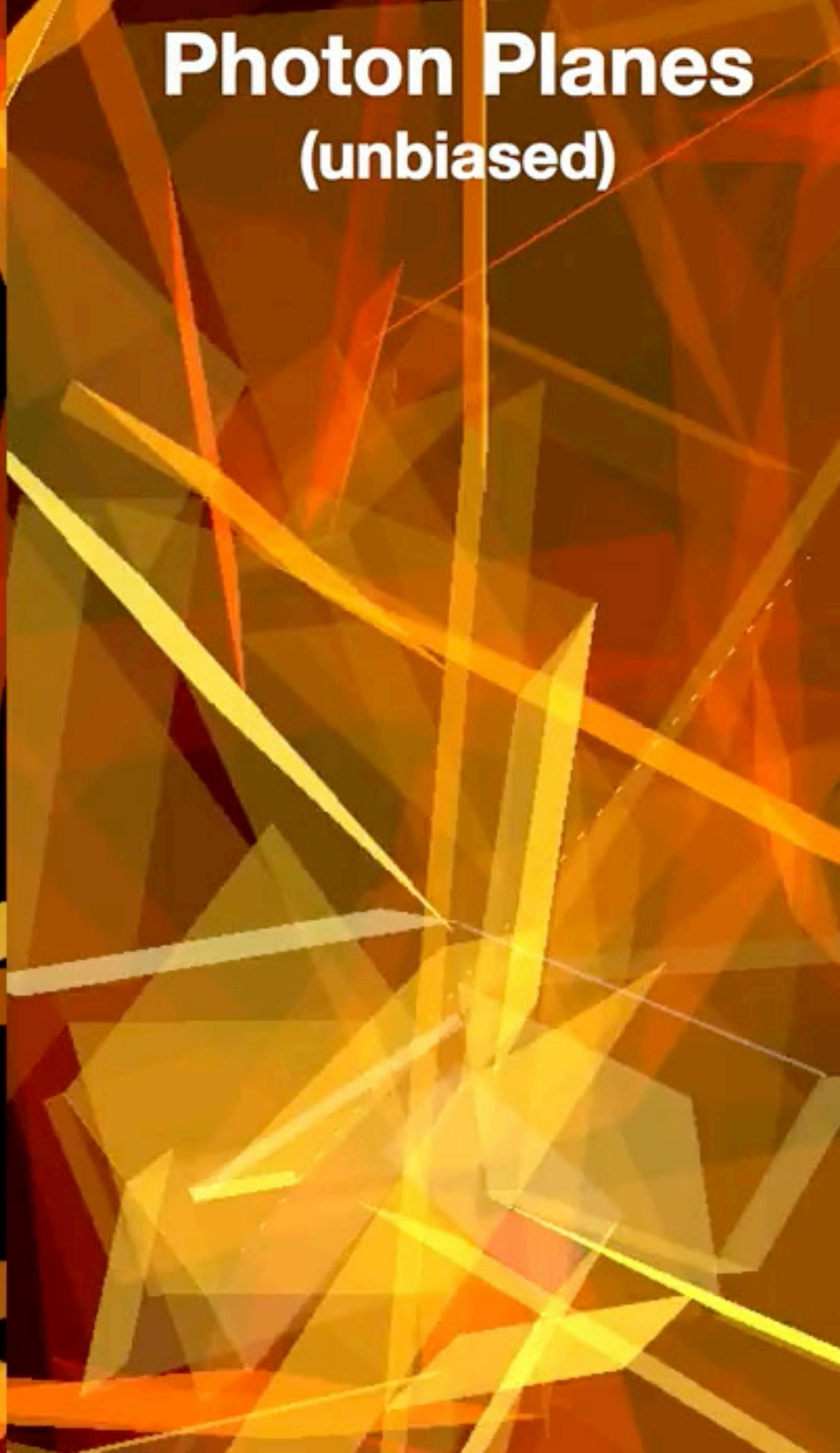
**Photon Points**  
(biased)



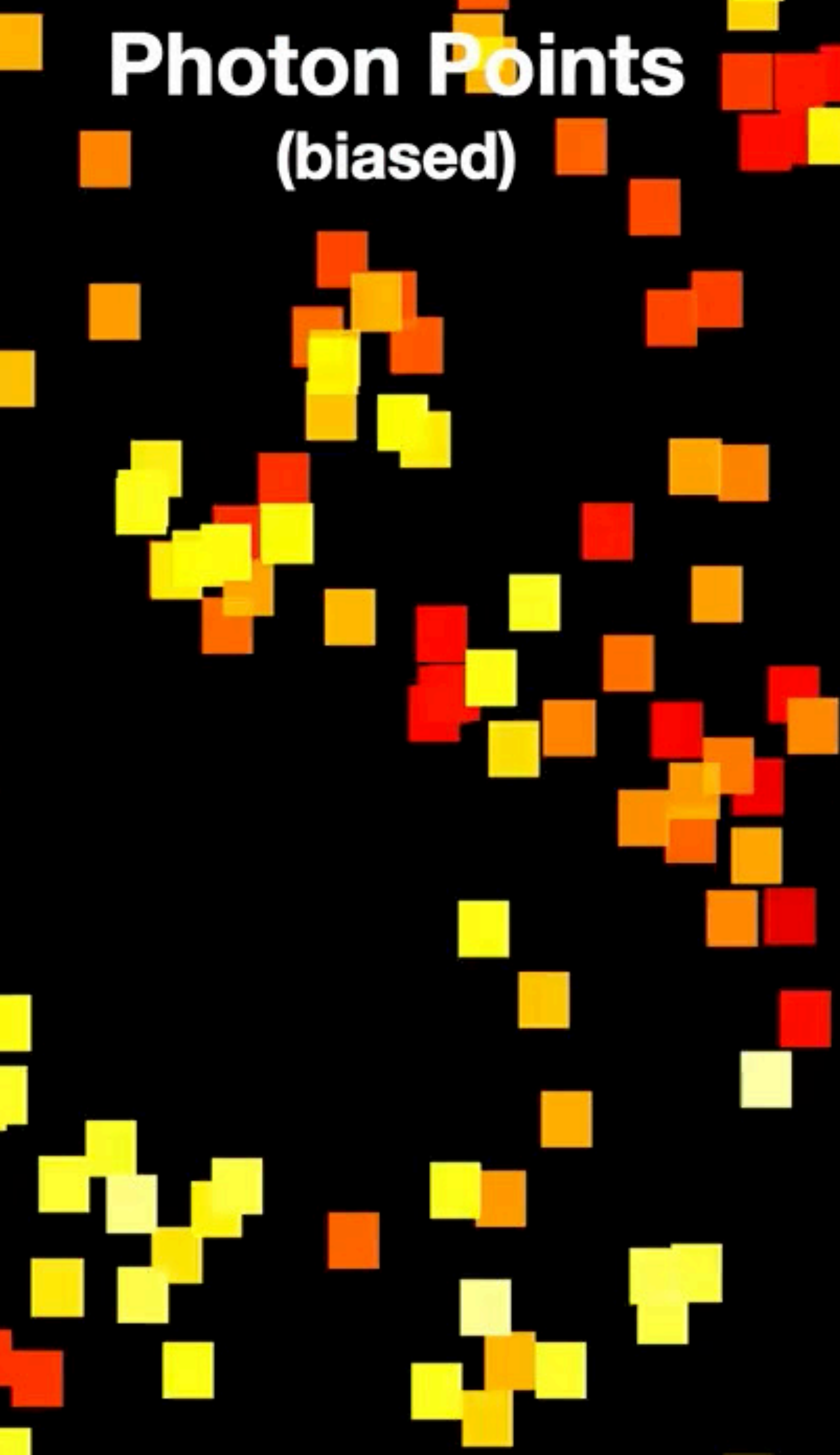
**Photon Beams**  
(biased)



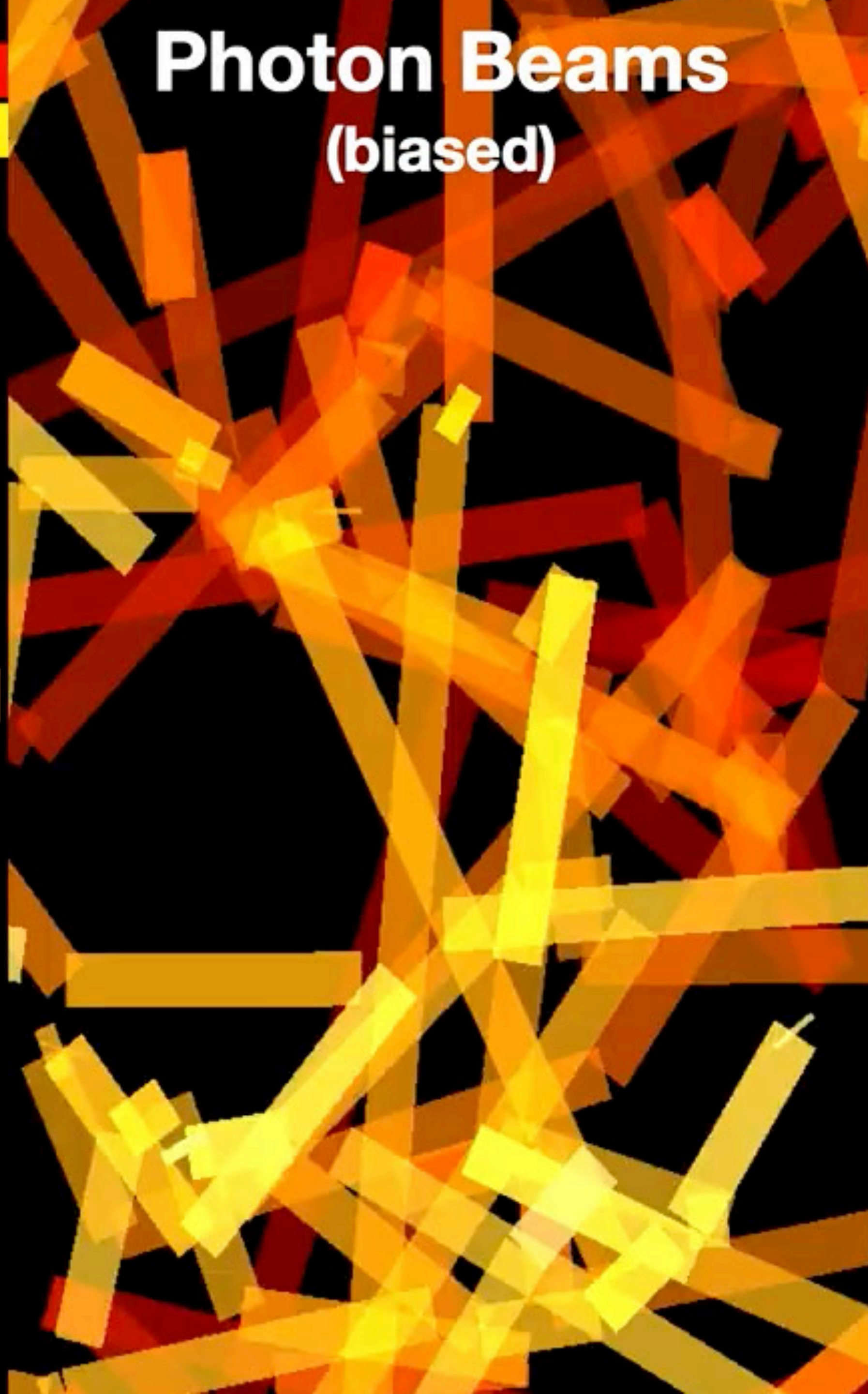
**Photon Planes**  
(unbiased)



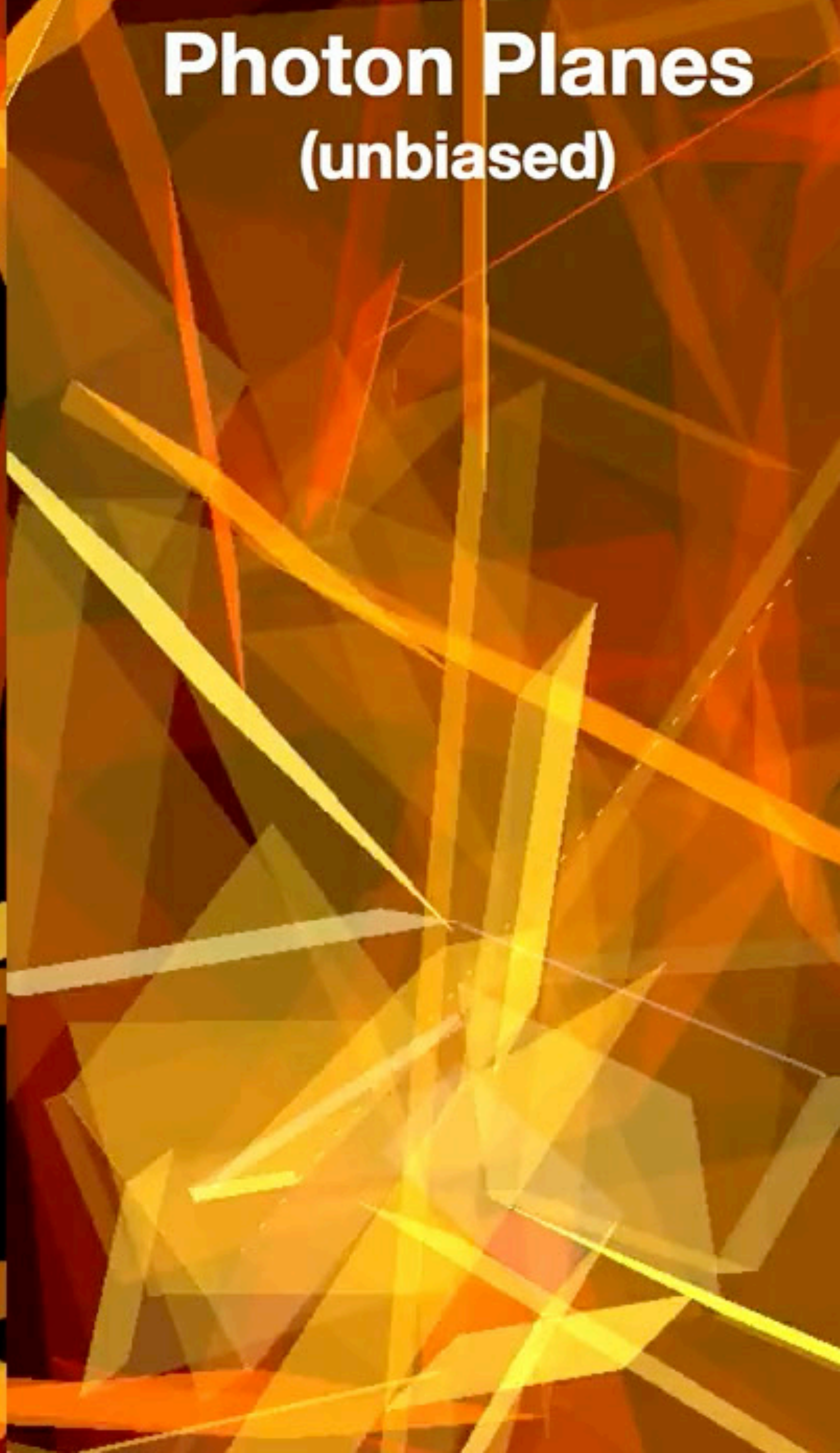
**Photon Points**  
(biased)



**Photon Beams**  
(biased)

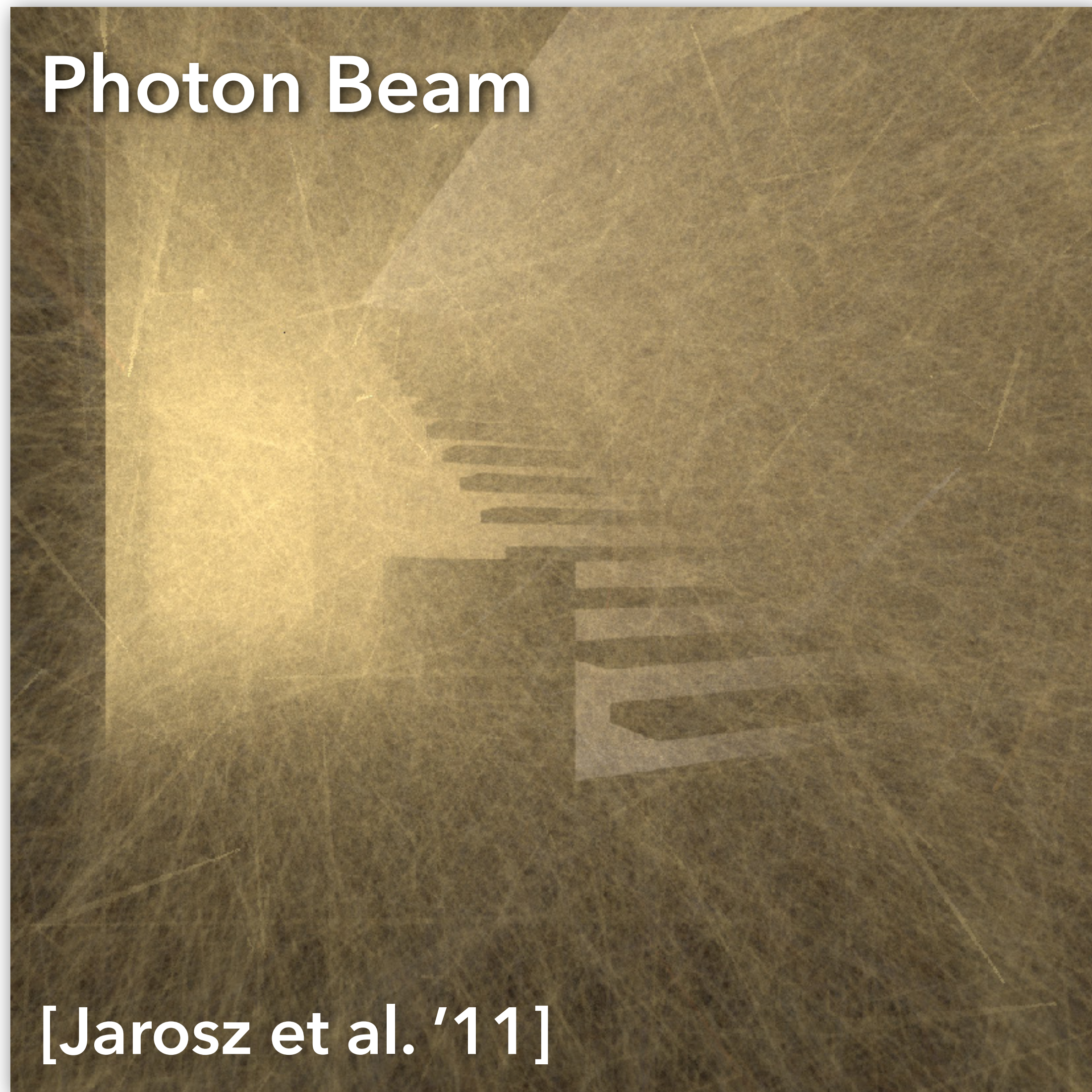


**Photon Planes**  
(unbiased)



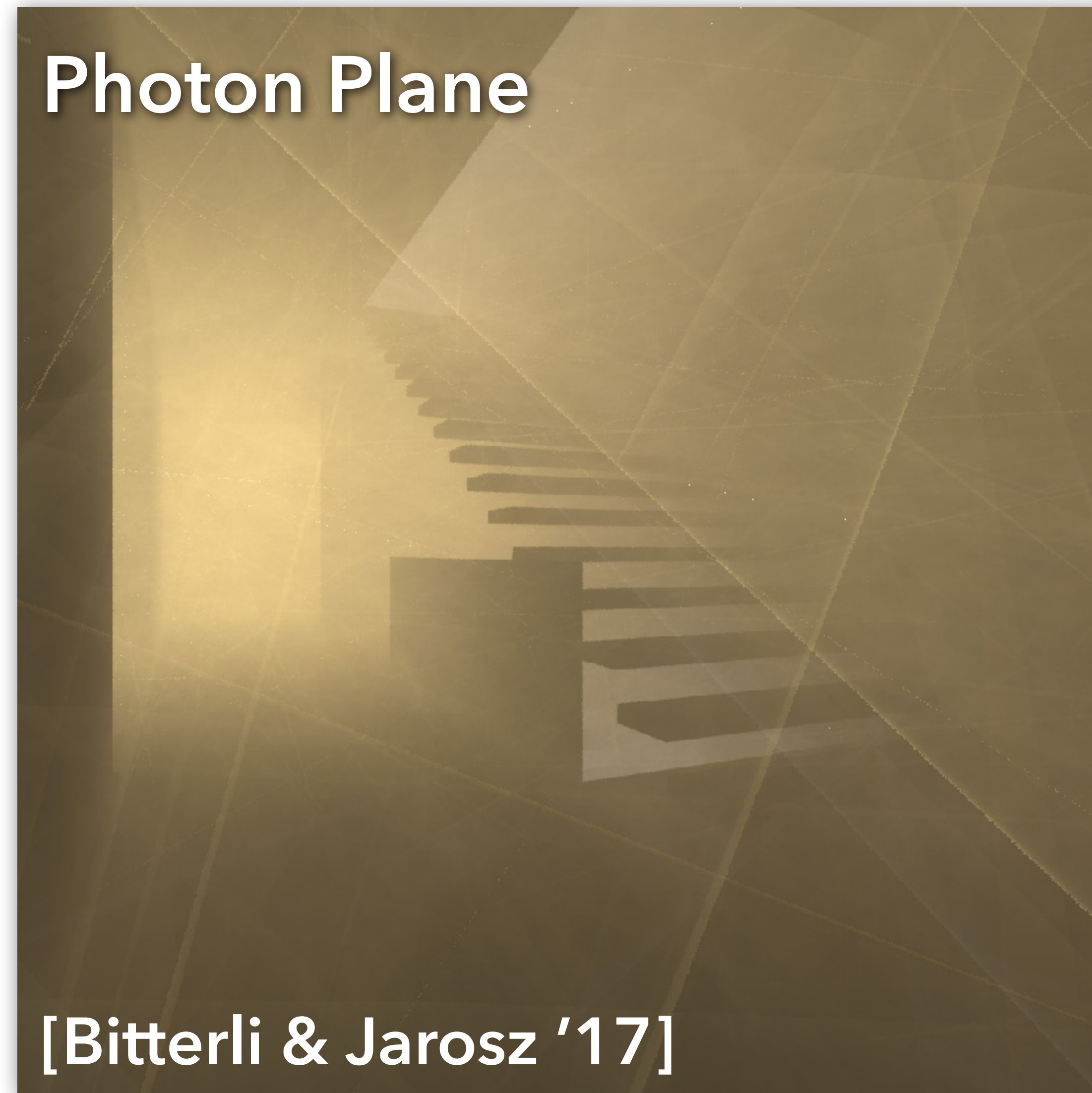
# Equal-time comparison

Photon Beam



[Jarosz et al. '11]

Photon Plane



[Bitterli & Jarosz '17]

# The limitations of photon planes

[Bitterli and Jarosz. '17]

# The limitations of photon planes

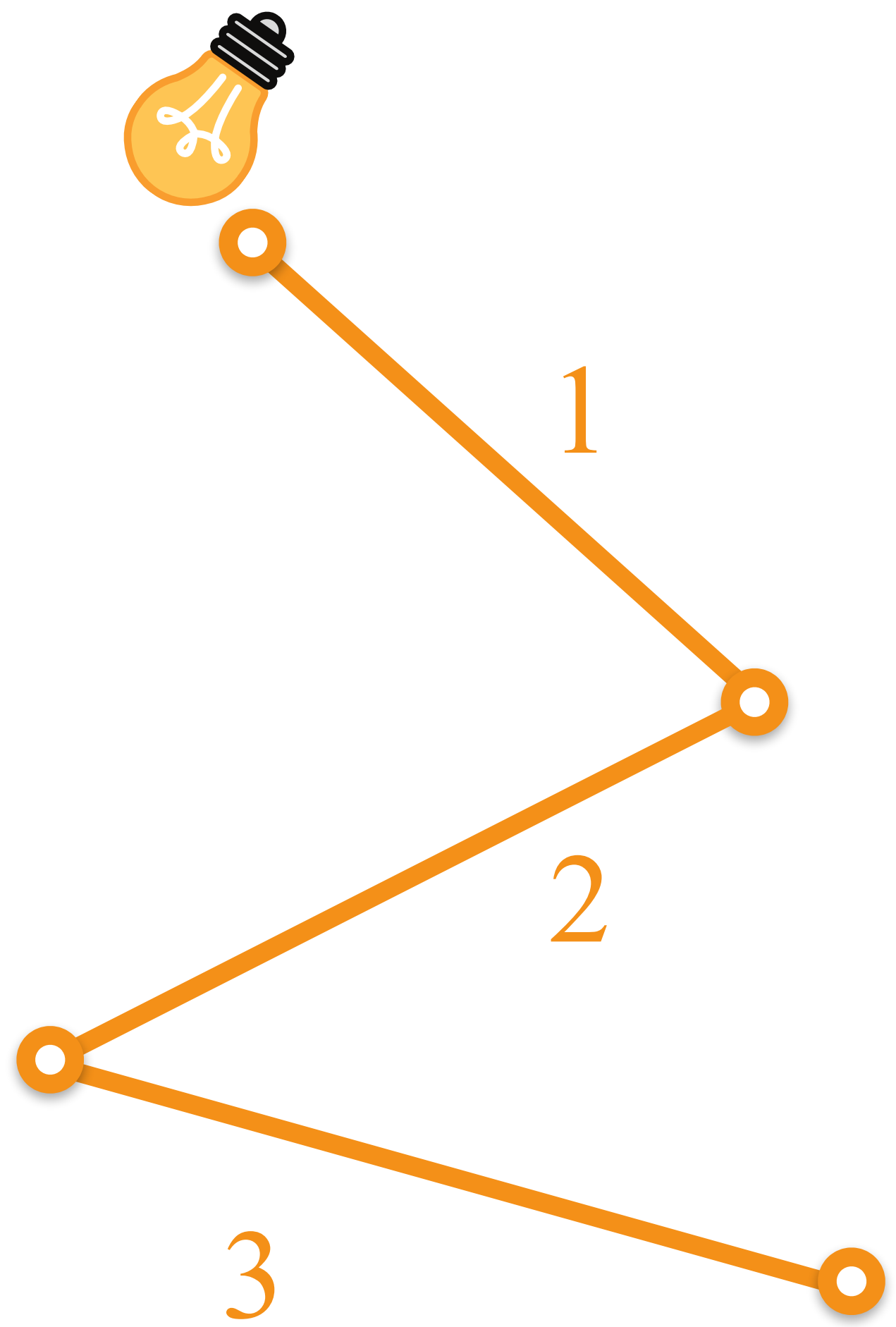
[Bitterli and Jarosz. '17]

# The limitations of photon planes

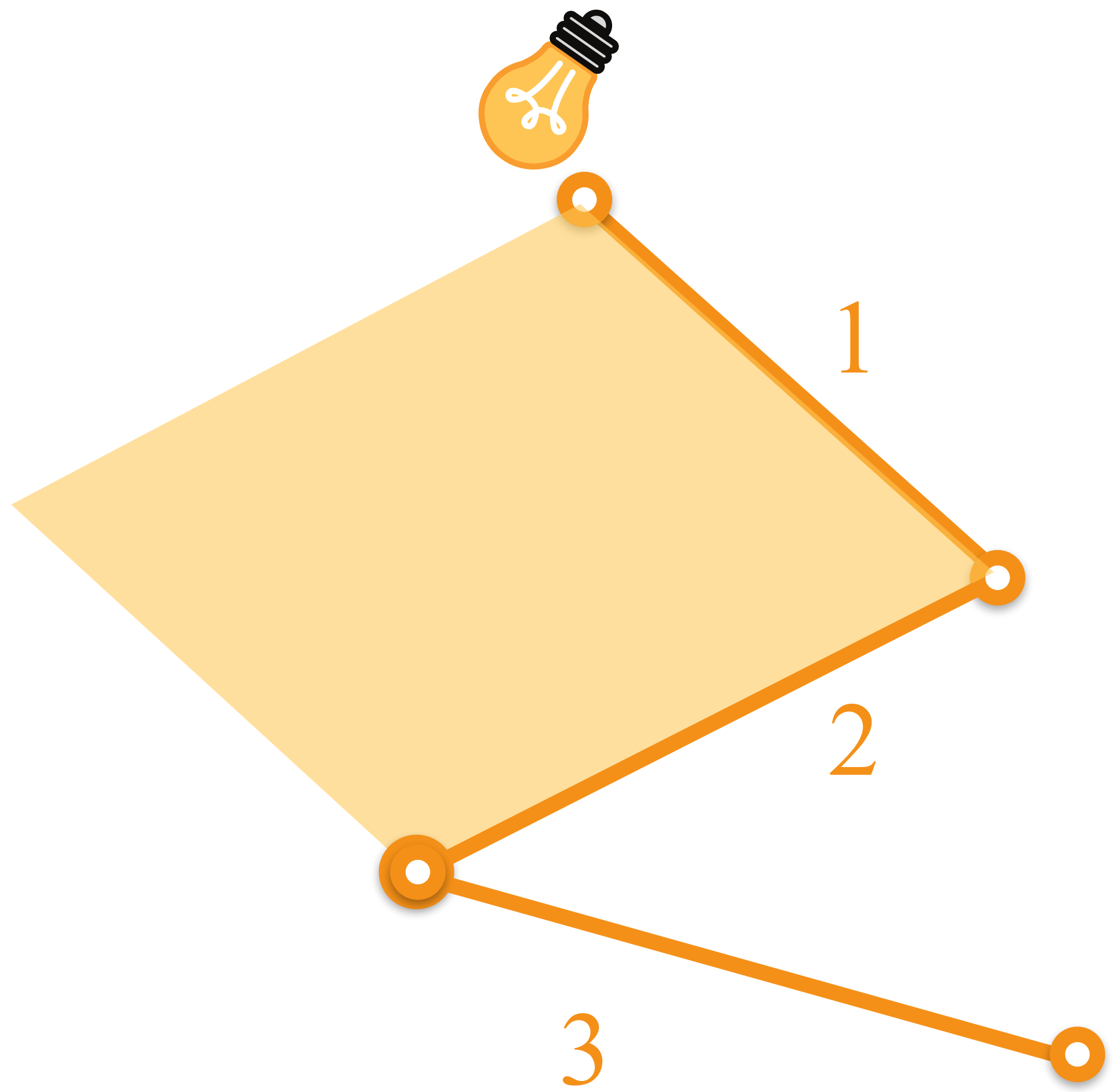
## Singularities

[Bitterli and Jarosz. '17]

# The limitations of photon planes

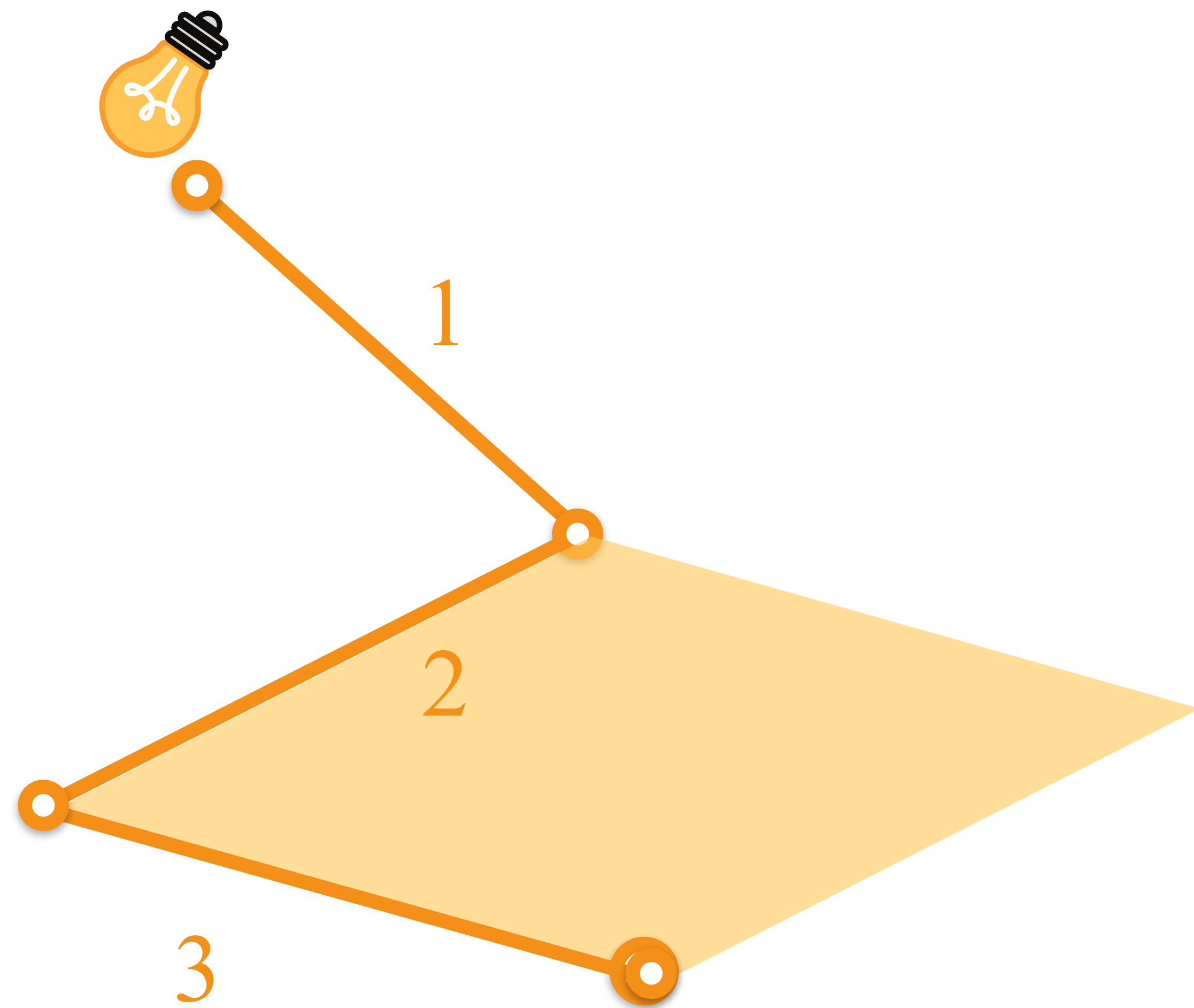


# The limitations of photon planes

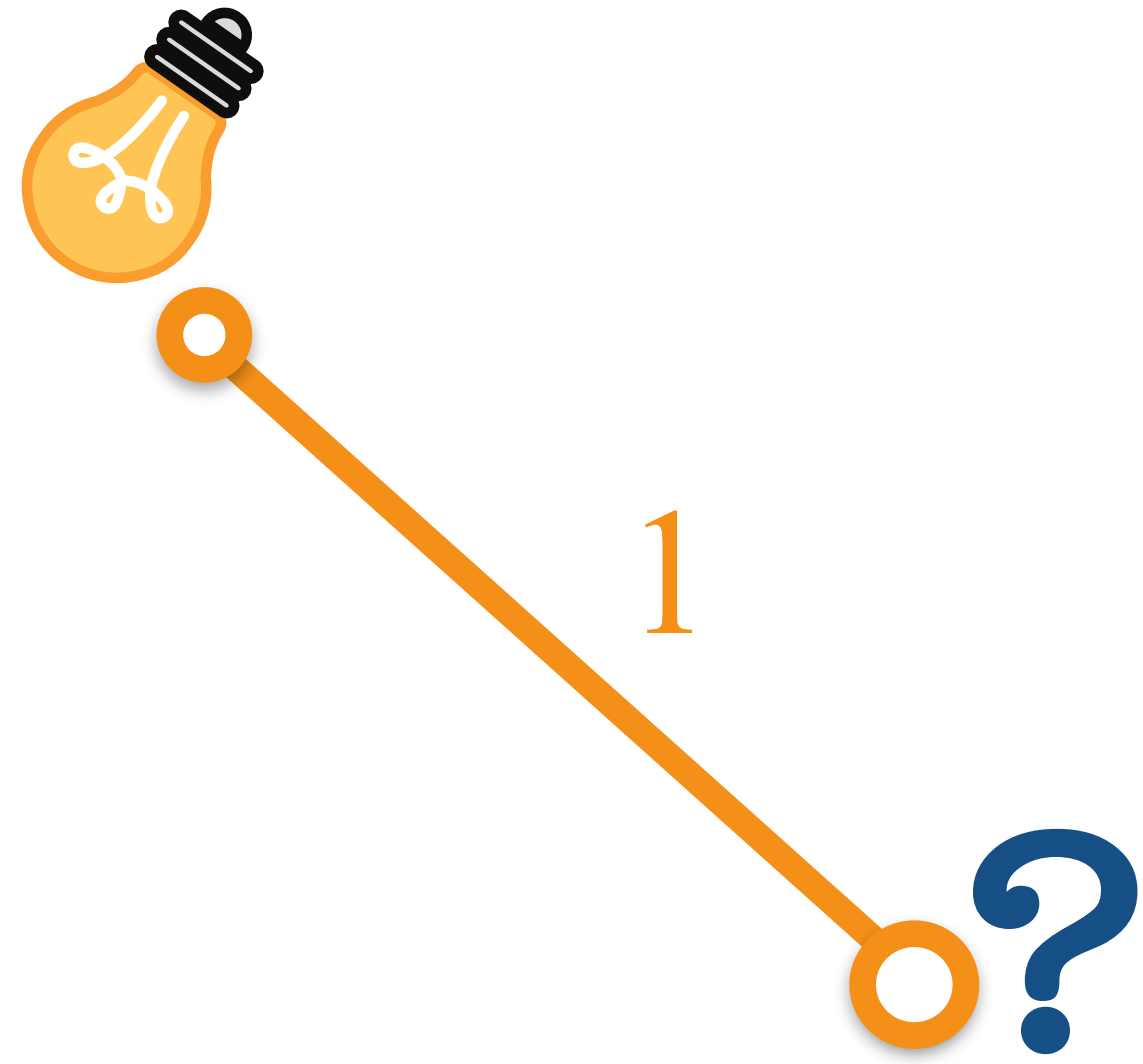




# The limitations of photon planes

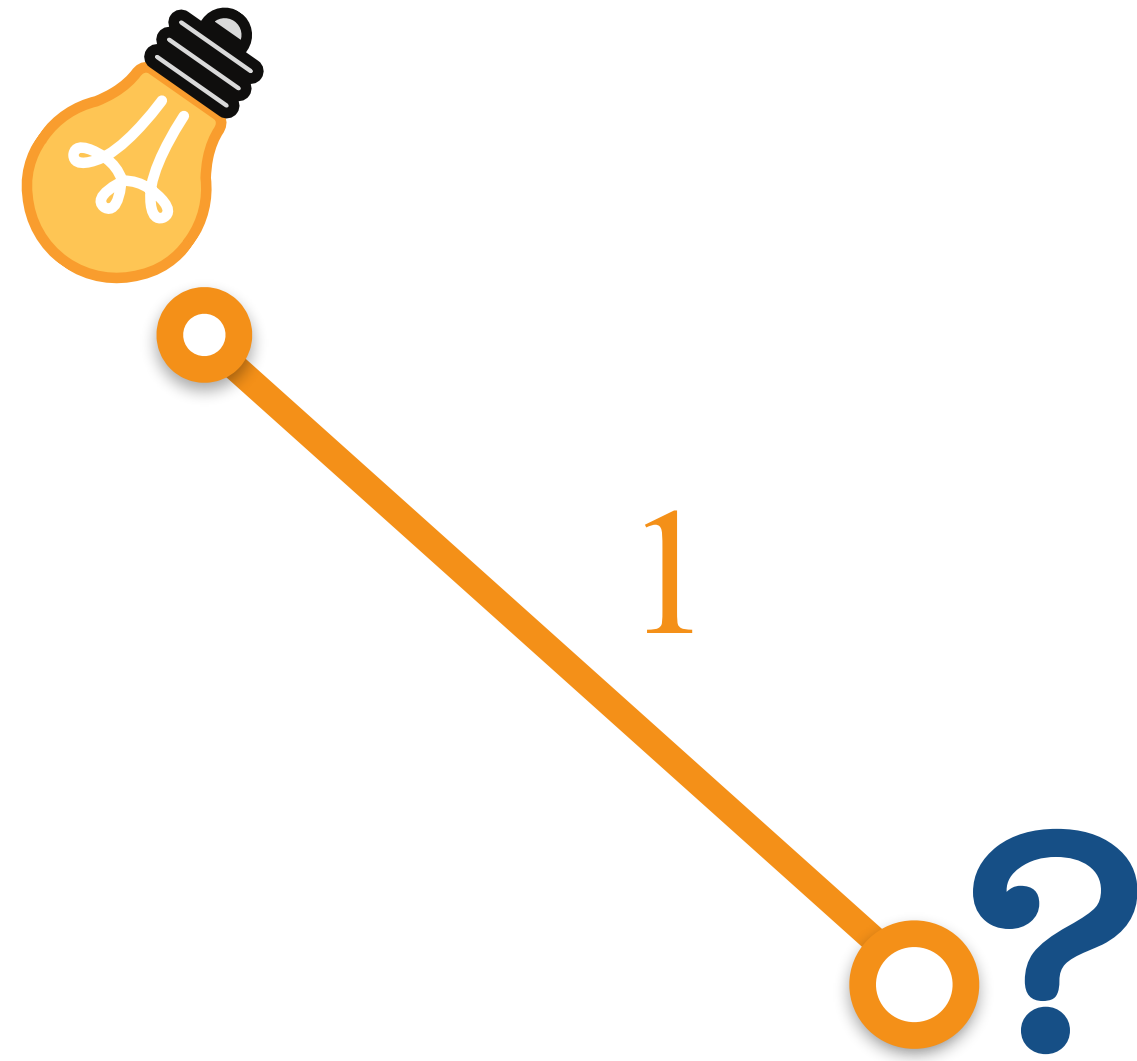


# The limitations of photon planes

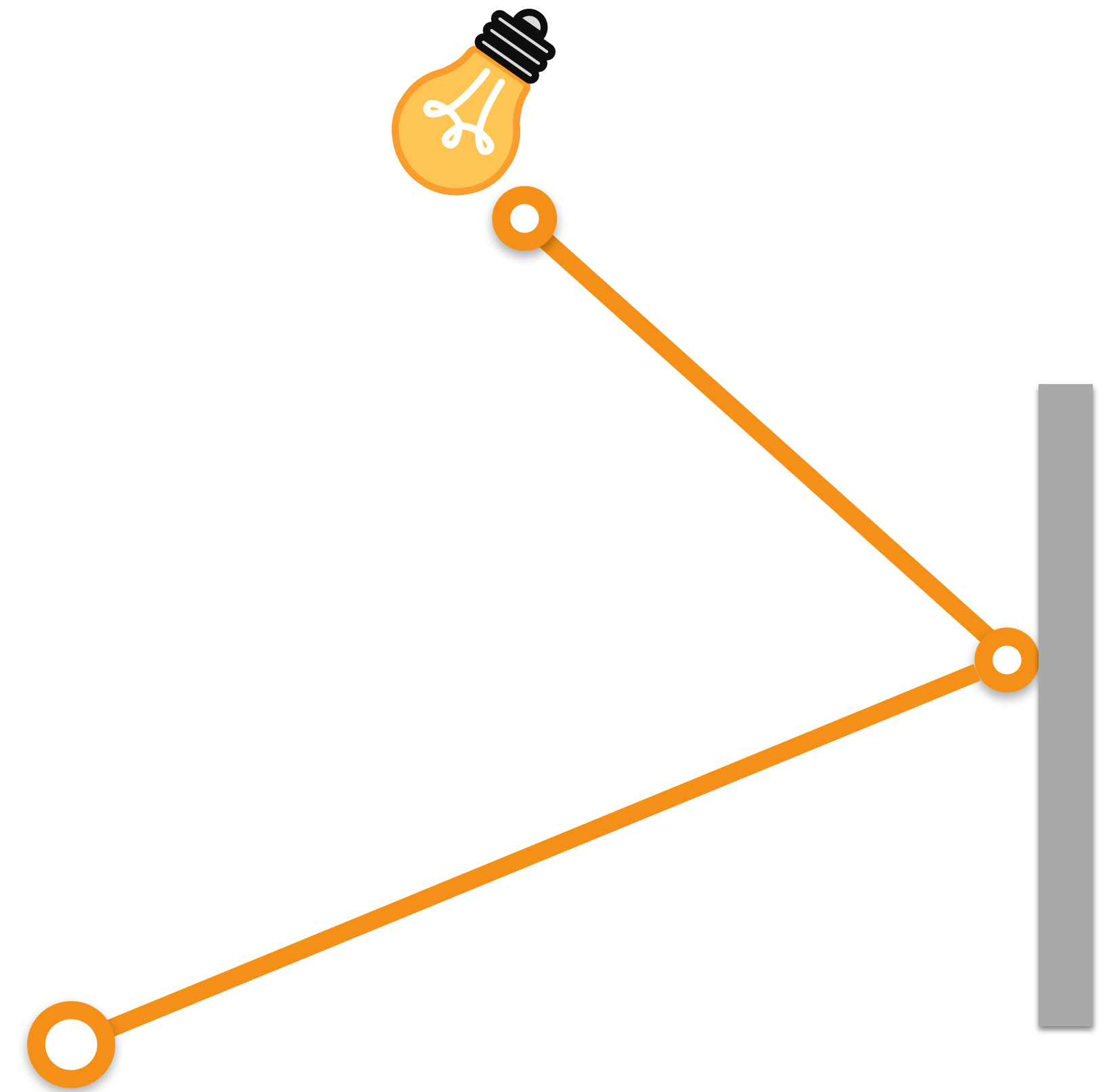


**single scattering**

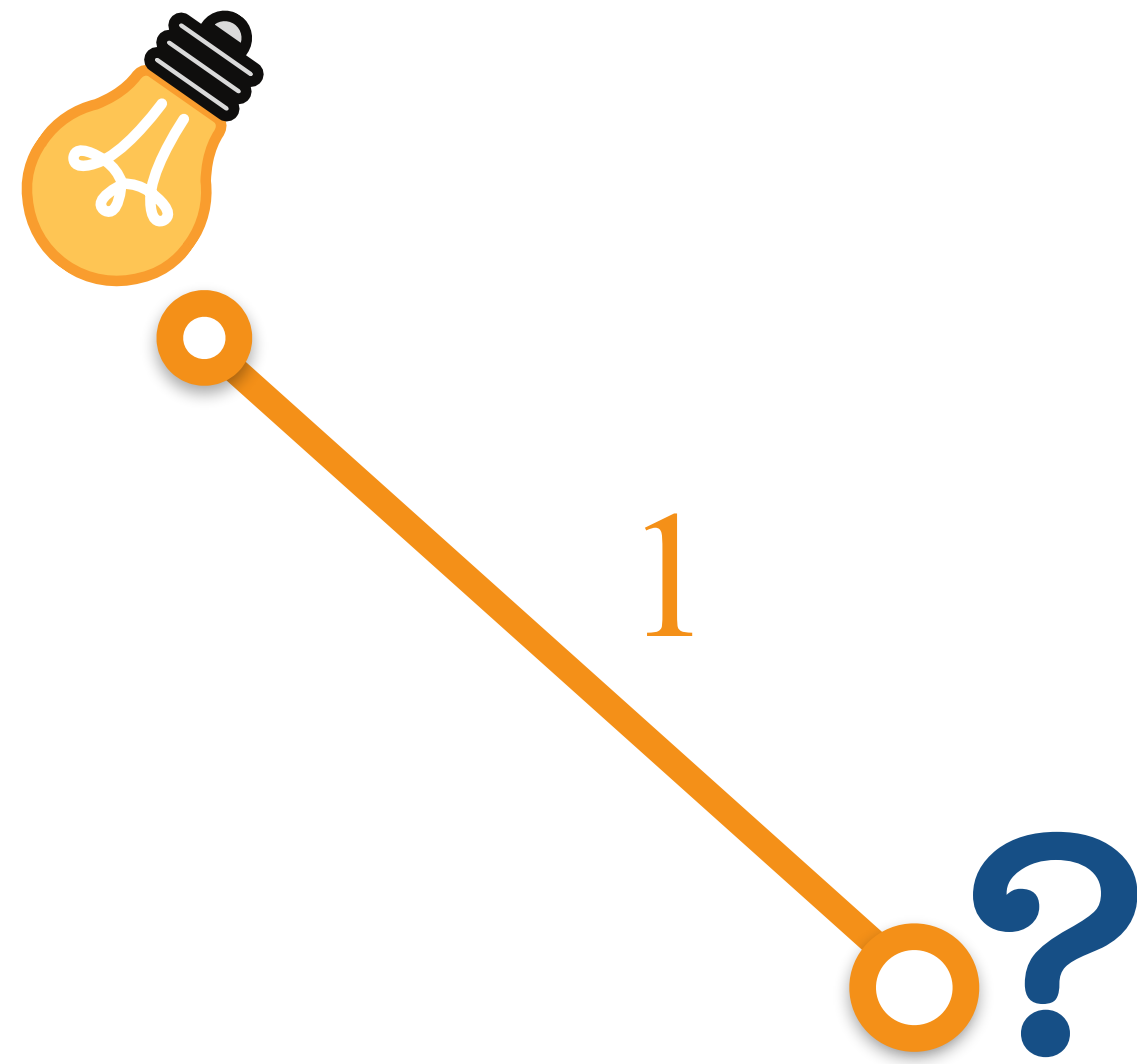
# The limitations of photon planes



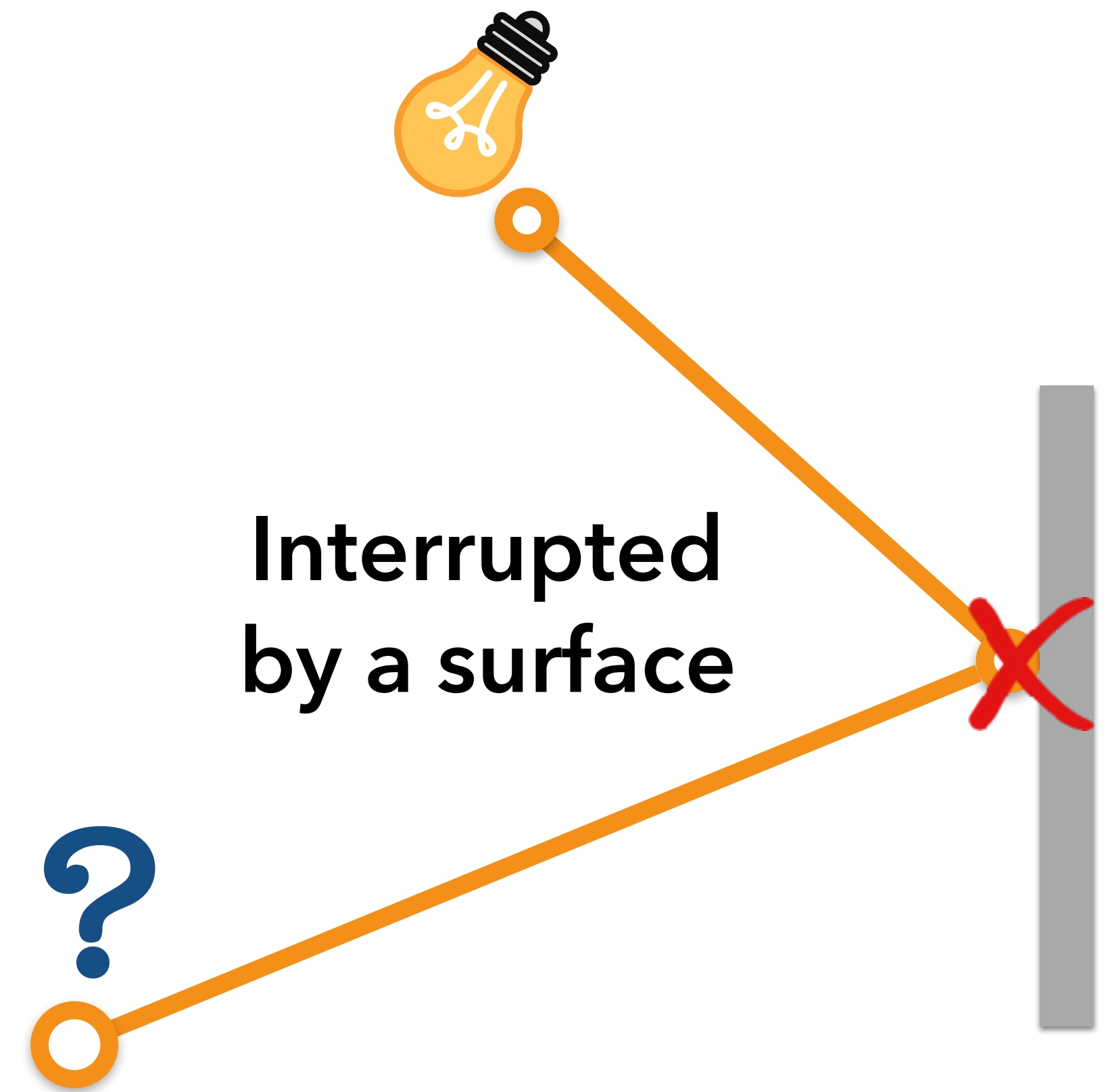
**single scattering**



# The limitations of photon planes



**single scattering**



**surface-medium**

# OUR WORK

# Contributions

## 1. New derivation

$$+ \int_{\mathbf{x}}^{\mathbf{x}_t} \text{Tr}(\mathbf{x}_{t'}, \mathbf{x}_t) \sigma_s(\mathbf{x}_{t'}) \left( \int_{S^2} \rho_p(\mathbf{x}_{t'}, \omega, \omega_i) L(\mathbf{x}_{t'}, \omega_i) d\omega_i \right) d\mathbf{x}_{t'}$$

$$I = \int_{\Omega} f(\bar{\zeta}) K(\mathbf{g}(\bar{\zeta})) f_{\omega}^{1,1} d\bar{\zeta}$$

$$L(\mathbf{x}_t, \omega) = \sum_{i=k}^n t_i \omega_i \quad \mathbf{y}_0 = \mathbf{y}_l + \sum_{i=k}^l s_i \psi_i \quad I = \int f(\bar{\zeta}) K(\mathbf{g}(\bar{\zeta})) f_{\omega}^{1,1} d\bar{\zeta} + \text{Tr}(\mathbf{x}, \mathbf{x}_t) L(\mathbf{x}, \omega)$$

$$\mathbf{g}(\bar{\zeta}) = \mathbf{x}_0(\bar{\zeta}) - \mathbf{y}_0(\bar{\zeta}) = \left( \mathbf{x}_l + \sum_{i=1}^l t_i \omega_i \right) - \left( \mathbf{y}_k + \sum_{i=1}^k s_i \psi_i \right)$$

$$K(\mathbf{g}) = \delta^3(\mathbf{g}) = \delta^1(x(\mathbf{g})) \delta^1(y(\mathbf{g})) \delta^1(\mathbf{g}(\bar{\zeta}_a)) \text{ is a shorthand for } \mathbf{g}(\bar{\zeta}_a, \bar{\zeta}_n)$$

$$L_0(\mathbf{x}, \omega_0) = L_e(\mathbf{x}, \omega_0) + L_s(\mathbf{x}, \omega_0) \quad \int_{\mathbf{x}}^{\mathbf{x}_t} \text{Tr}(\mathbf{x}_{t'}, \mathbf{x}_t) L_e(\mathbf{x}_{t'}, \omega) d\mathbf{x}_{t'}$$

$$= L_e(\mathbf{x}, \omega_0) + \int_{S^2} \rho_s(\mathbf{x}, \omega_i, \omega_0) L(\mathbf{x}, \omega_i) |\mathbf{n}(\mathbf{x}) \cdot \omega_i| d\omega_i$$

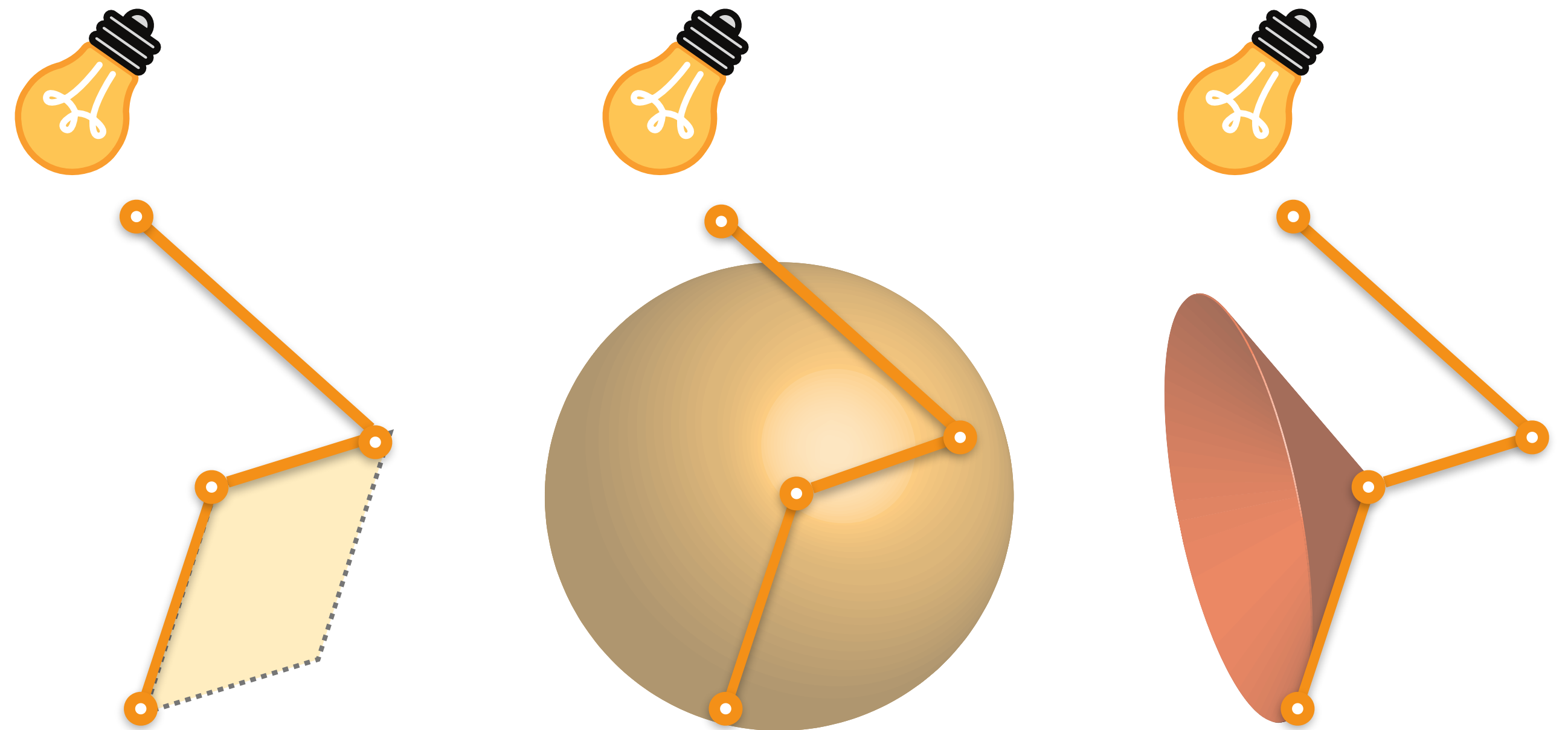
$$I = \int_{\Omega} f(\bar{\zeta}_n) \underbrace{\int_{\Omega_a(\bar{\zeta}_n)} f(\bar{\zeta}_a) \delta^3(\mathbf{g}(\bar{\zeta}_a)) f_{\omega}^{1,1} d\bar{\zeta}_a}_{I_a} d\bar{\zeta}_n$$

$$I = \int_A \int_{H^2} W_e(\mathbf{x}, \omega) L(\mathbf{x}, \omega) d\omega d\mathbf{x}$$

$$I = \frac{f(\bar{\zeta}_n) I_a(\bar{\zeta}_n)}{p(\bar{\zeta}_n)}$$

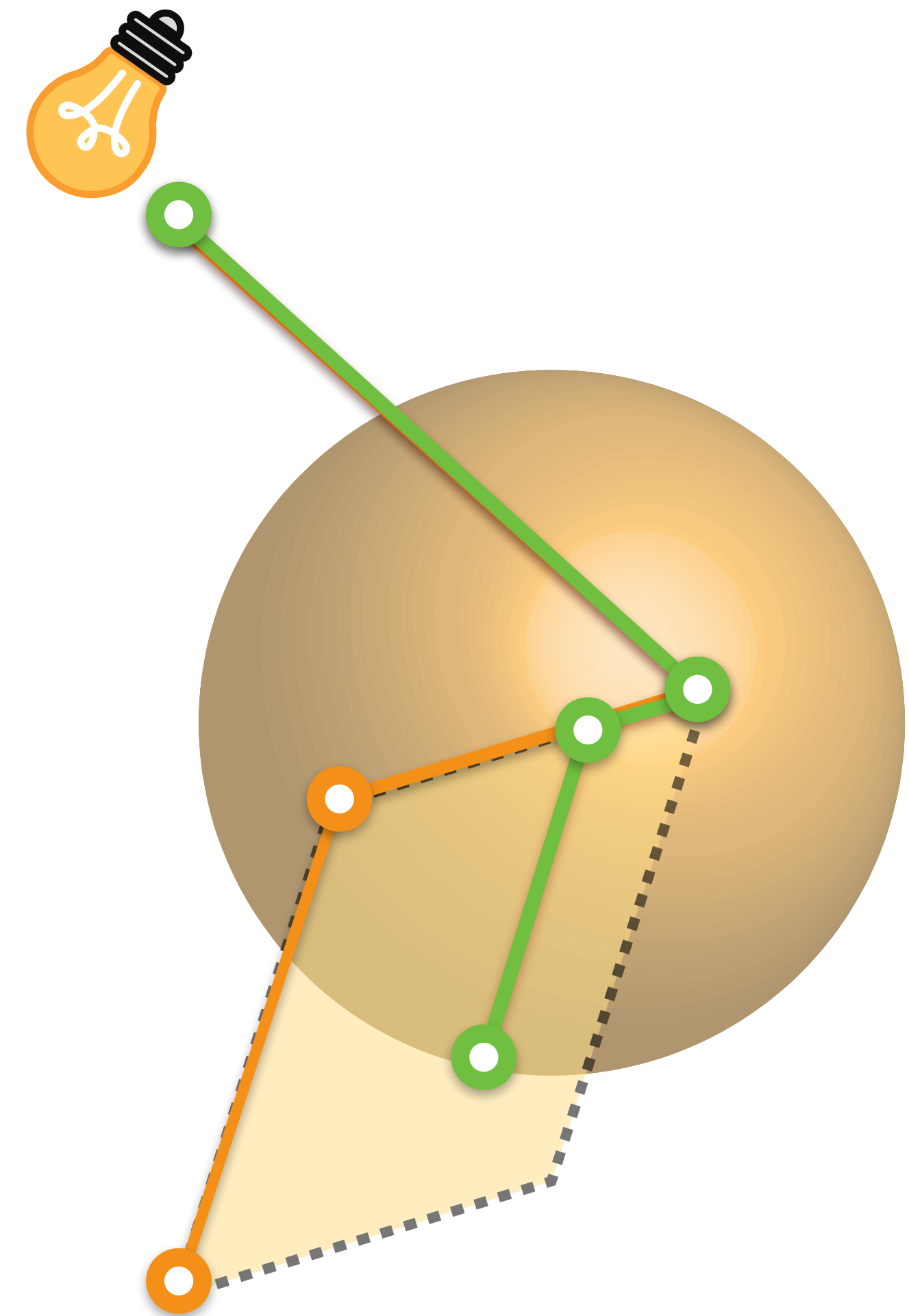
# Contributions

1. New derivation
2. Photon surfaces



# Contributions

1. New derivation
2. Photon surfaces
3. Combine using MIS





Photon planes

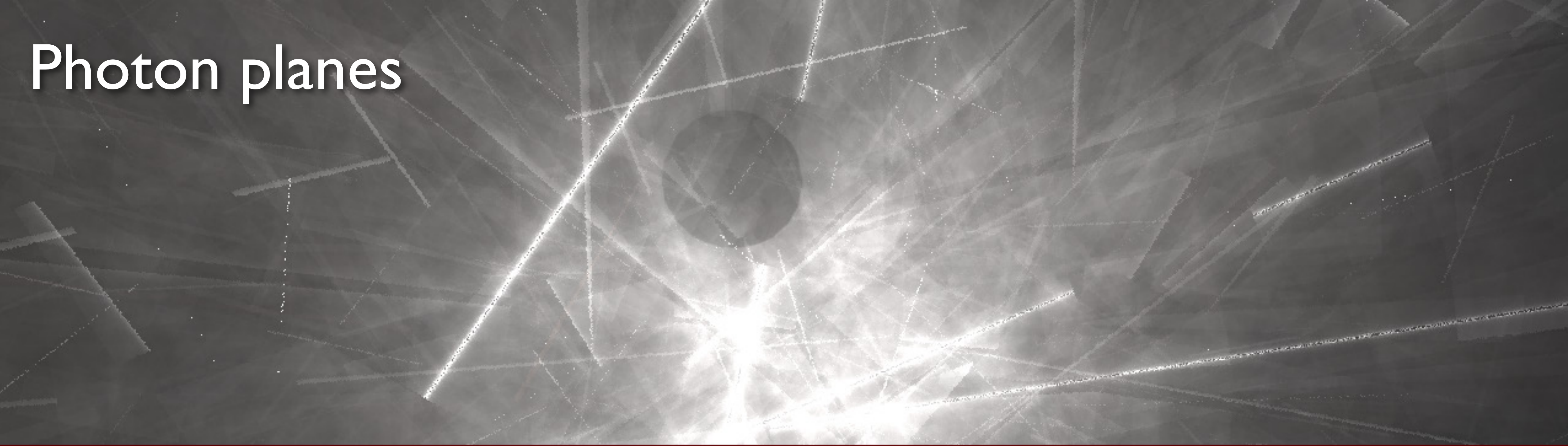


Ours

Var: 0.209x



Photon planes



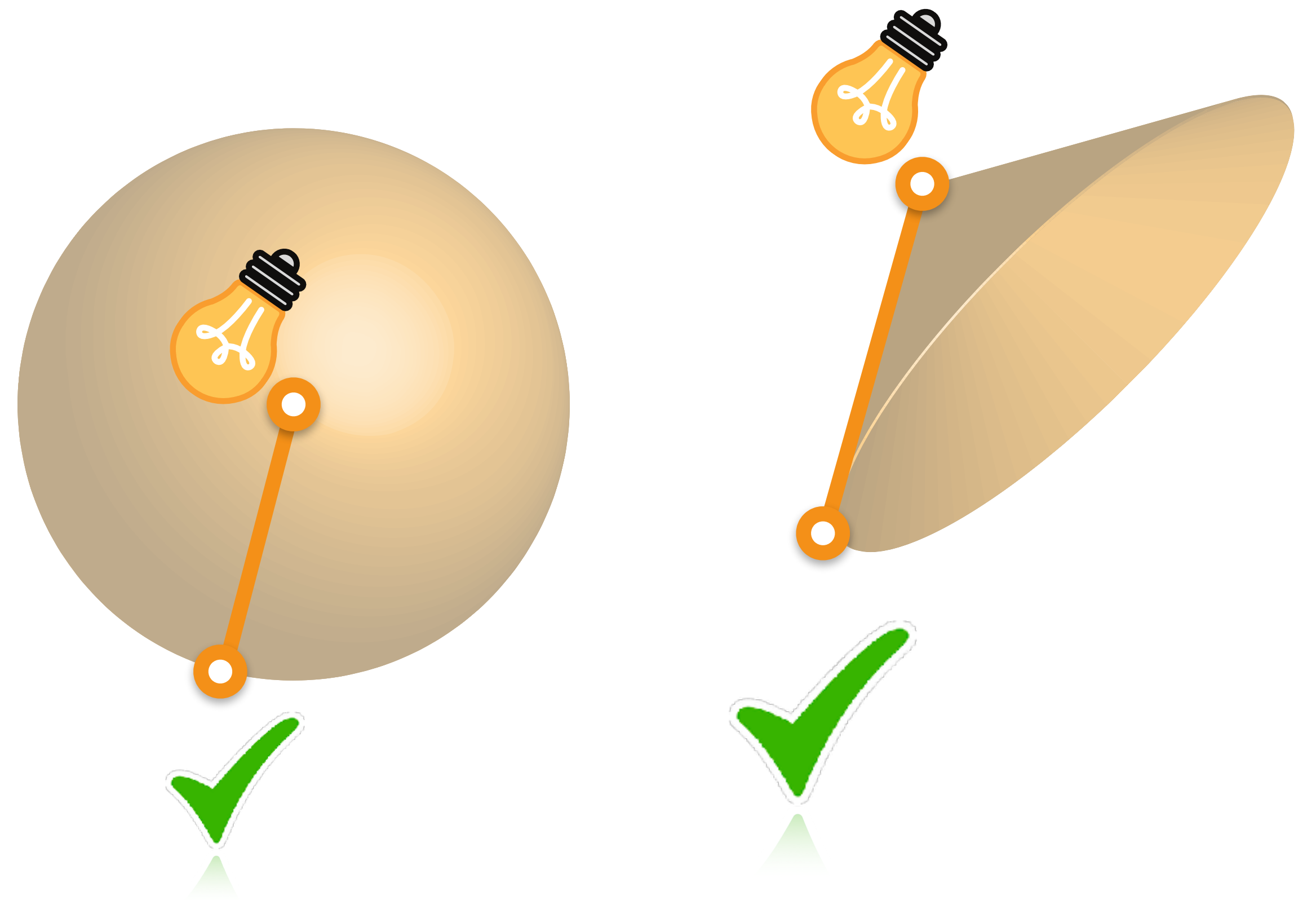
Ours

Var: 0.209x

The combination resolves singularities

# Contributions

1. New derivation
2. Photon surfaces
3. Combine using MIS
4. Single scattering



# Photon beams



Ours

Var: 0.155x



# Photon beams



Ours

Var: 0.155x



Single-scattering is supported

# Theory

# Geometric interpretation

Photon surfaces for robust, unbiased volumetric density estimation • 465

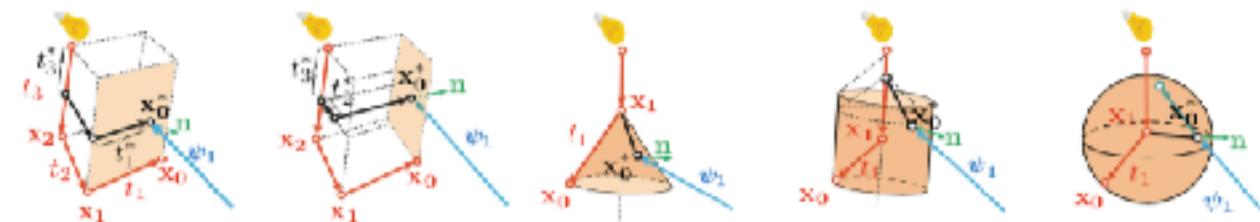


Fig. 2. Our framework allows us to derive new density estimators depending on which dimensions are analytically integrated. Integrating non-consecutive distance dimensions lead to generalized photon planes (left two); integrating azimuth and distance leads to photon cones and cylinders (middle, middle-right); integrating azimuth and inclination leads to photon spheres (right).

photon subpath. The Jacobian is then

$$J_{\xi_a}^u(\xi_a^*) = \det \begin{bmatrix} \frac{\partial x_0}{\partial \xi_{a1}}(\xi_a^*) & \frac{\partial x_0}{\partial \xi_{a2}}(\xi_a^*) & \frac{\partial y_0}{\partial \xi_{a1}}(\xi_a^*) \\ \frac{\partial x_0}{\partial \xi_{a1}}(\xi_a^*) & \frac{\partial x_0}{\partial \xi_{a2}}(\xi_a^*) & \frac{\partial y_0}{\partial \xi_{a2}}(\xi_a^*) \end{bmatrix} = |\mathbf{n}(\xi_a^*) \cdot \psi_1|, \quad (22)$$

where  $\mathbf{n} = \partial x_0 / \partial \xi_{a1} \times \partial x_0 / \partial \xi_{a2}$  can be interpreted as a scaled surface normal of the photon primitive that arises from “sweeping”  $x_0$  for all values of  $\xi_{a1}$  and  $\xi_{a2}$ . The Jacobian of such an estimator is then simply the dot product of the surface normal and the query ray direction. Intuitively, this means that photon surfaces will become brighter when viewed at grazing angles.

It is worth noting that the normal  $\mathbf{n}$  is not necessarily of unit length. Indeed, its length encodes how a differential 2D element stretches and squishes as we map from parametric space  $(\xi_{a1}, \xi_{a2})$  to a differential area on the photon surface. For example, in the case of a photon plane, a sheared plane will be brighter as it leads to a shorter normal and thus smaller Jacobian.

By interpreting these estimators as ray-surface intersections, we can easily see that it is possible for  $g$  to have multiple roots. For example, in the case of a spherical photon surface (Fig. 2, right), a ray can hit both the front- and back-side of the sphere. This requires a more general form of Eq. (18) that holds for multiple roots:

$$\delta^3(g(\xi_a^*)) = \sum_r \frac{\delta^3(\xi_{a1}^* - \xi_{a1}^{*r})}{\left| \frac{\partial g}{\partial \xi_{a1}}(\xi_a^*) \right|}, \quad (23)$$

where the summation is over all roots  $\xi_{a1}^{*r}$ . In practical terms, this means that the total contribution of such an estimator is simply the sum of contributions over all hitpoints. This results in a remarkably simple formula for the estimator contribution, generalizing the photon plane estimator (21) to photon surfaces:

$$t_a = \sum_r \frac{f^{1,1}(\xi_a^*)}{|\mathbf{n}(\xi_a^*) \cdot \psi_1|}. \quad (24)$$

In the following subsection, we will instantiate this estimator for specific choices of  $\xi_{a1}$  and  $\xi_{a2}$ , listing the required scaled surface normals for Eq. (24) to obtain new photon cone, sphere, cylinder estimators and more.

## 5.1 Multiple-Scattering Photon Surfaces

**Generalized Photon Planes.** Previously we integrated out the last two propagation distance dimensions  $\xi_{a1} = t_1$  and  $\xi_{a2} = t_2$  to obtain an unbiased photon plane (Eq. (21)). We can create a larger family of generalized, multi-scattering photon plane estimators by simply choosing any two distance dimensions from  $\vec{t}$ . Let us denote these  $\xi_{a1} = t_i$  and  $\xi_{a2} = t_j$  where  $i \geq 1 > j \geq 1$ . Intuitively, integrating

Table 1. Each choice of two dimensions  $\xi_{a1}, \xi_{a2}$  leads to a different multiple-scattering photon surface with corresponding scaled surface normal.

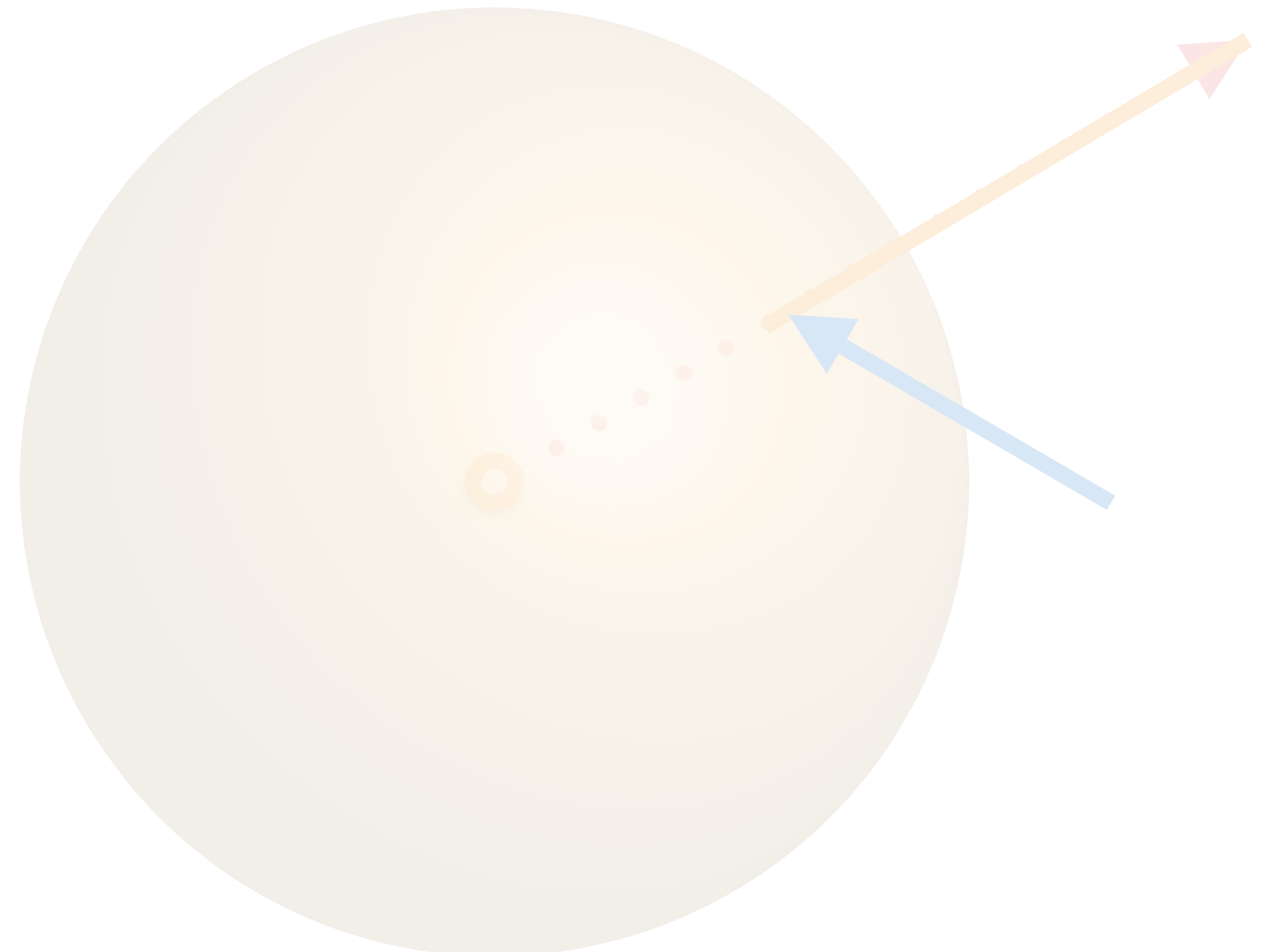
Primitive	$[\xi_{a1}, \xi_{a2}]$	Scaled surface normal $\mathbf{n}(\xi_a^*)$
Plane	$[t_1, t_2]$	$(a_1 \times a_2)$
Sphere	$[\cos \theta_1, \phi_1]$	$a_1^2 a_2^2$
Cone	$[t_1, \phi_1]$	$(a_1^2 \times (a_1^2 \times a_2)) t_1^2$
Cylinder	$[t_2, \phi_1]$	$(a_2 \times (a_1^2 \times a_2)) t_1$
Disk	$[t_1, \cos \theta_1]$	$\frac{a_1^2 a_2^2 \sin \theta_1}{\cos \theta_1}$
Toroid	$[t_1, t_2]$	$\frac{(a_1 \times a_2) \cos(\theta_1) a_1^2 a_2^2 \sin(\theta_1) \cos(\theta_1) \sin(\theta_1)}{a_1^2 a_2^2 \sin(\theta_1)}$
Hyperboloid	$[\phi_1, t_1]$	$\frac{(a_1 \times a_2) \cos(\theta_1) a_1^2 a_2^2 \sin(\theta_1) \cos(\theta_1) \sin(\theta_1)}{\sin \theta_1}$

out these two dimensions sweeps the photon position  $x_0$  over the plane spanned by the corresponding directions  $a_1$  and  $a_2$  (see Fig. 2, left). We provide a full derivation in the supplemental document, but the resulting scaled surface normal is simply  $a_1 \times a_2$  (instead of  $a_2 \times a_1$ ), as listed in Table 1. Inserting this surface normal into Eq. (24) gives us a generalized photon plane estimator.

**Photon Spheres.** It is also possible to integrate out directional dimensions. For instance, analytically integrating  $a_1$  can be thought of as sweeping the photon position  $x_0$  over all 4 $\pi$  steradians centered at  $x_1$ , resulting in a spherical photon surface (see Fig. 2, right). The surface normal (see Table 1) is simply the normal of the sphere  $a_1^2$ , but scaled by its squared radius  $t_1^2$  to account for the change from solid angle (unit sphere) to the surface area of a non-unit sphere.<sup>4</sup> Just as with generalized photon planes, we are not limited to choosing the last directional domain, but can choose an arbitrary  $a_2$  along the path to obtain different photon sphere estimators. We include a full derivation in the supplemental document and list the resulting normal that needs to be inserted into Eq. (24) in Table 1.

**Photon Cones, Cylinders, and beyond.** By decomposing a direction  $a_2$  into spherical angles  $\phi_1, \cos \theta_1$ , we can mix and match an analytic distance dimension with a polar or azimuthal angle. Simultaneously spinning the photon  $x_0$  along  $\phi_1$  and sweeping along distance  $t_1$  results in a photon cone, while choosing  $\phi_1$  and the distance along the previous segment  $t_{i+1}$  leads to a photon cylinder (see Fig. 2, middle). In general, it is possible to mix and match the angles and distances from different bounces, producing more general photon toroids and hyperboloids. We list the scaled surface normals for several options in Table 1.

<sup>4</sup>Integrating both direction and distance at a vertex leads to an infinite “spherical photon volume”, whose Jacobian is exactly equivalent to the standard geometry term for nearest estimator,  $\text{Vol}(x_0)$  or shadow rays  $\times \text{Vol}$ . The more general photon techniques can be derived from our framework, for a particular choice of integration dimensions.



# Theory

# Geometric interpretation

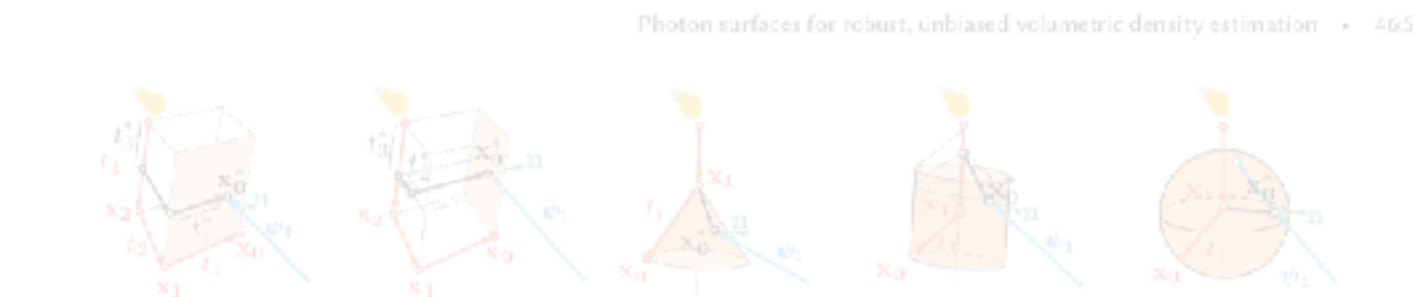


Fig. 2. Our framework allows us to derive new density estimators depending on which dimensions are analytically integrated. Integrating non-conservative distance dimensions leads to generalized photon planes (left two), integrating axis  $z$  and distance leads to photon cones and cylinders (middle, middle right), integrating azimuthal inclination leads to photon spheres (right).

photon subpath. The Jacobian is then

$$J_{\xi}(\vec{r}_a) = \det \begin{bmatrix} \frac{\partial x_0}{\partial \xi_{a1}} & \frac{\partial x_0}{\partial \xi_{a2}} \\ \frac{\partial y_0}{\partial \xi_{a1}} & \frac{\partial y_0}{\partial \xi_{a2}} \end{bmatrix} = |\mathbf{n}(\vec{r}_a) \cdot \hat{\psi}|, \quad (20)$$

where  $\mathbf{n} = \partial \mathbf{x}_0 / \partial \xi_{a1} \times \partial \mathbf{x}_0 / \partial \xi_{a2}$  can be interpreted as a scaled surface normal of the photon primitive that arises from “sweeping”  $\mathbf{x}_0$  for all values of  $\xi_{a1}$  and  $\xi_{a2}$ . The Jacobian of such an estimator is then simply the dot product of the surface normal and the query ray direction. Intuitively, this means that photon surfaces will become brighter when viewed at grazing angles.

It is worth noting that the normal  $\mathbf{n}$  is not necessarily of unit length. Indeed, its length encodes how a differential 2D element stretches and squishes as we map from parametric space  $(\xi_{a1}, \xi_{a2})$  to a differential area on the photon surface. For example, in the case of a photon plane, a sheared plane will be brighter as it leads to a shorter normal and thus smaller Jacobian.

By interpreting these estimators as ray-surface intersections, we can easily see that it is possible for  $g$  to have multiple roots. For example, in the case of a spherical photon surface (Fig. 2, right), a ray can hit both the front- and back-side of the sphere. This requires a more general form of Eq. (18) that holds for multiple roots:

$$g^2(\vec{r}_a) = \sum_r \frac{\delta^3(\vec{r}_a - \vec{r}_a^r)}{\left| \frac{\partial \mathbf{g}}{\partial \xi_a}(\vec{r}_a^r) \right|}, \quad (21)$$

where the summation is over all roots  $\vec{r}_a^r$ . In practical terms, this means that the total contribution of such an estimator is simply the sum of contributions over all hitpoints. This results in a remarkably simple formula for the estimator contribution, generalizing the photon plane estimator (21) to photon surfaces:

$$I_a = \sum_r \frac{r^{3-1} f(r_a^r)}{\left| \mathbf{n}(\vec{r}_a^r) \cdot \hat{\psi} \right|}. \quad (24)$$

In the following subsection, we will instantiate this estimator for specific choices of  $\xi_{a1}$  and  $\xi_{a2}$ , listing the required scaled surface normals for Eq. (24) to obtain new photon cone, sphere, cylinder estimators and more.

## 5.1 Multiple-Scattering Photon Surfaces

**Generalized Photon Planes.** Previously we integrated out the last two propagation distance dimensions  $\xi_{a1} = t_1$  and  $\xi_{a2} = t_2$  to obtain an unbiased photon plane (Eq. (21)). We can create a larger family of generalized, multi-scattering photon plane estimators by simply choosing any two distance dimensions from  $\vec{t}$ . Let us denote these  $\xi_{a1} = t_i$  and  $\xi_{a2} = t_j$  where  $i \geq 1 > j \geq 1$ . Intuitively, integrating

Table 1. Each choice of two dimensions  $\xi_{a1}, \xi_{a2}$  leads to a differential multi-scattering photon surface with corresponding scaled surface normal.

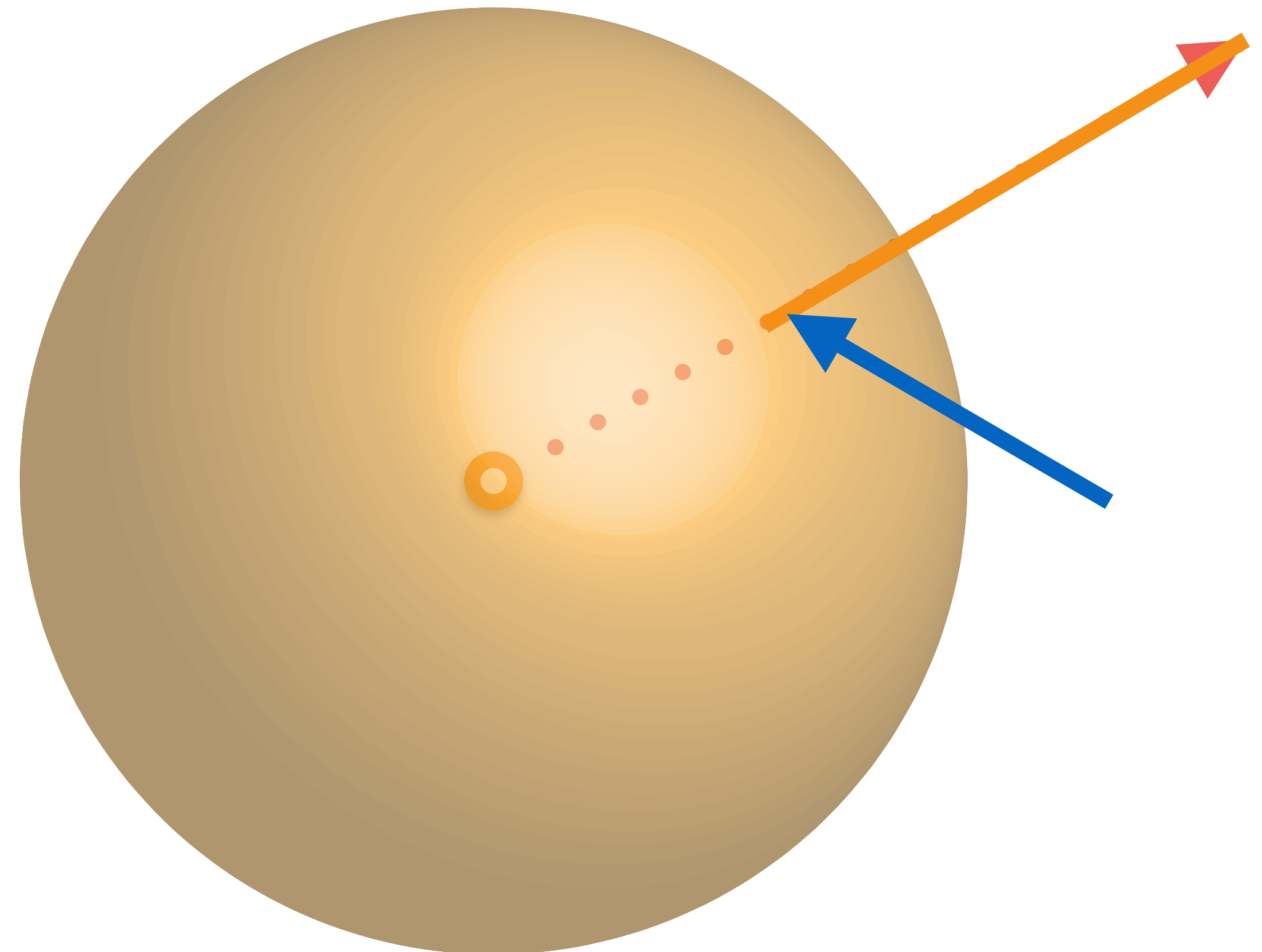
Primitive	$[\xi_{a1}, \xi_{a2}]$	Scaled surface normal: $\mathbf{n}(\vec{r})$
Plane	$[t_1, t_2]$	$(\omega_1 \times \omega_2)$
Sphere	$(\cos \theta_1, \phi_1)$	$\omega_1^2 \hat{\psi}$
Cone	$[t_1, \phi_1]$	$(\omega_1^2 \times (\omega_1^2 \times \omega_1)) \hat{\psi}$
Cylinder	$[t_1, \phi_1]$	$(\omega_2 \times (\omega_1^2 \times \omega_1)) \hat{\psi}$
Disk	$[t_1, \cos \theta_1]$	$\frac{\omega_1 \omega_1 \omega_1^2}{\sin \theta_1} \hat{\psi}$
Torus	$[t_1, \phi_1]$	$\frac{(\omega_1 \times \omega_2) \omega_1 \omega_1^2 \times (\omega_1 \times \omega_2) \omega_1 \omega_1^2}{\sin \theta_1 \sin \theta_1} \hat{\psi}$
Hyperboloid	$[t_1, t_2]$	$\frac{(\omega_1 \times \omega_2) \omega_1 \omega_1^2 \times (\omega_1 \times \omega_2) \omega_1 \omega_1^2}{\sin \theta_1 \sin \theta_1} \hat{\psi}$

out these two dimensions sweeps the photon position  $\mathbf{x}_0$  over the plane spanned by the corresponding directions  $\omega_1$  and  $\omega_2$  (see Fig. 2, left). We provide a full derivation in the supplemental document, but the resulting scaled surface normal is simply  $\omega_1 \times \omega_2$  (instead of  $\omega_2 \times \omega_1$ ), as listed in Table 1. Inserting this surface normal into Eq. (24) gives us a generalized photon plane estimator.

**Photon Spheres.** It is also possible to integrate out directional dimensions. For instance, analytically integrating  $\omega_1$  can be thought of as sweeping the photon position  $\mathbf{x}_0$  over all 4 $\pi$  steradians centered at  $\mathbf{x}_1$ , resulting in a spherical photon surface (see Fig. 2, right). The surface normal (see Table 1) is simply the normal of the sphere  $\omega_1^2$ , but scaled by its squared radius  $r^2$  to account for the change from solid angle (unit sphere) to the surface area of a non-unit sphere.<sup>4</sup> Just as with generalized photon planes, we are not limited to choosing the last directional dimension, but can choose an arbitrary  $\omega_j$  along the path to obtain different photon sphere estimators. We include a full derivation in the supplemental document and list the resulting normal that needs to be inserted into Eq. (24) in Table 1.

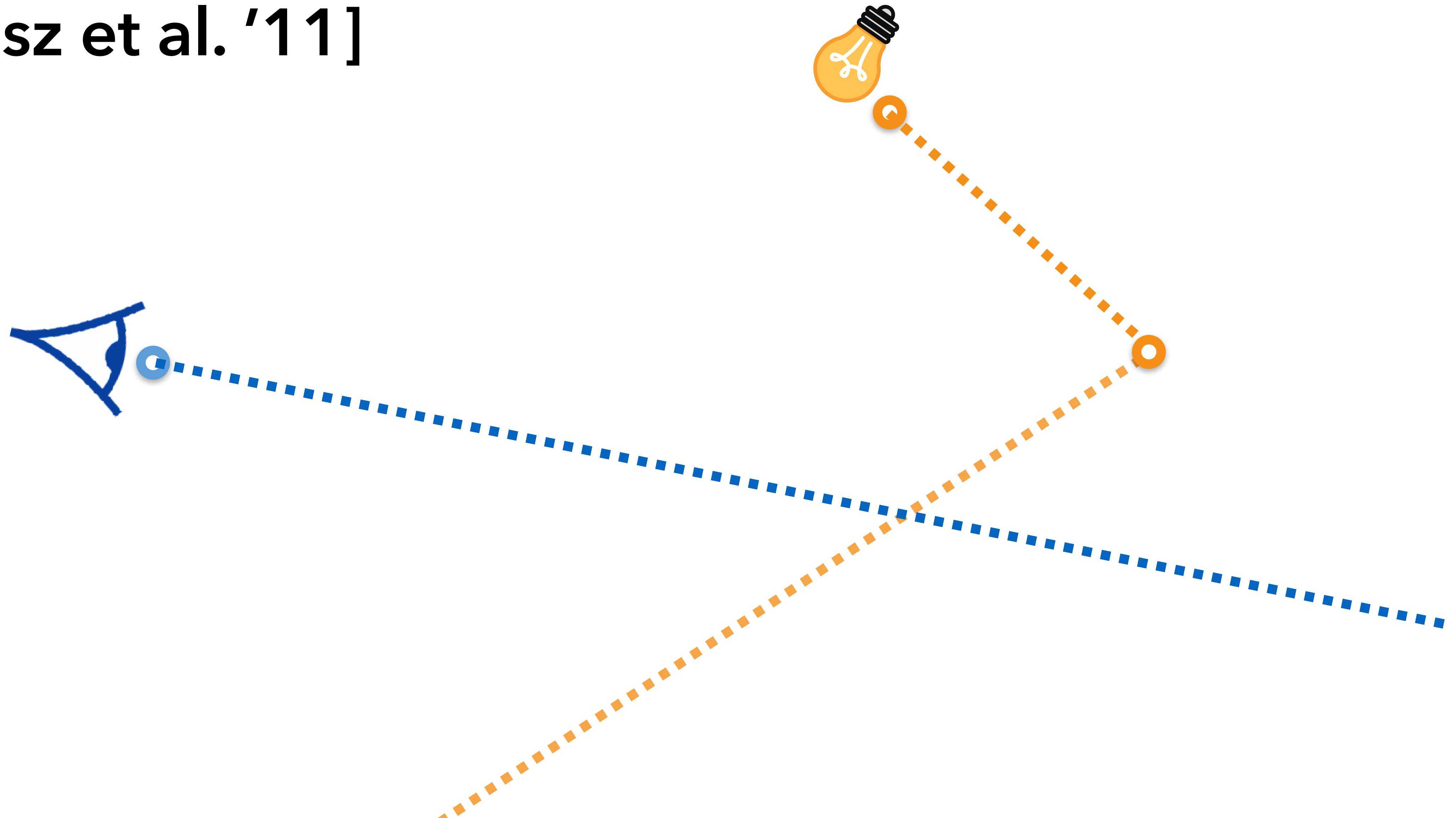
**Photon Cones, Cylinders, and beyond.** By decomposing a direction  $\omega_1$  into spherical angles  $\phi_1, \cos \theta_1$ , we can mix and match an analytic distance dimension with a polar or azimuthal angle. Simultaneously spinning the photon  $\mathbf{x}_0$  along  $\phi_1$  and sweeping along distance  $t_1$  results in a photon cone, while choosing  $\phi_1$  and the distance along the previous segment  $t_{j-1}$  leads to a photon cylinder (see Fig. 2, middle). In general, it is possible to mix and match the angles and distances from different bounces, producing more general photon toroids and hyperboloids. We list the scaled surface normals for several options in Table 1.

<sup>4</sup>Integrating both directional dimensions over an infinite “spherical photon volume”, whose Jacobian is exactly equivalent to the solid angle term for nearest estimator, yields a photon ray  $\omega_1^2 \hat{\psi}$ . This more general estimator can be derived from our framework, for a particular choice of integration dimensions.



# Derivation in previous work

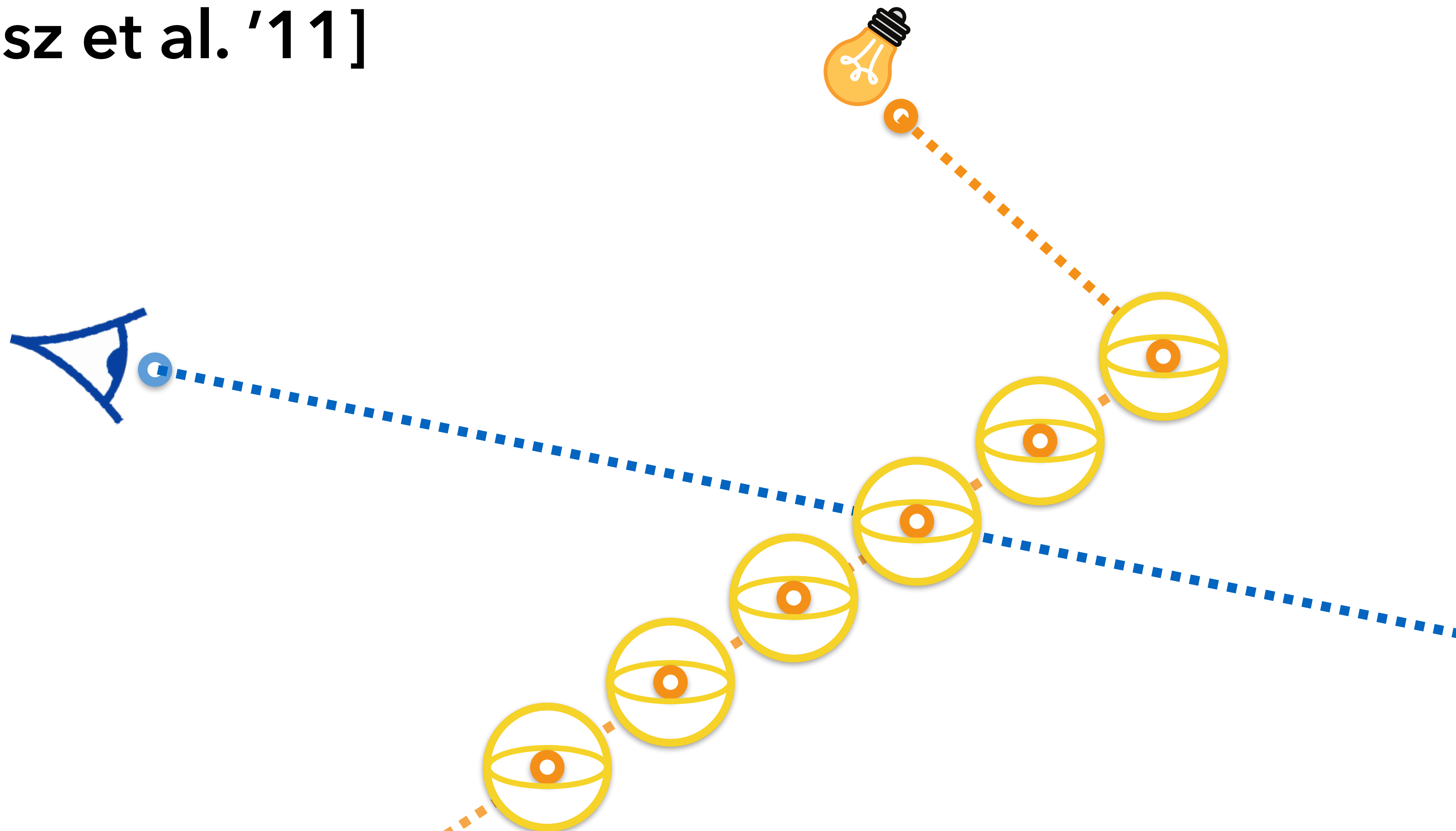
[Jarosz et al. '11]





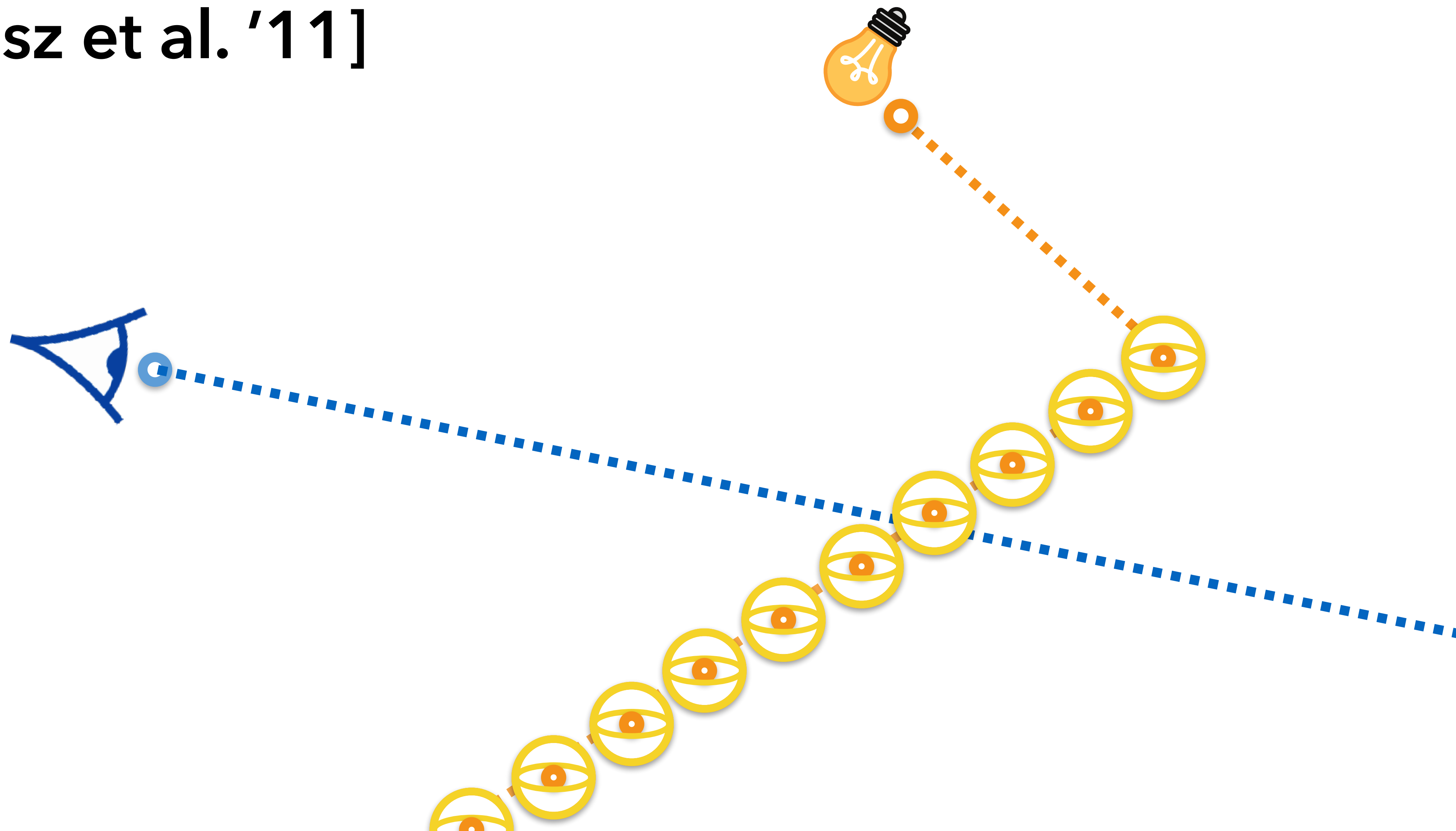
# Derivation in previous work

[Jarosz et al. '11]



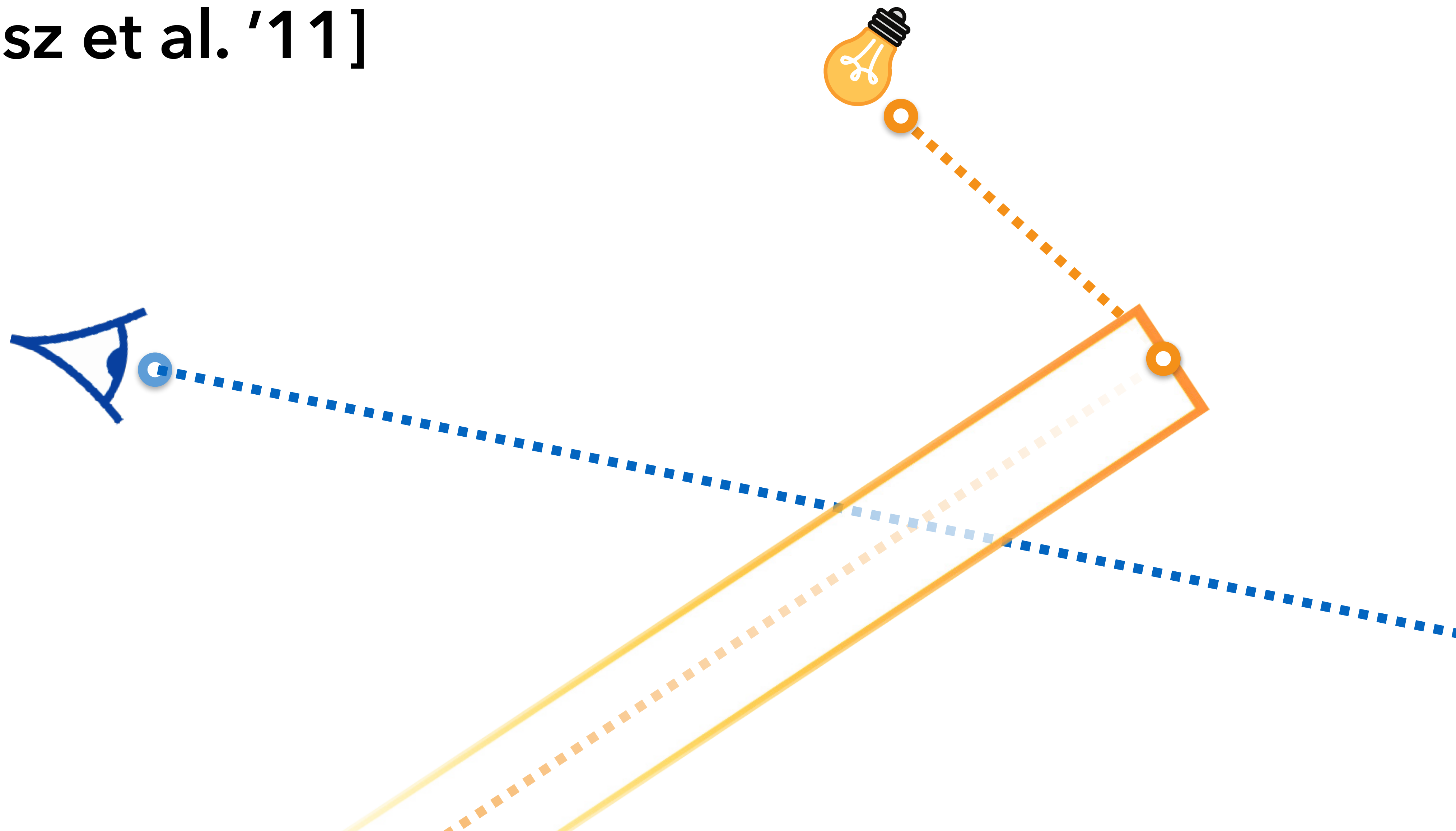
# Derivation in previous work

[Jarosz et al. '11]



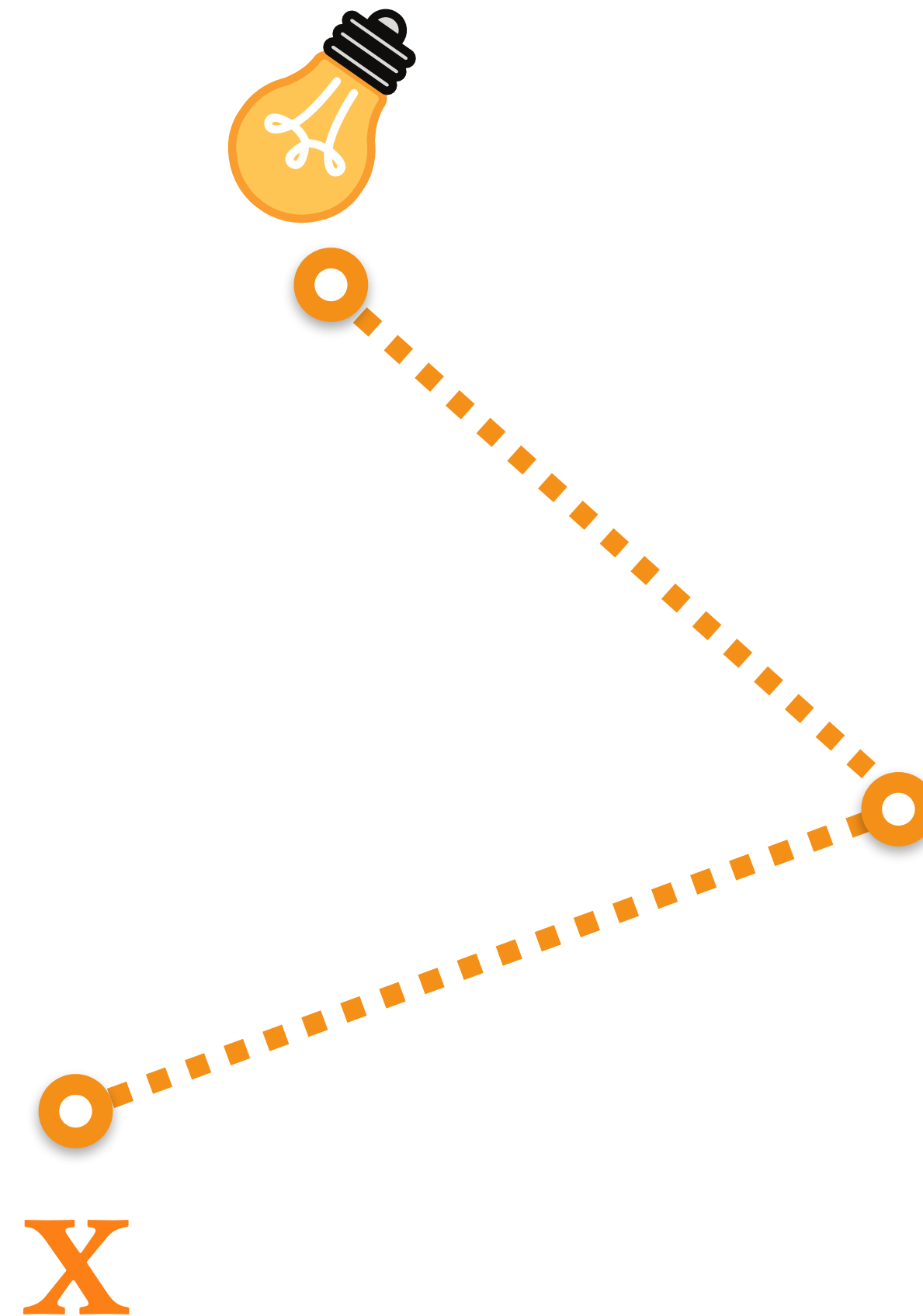
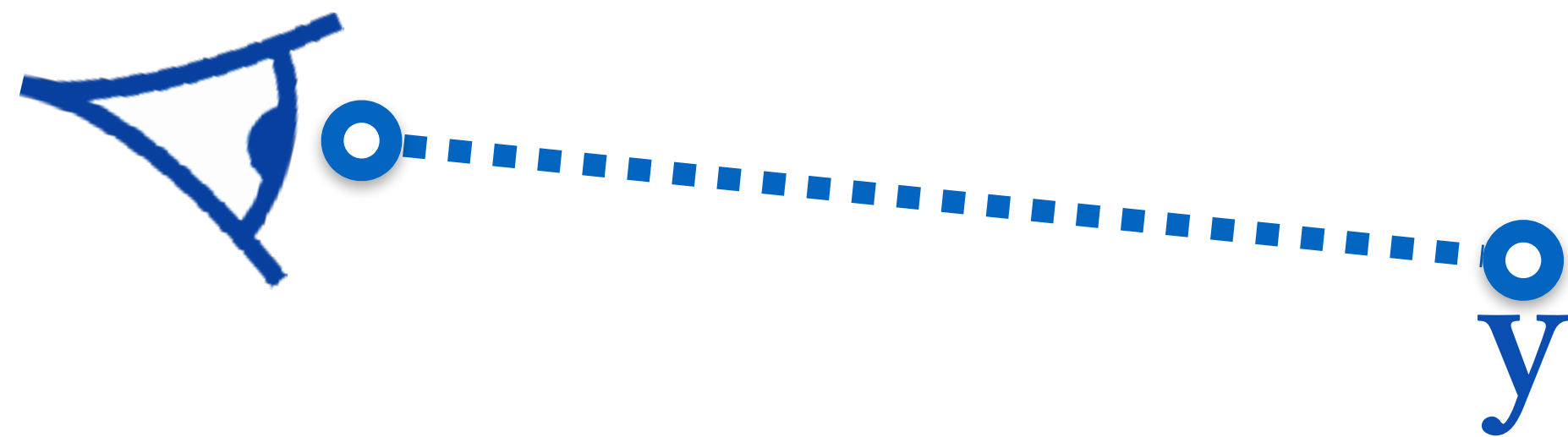
# Derivation in previous work

[Jarosz et al. '11]



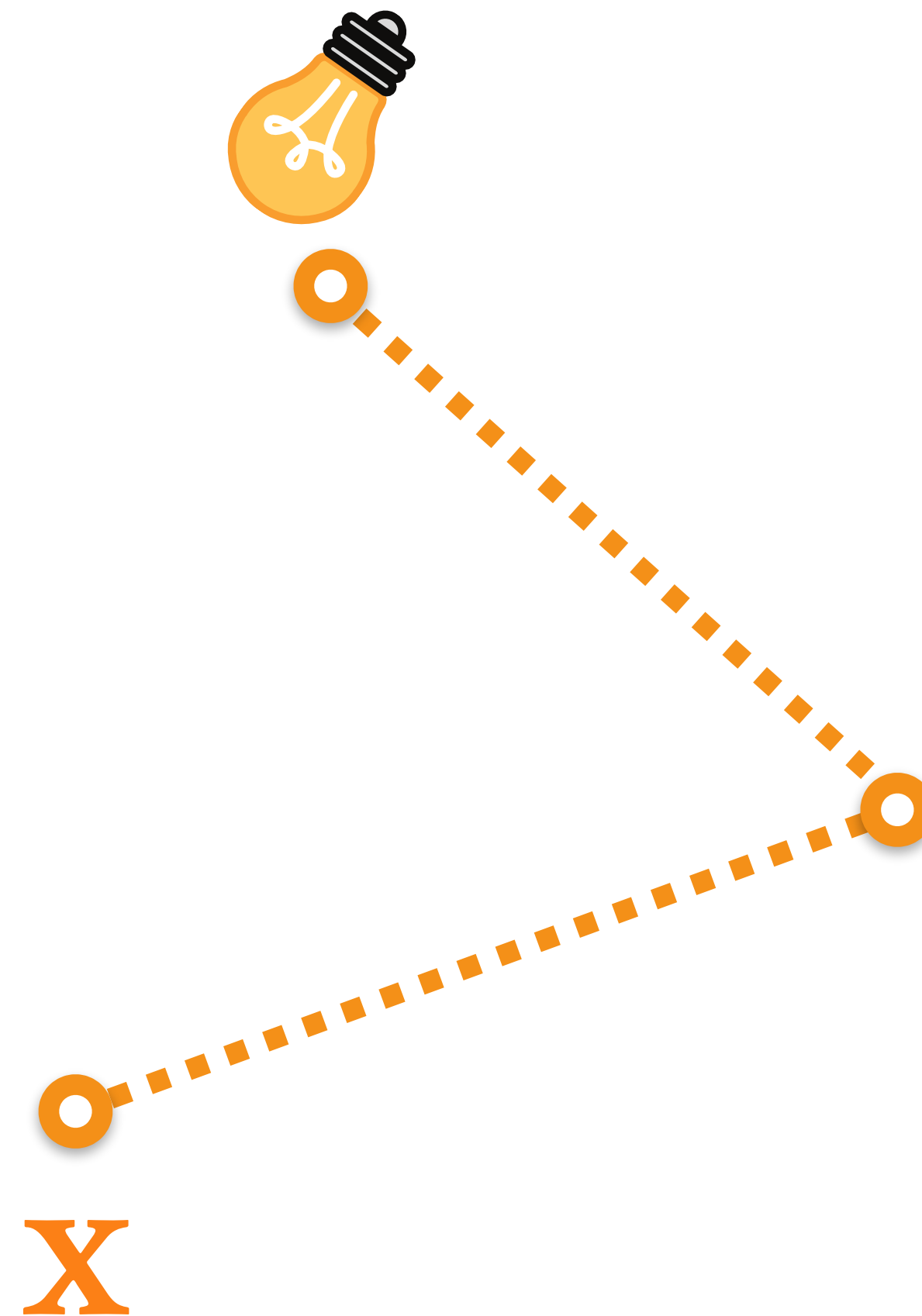
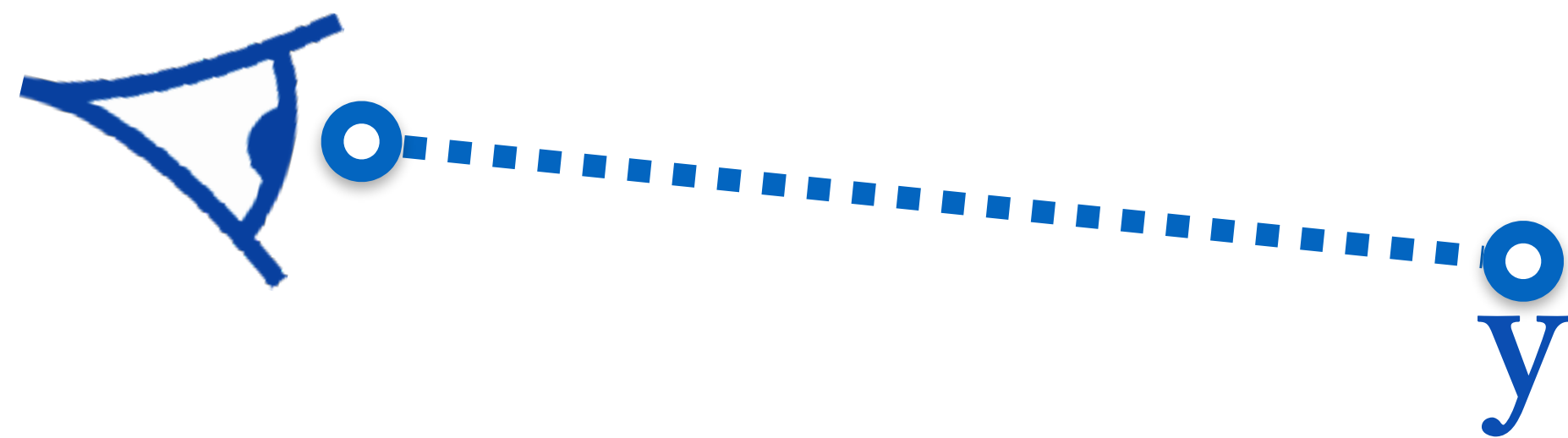
# Our approach

$$I = \int f(\bar{y}) K_{3D}(\mathbf{y} - \mathbf{x}) f(\bar{x}) d\mu$$



# Our approach

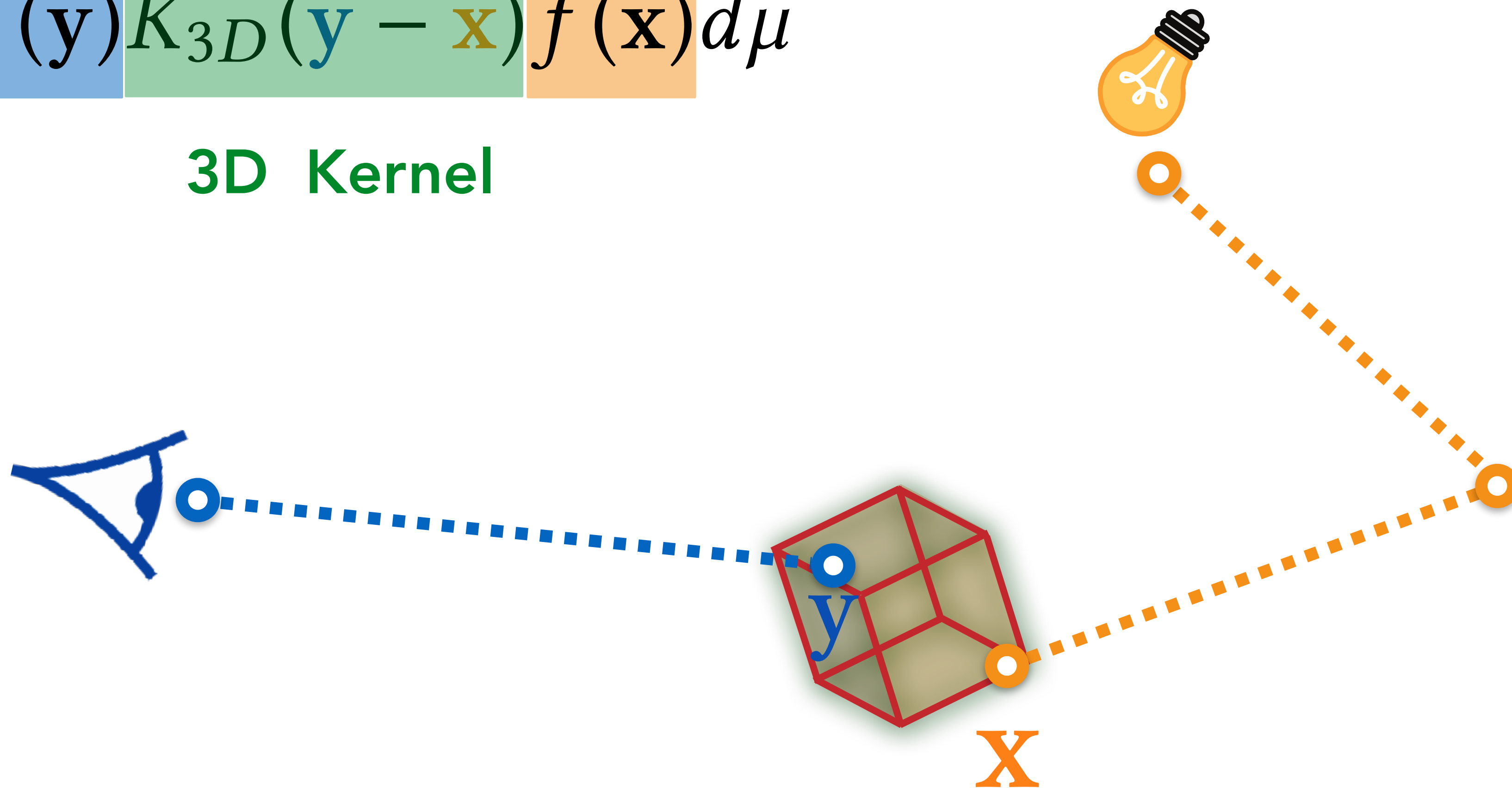
$$I = \int f(\bar{\mathbf{y}}) K_{3D}(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$



# Our approach

$$I = \int f(\bar{y}) K_{3D}(y - x) f(\bar{x}) d\mu$$

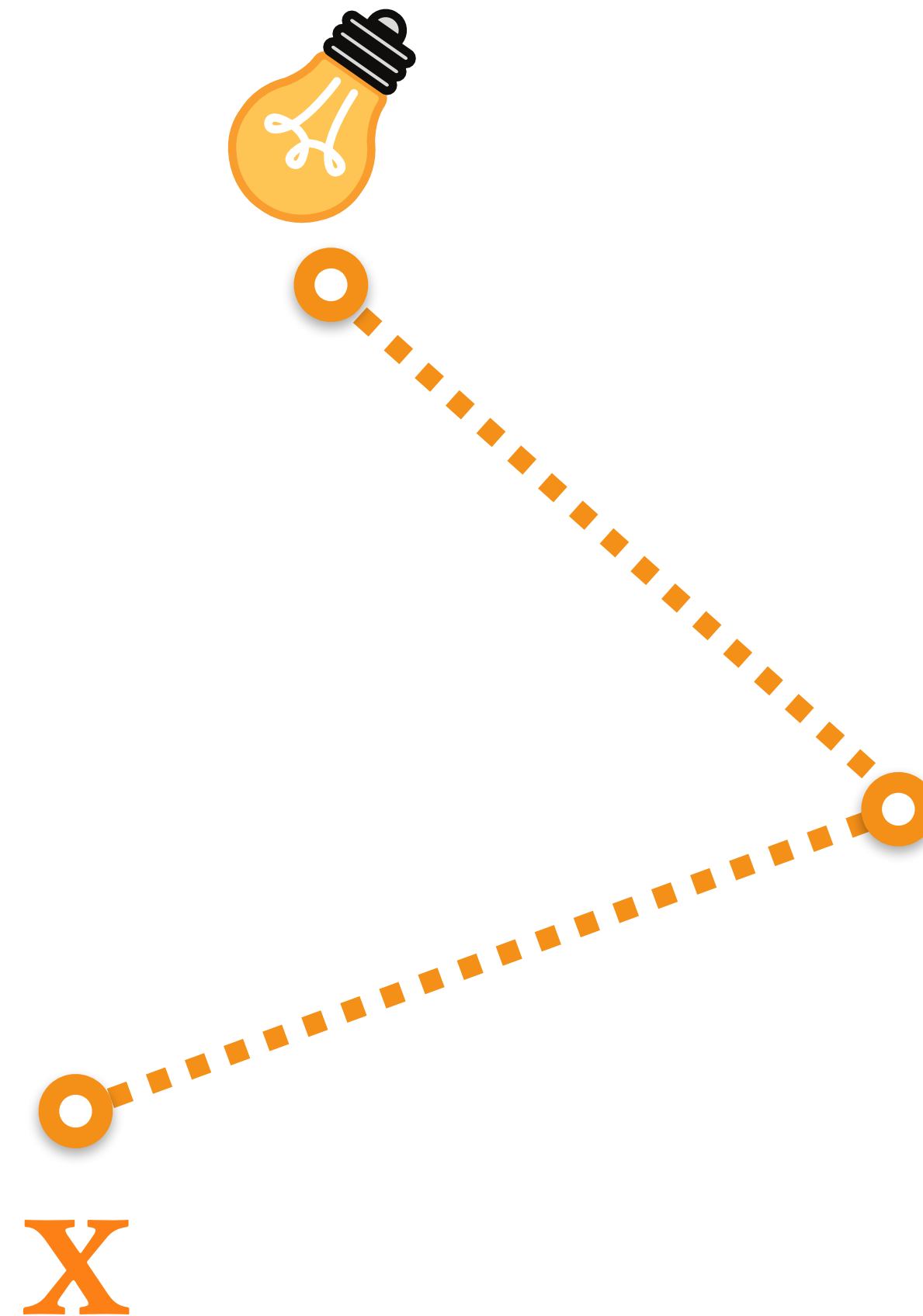
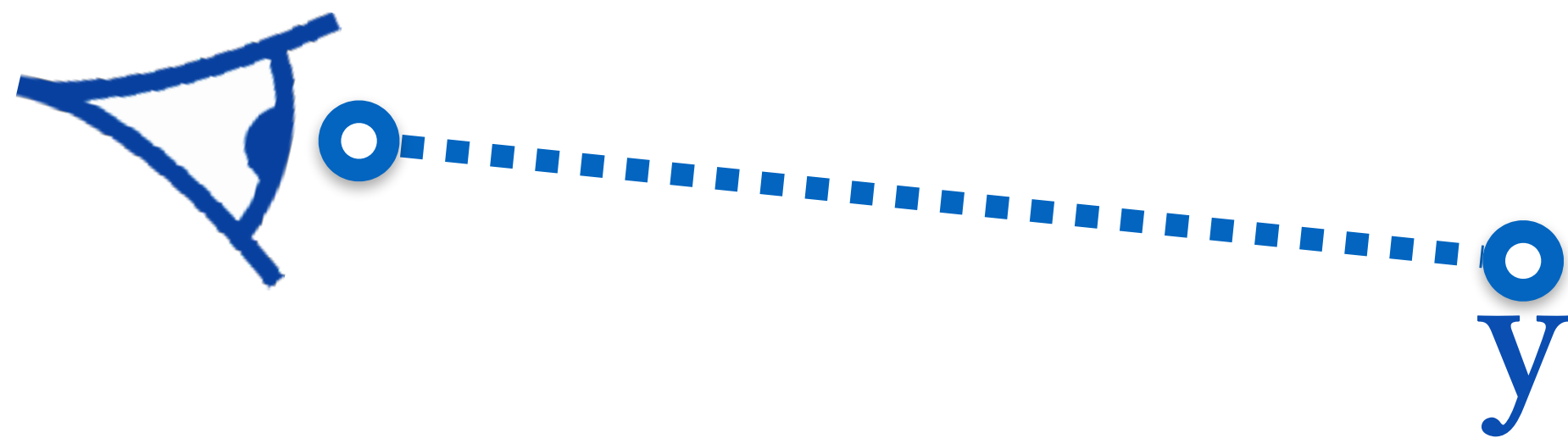
3D Kernel



# Our approach

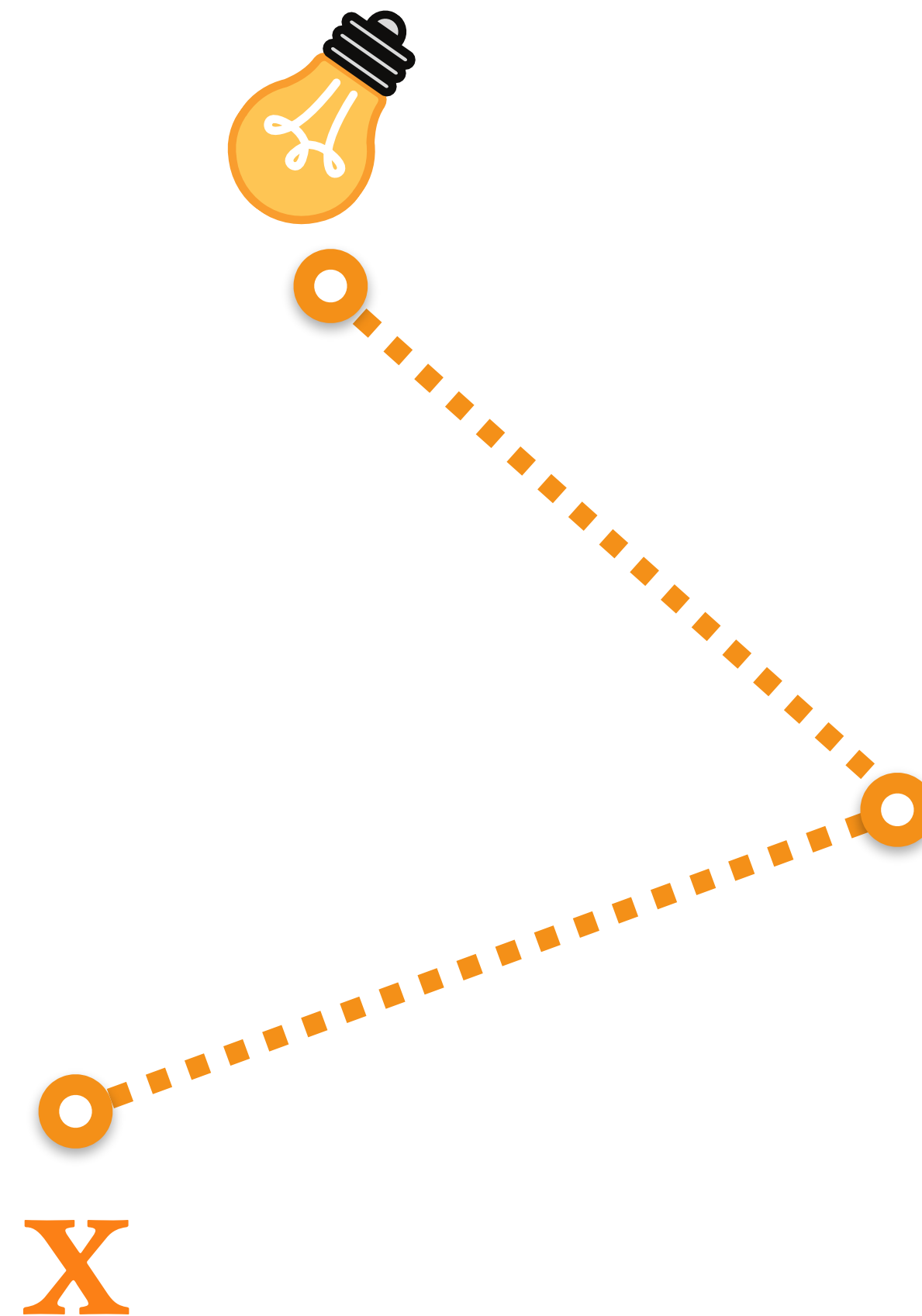
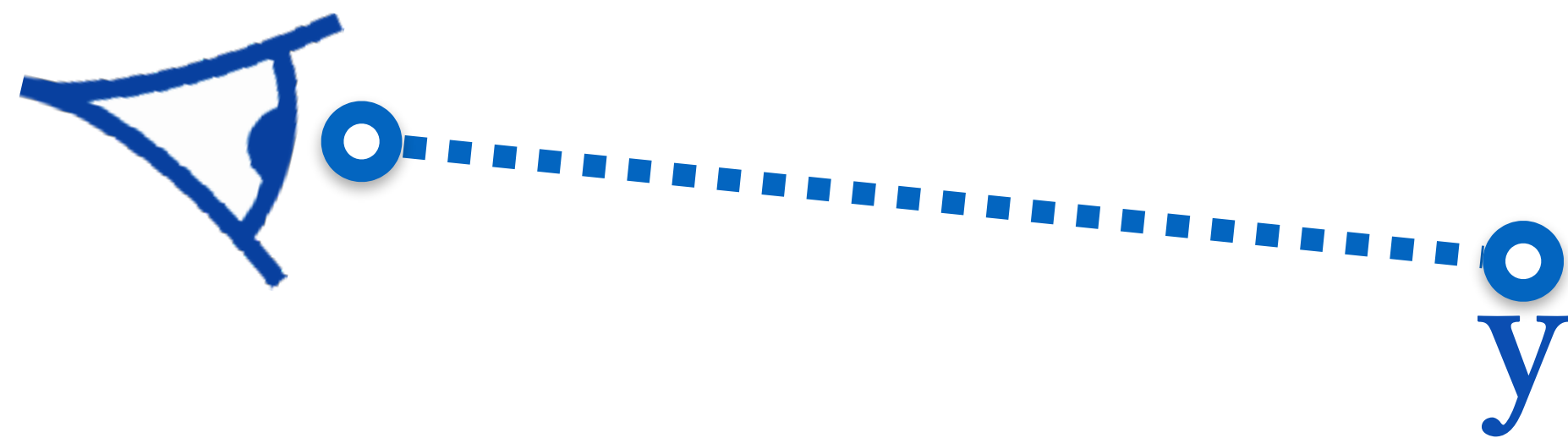
$$I = \int f(\bar{y}) K_{3D}(\mathbf{y} - \mathbf{x}) f(\bar{x}) d\mu$$

3D Kernel



# Our approach

$$I = \int f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$

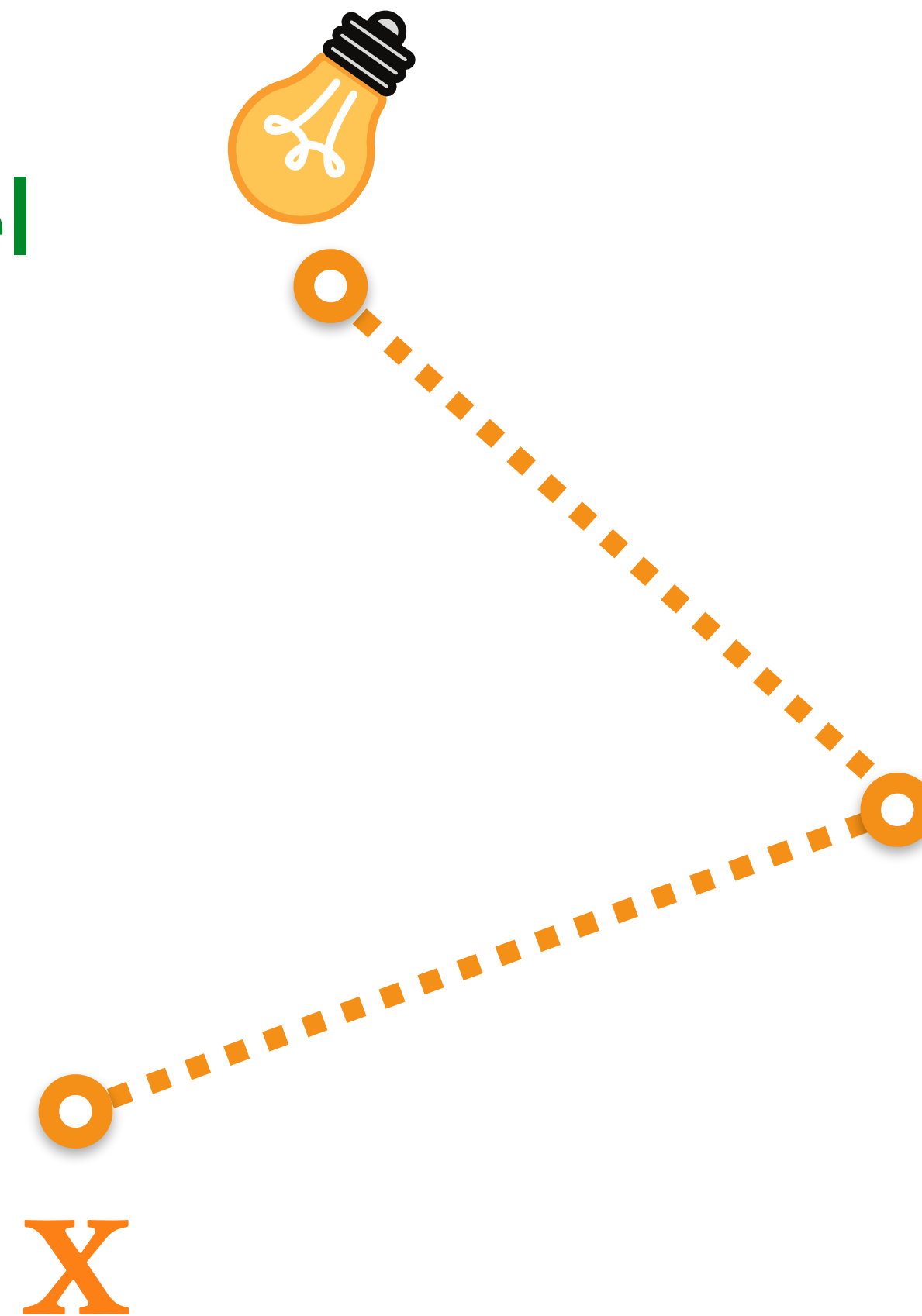
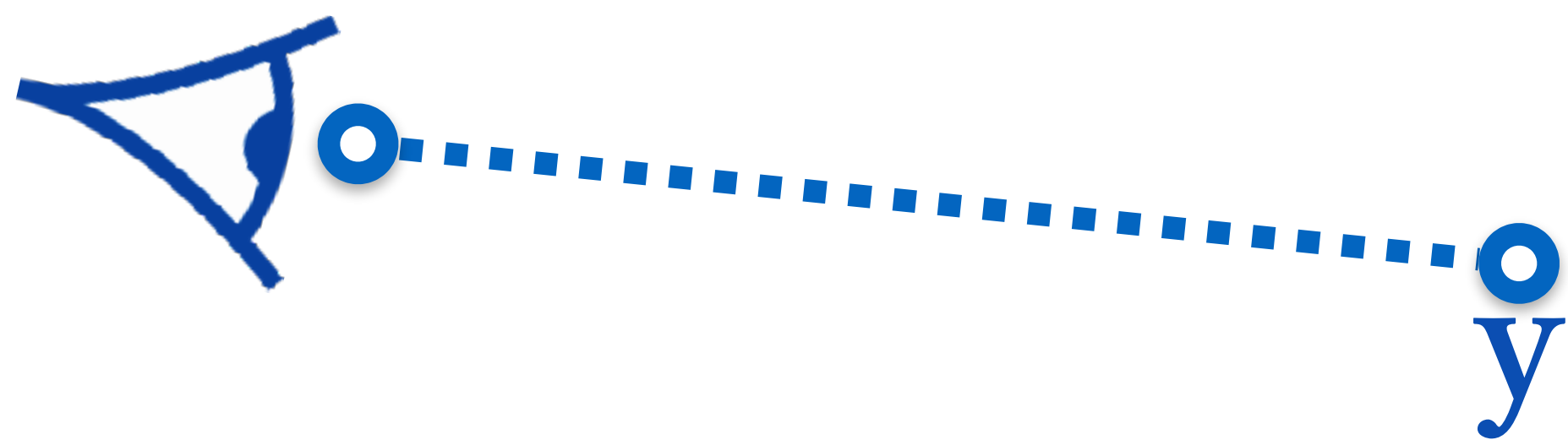




# Our approach

$$I = \int f(\bar{y}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$

3D delta kernel

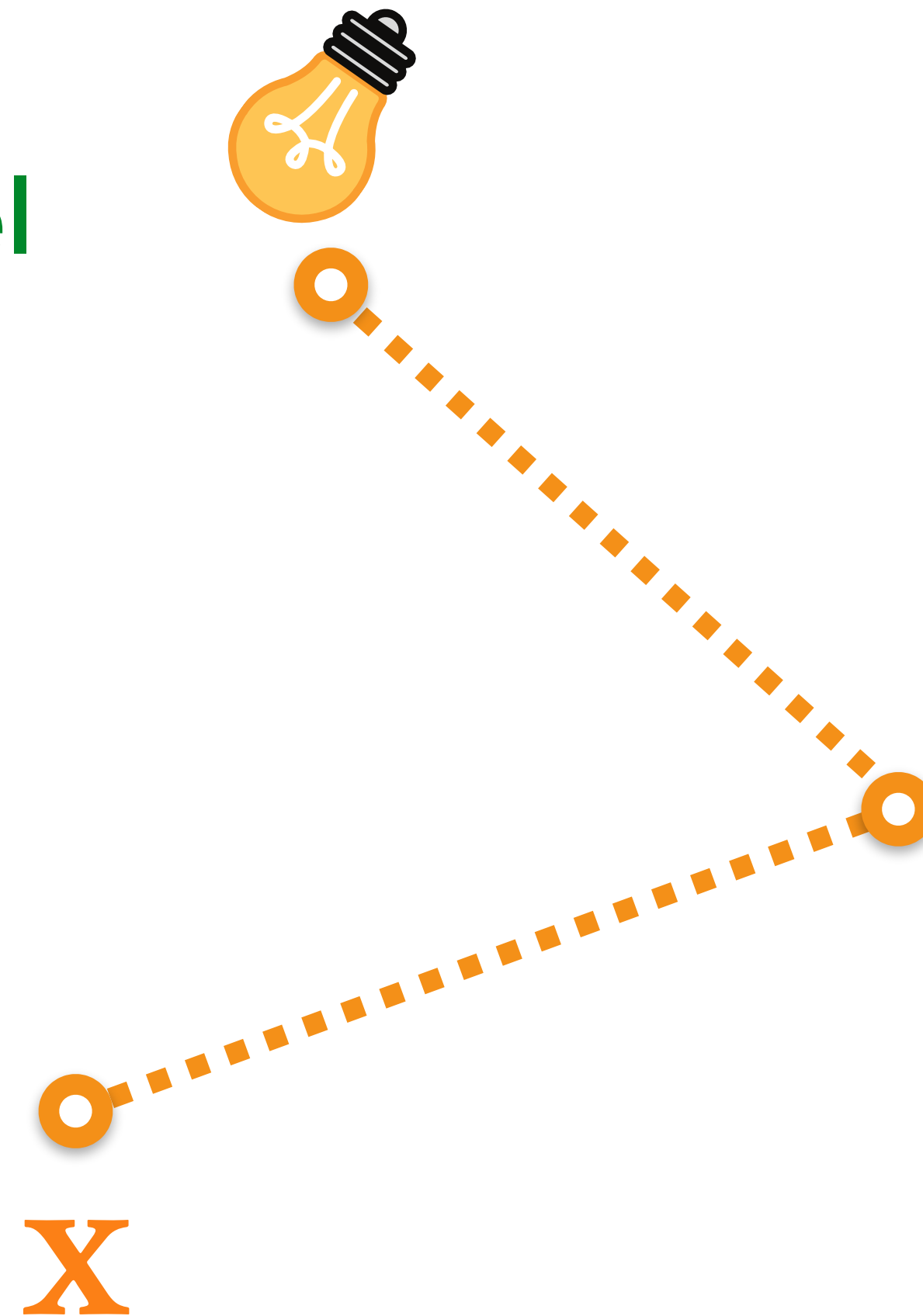
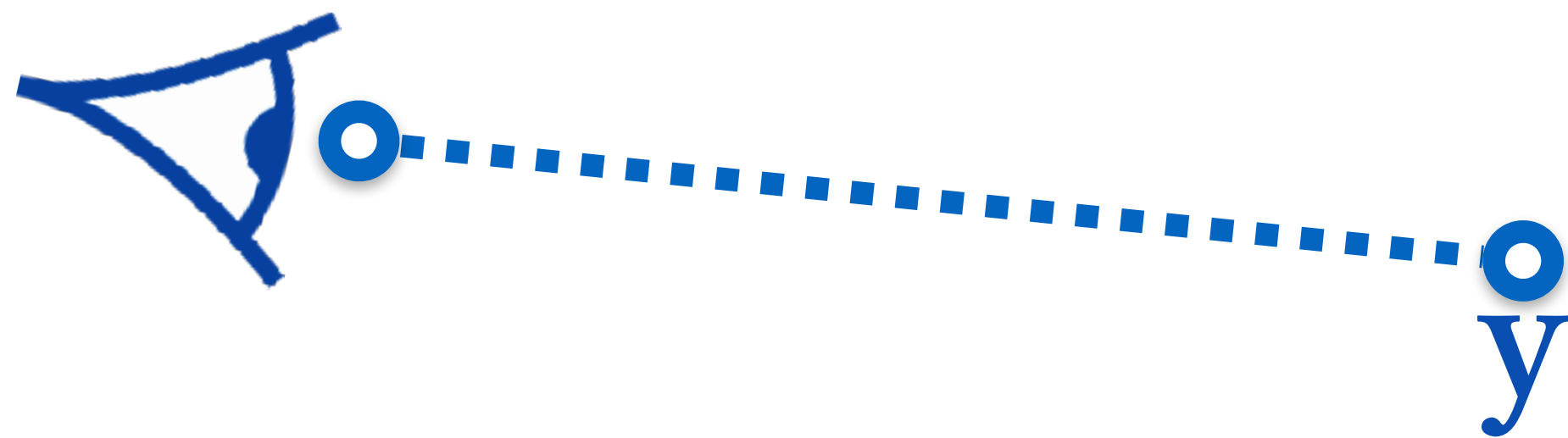


# Our approach

$$I = \int f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$

3D delta kernel

$$\approx \frac{f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}})}{p(\bar{\mathbf{y}}) p(\bar{\mathbf{x}})}$$



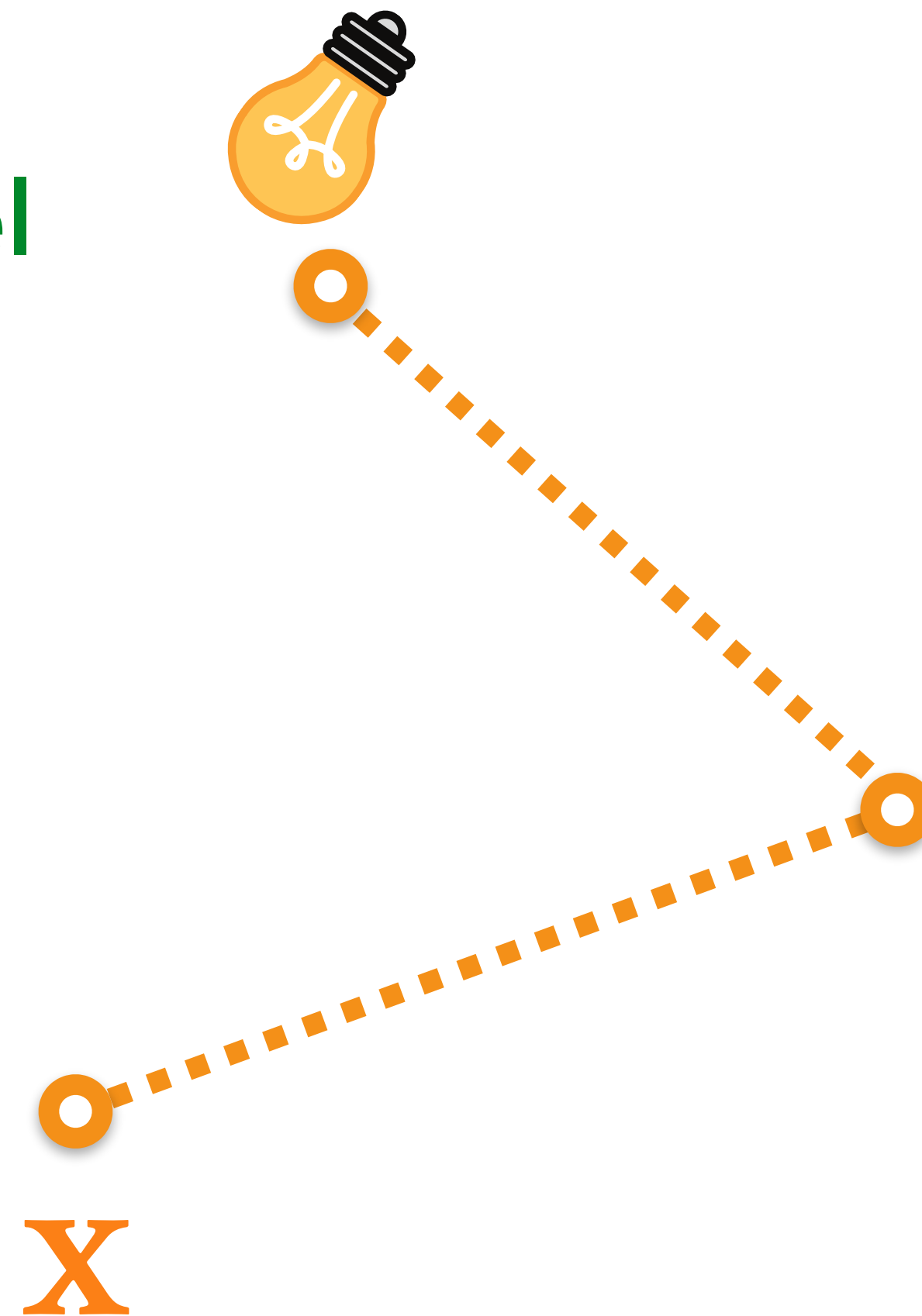
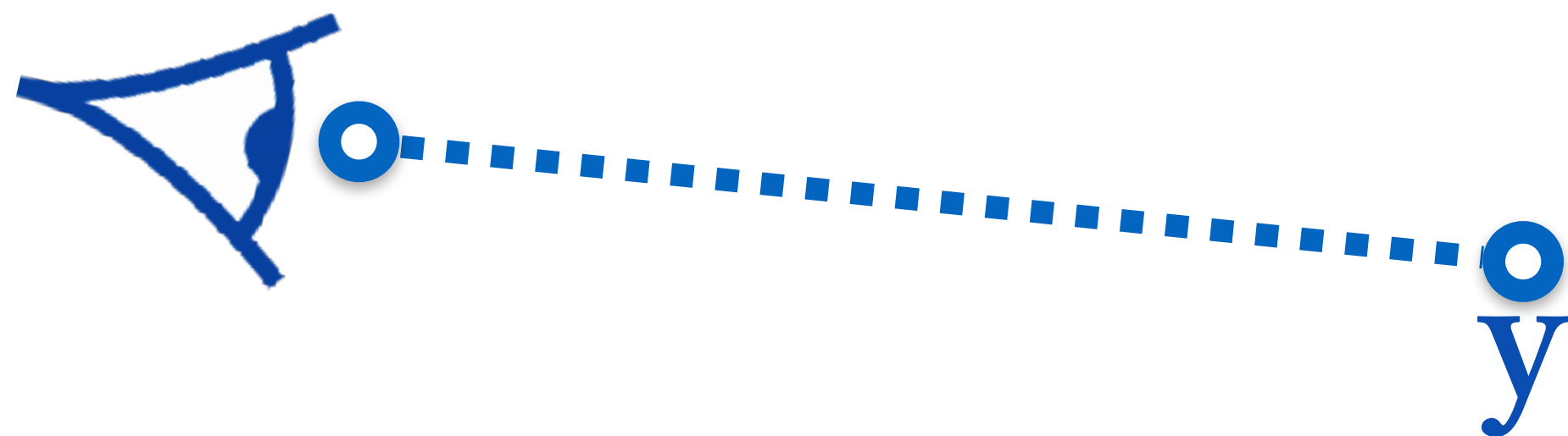
# Our approach

$$I = \int f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$

3D delta kernel

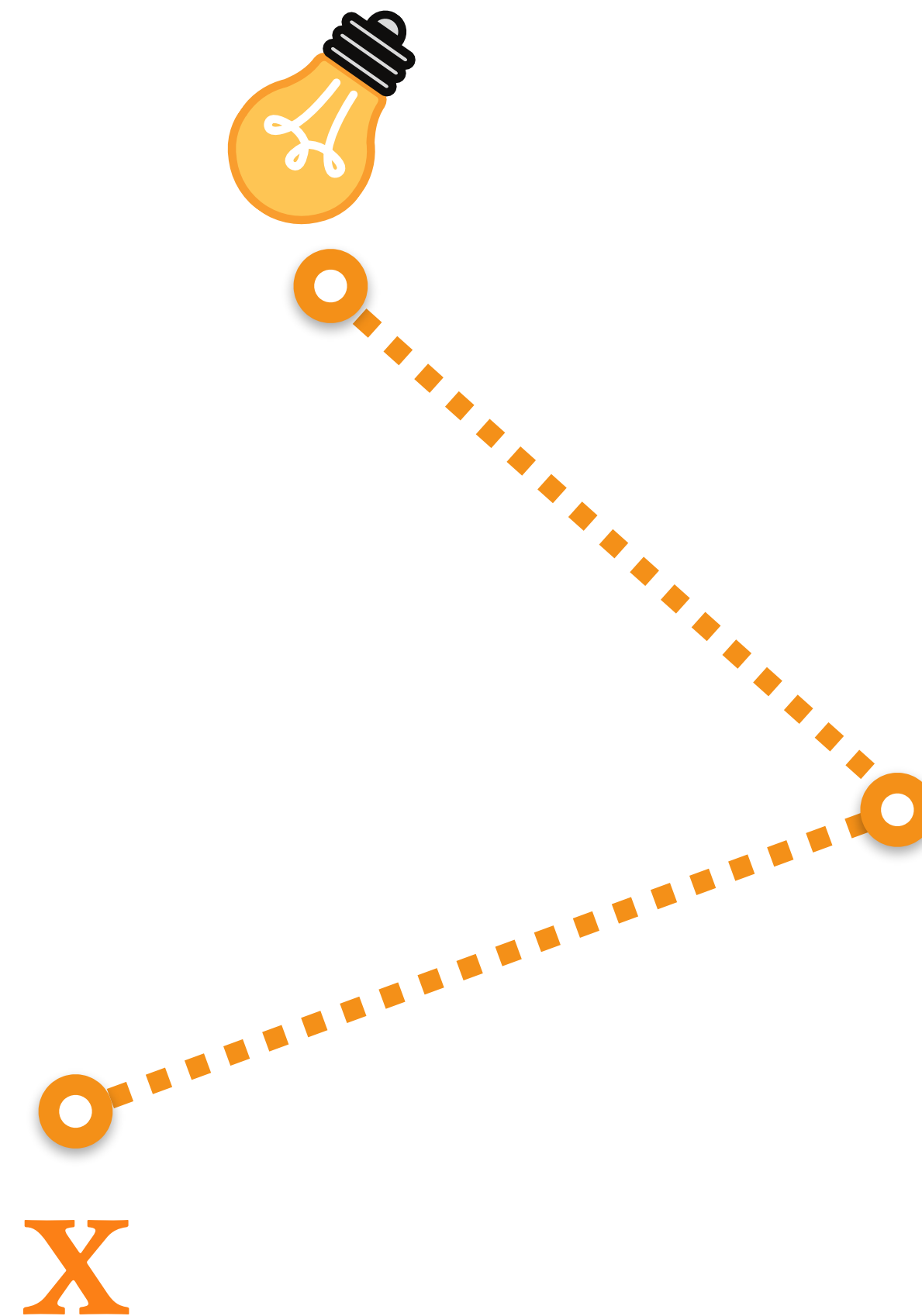
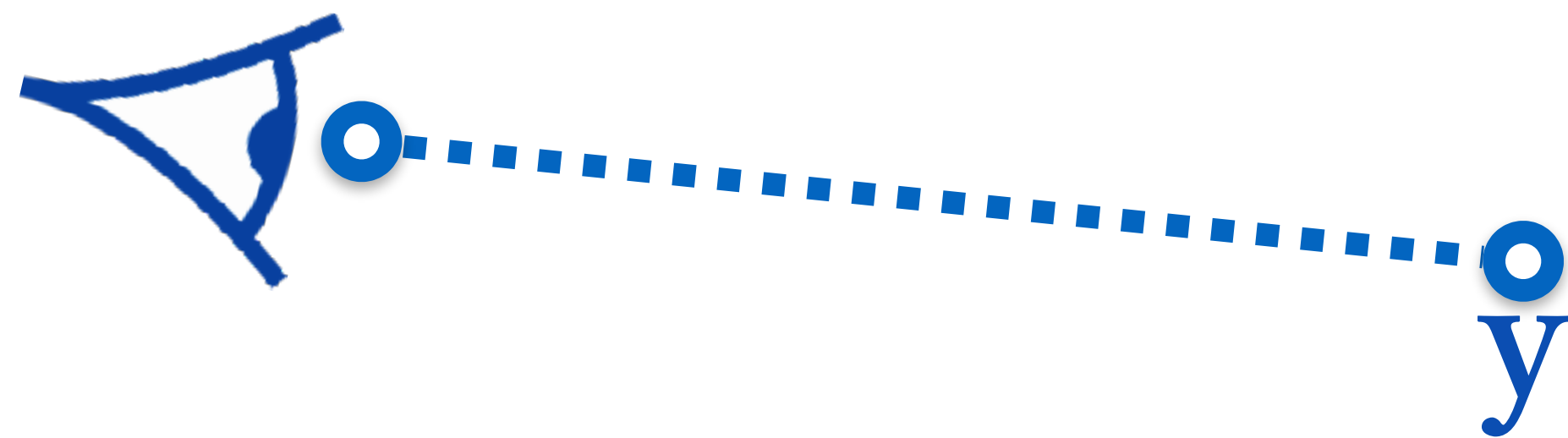
$$\approx \frac{f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}})}{p(\bar{\mathbf{y}}) p(\bar{\mathbf{x}})}$$

0 probability



# Our approach

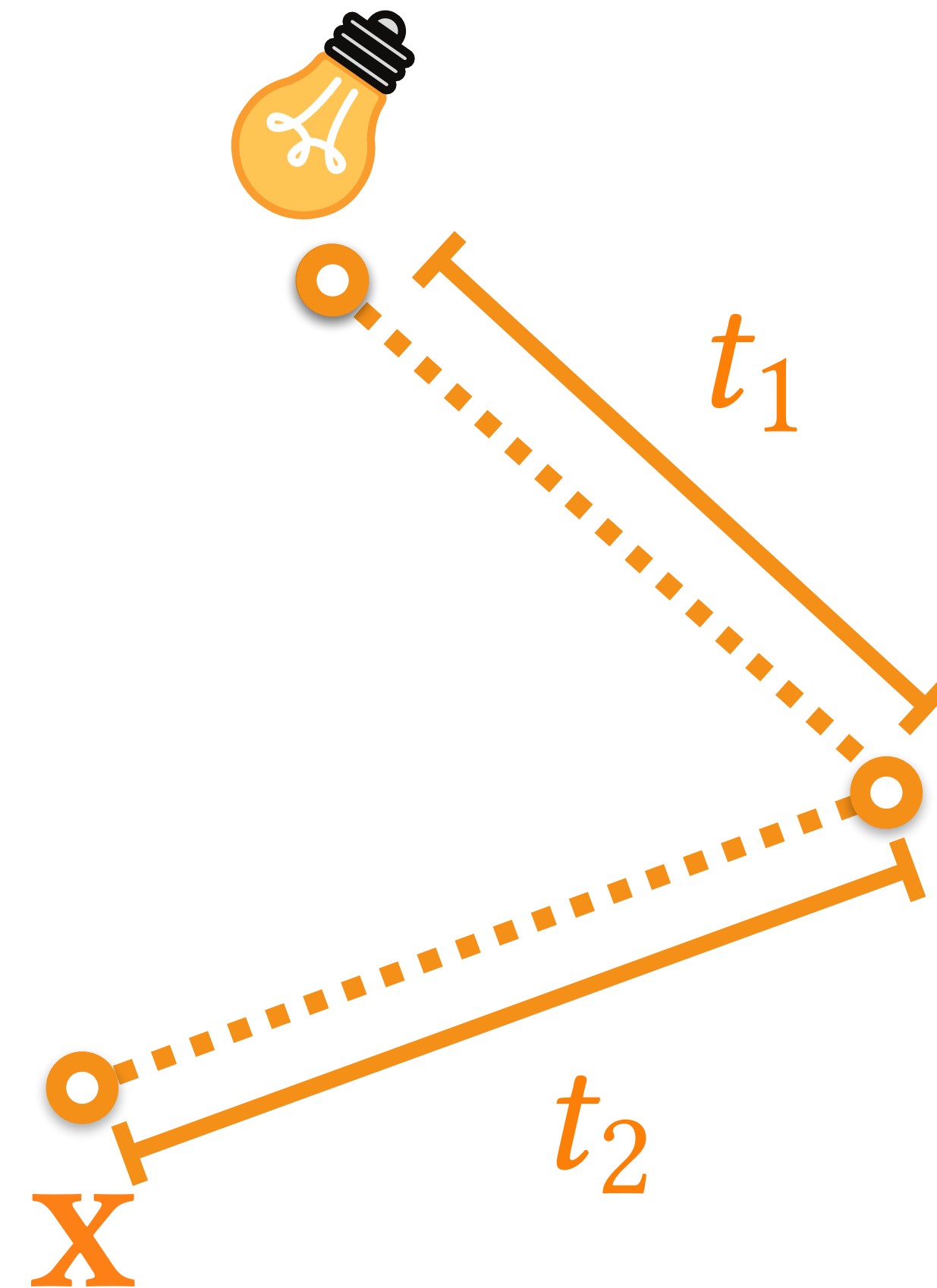
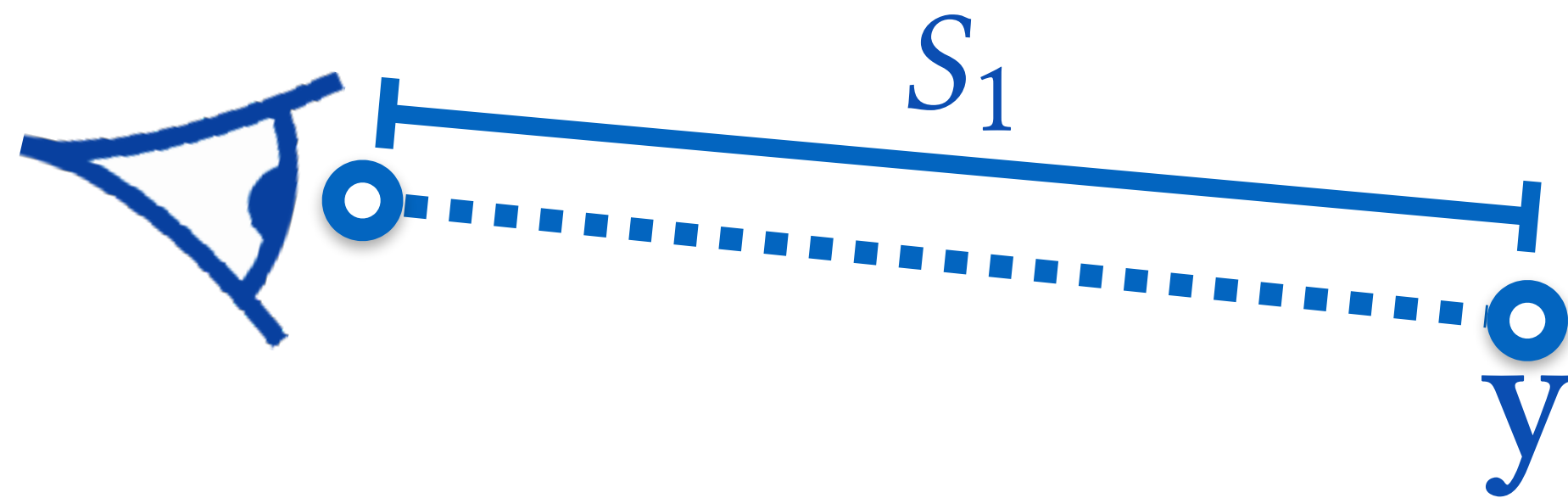
$$I = \int f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$



# Our approach

$$I = \int f(\bar{\mathbf{y}}) \delta(\mathbf{y} - \mathbf{x}) f(\bar{\mathbf{x}}) d\mu$$

split integral

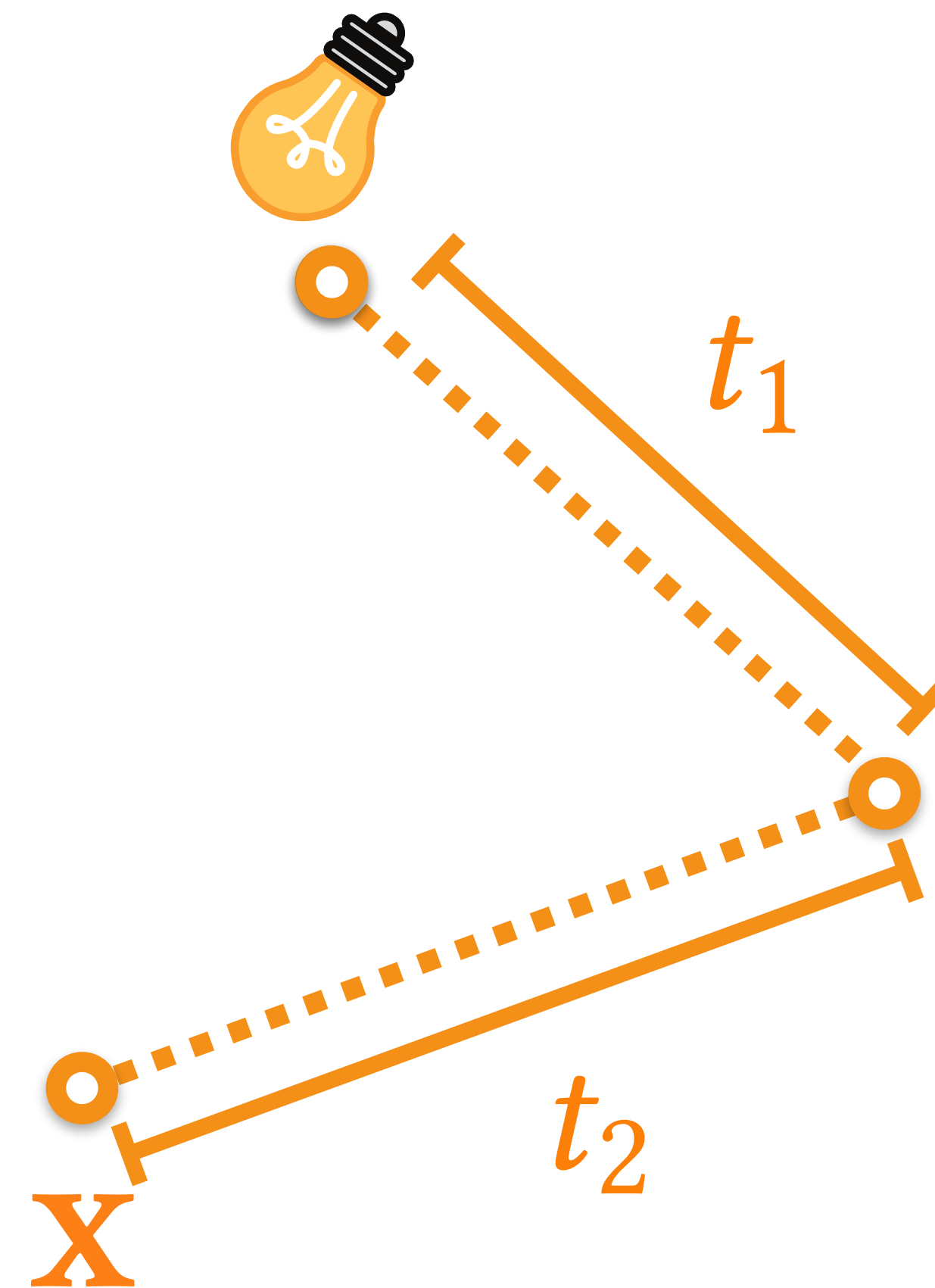
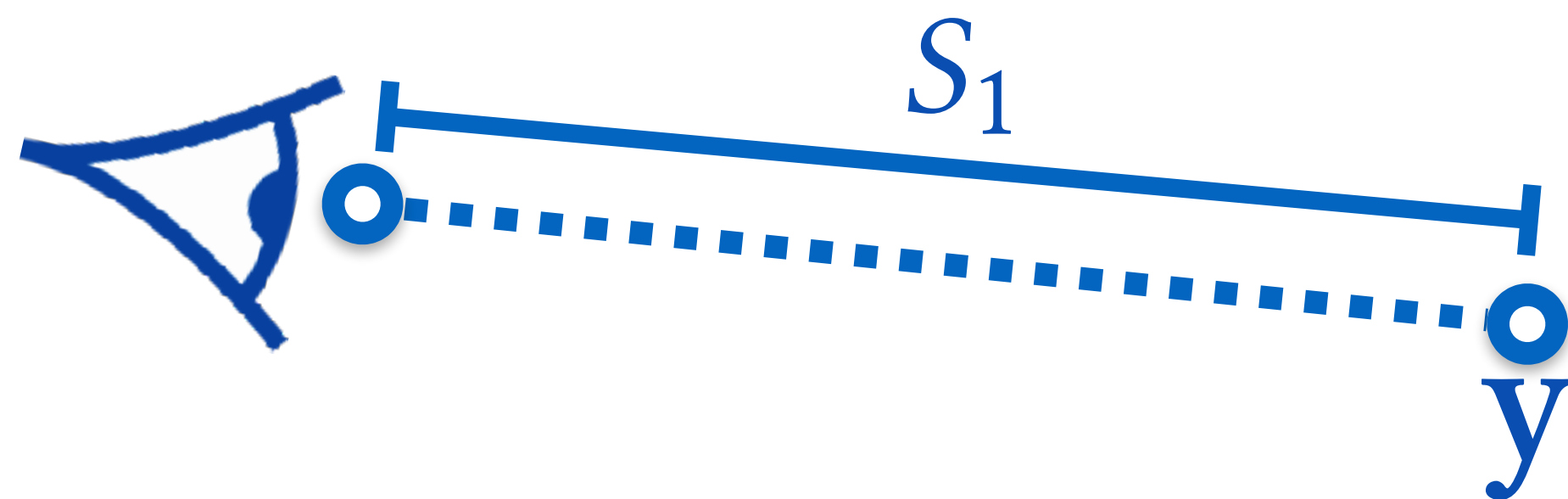


# Our approach

$$I = \int \cdots \int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1 \cdots d\mu'$$

split integral

pre-integrate  
3 dimensions

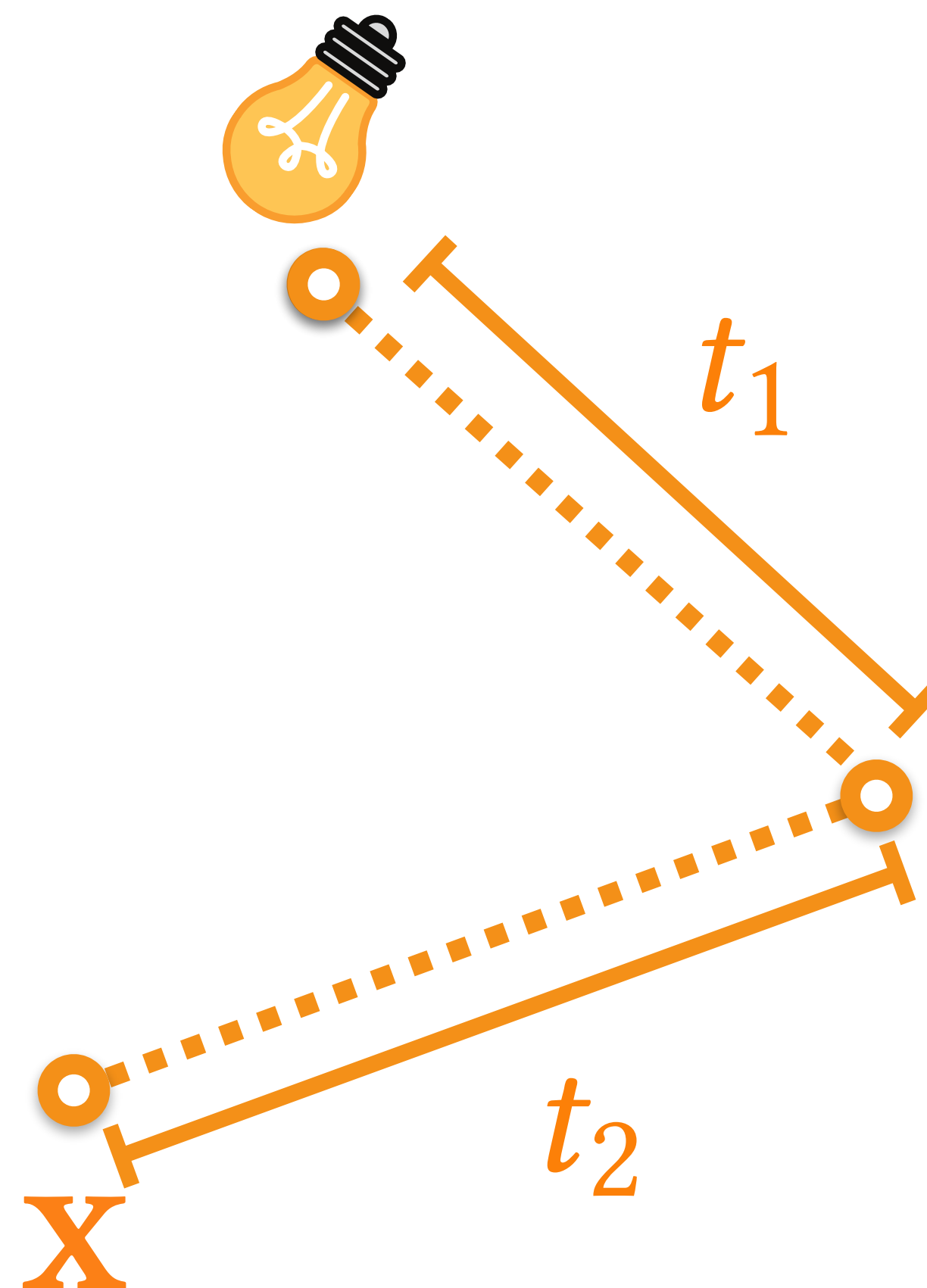
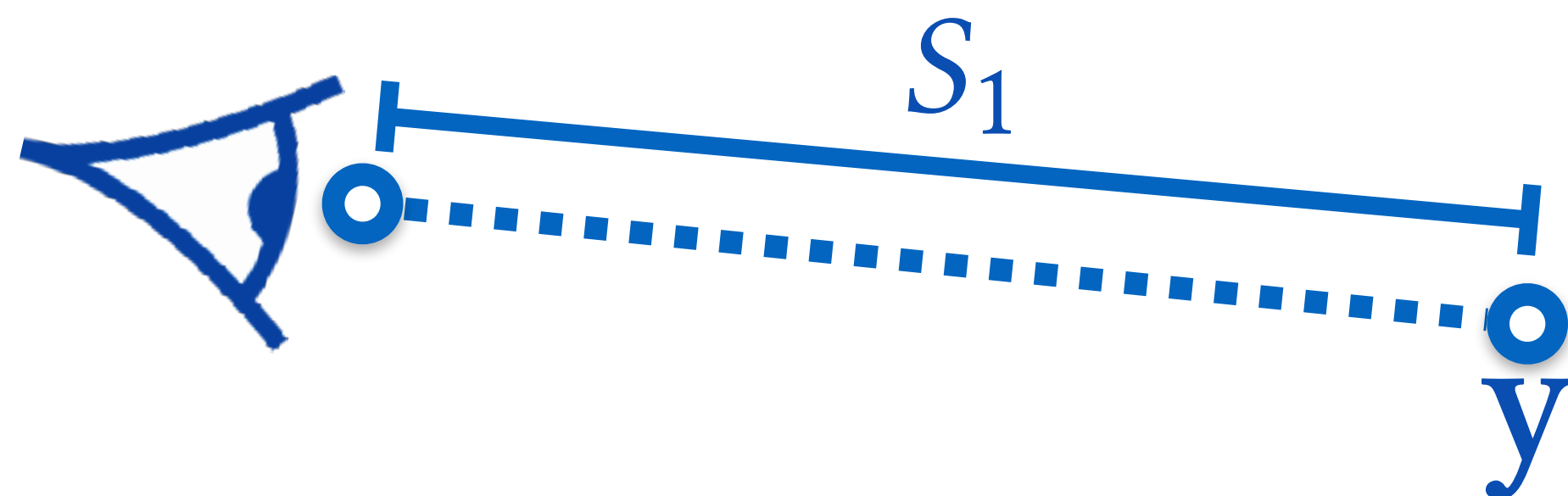


# Our approach

$$I = \int \dots J \dots d\mu'$$

split integral

pre-integrate  
3 dimensions



# Our approach

$$I = \int \dots$$

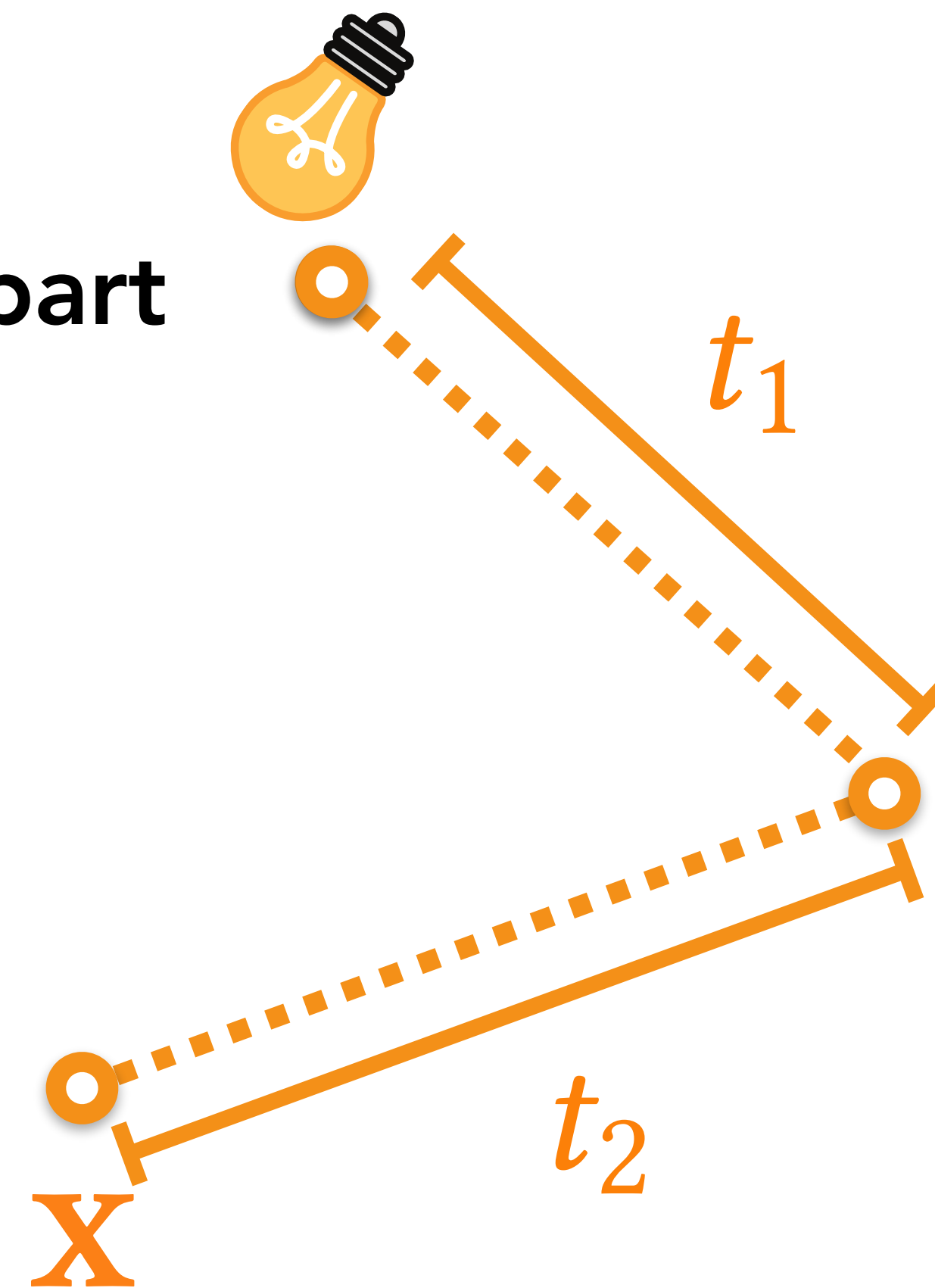
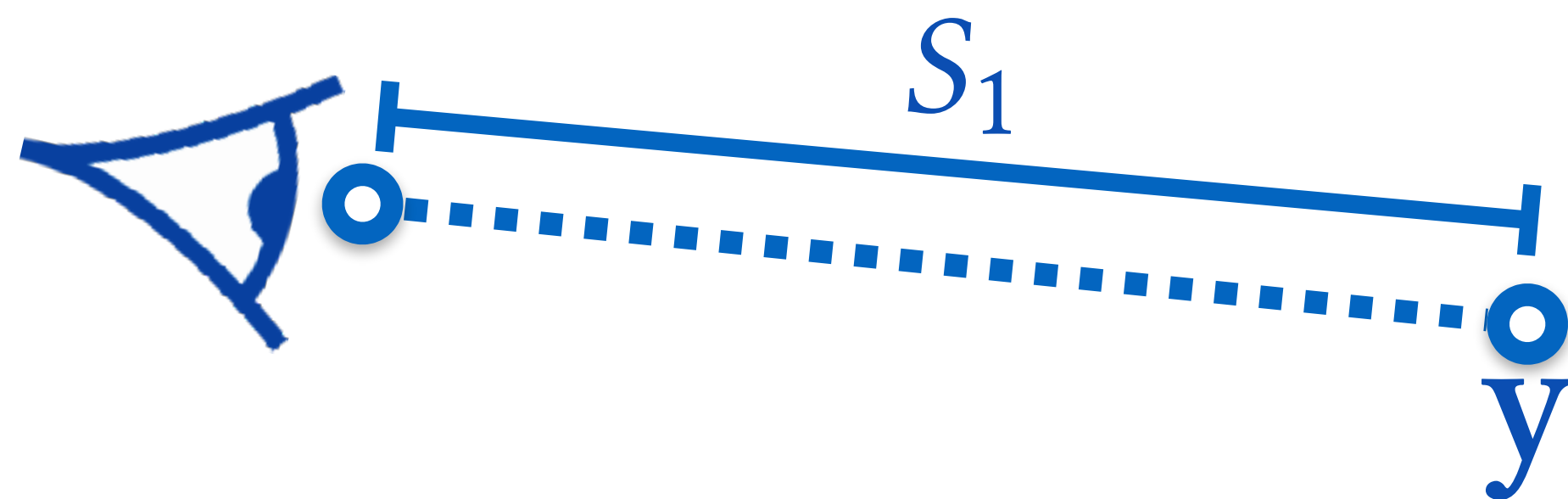
split integral

J

pre-integrate  
3 dimensions

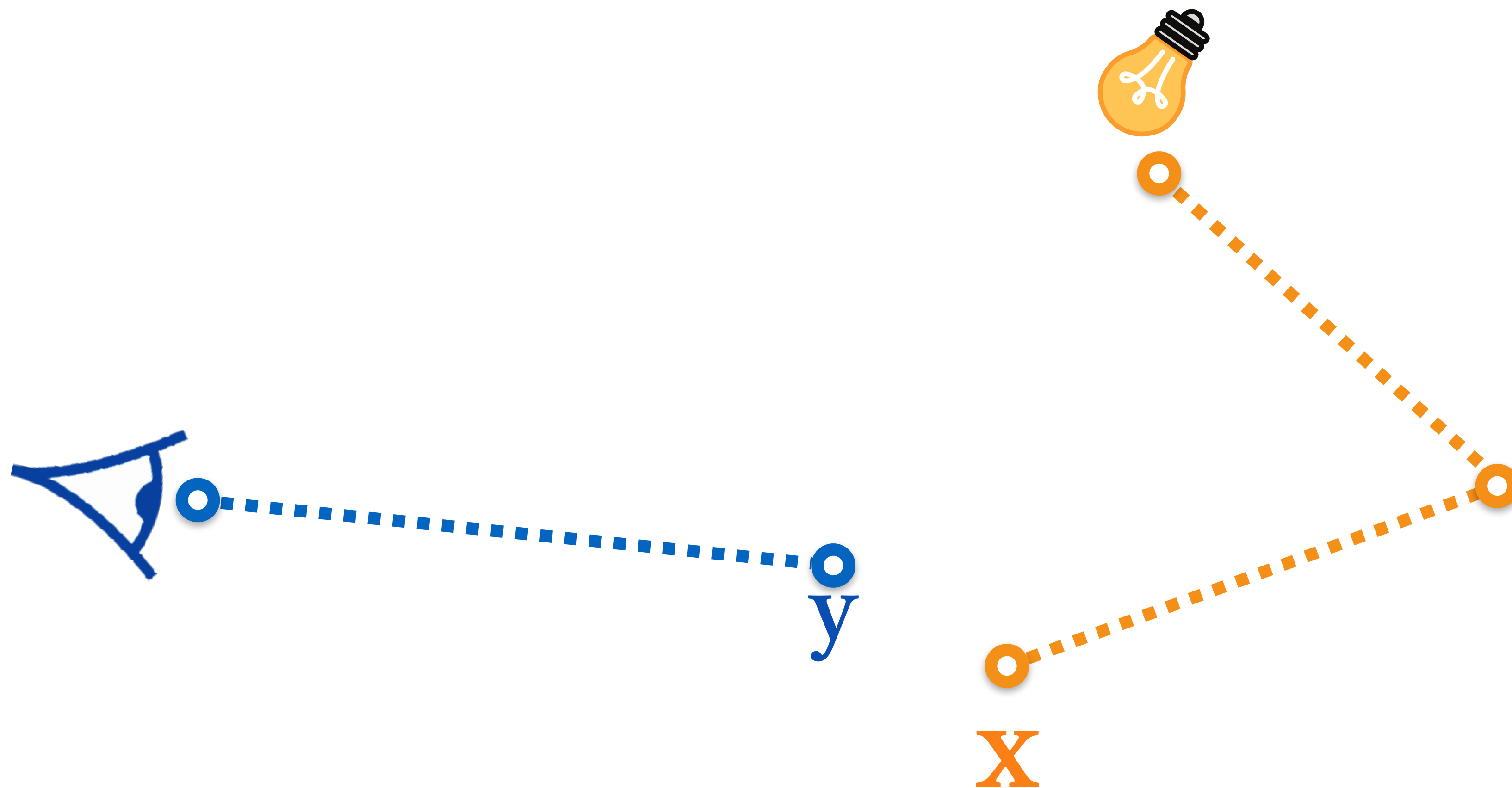
$$\dots d\mu'$$

remain part



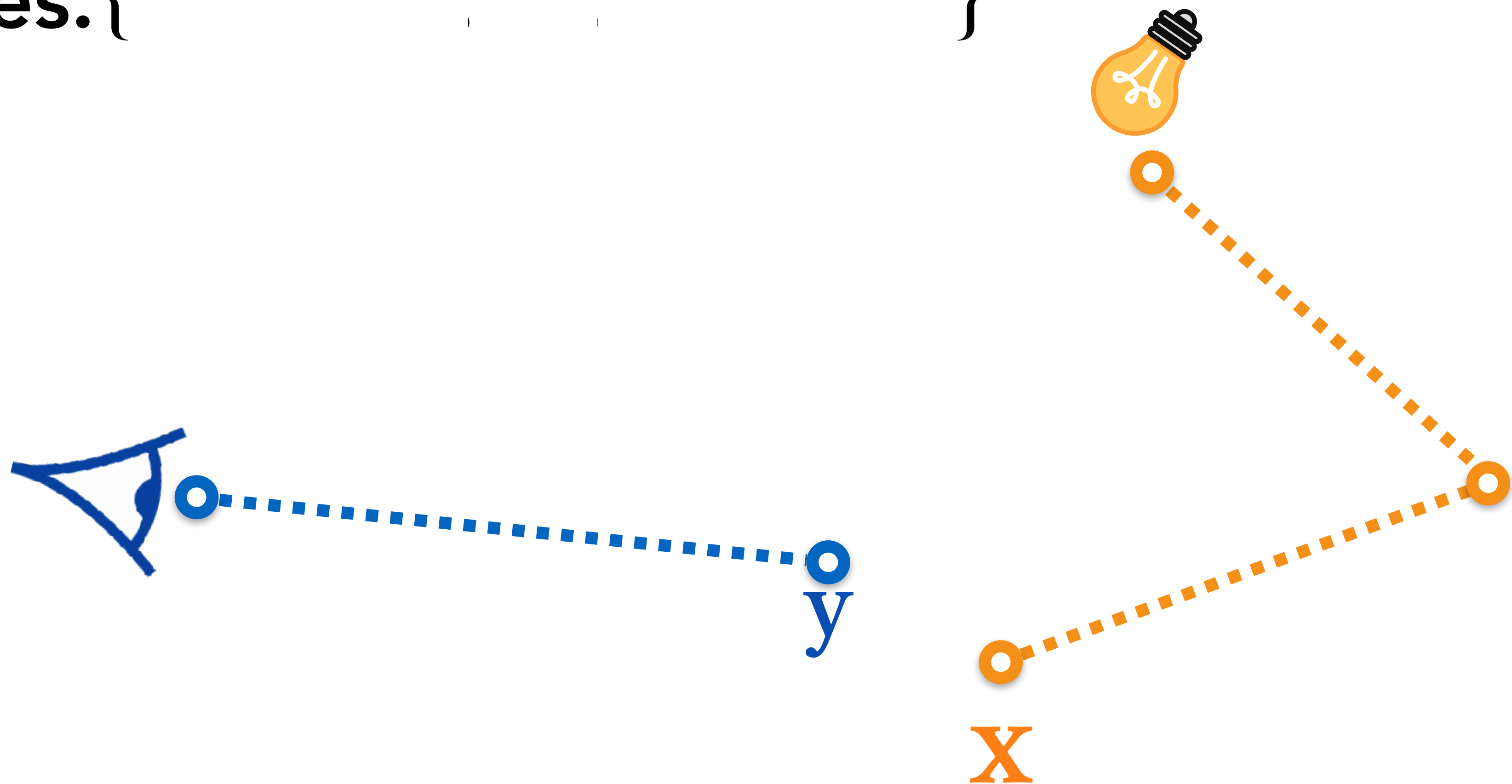


# Our approach



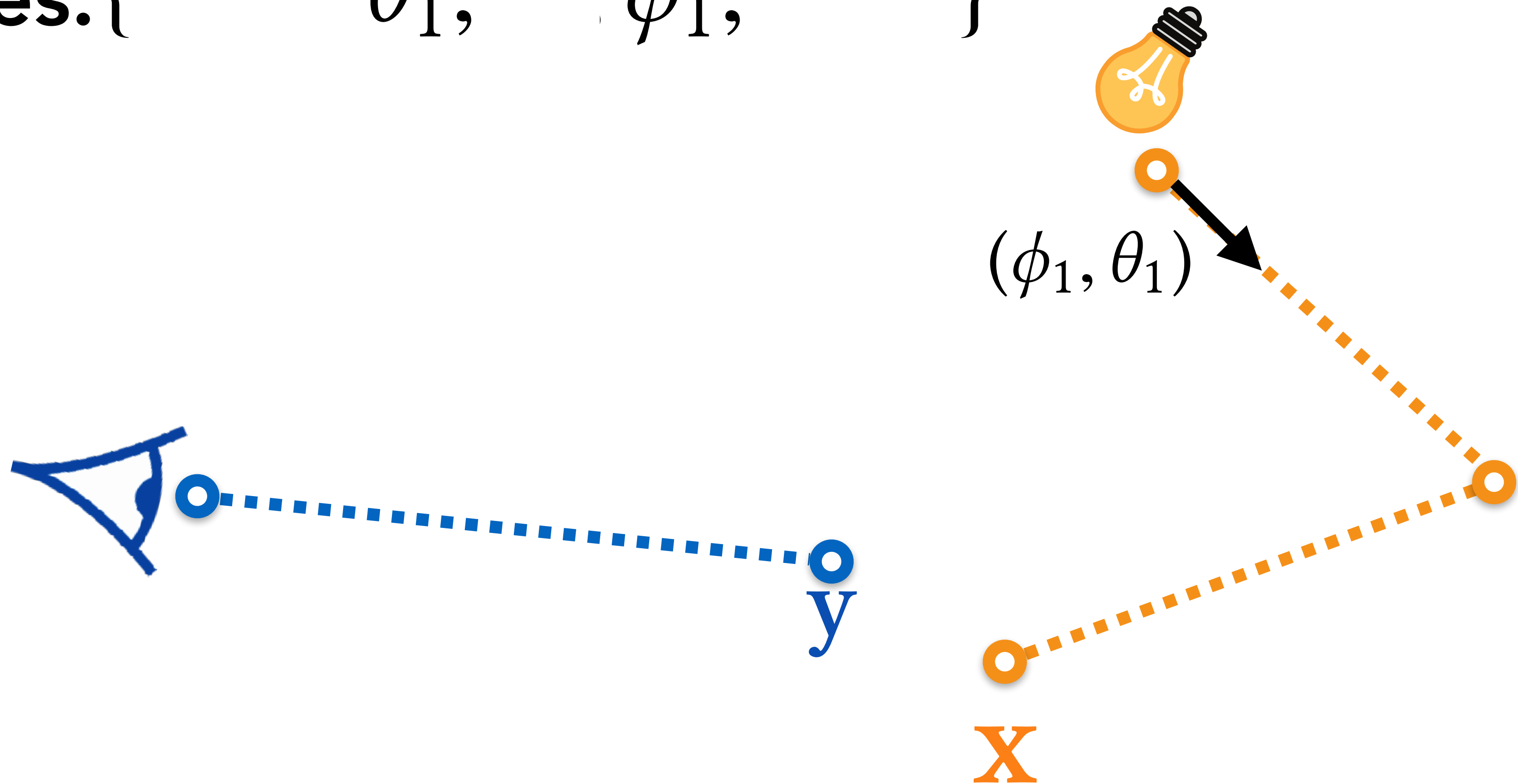
# Our approach

Variables: { , , } }



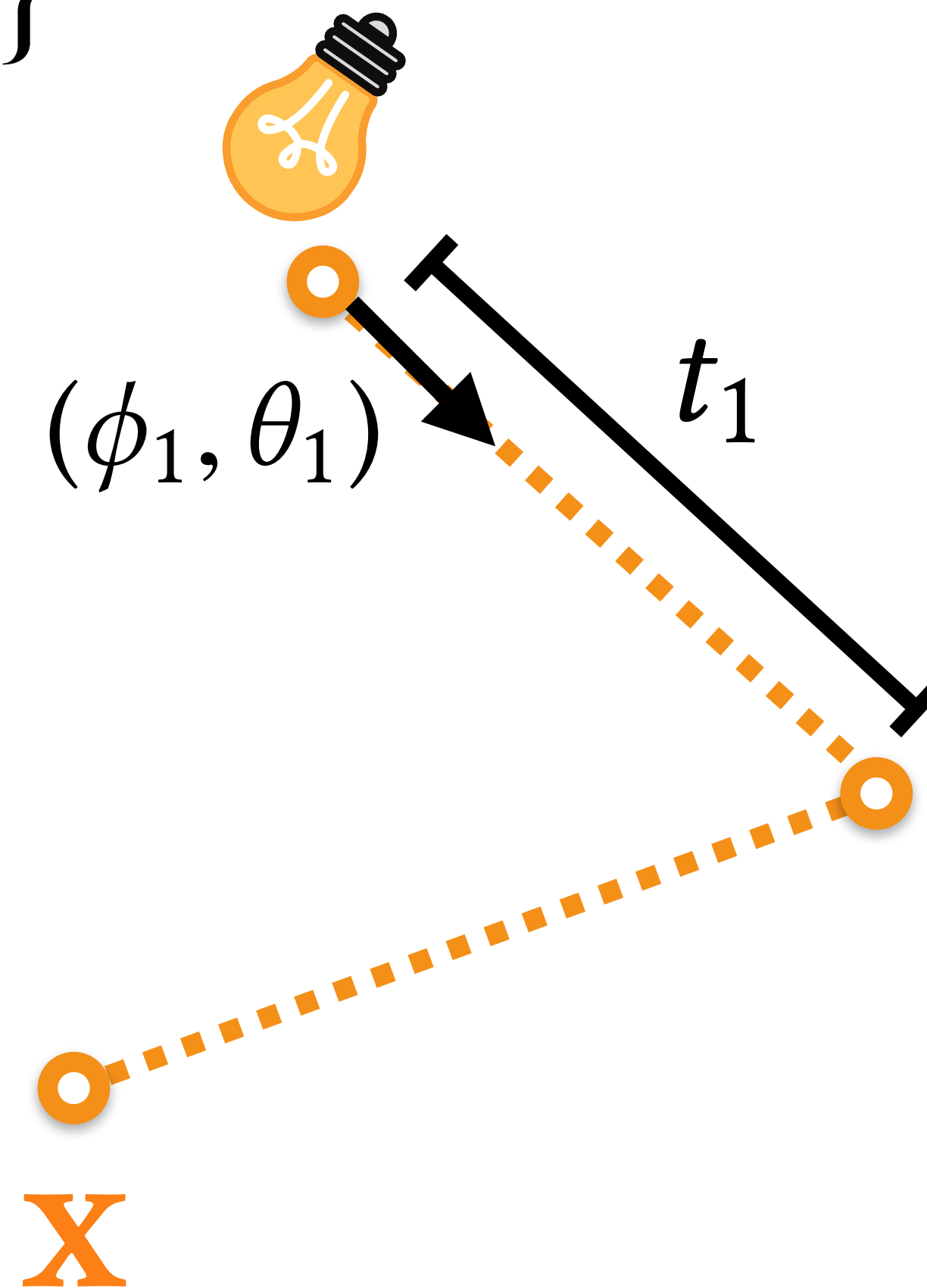
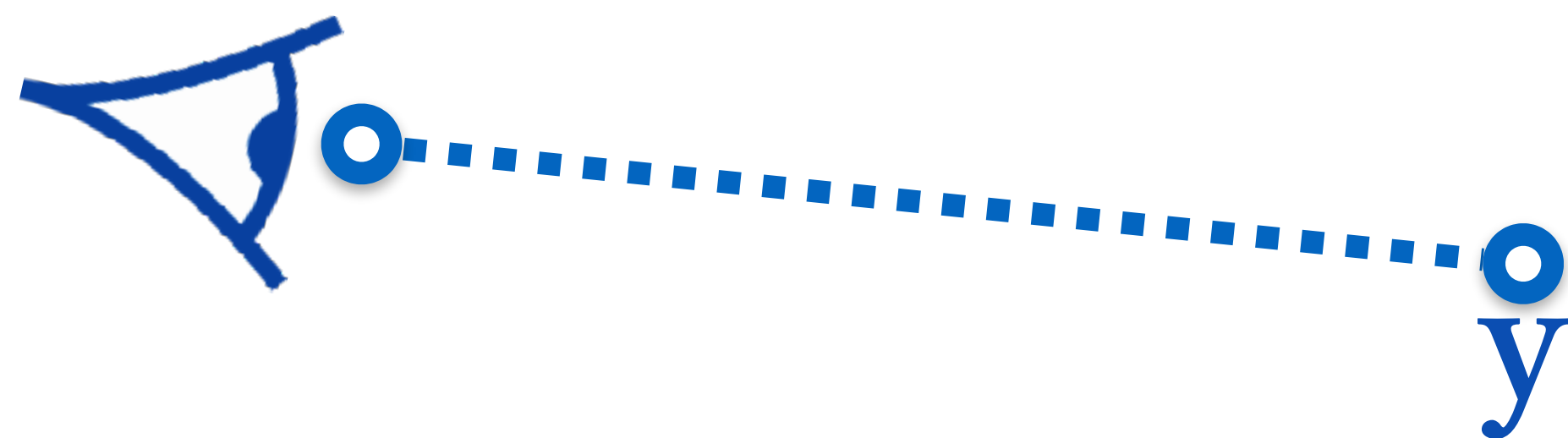
# Our approach

Variables:  $\{ \theta_1, \phi_1, \}$



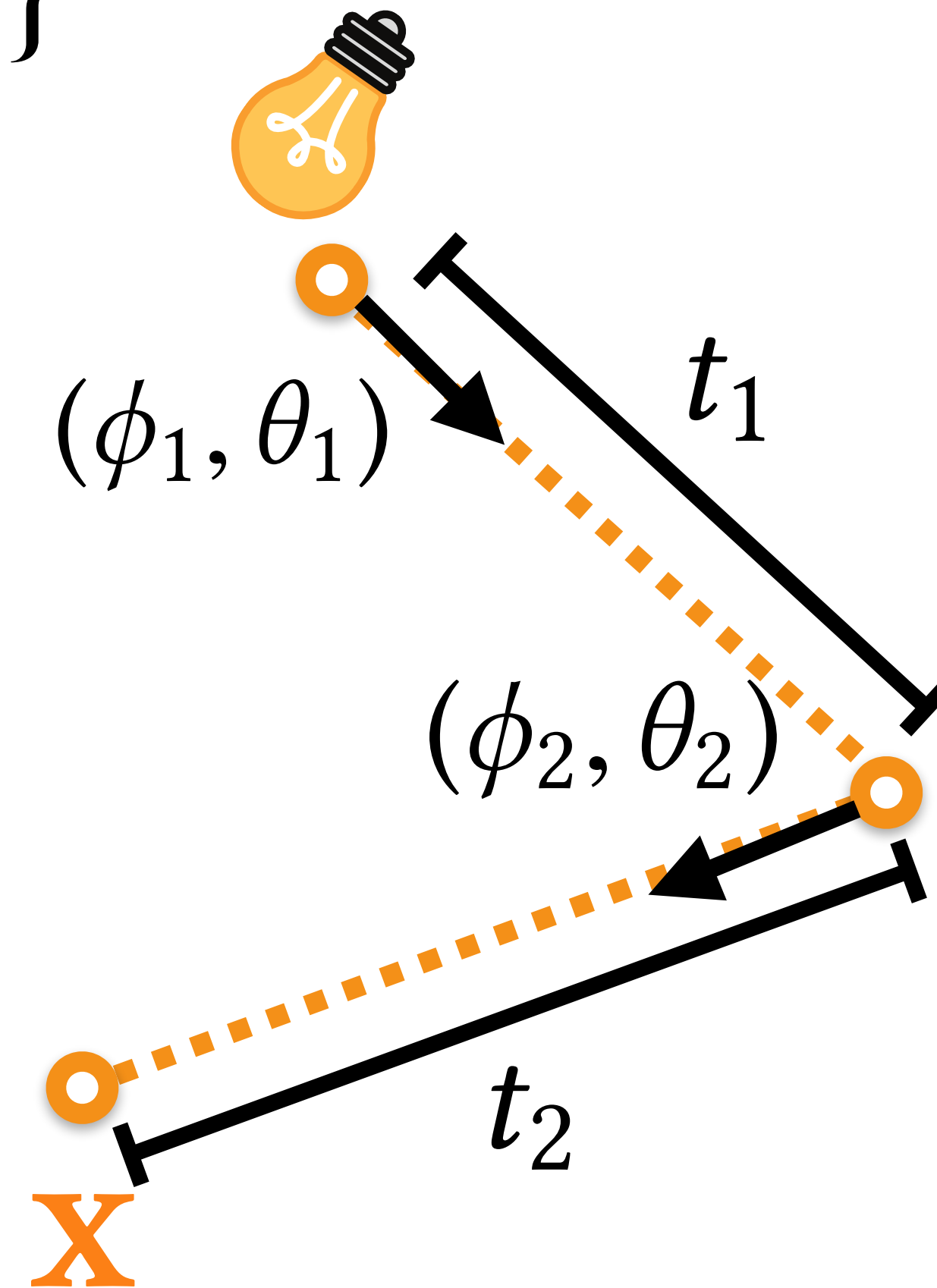
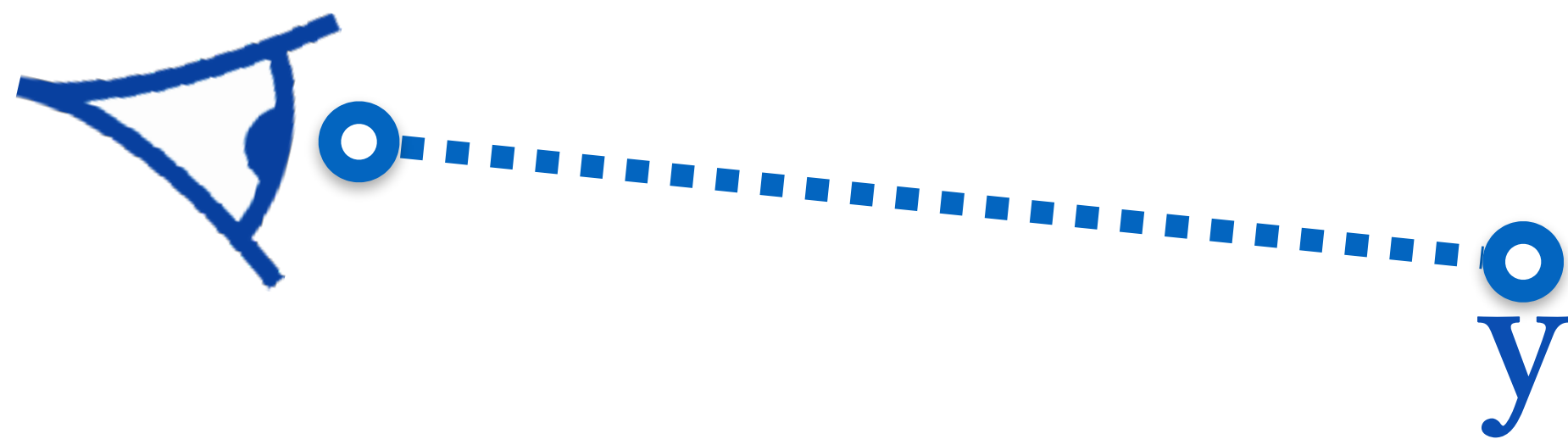
# Our approach

Variables:  $\{t_1, \theta_1, \phi_1, y\}$



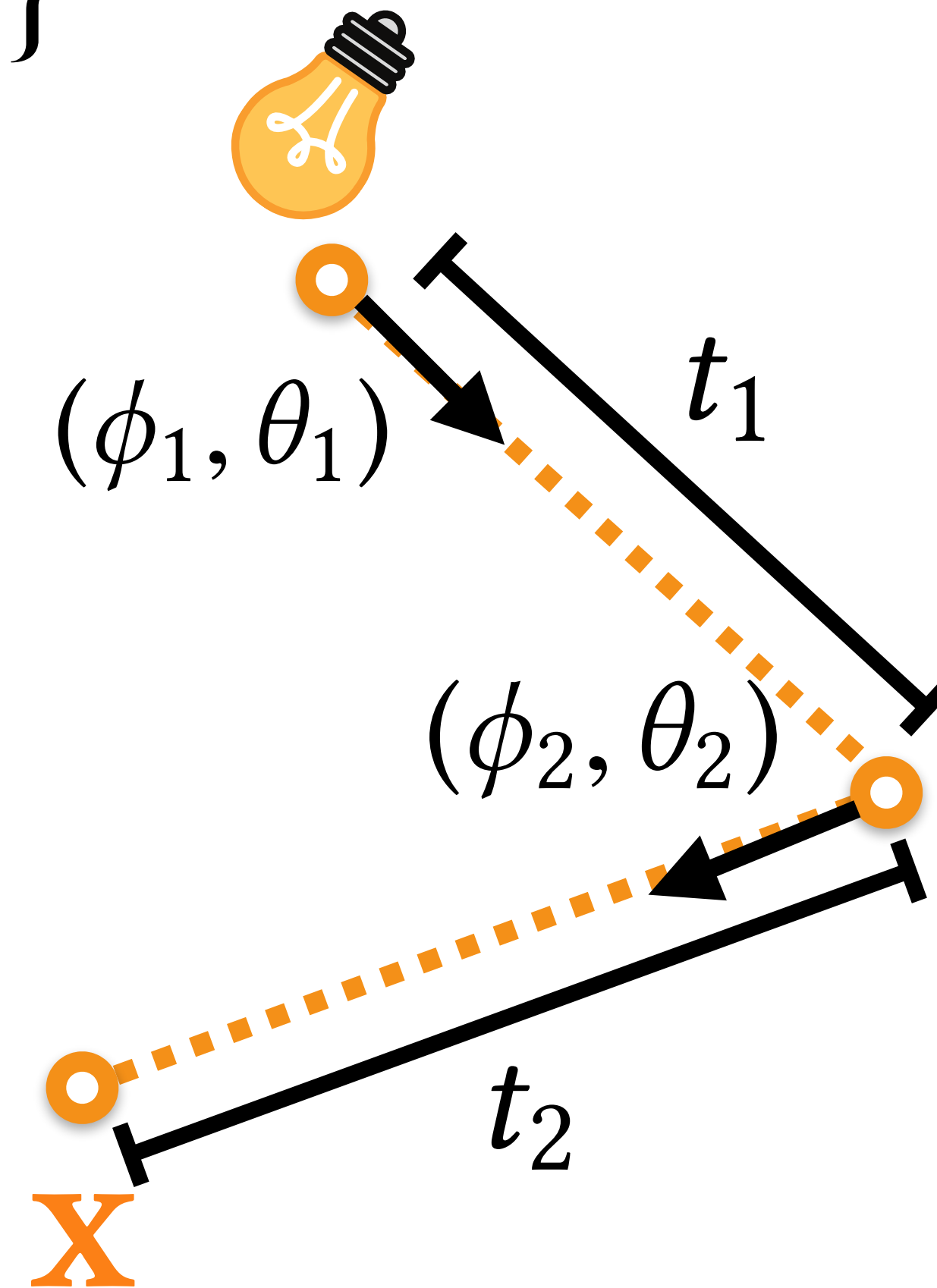
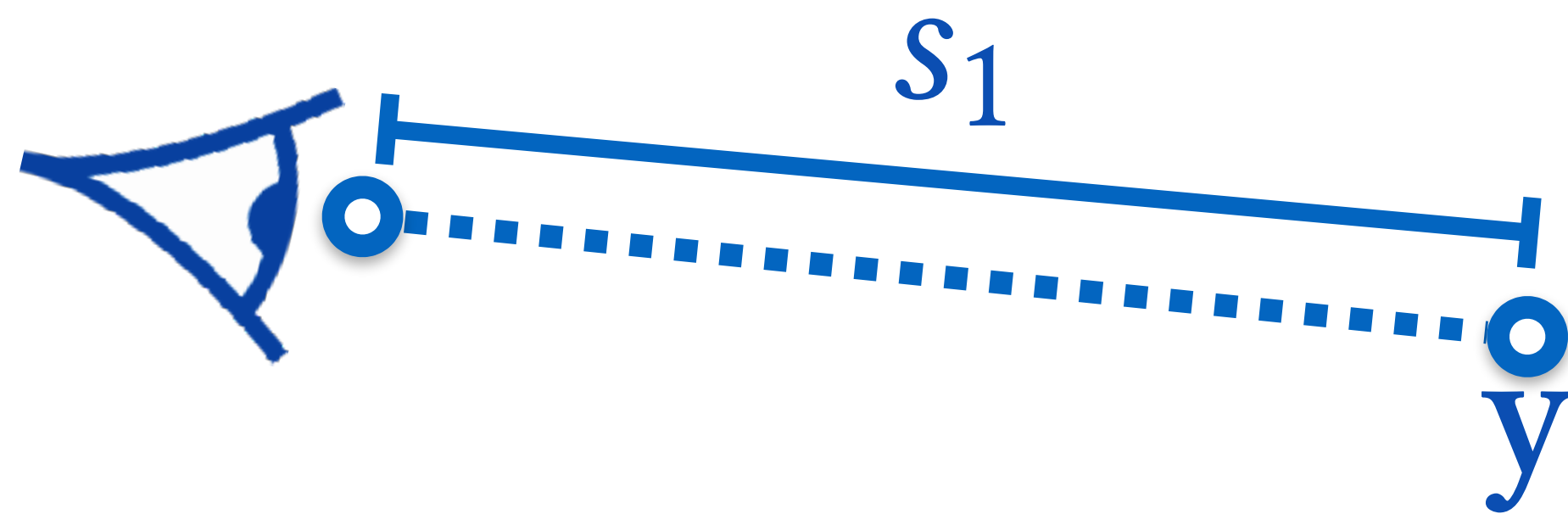
# Our approach

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, \}$



# Our approach

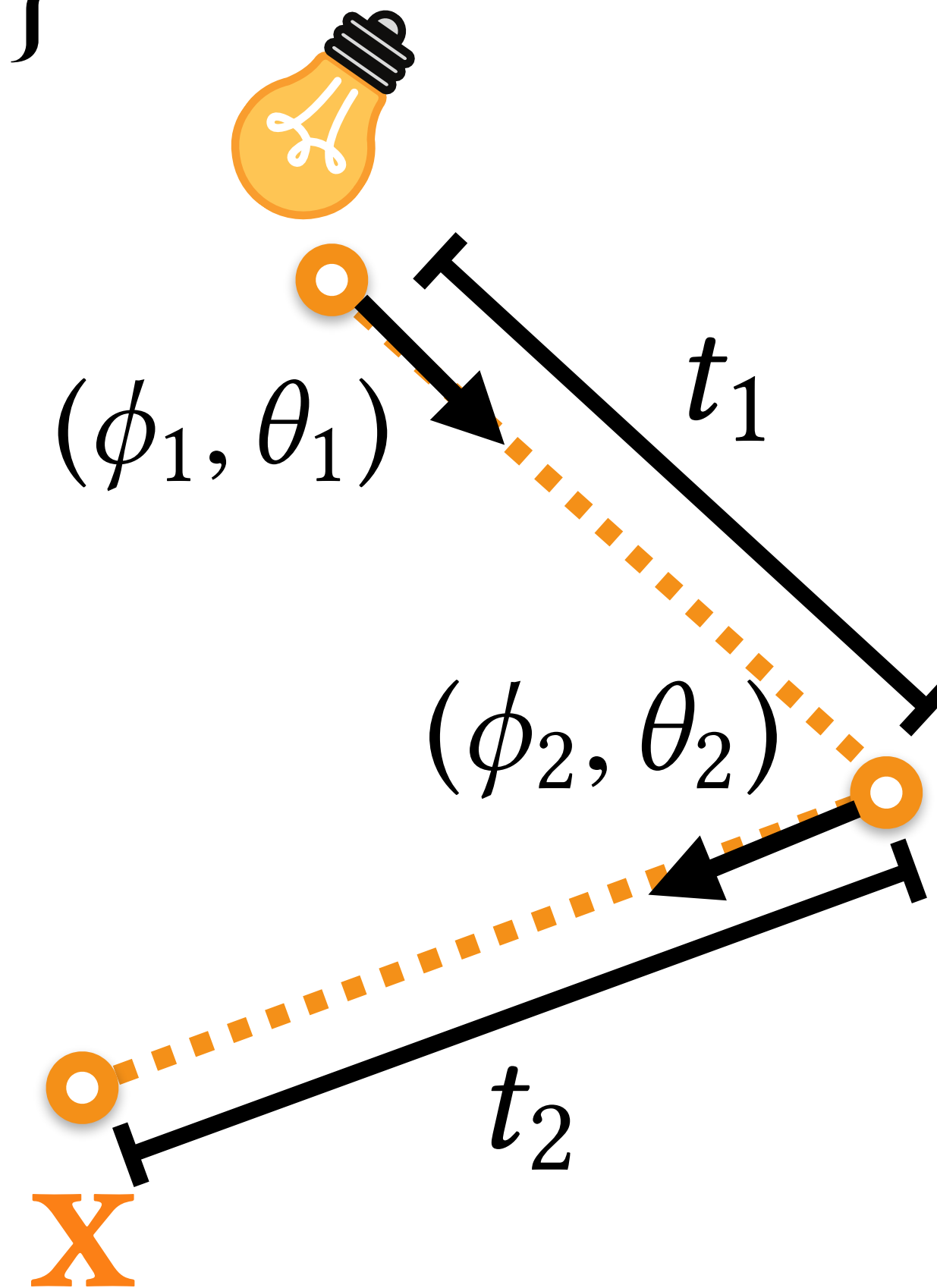
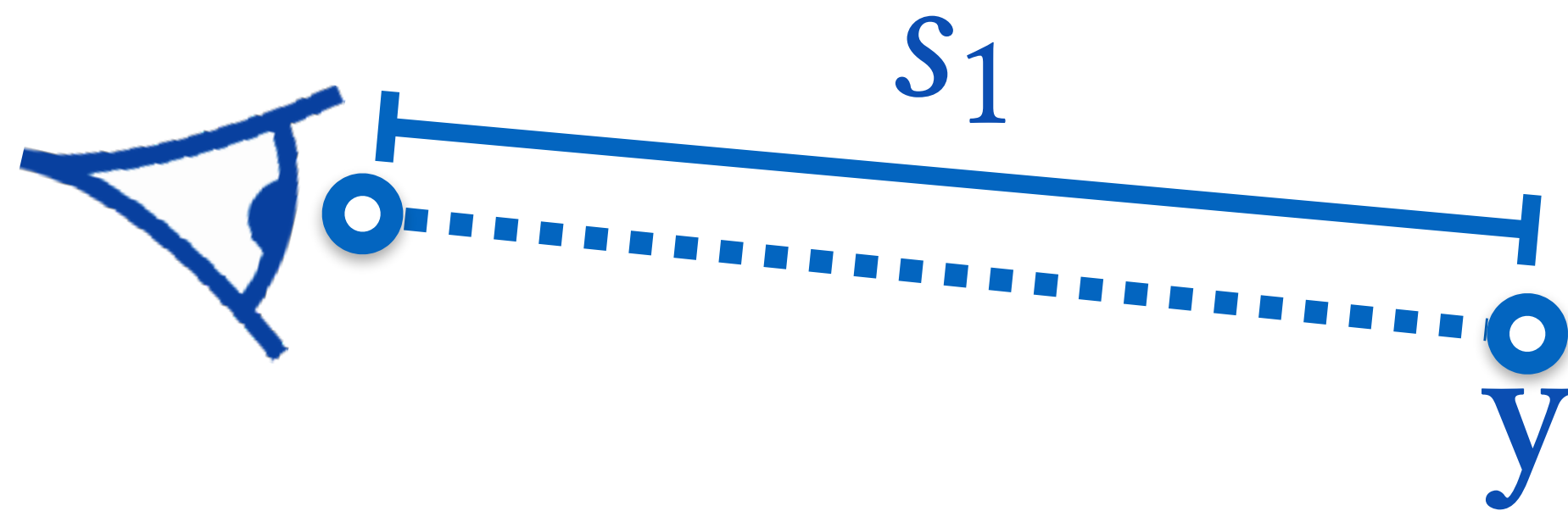
Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$



# Our approach

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

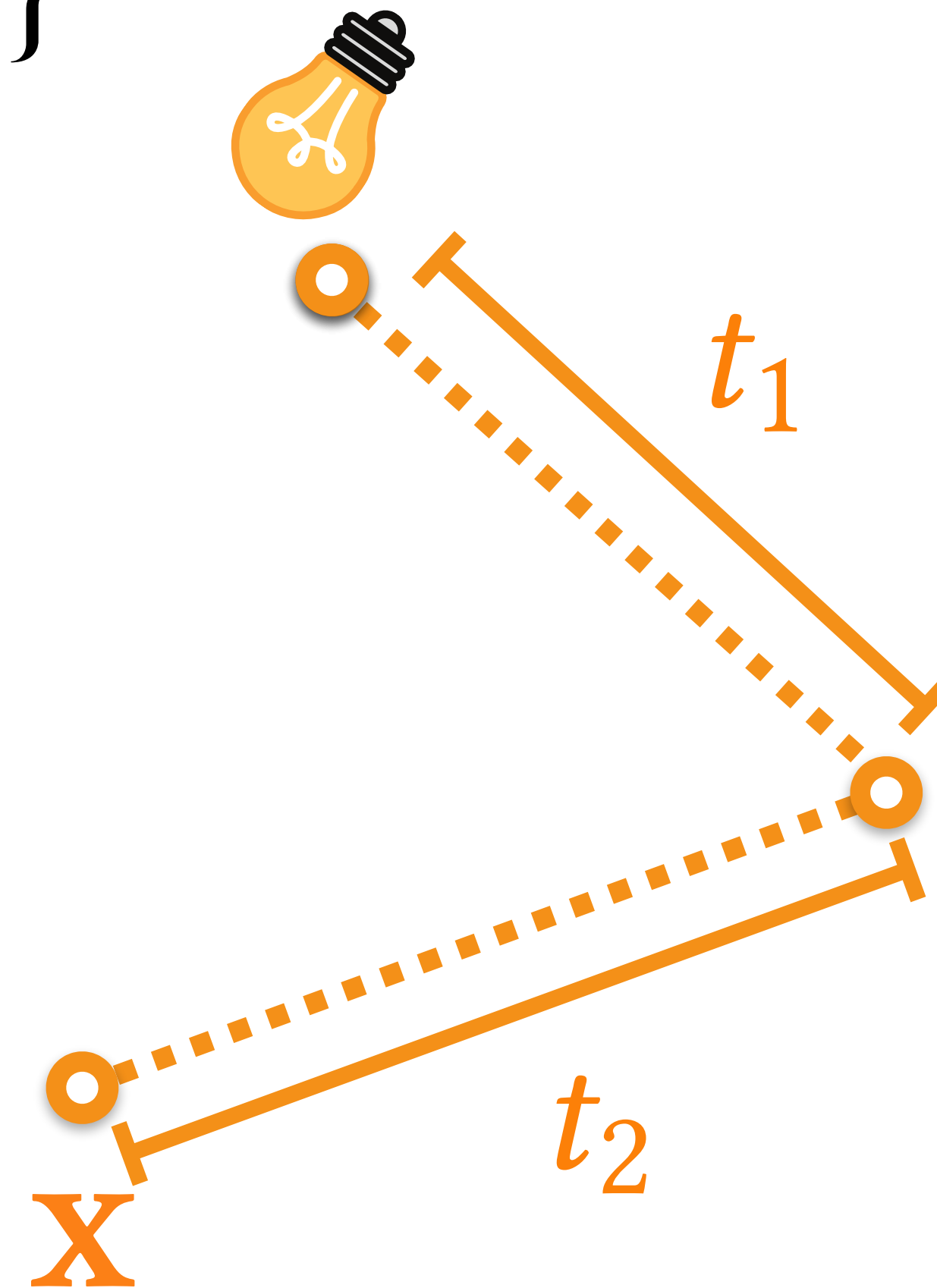
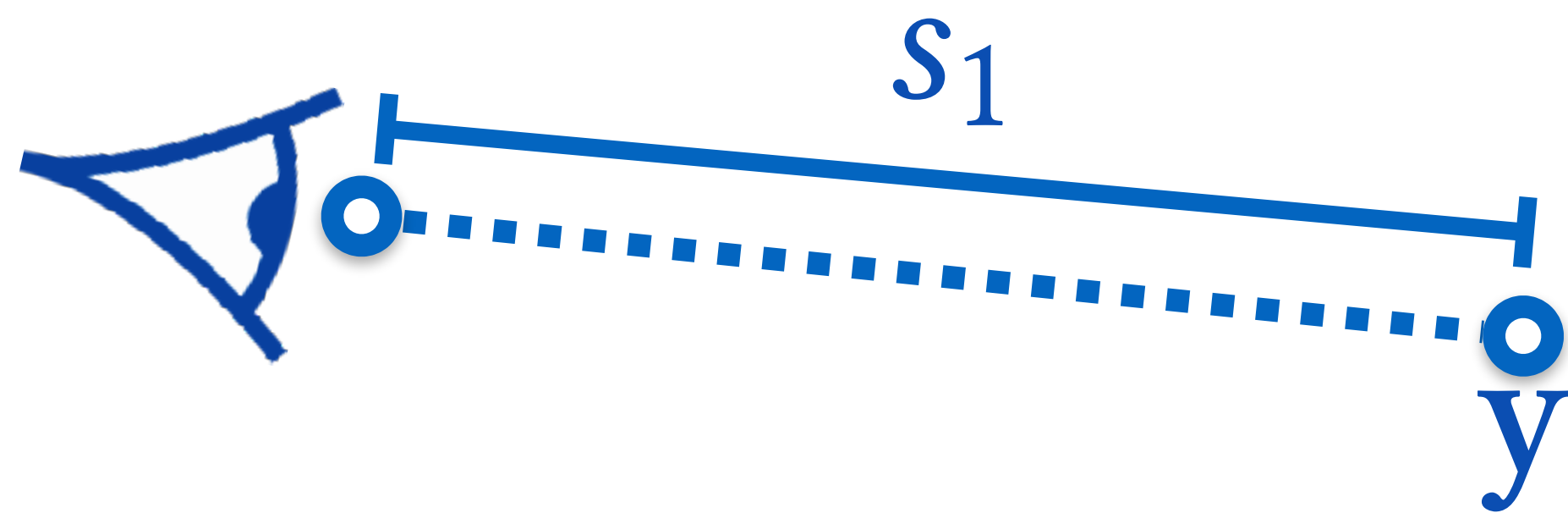
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$

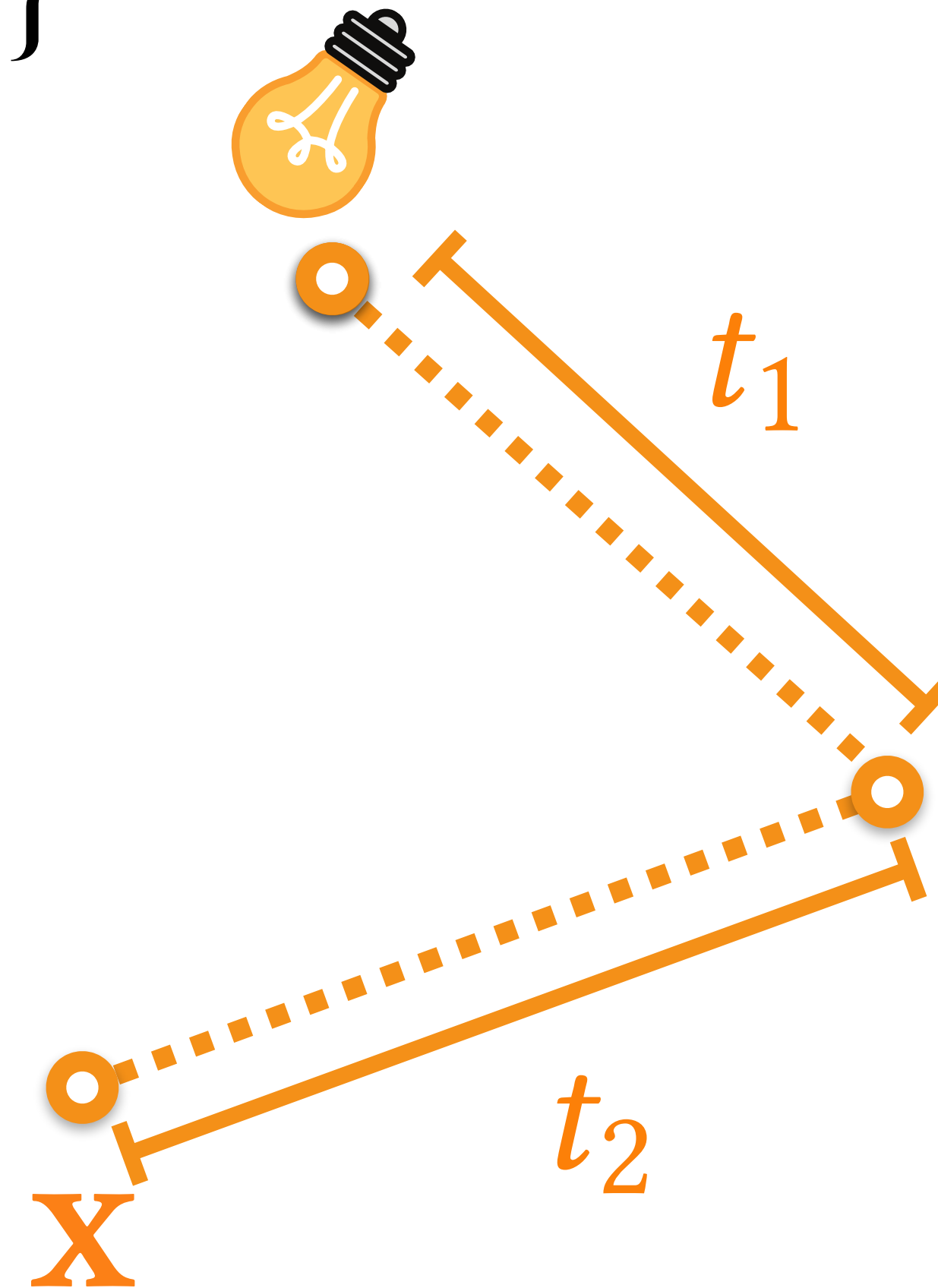
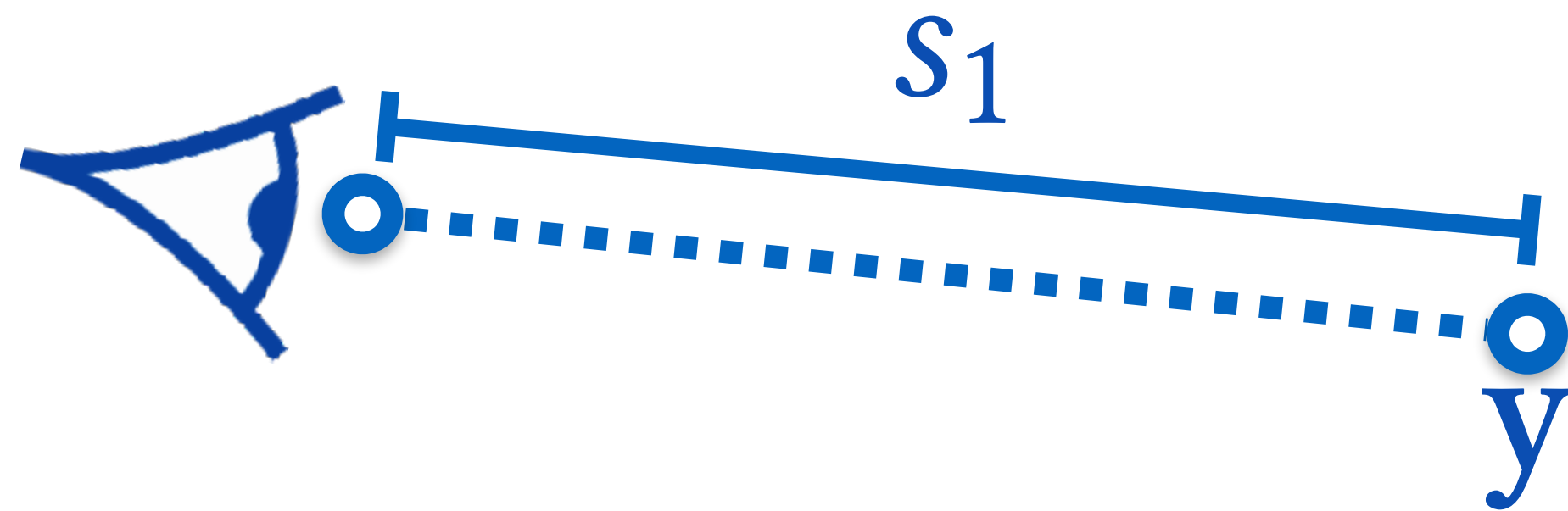




# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

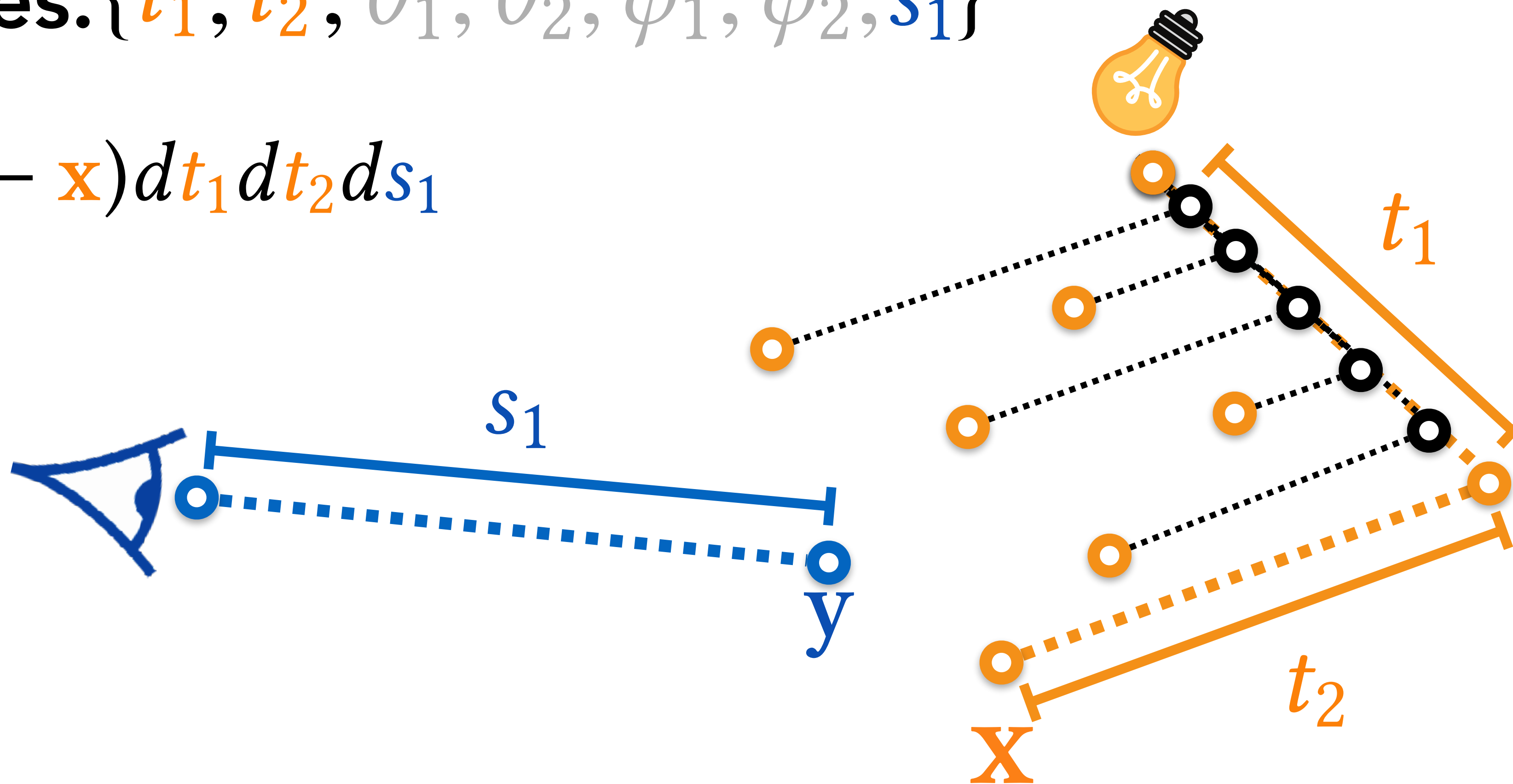
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

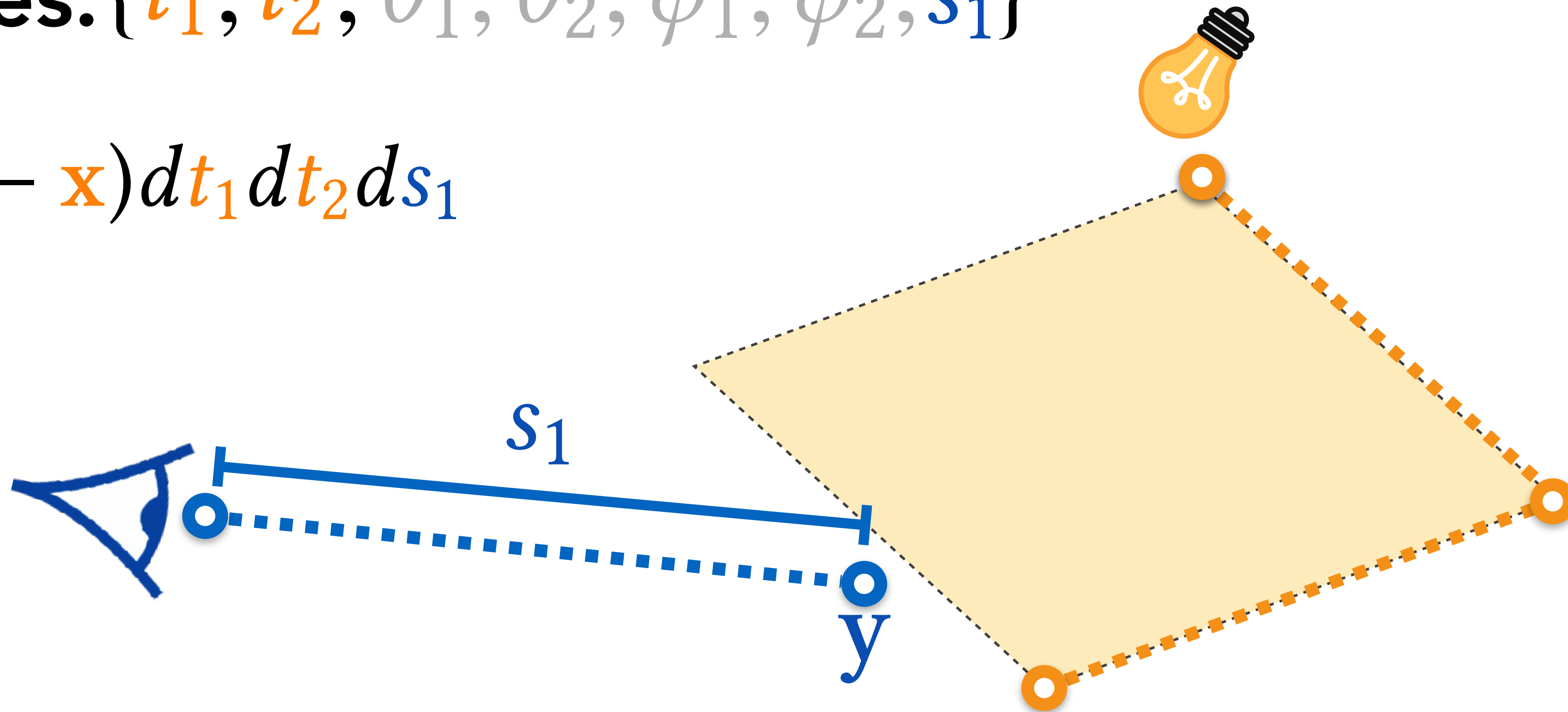
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

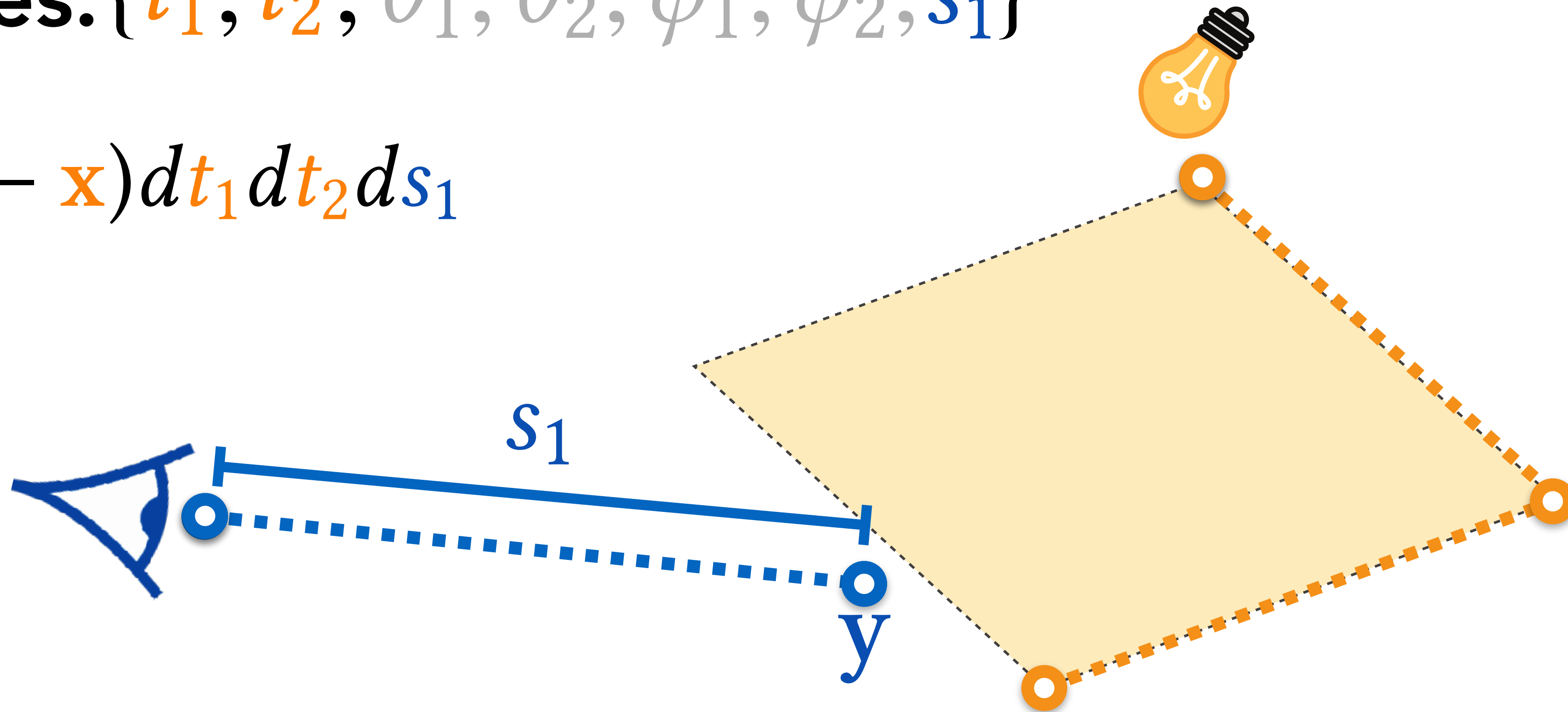
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

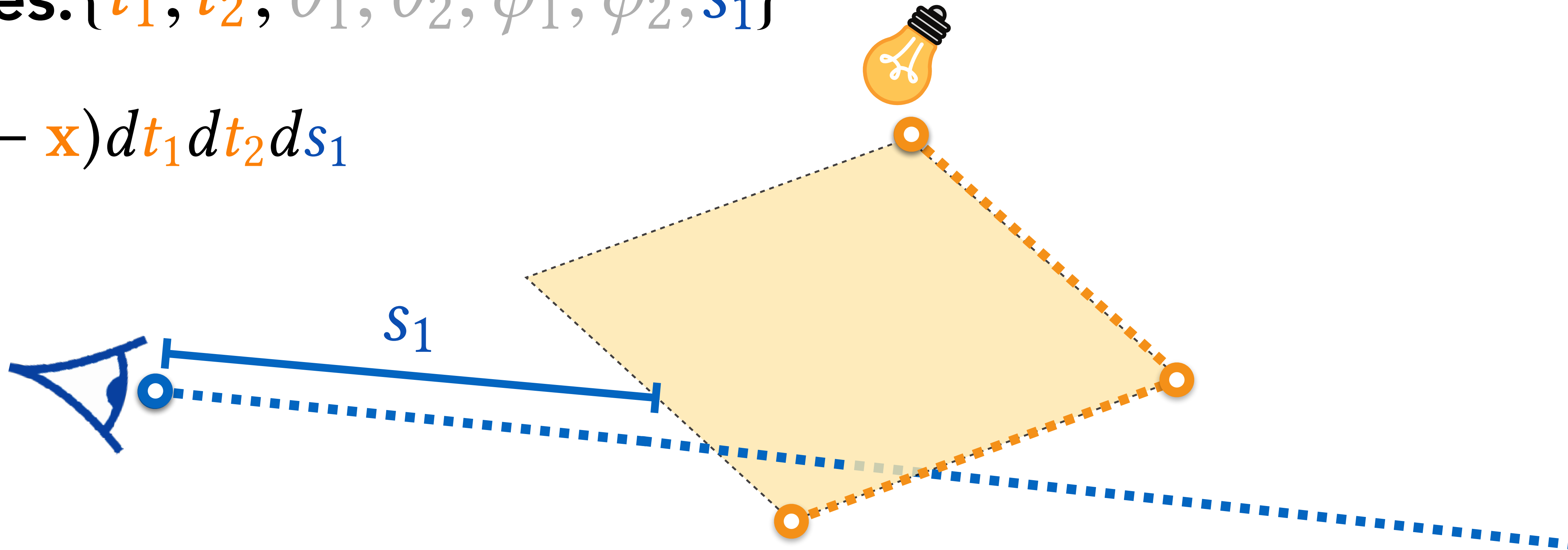
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

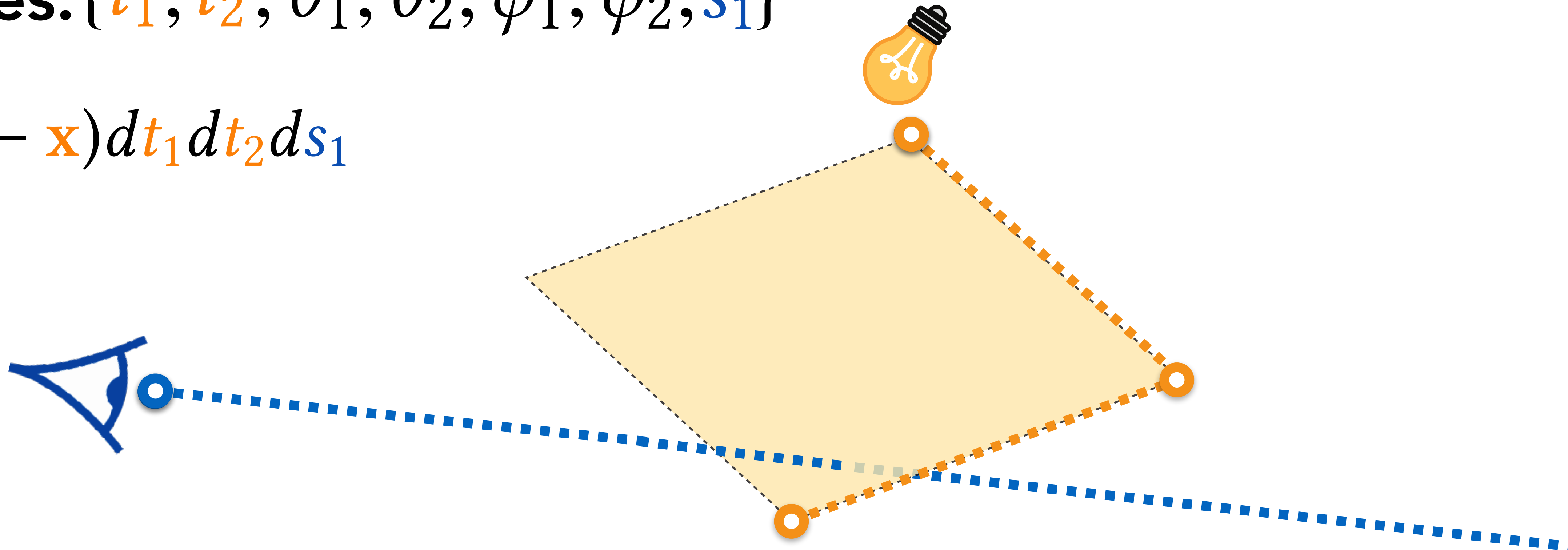
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

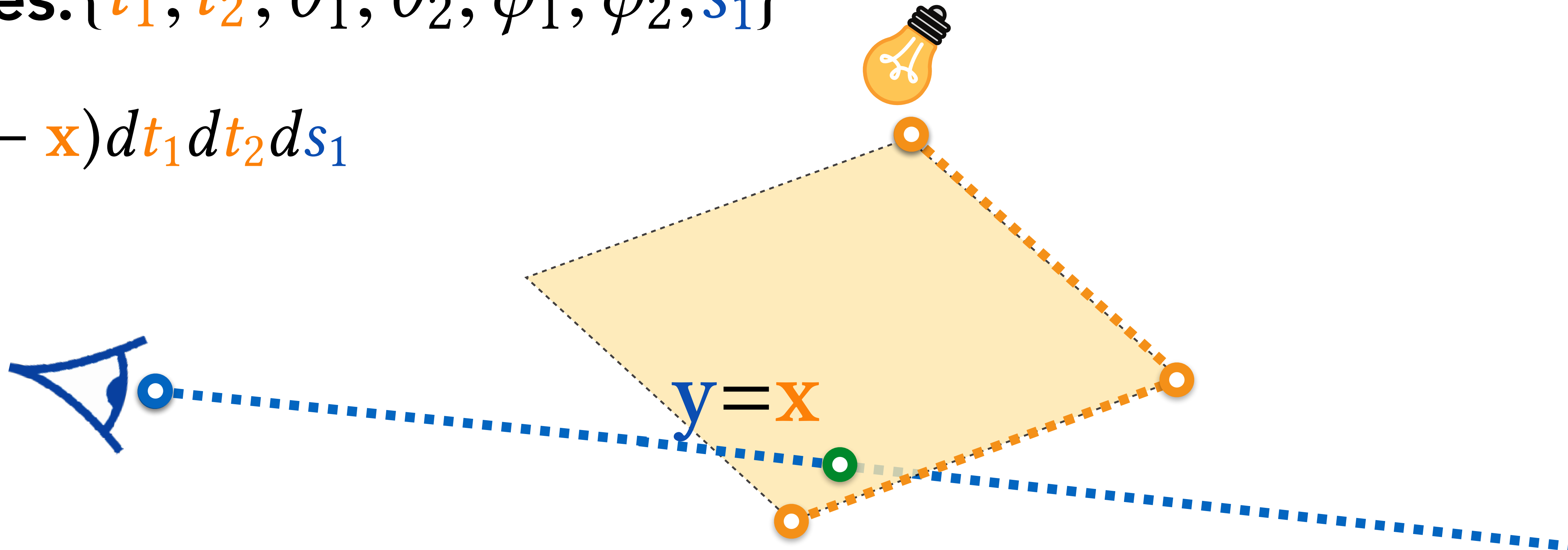
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

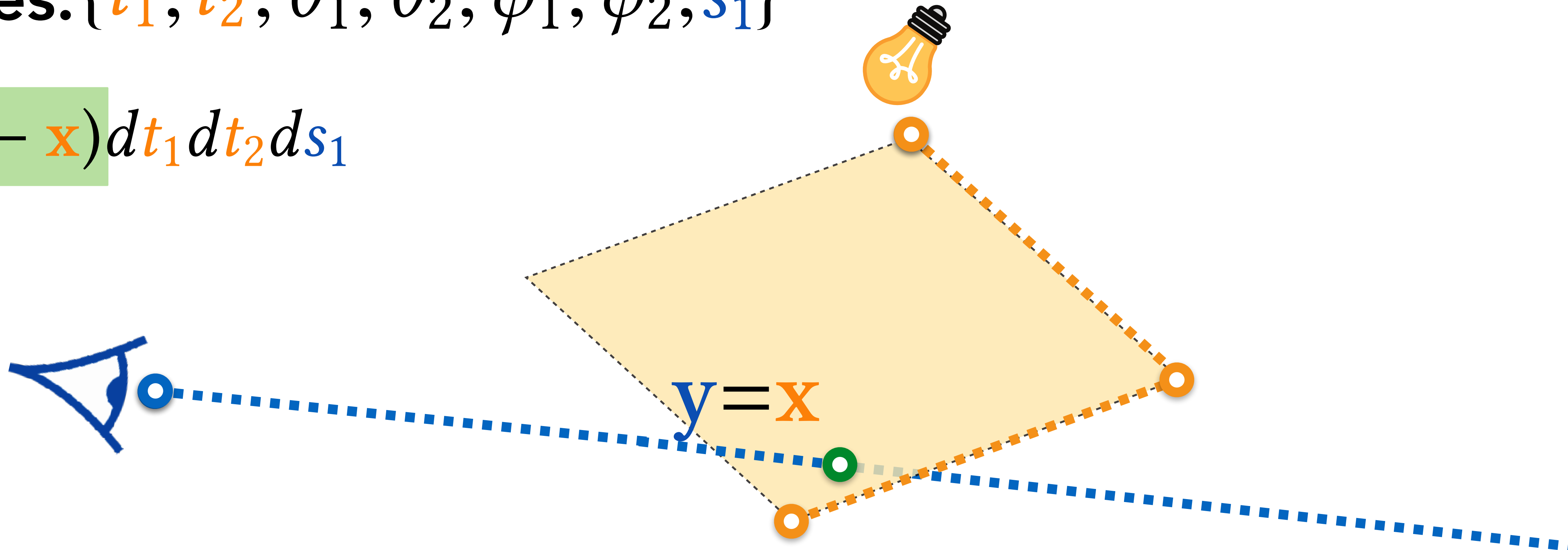
$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$



# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$

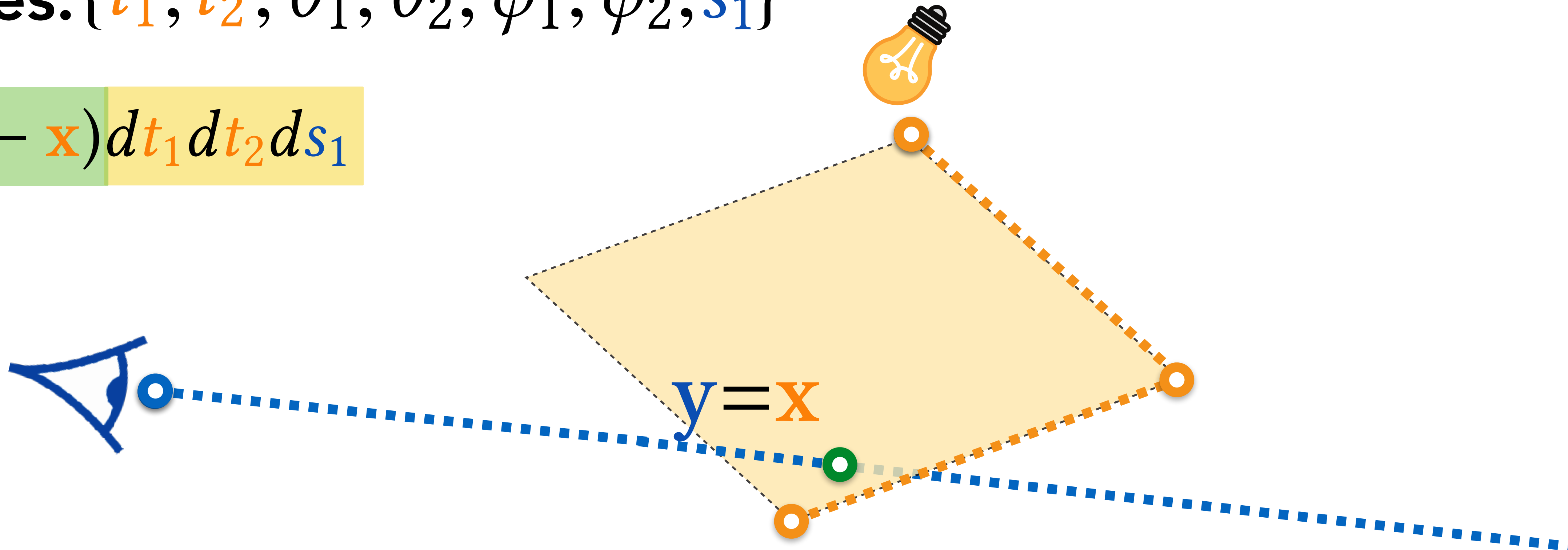




# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$

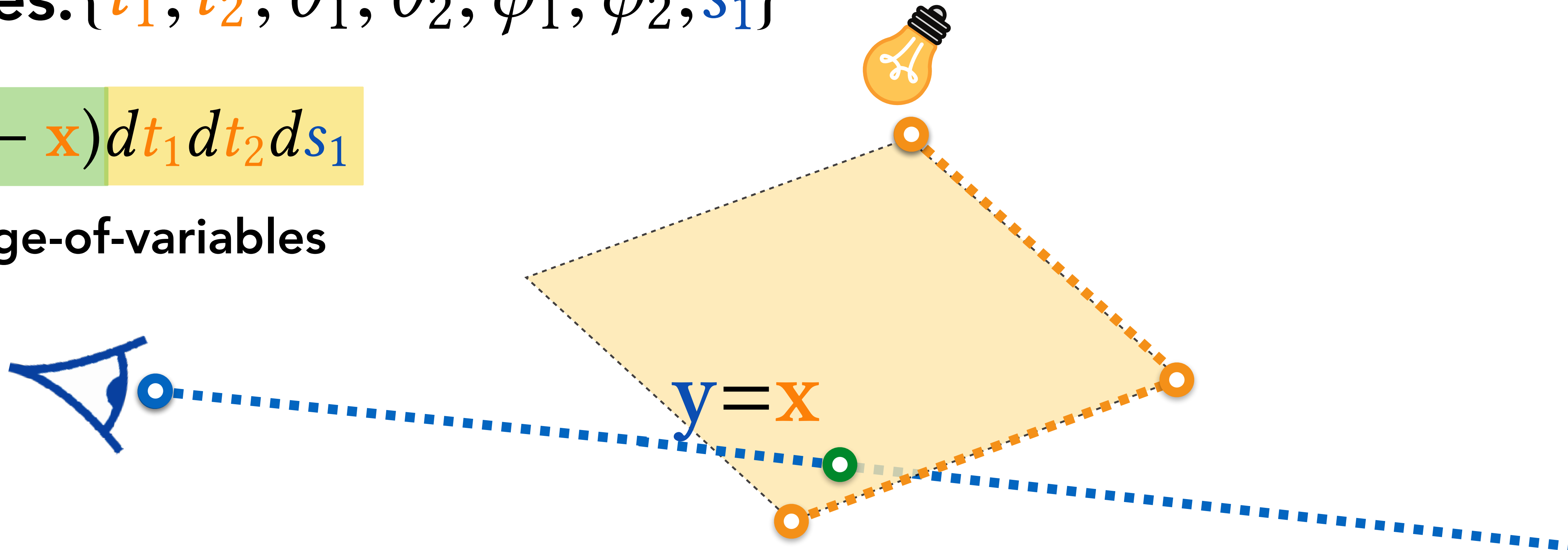


# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1$$

change-of-variables

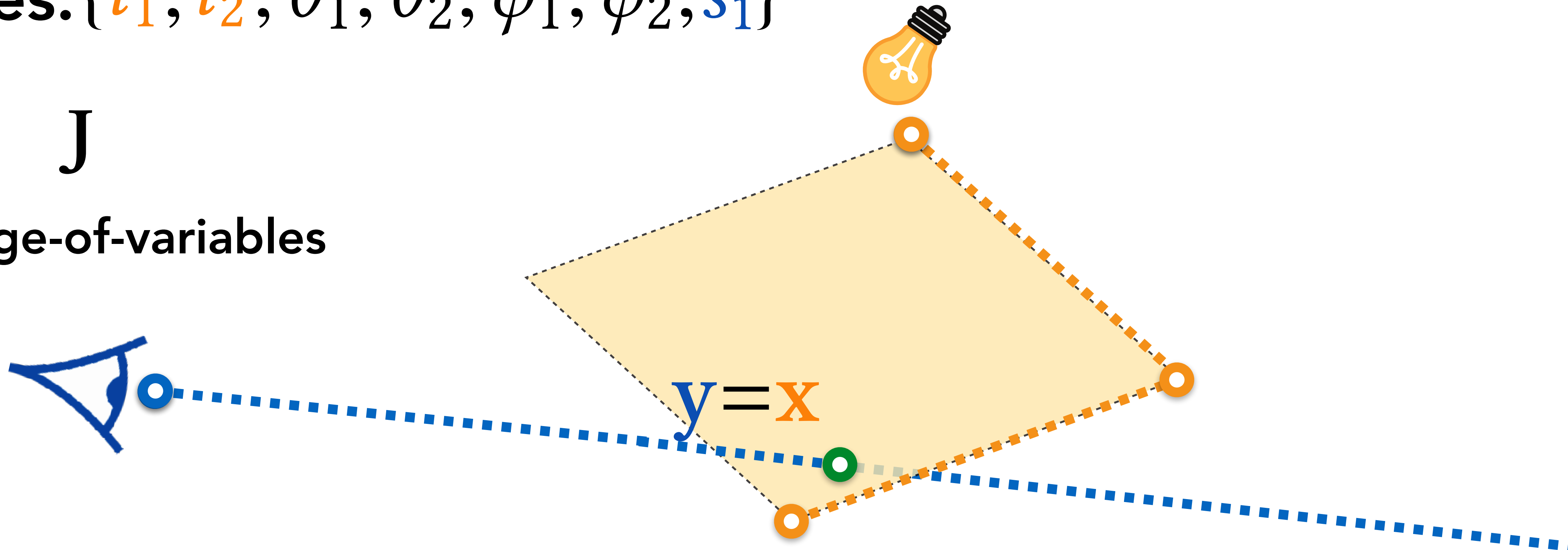


# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

J

change-of-variables

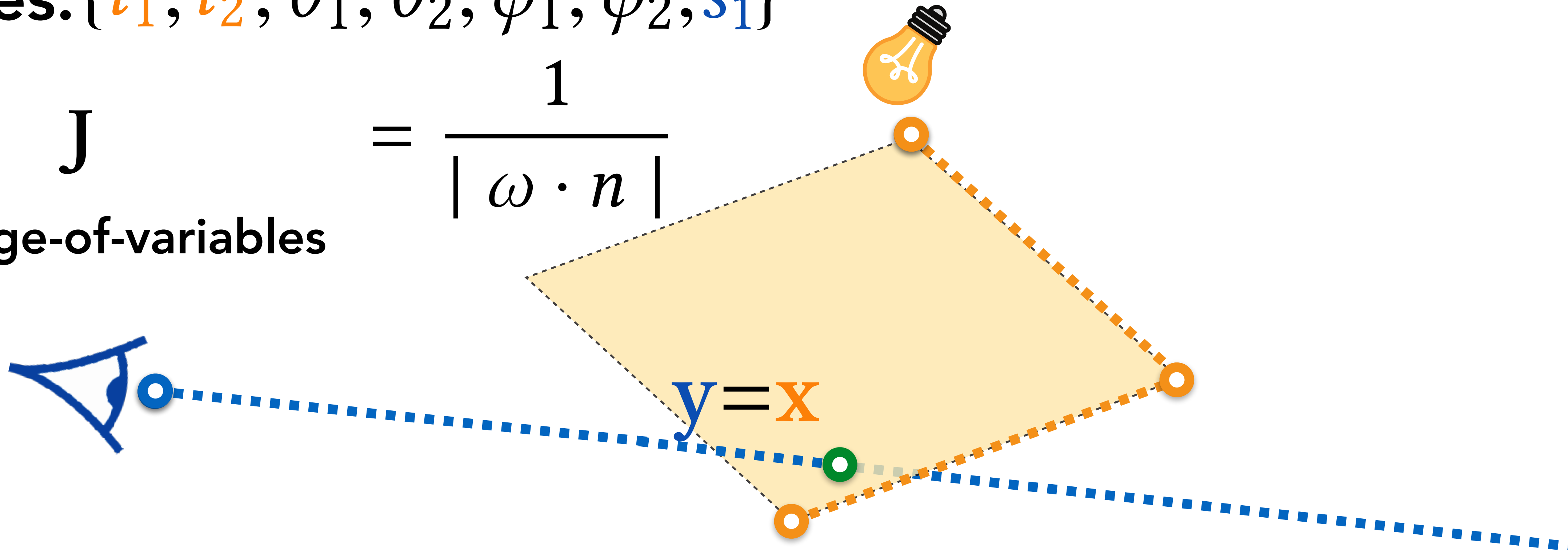


# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$J = \frac{1}{|\omega \cdot n|}$$

change-of-variables

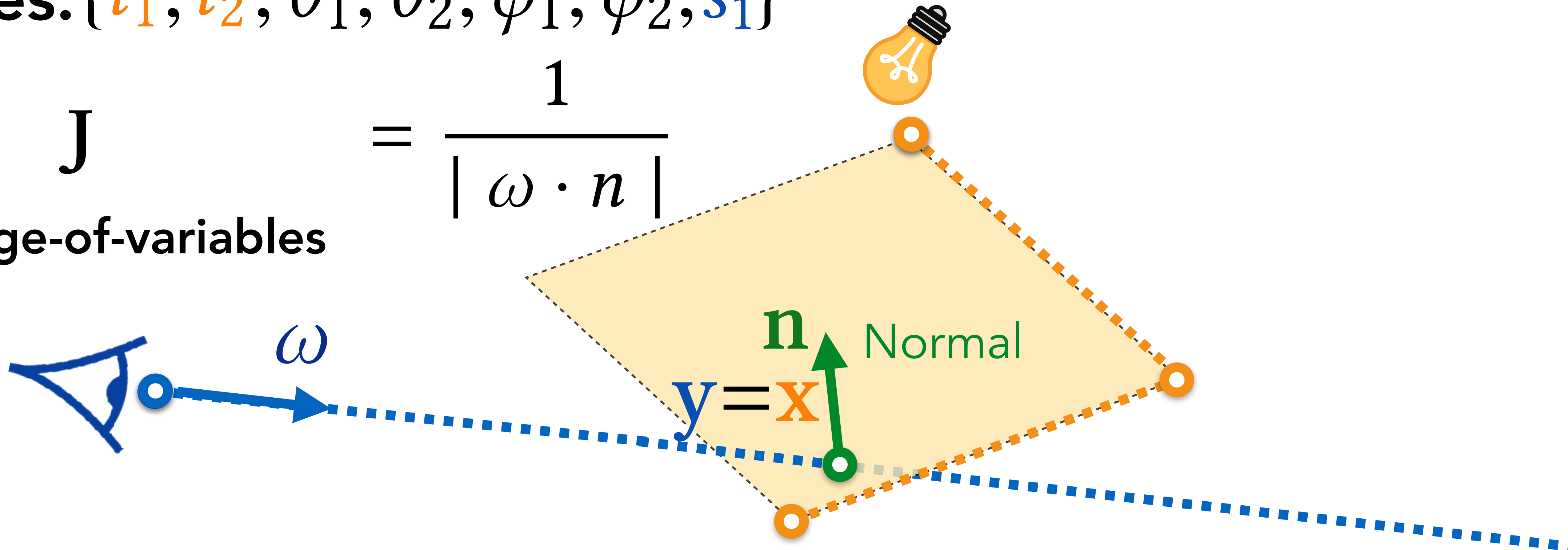


# Photon surfaces

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$J = \frac{1}{|\omega \cdot \mathbf{n}|}$$

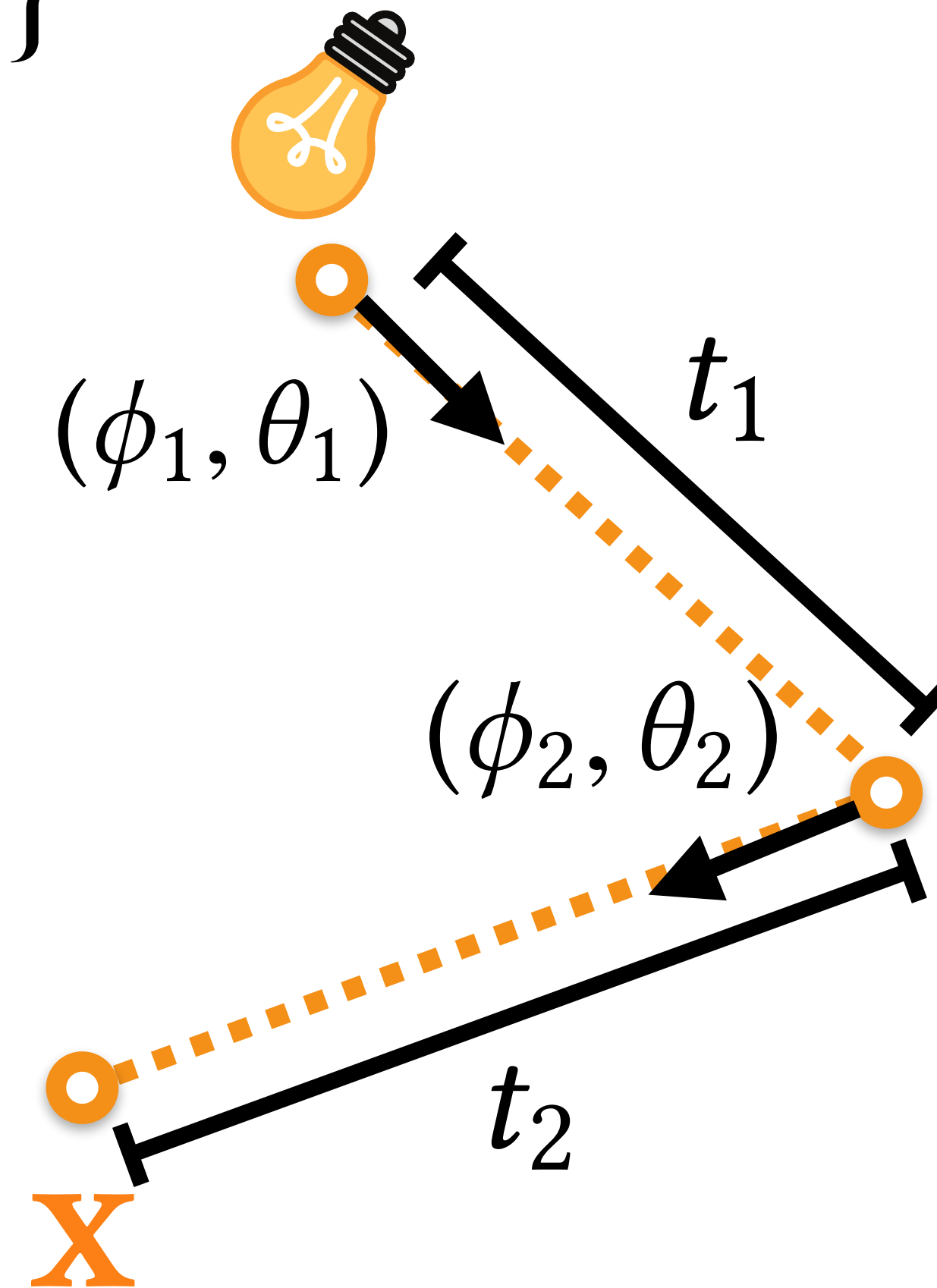
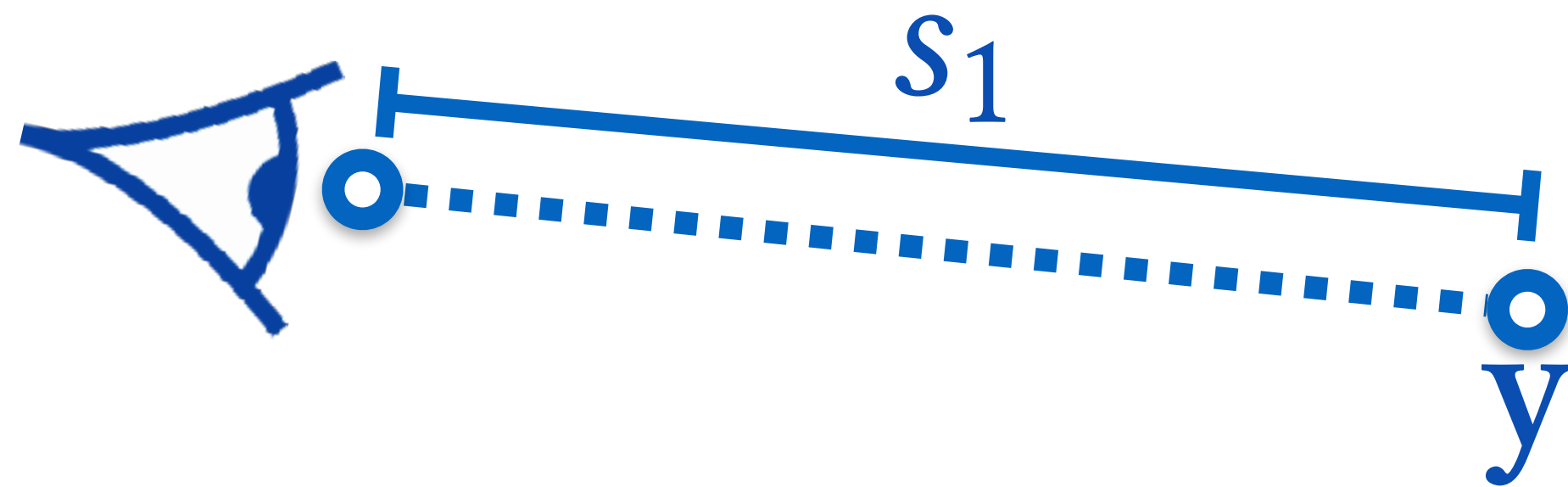
change-of-variables



# Photon surfaces

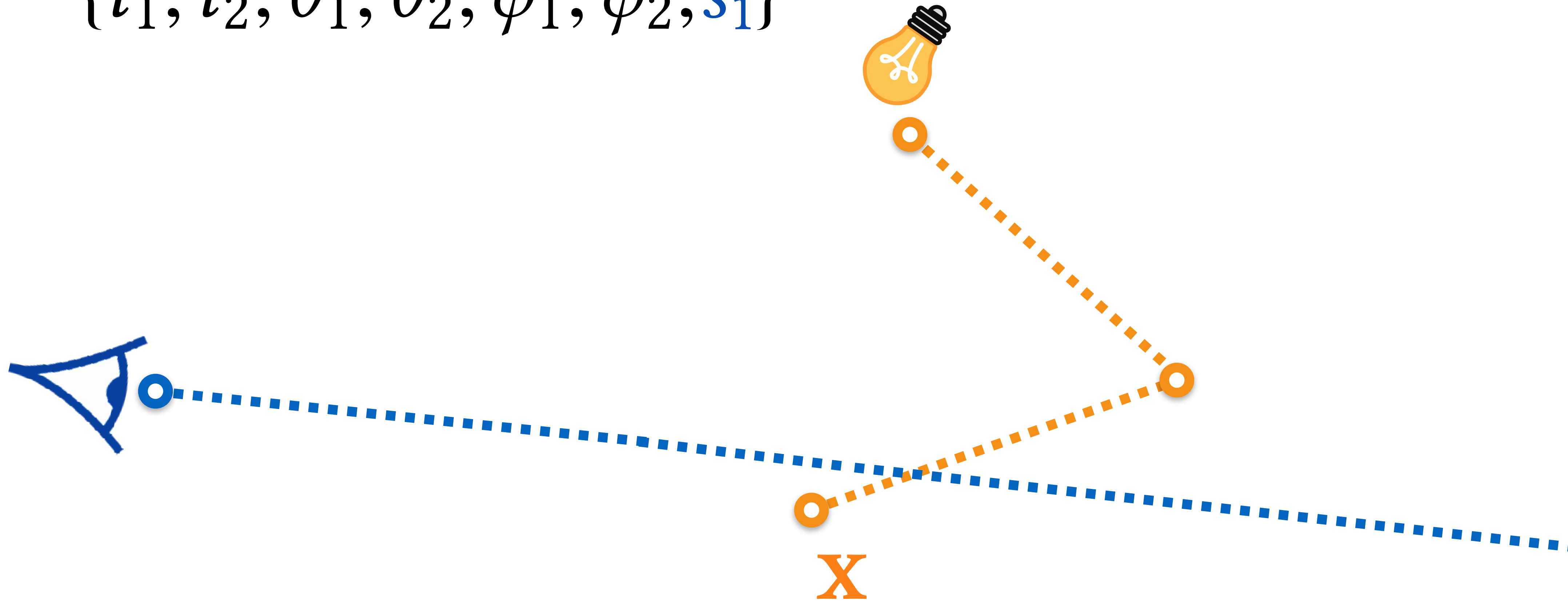
Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{y} - \mathbf{x}) dt_1 dt_2 ds_1 = \frac{1}{|\boldsymbol{\omega} \cdot \mathbf{n}|}$$



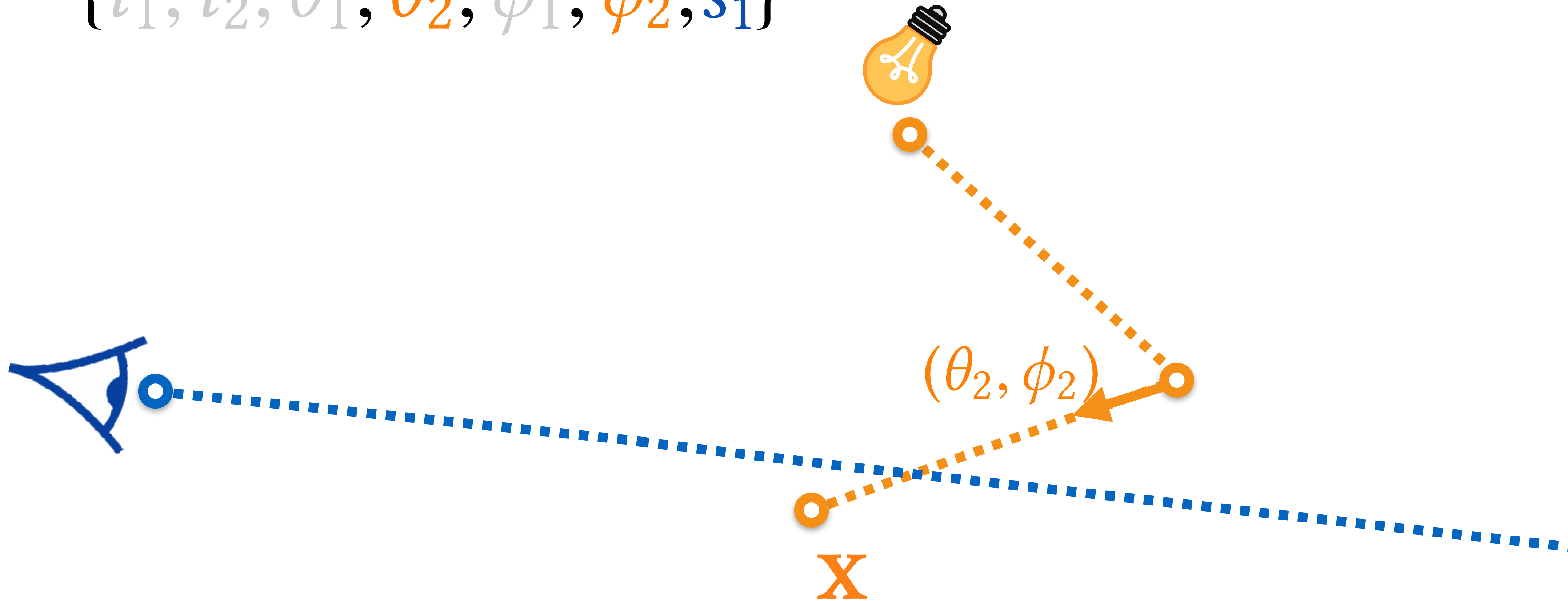
# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$



# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

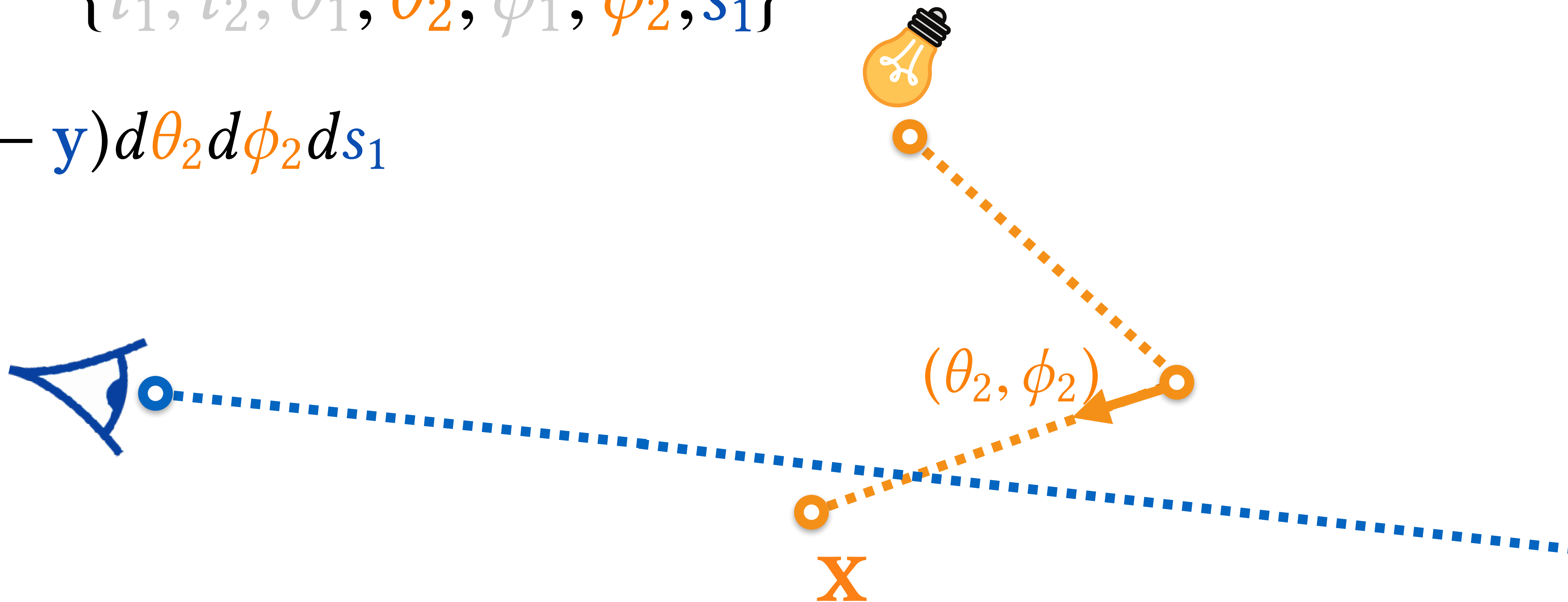




# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

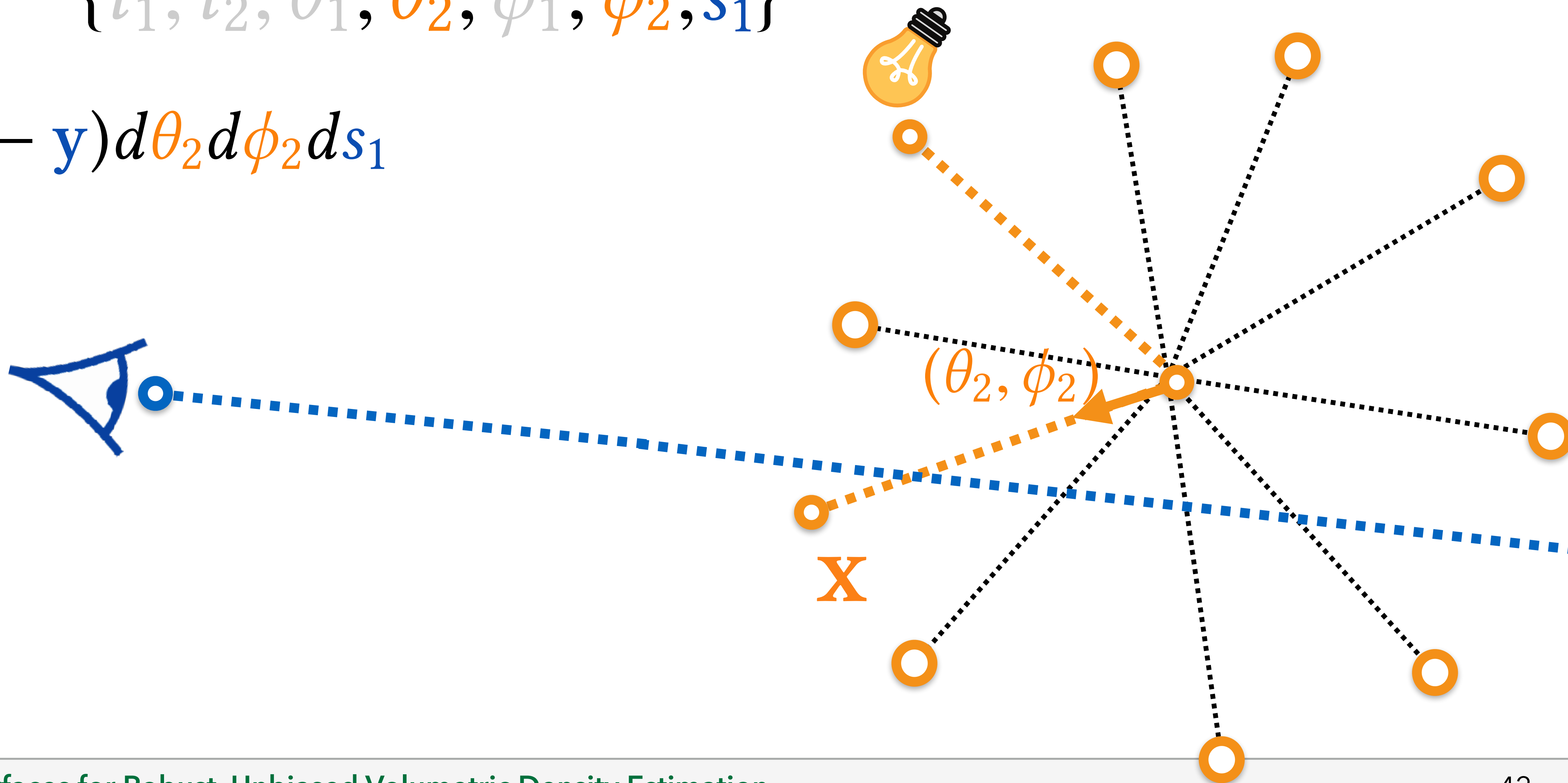
$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1$$



# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

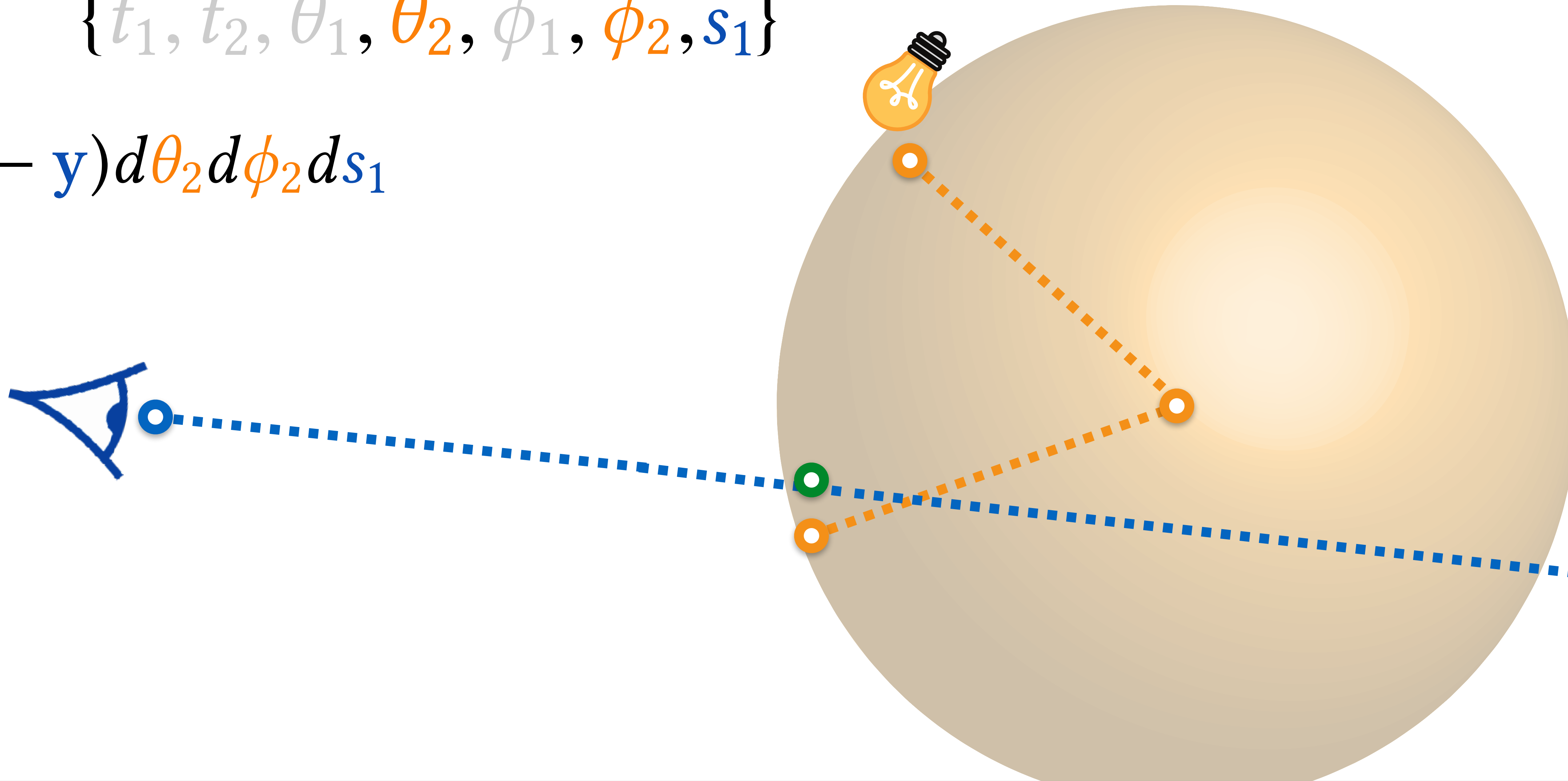
$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1$$



# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

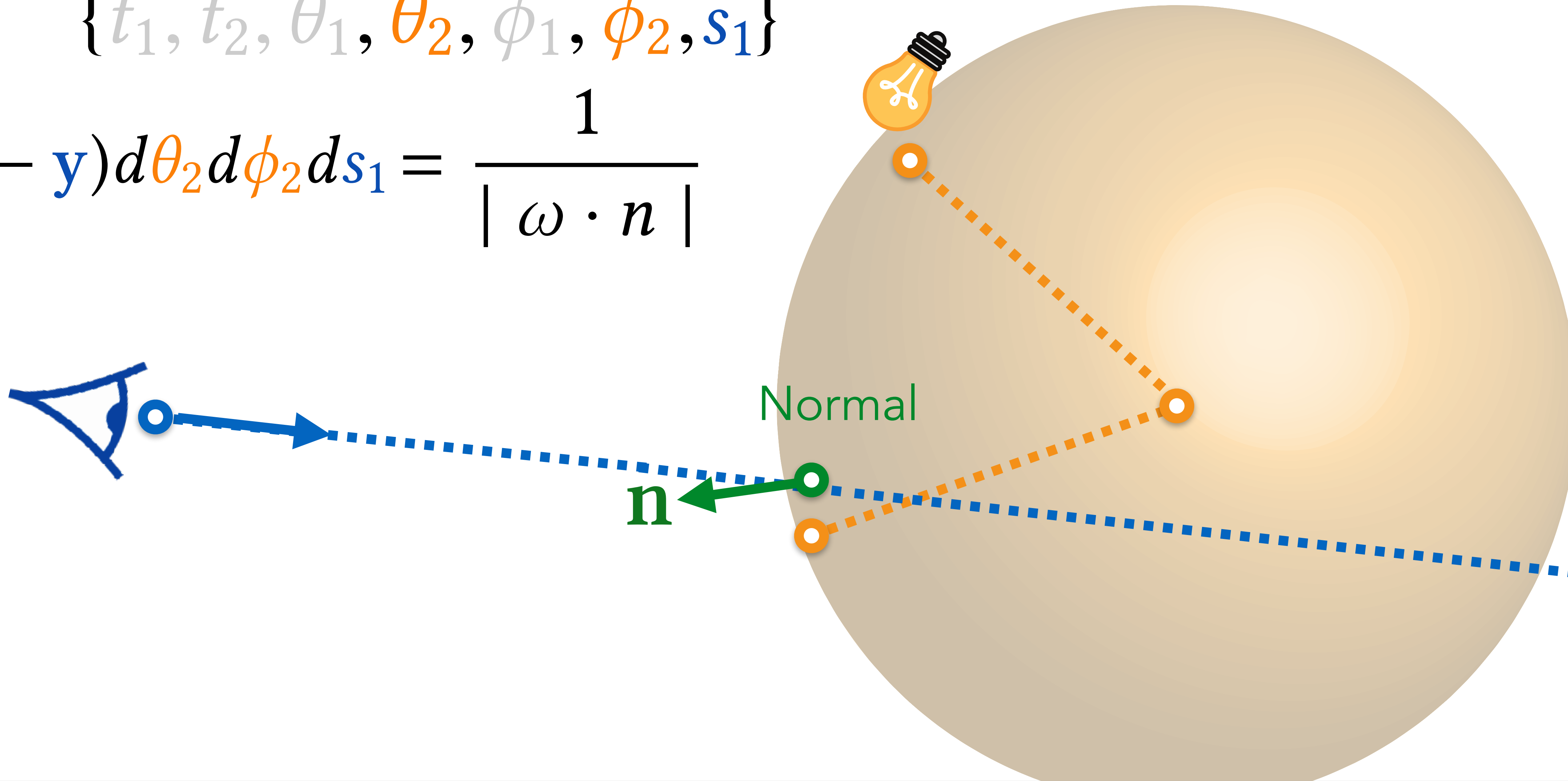
$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1$$



# Some possible estimators

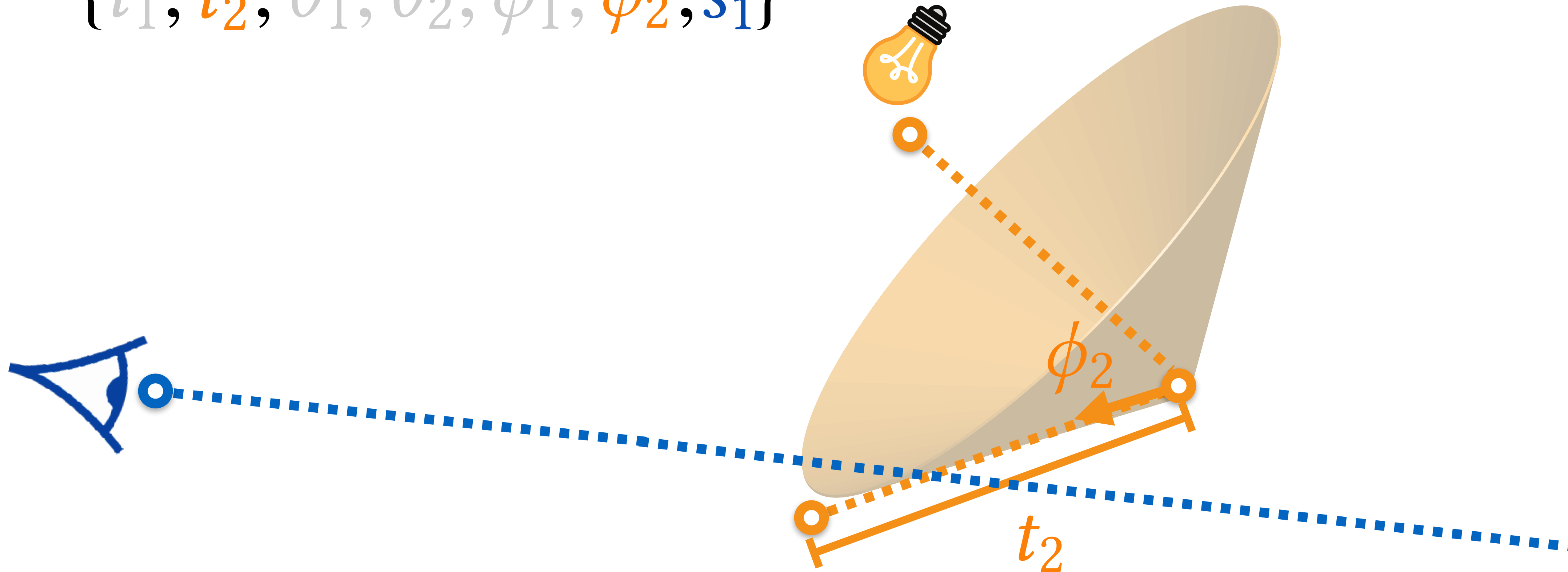
Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1 = \frac{1}{|\omega \cdot \mathbf{n}|}$$



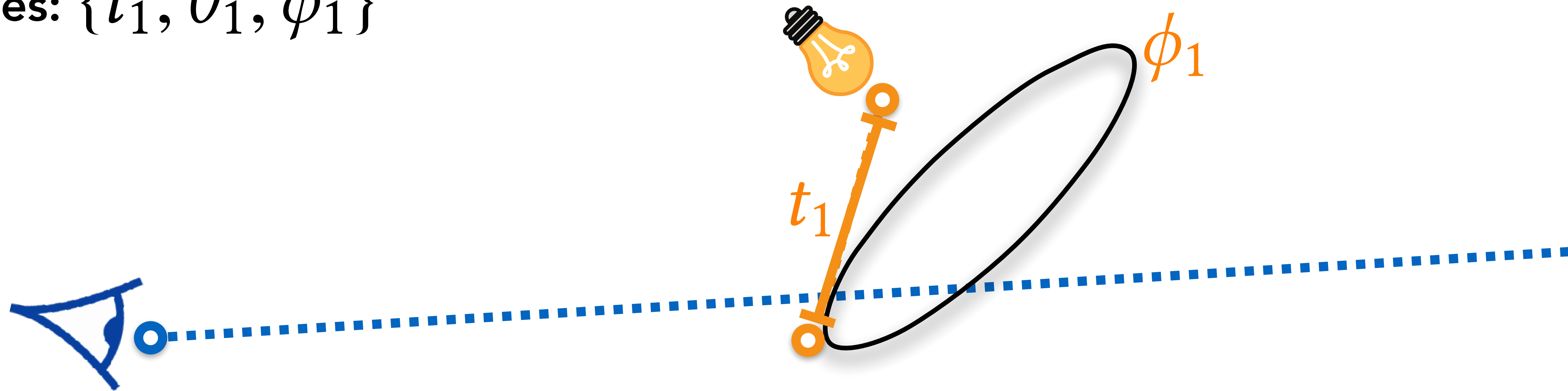
# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$



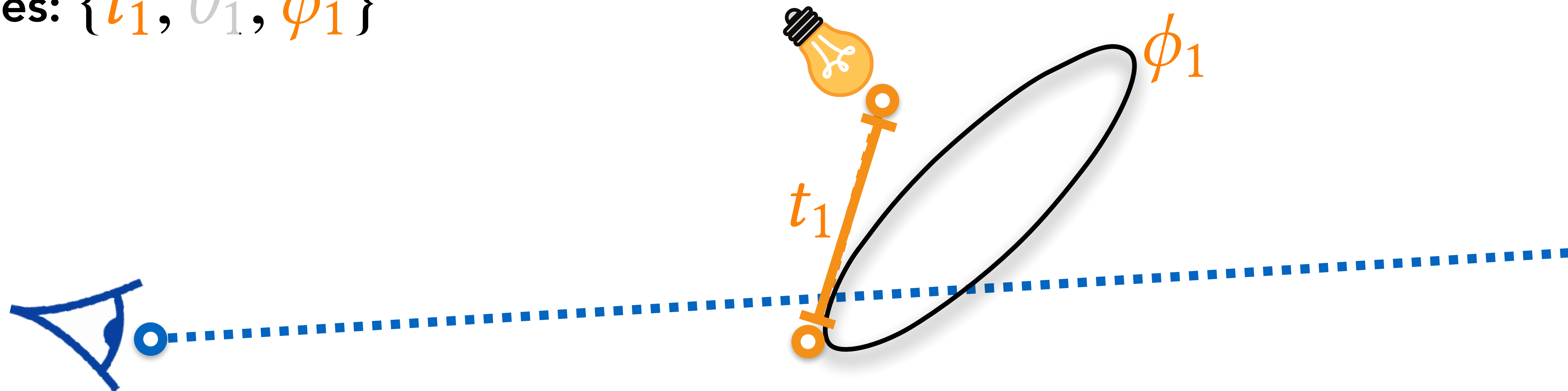
# What about single scattering?

Variables:  $\{t_1, \theta_1, \phi_1\}$



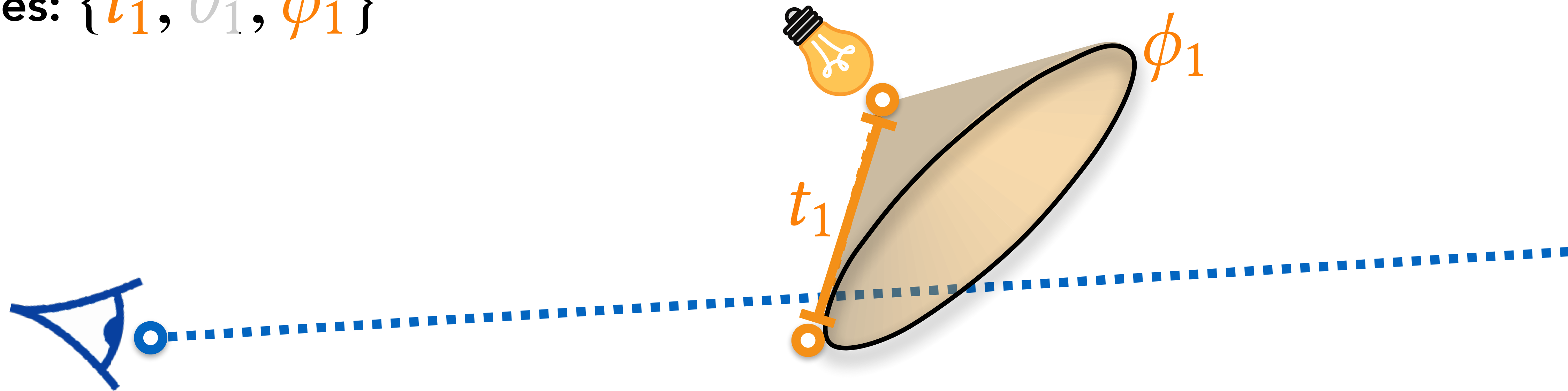
# What about single scattering?

Variables:  $\{t_1, \theta_1, \phi_1\}$



# What about single scattering?

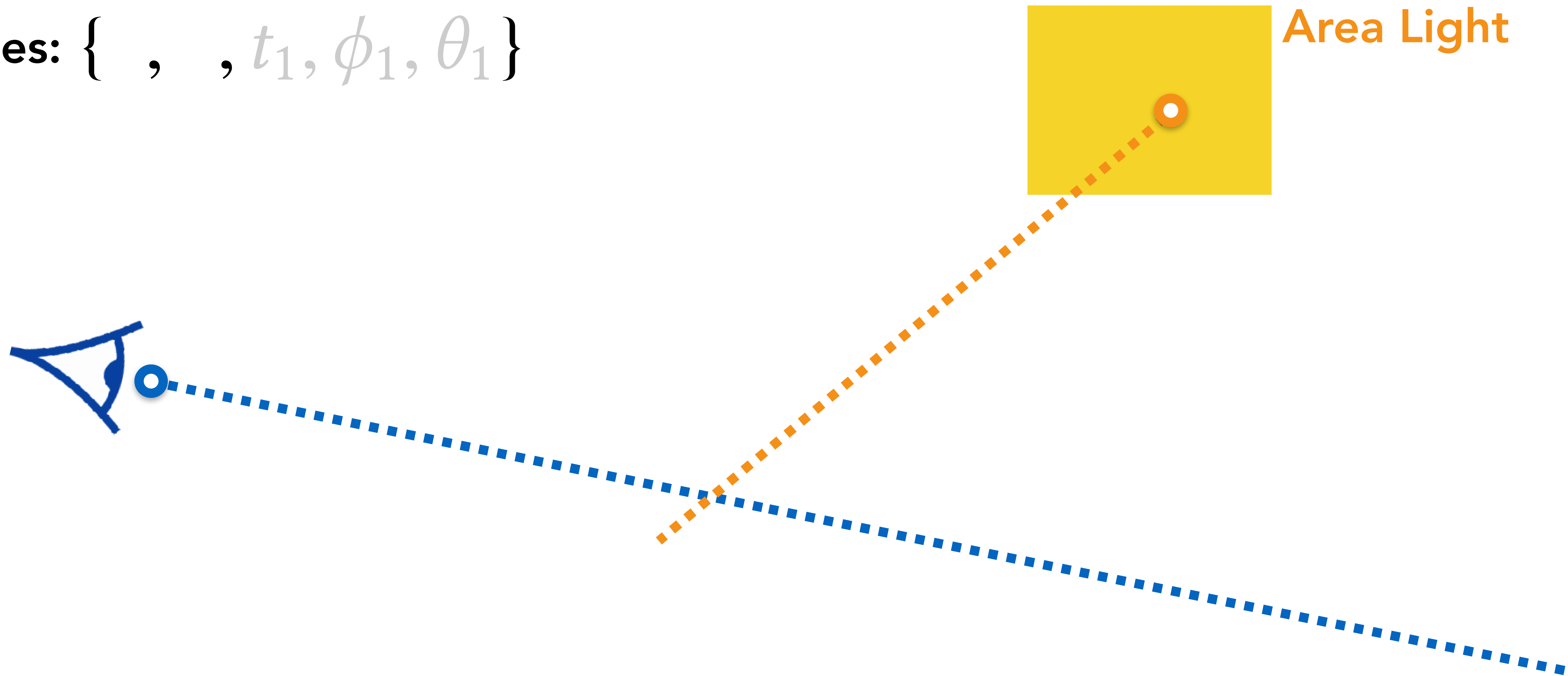
Variables:  $\{t_1, \theta_1, \phi_1\}$





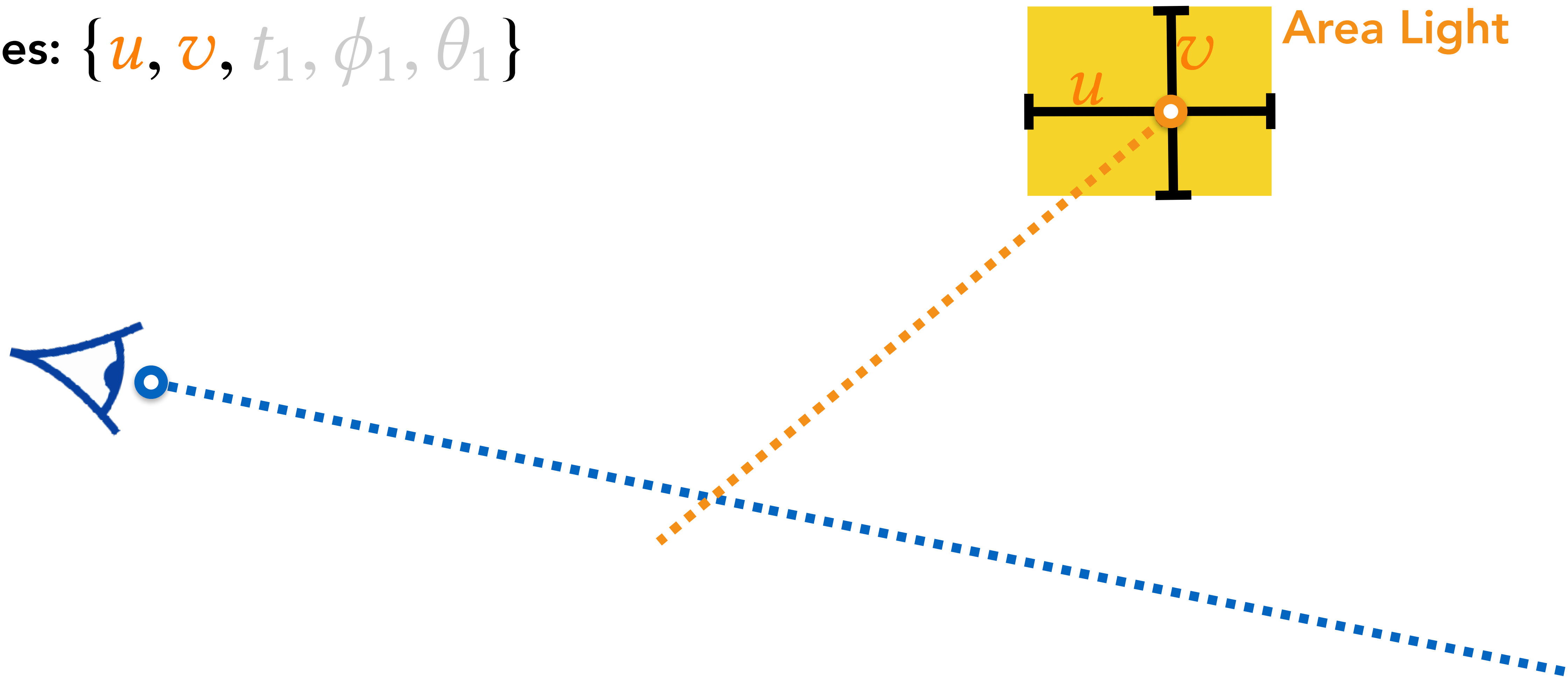
# More possible estimators

Variables:  $\{ \quad , \quad , t_1, \phi_1, \theta_1 \}$



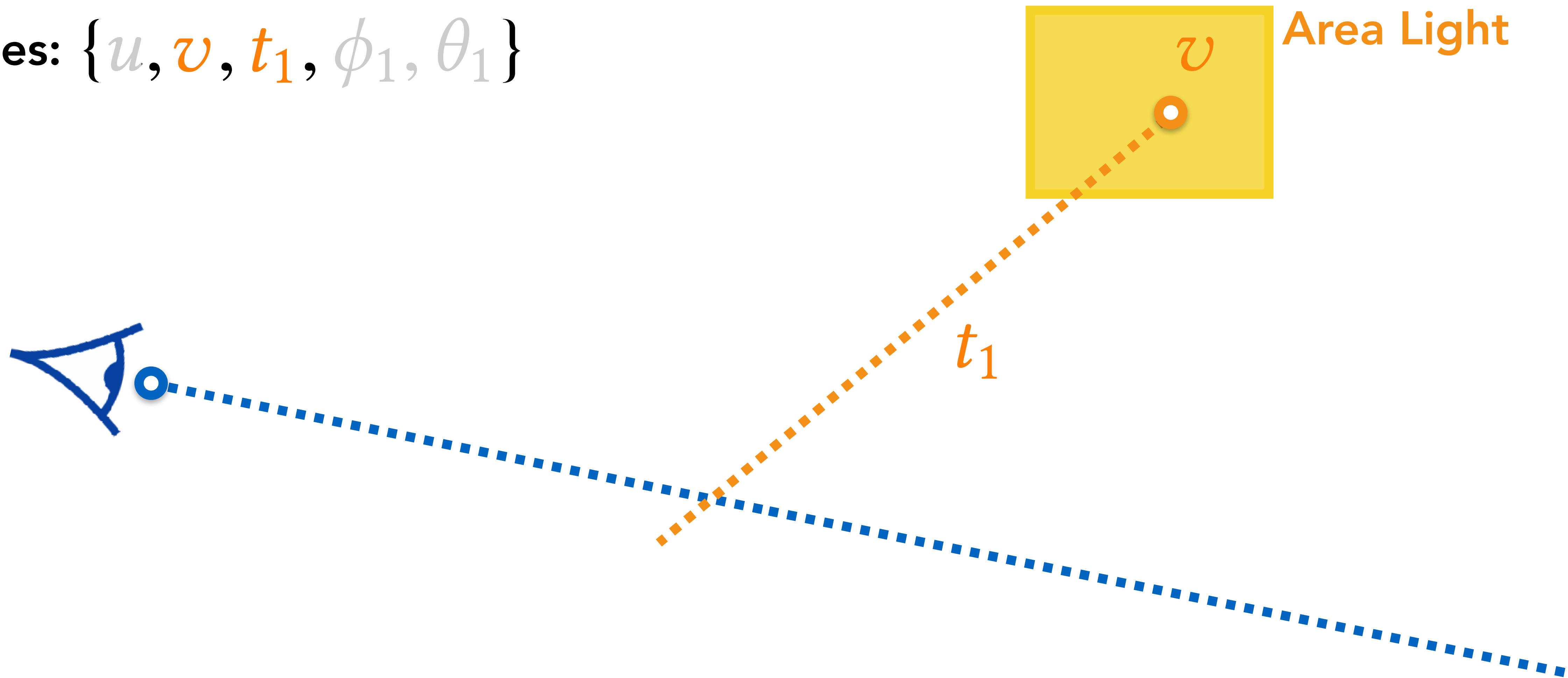
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



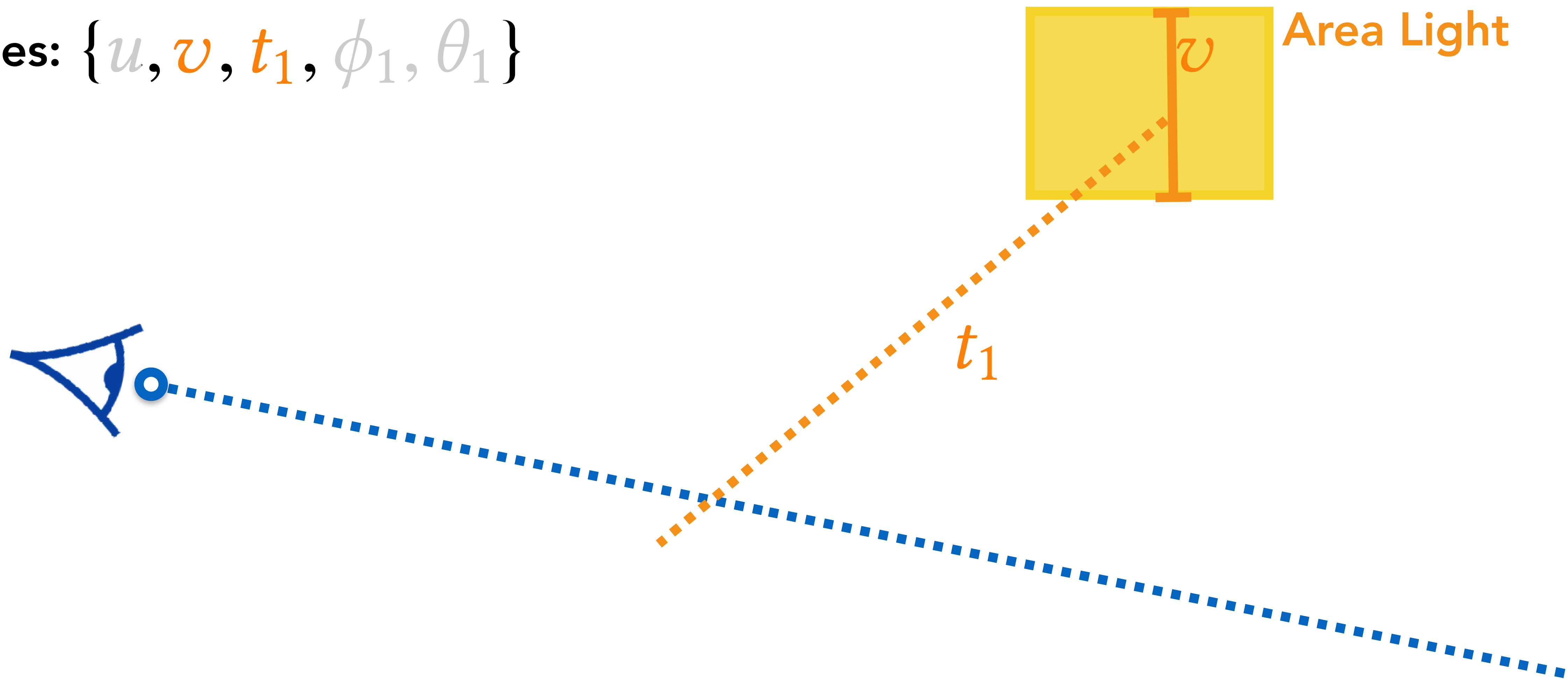
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



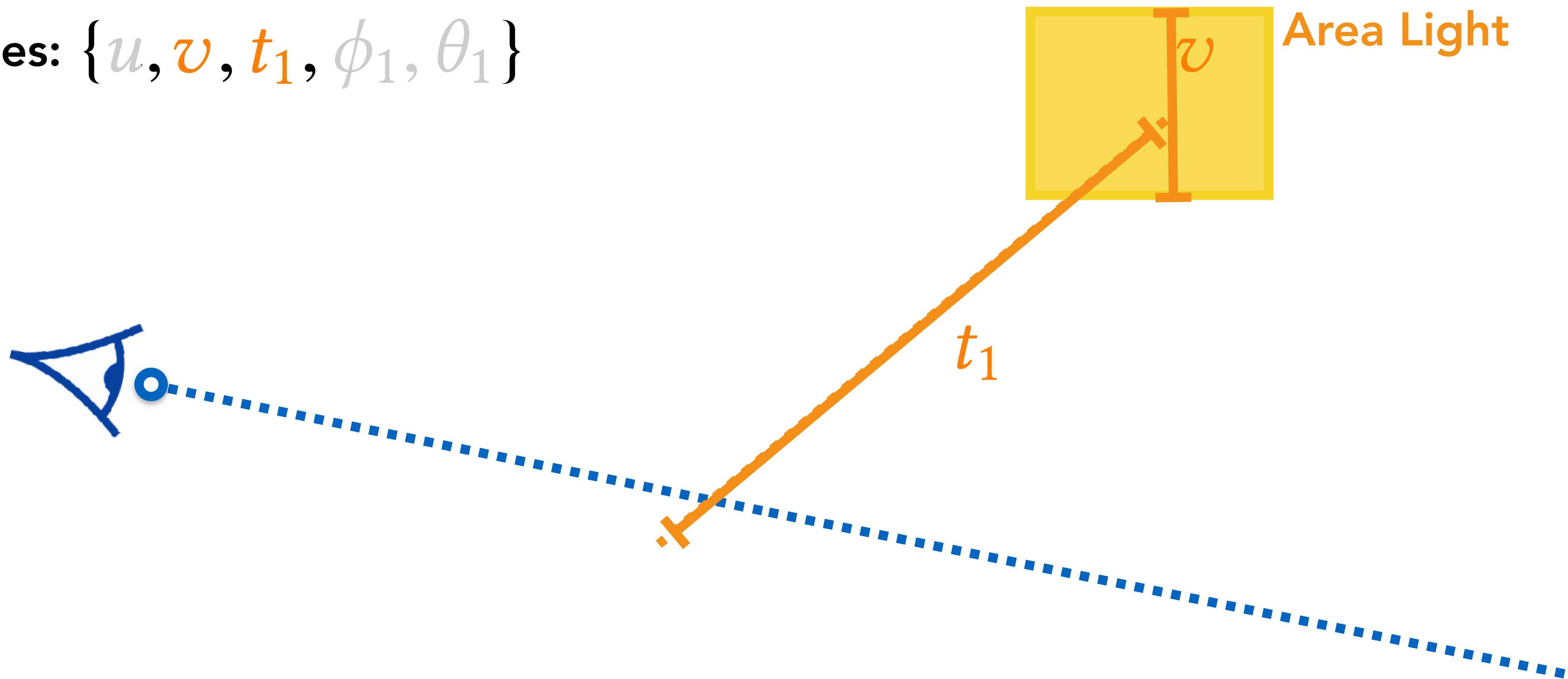
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



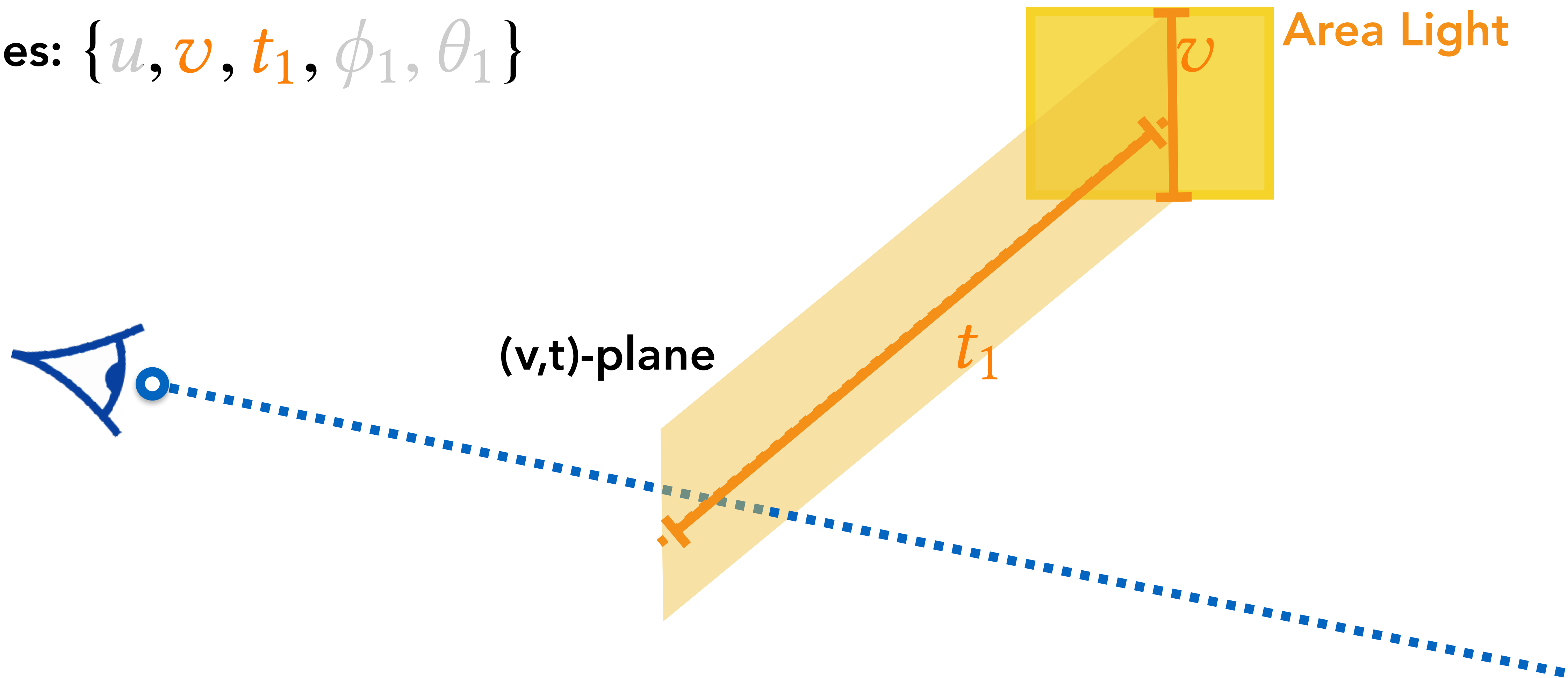
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



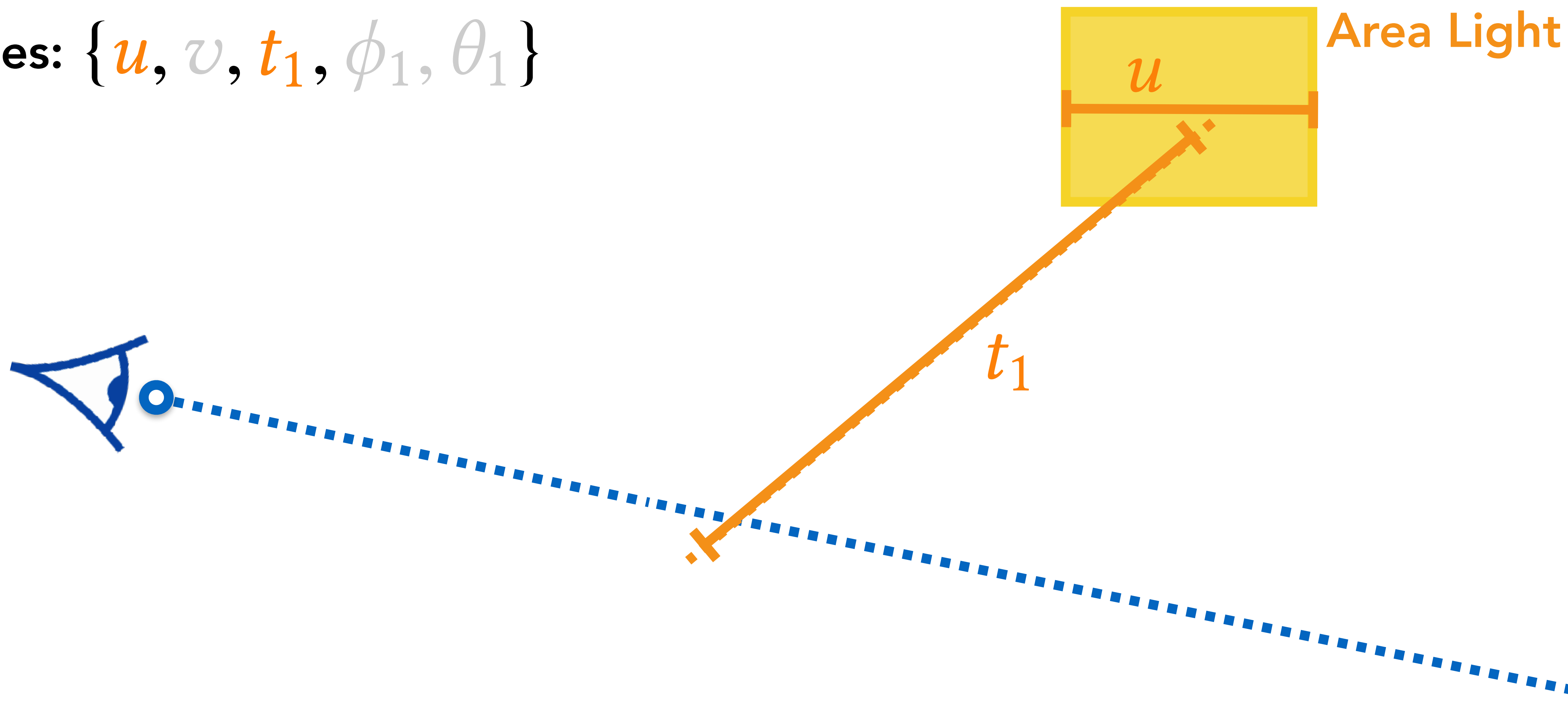
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



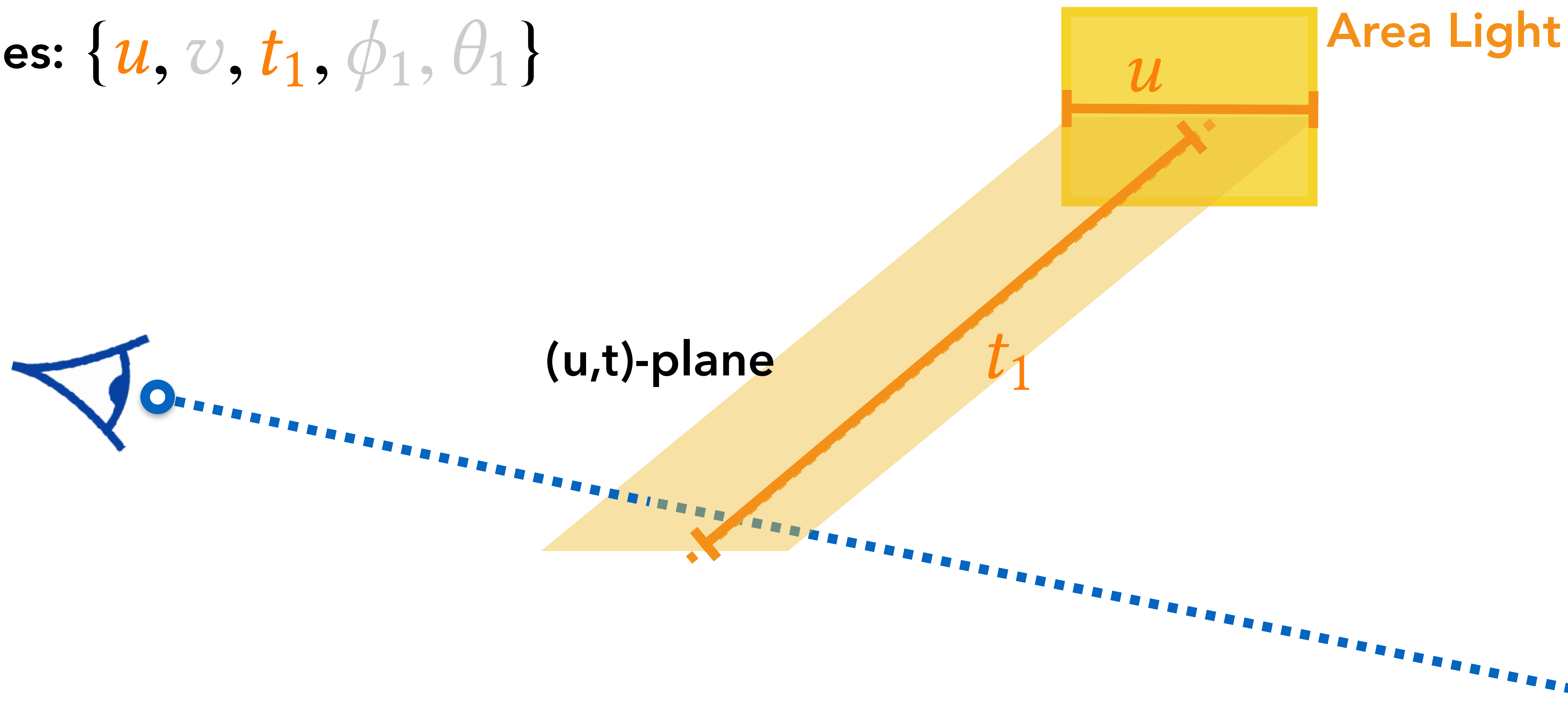
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



# More possible estimators

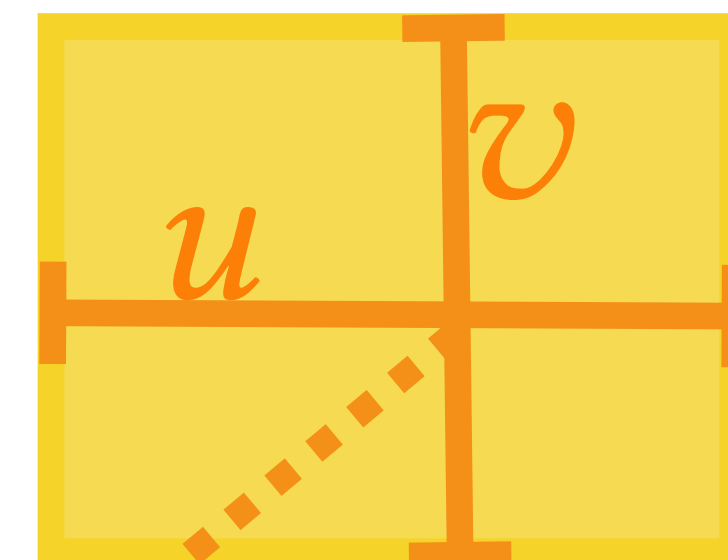
Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$





# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$

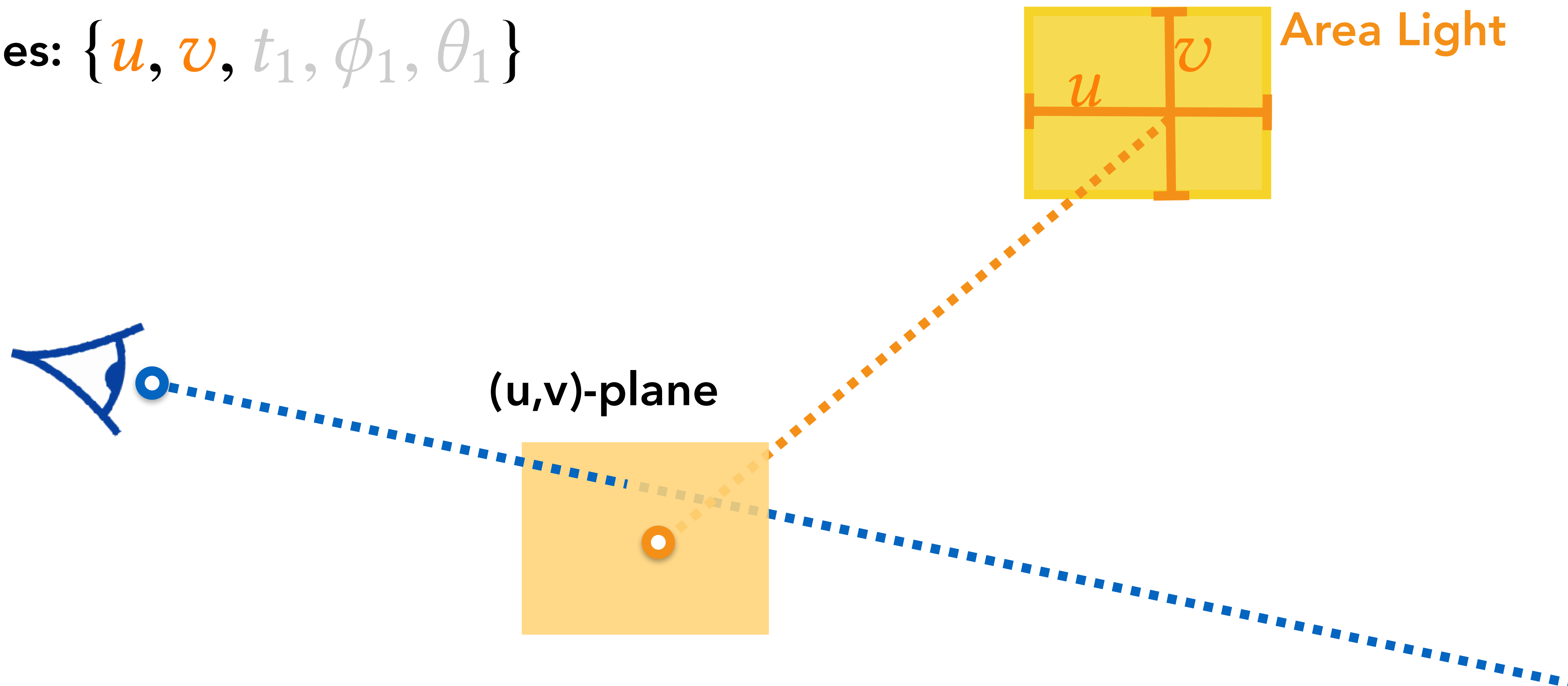


Area Light



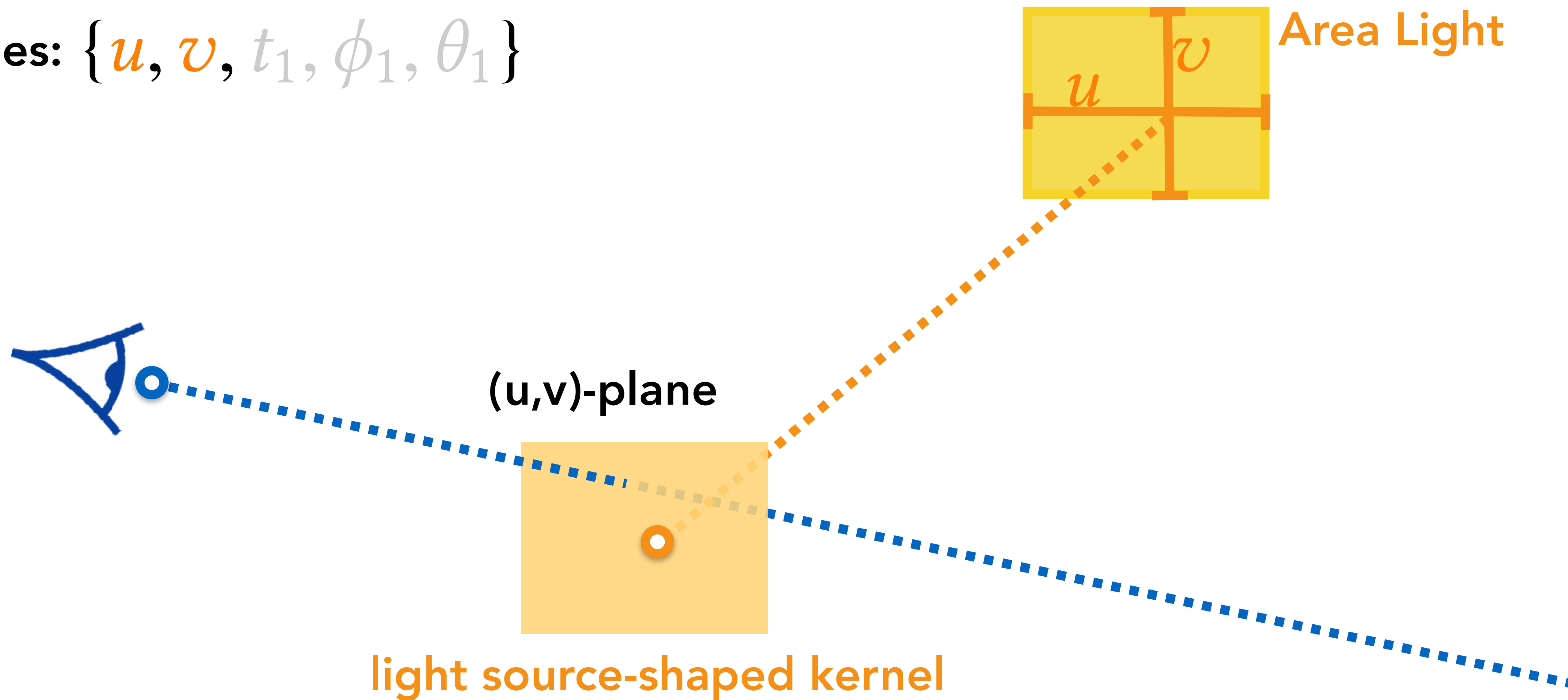
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



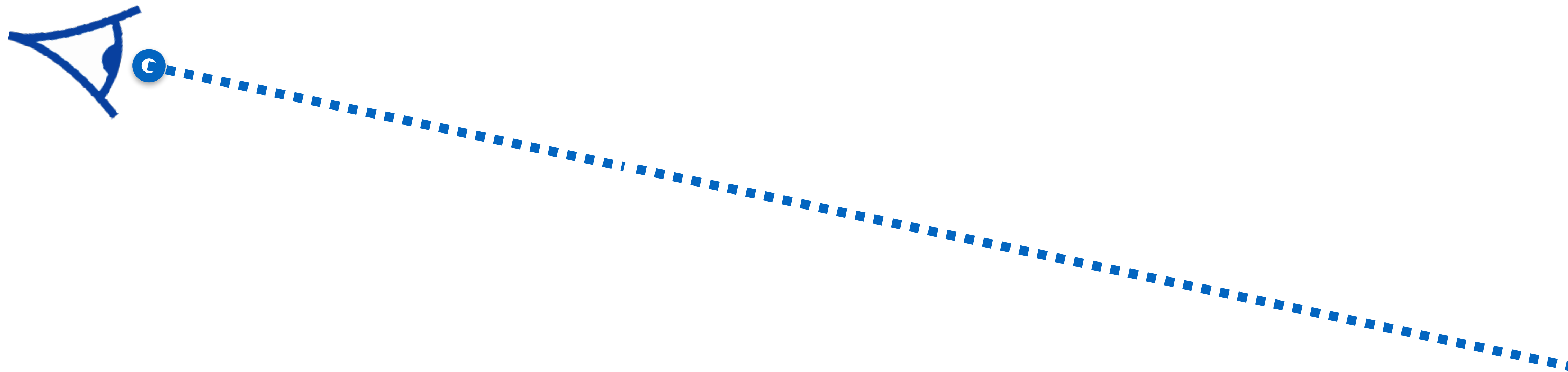
# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$

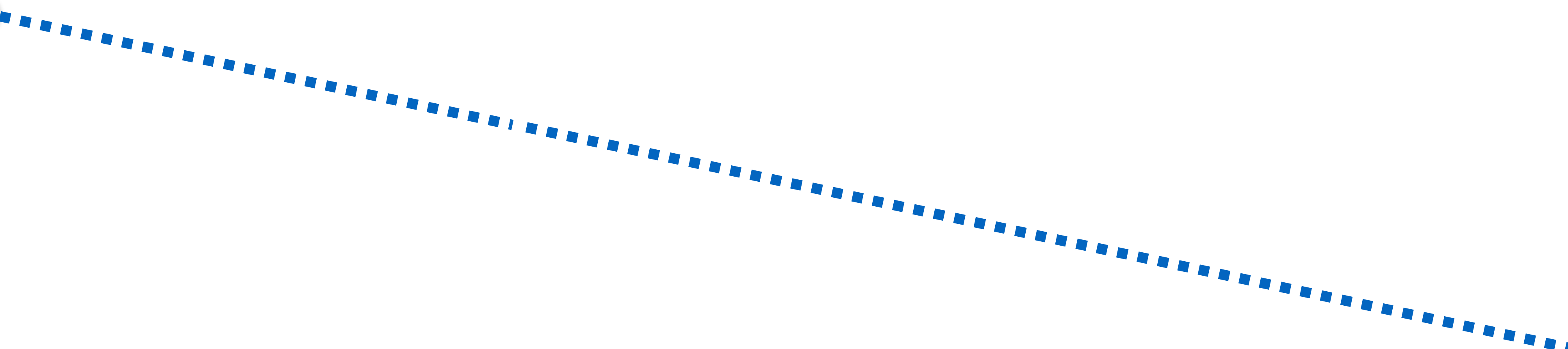


# More possible estimators

Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$

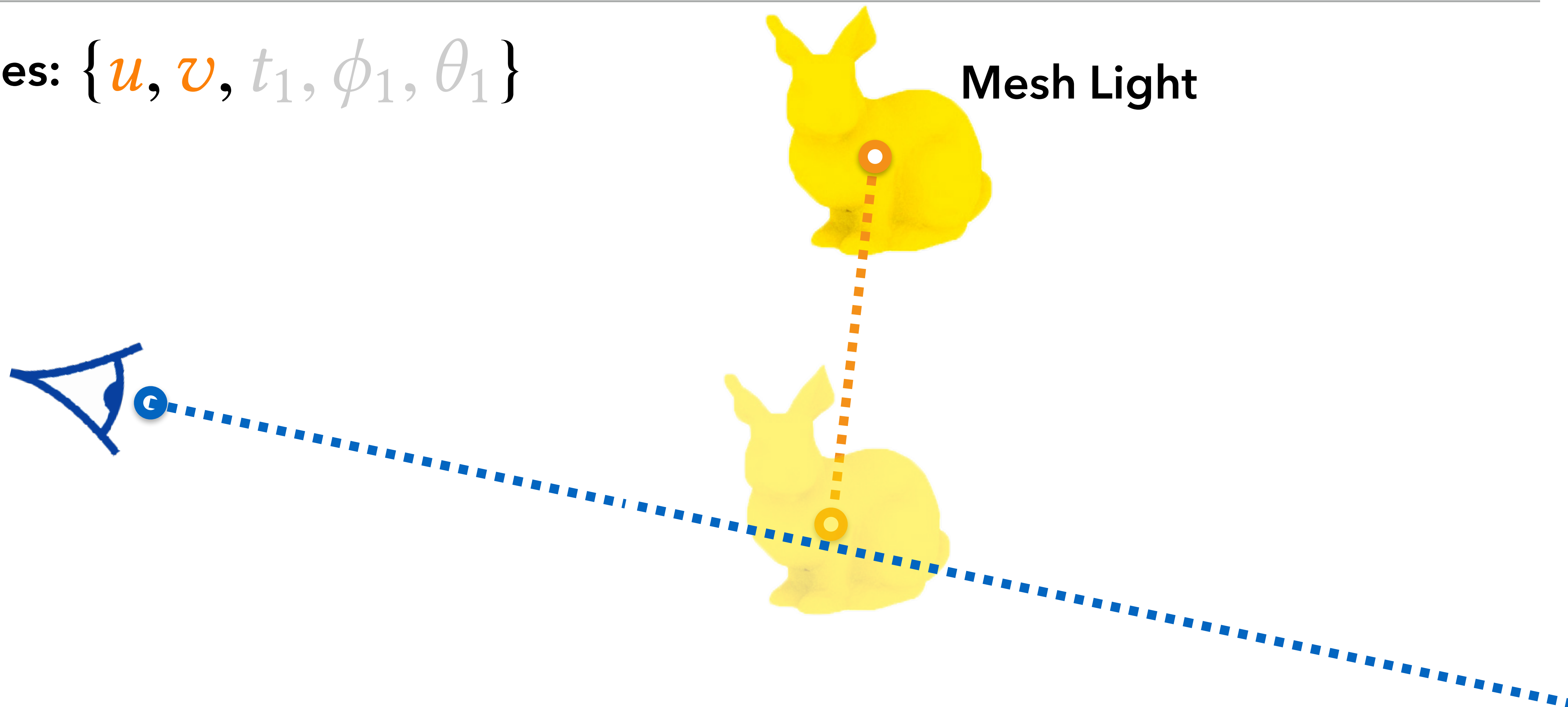


Mesh Light

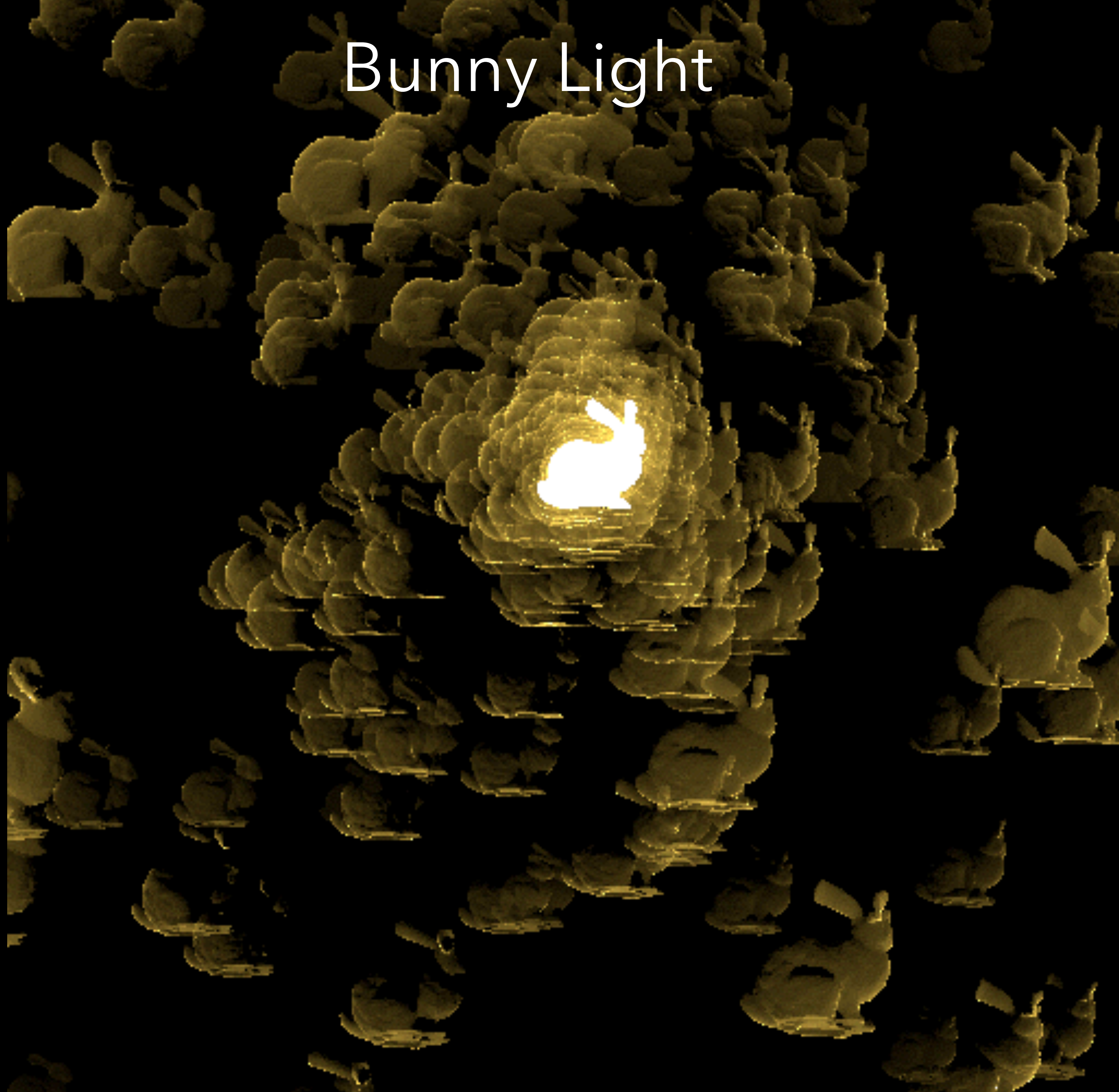


# More possible estimators

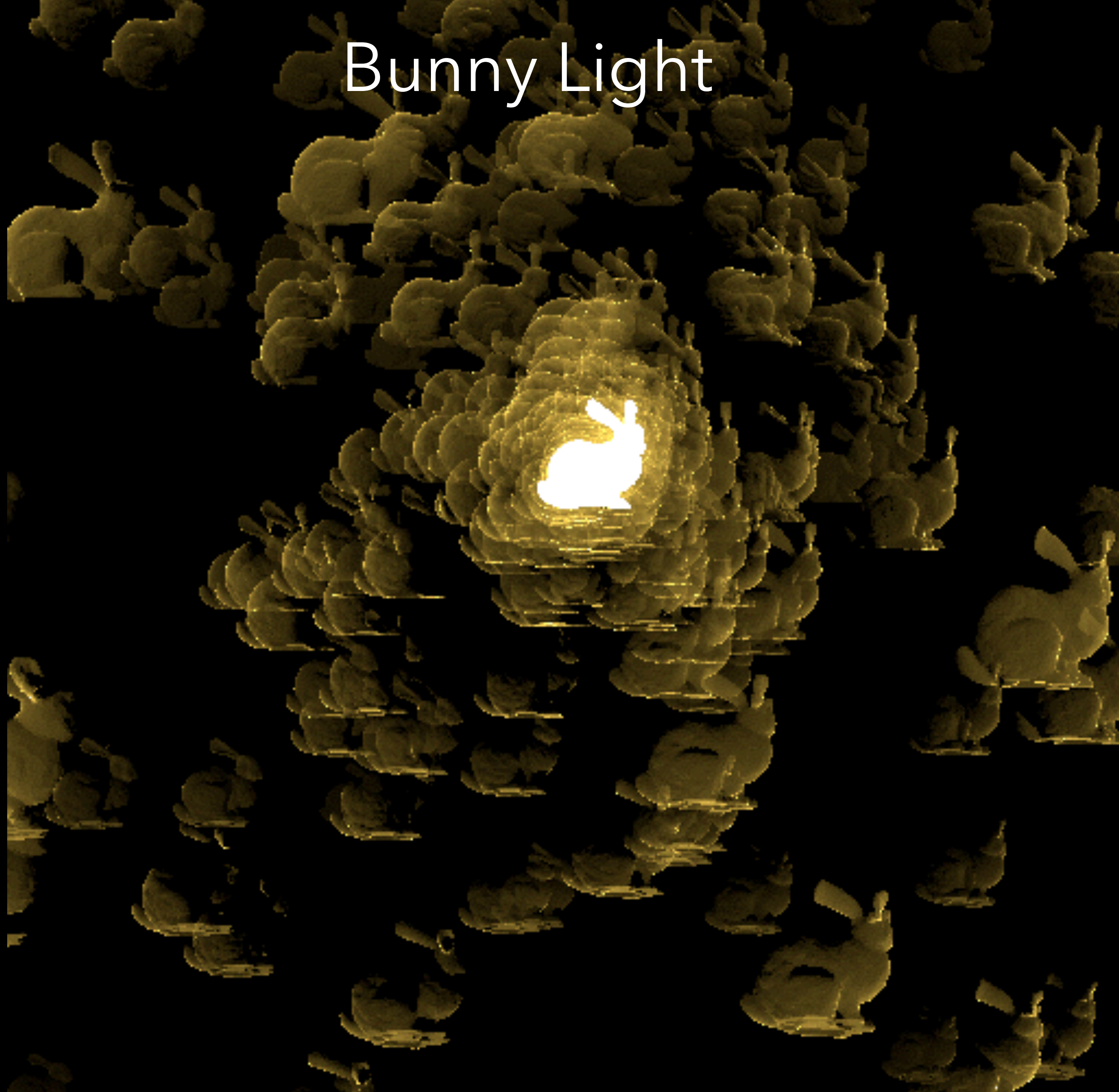
Variables:  $\{u, v, t_1, \phi_1, \theta_1\}$



# Bunny Light

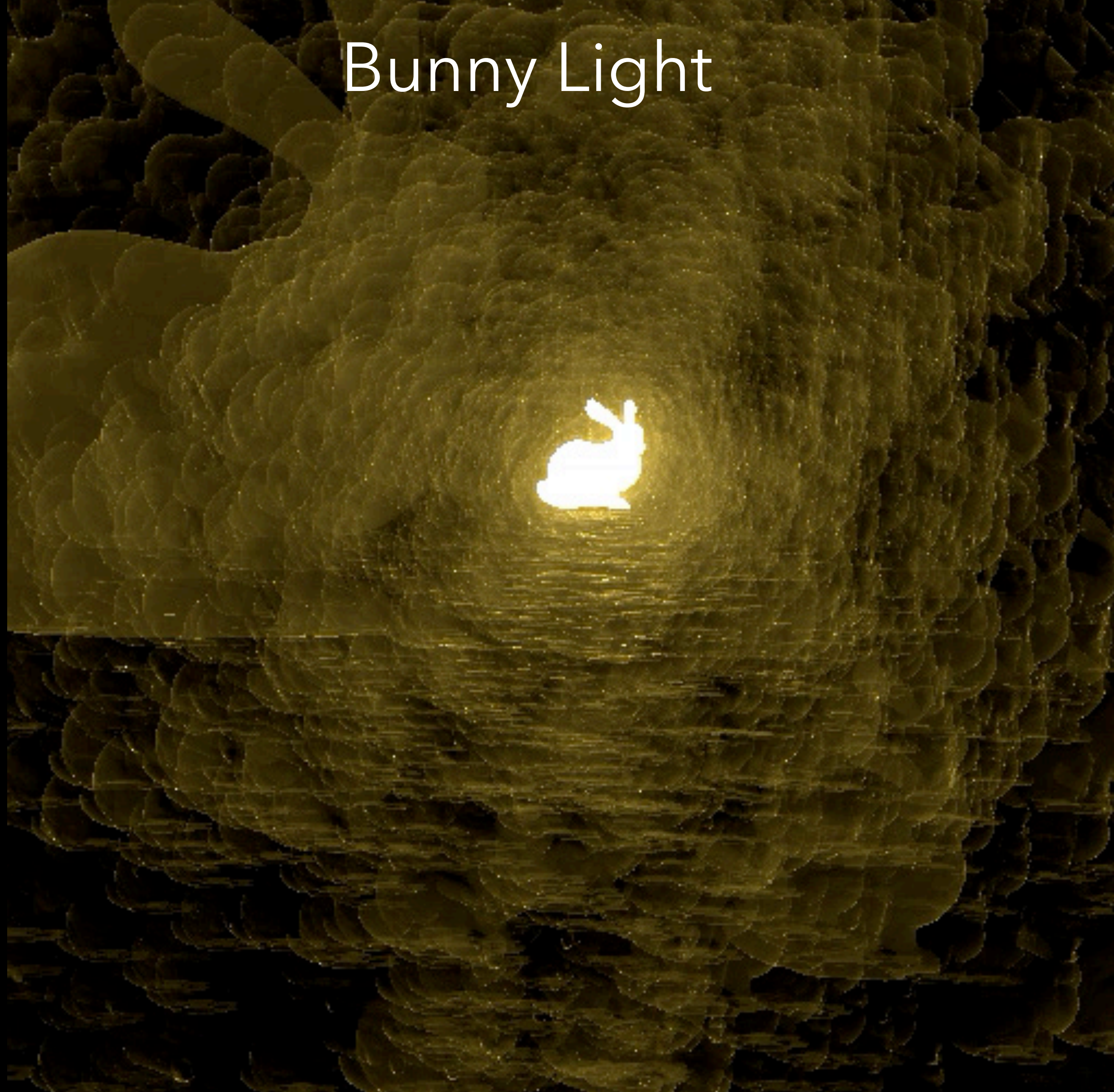


# Bunny Light

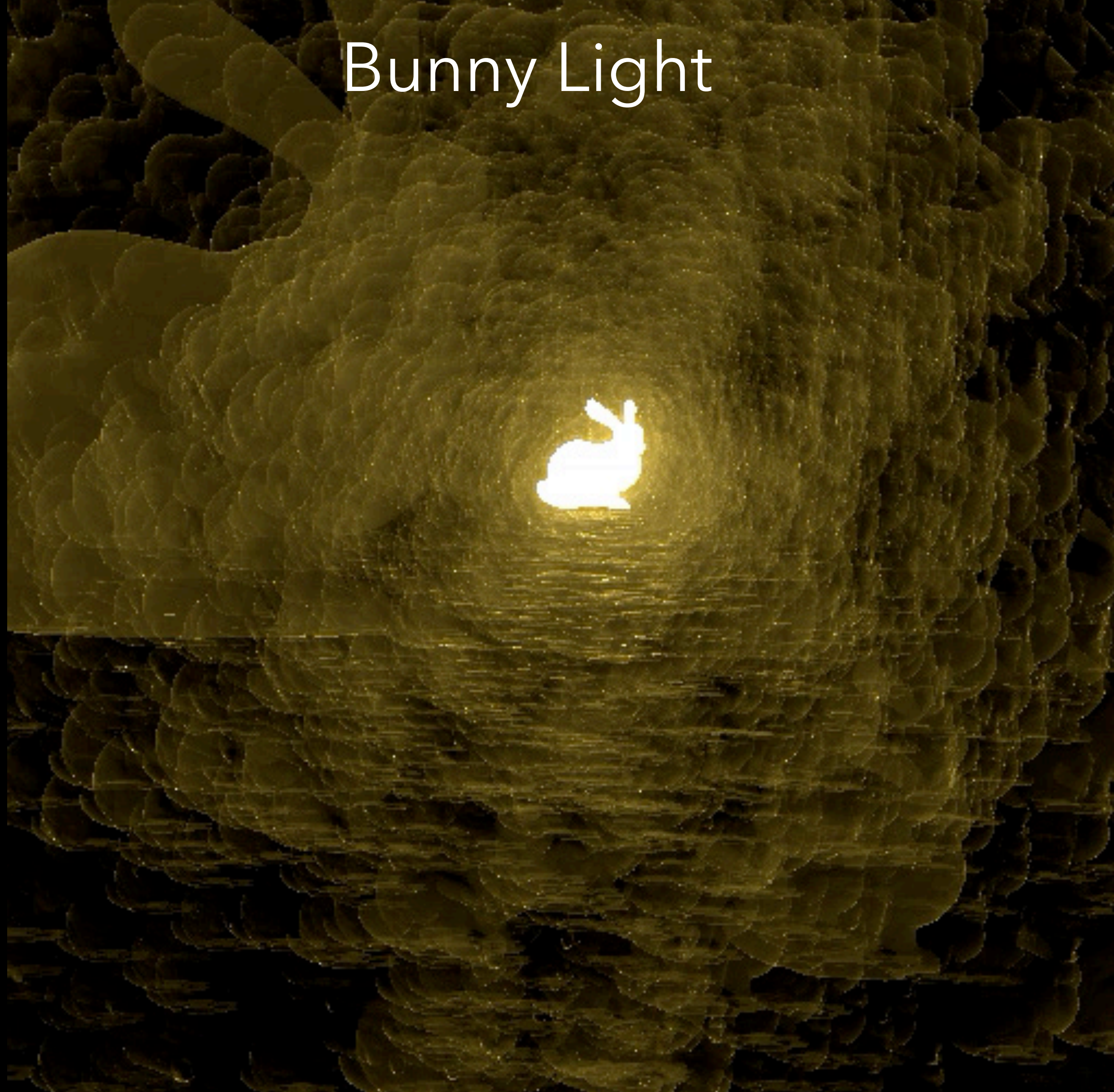




# Bunny Light

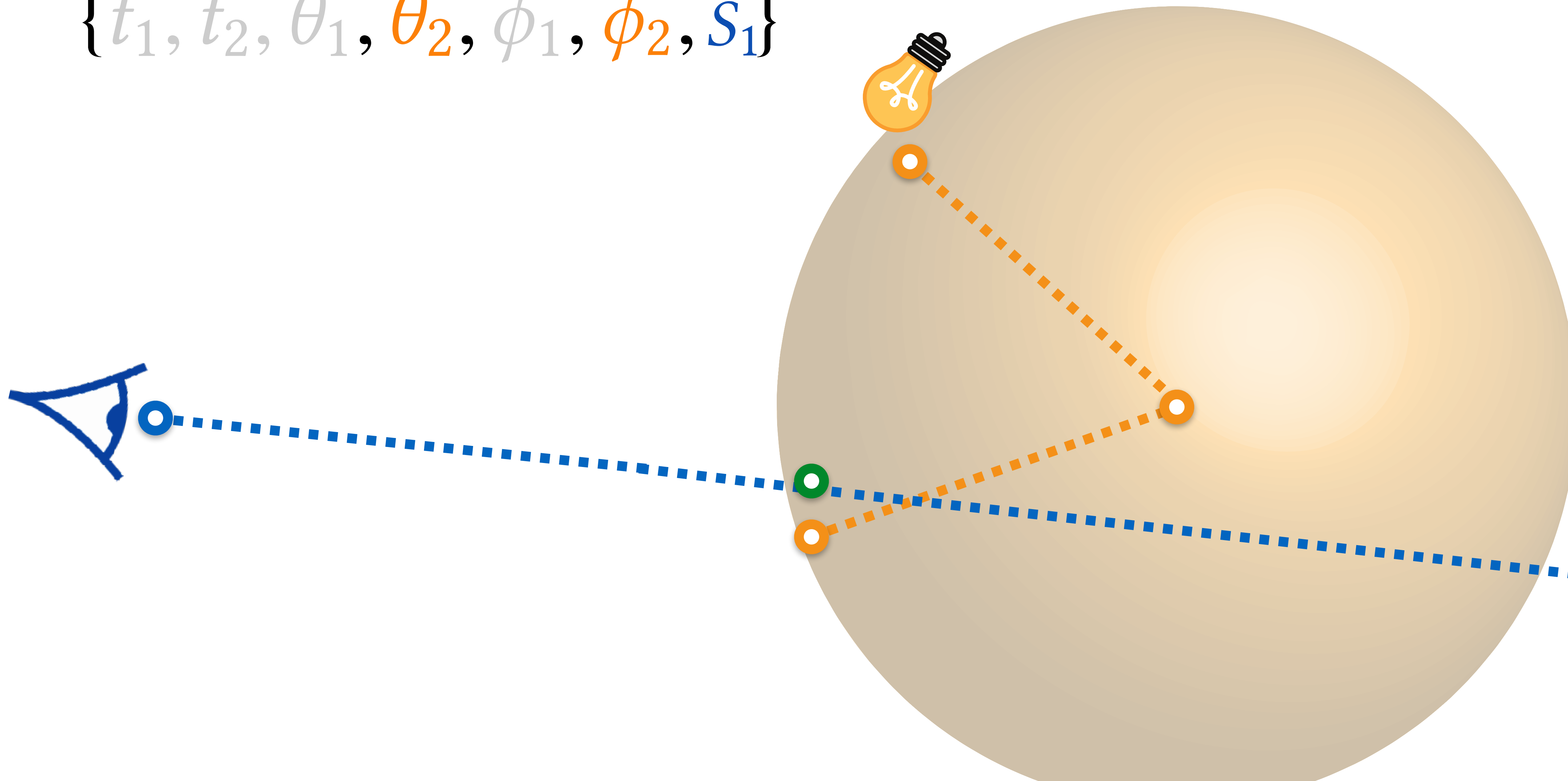


# Bunny Light



# Some possible estimators

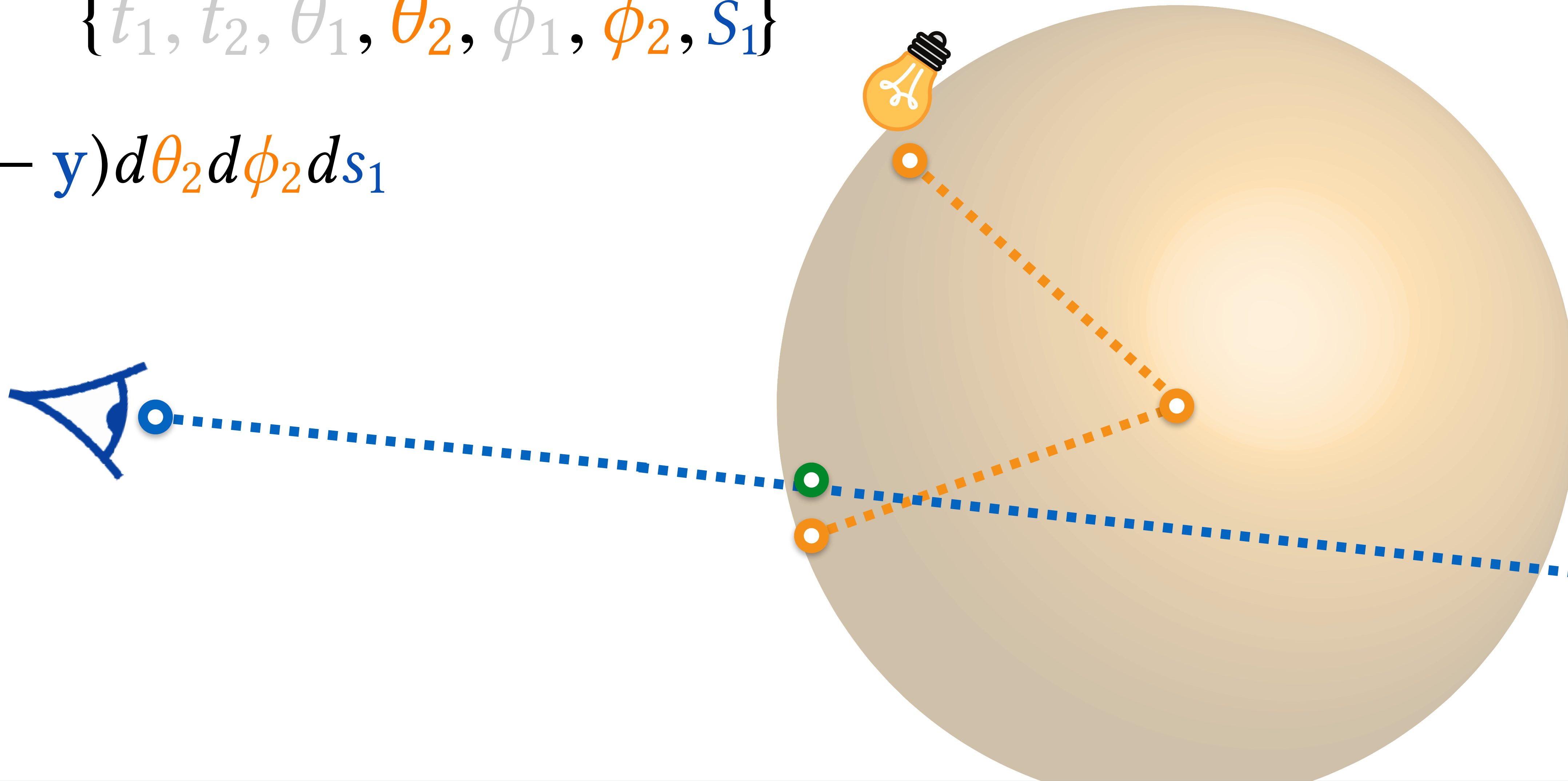
Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, S_1\}$



# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

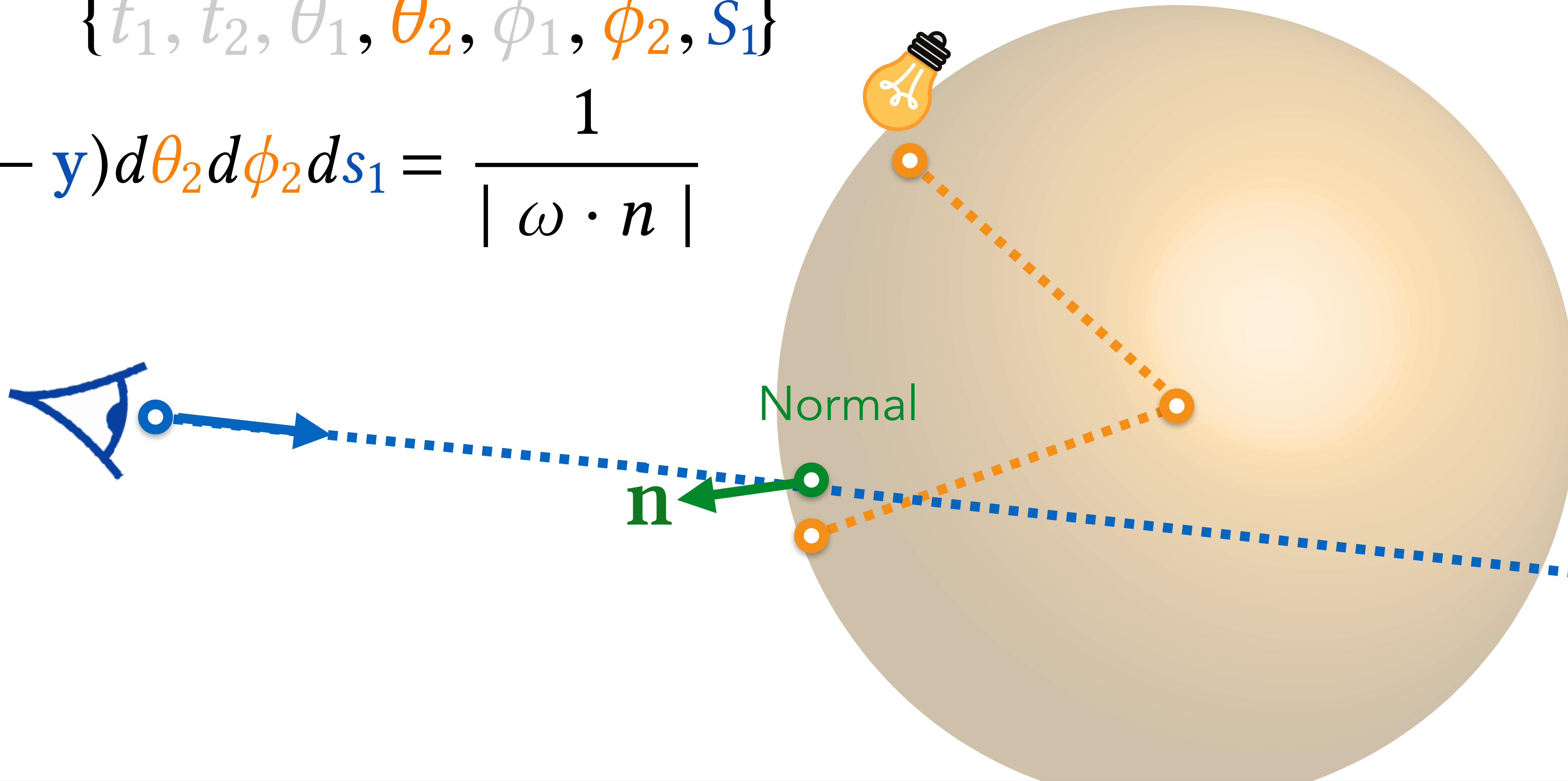
$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1$$



# Some possible estimators

Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

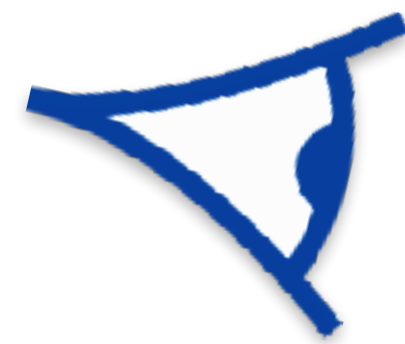
$$\int \delta(\mathbf{x} - \mathbf{y}) d\theta_2 d\phi_2 ds_1 = \frac{1}{|\omega \cdot \mathbf{n}|}$$



# PROCEDURE

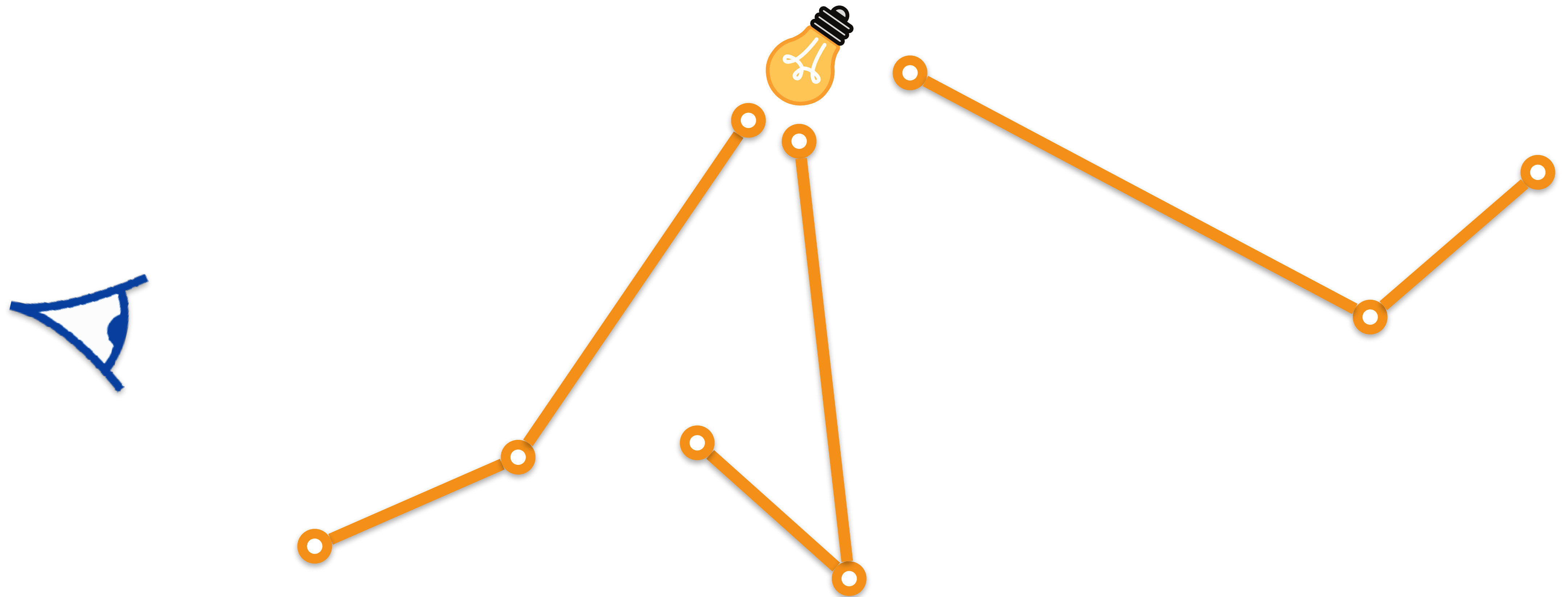
# How we render with photon surfaces

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# How we render with photon surfaces

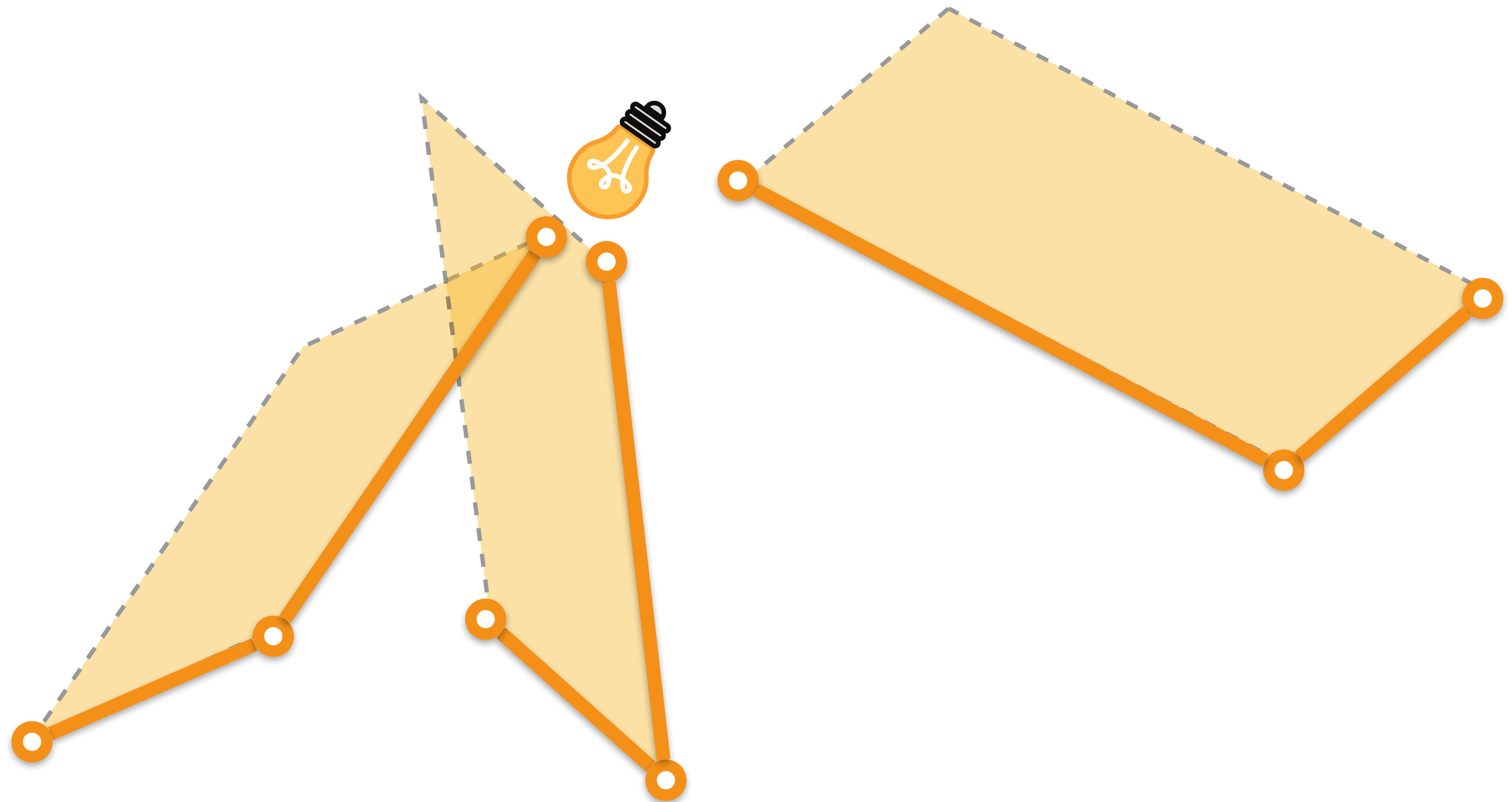
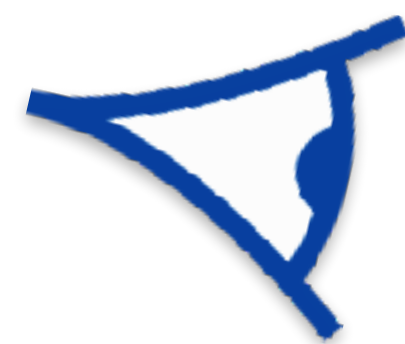
1: Trace





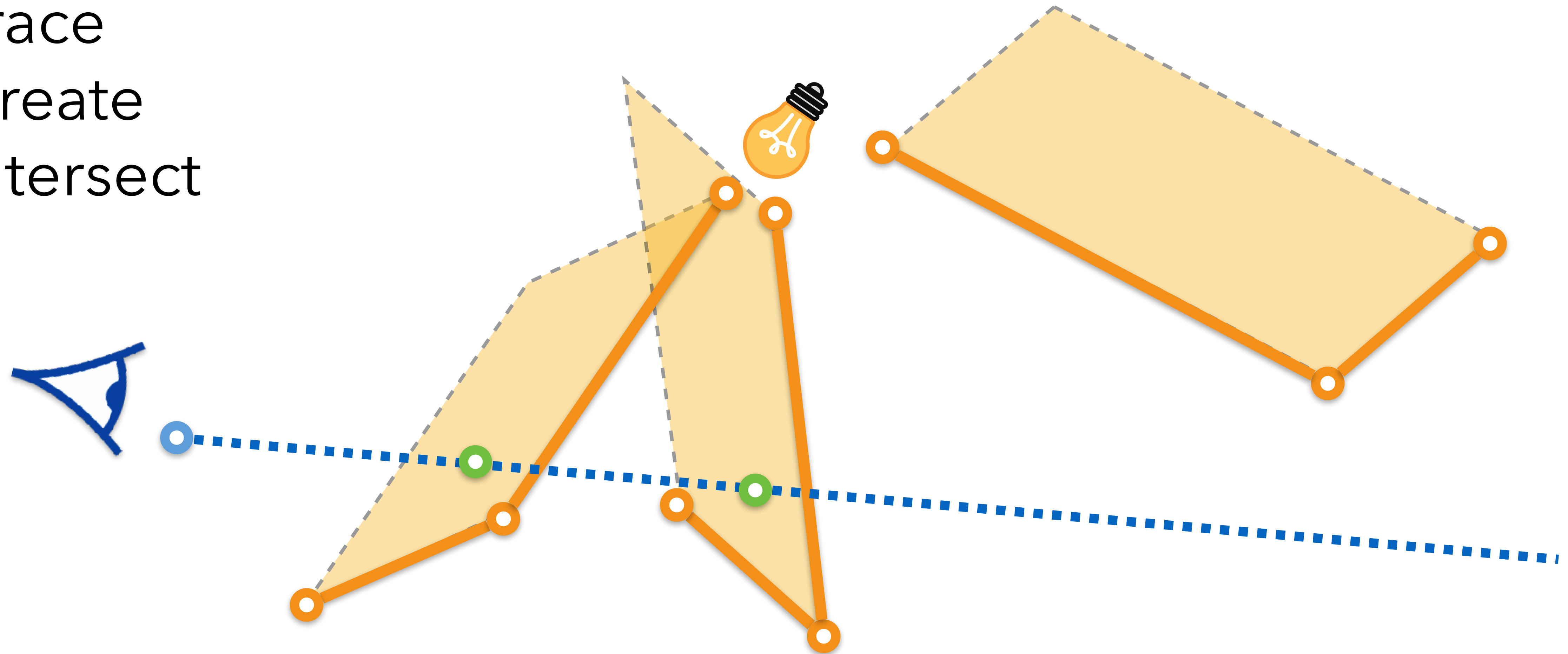
# How we render with photon surfaces

- 1: Trace
- 2: Create



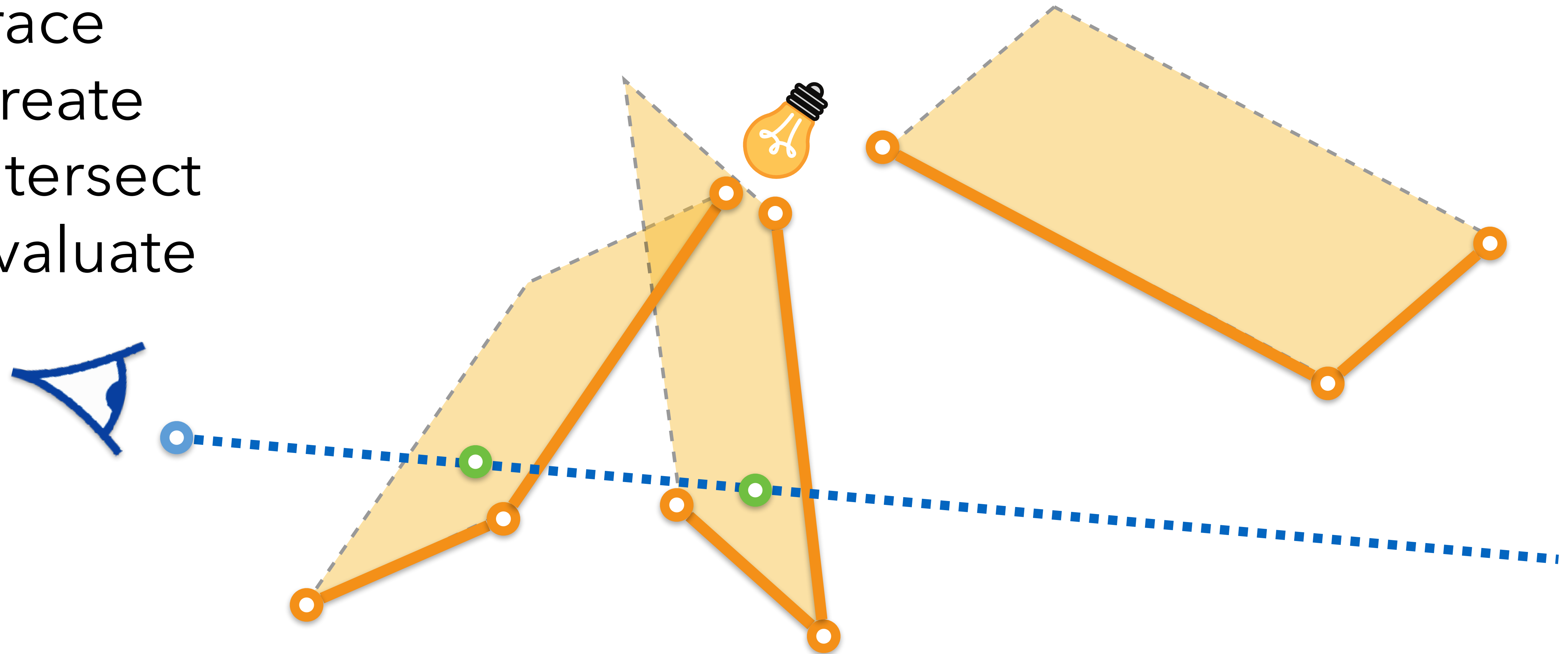
# How we render with photon surfaces

- 1: Trace
- 2: Create
- 3: Intersect



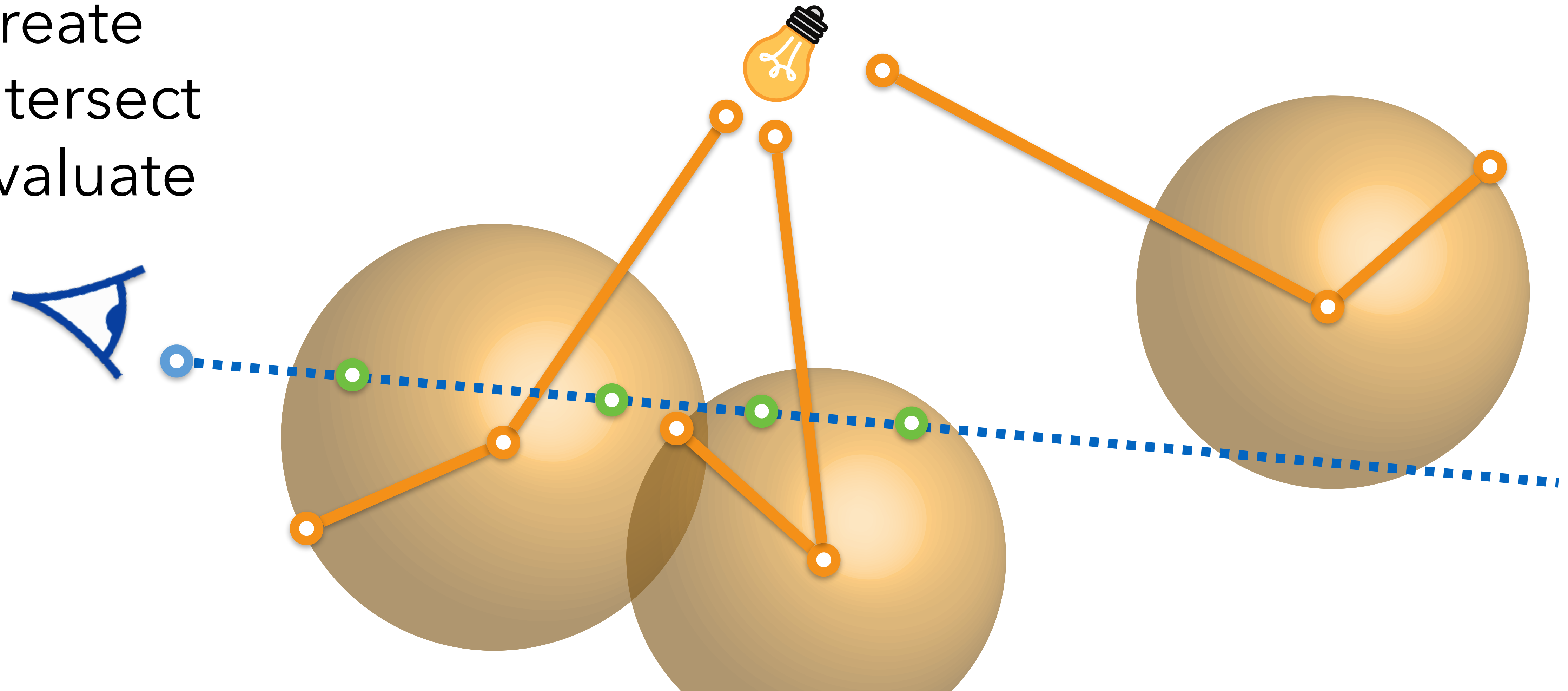
# How we render with photon surfaces

- 1: Trace
- 2: Create
- 3: Intersect
- 4: Evaluate

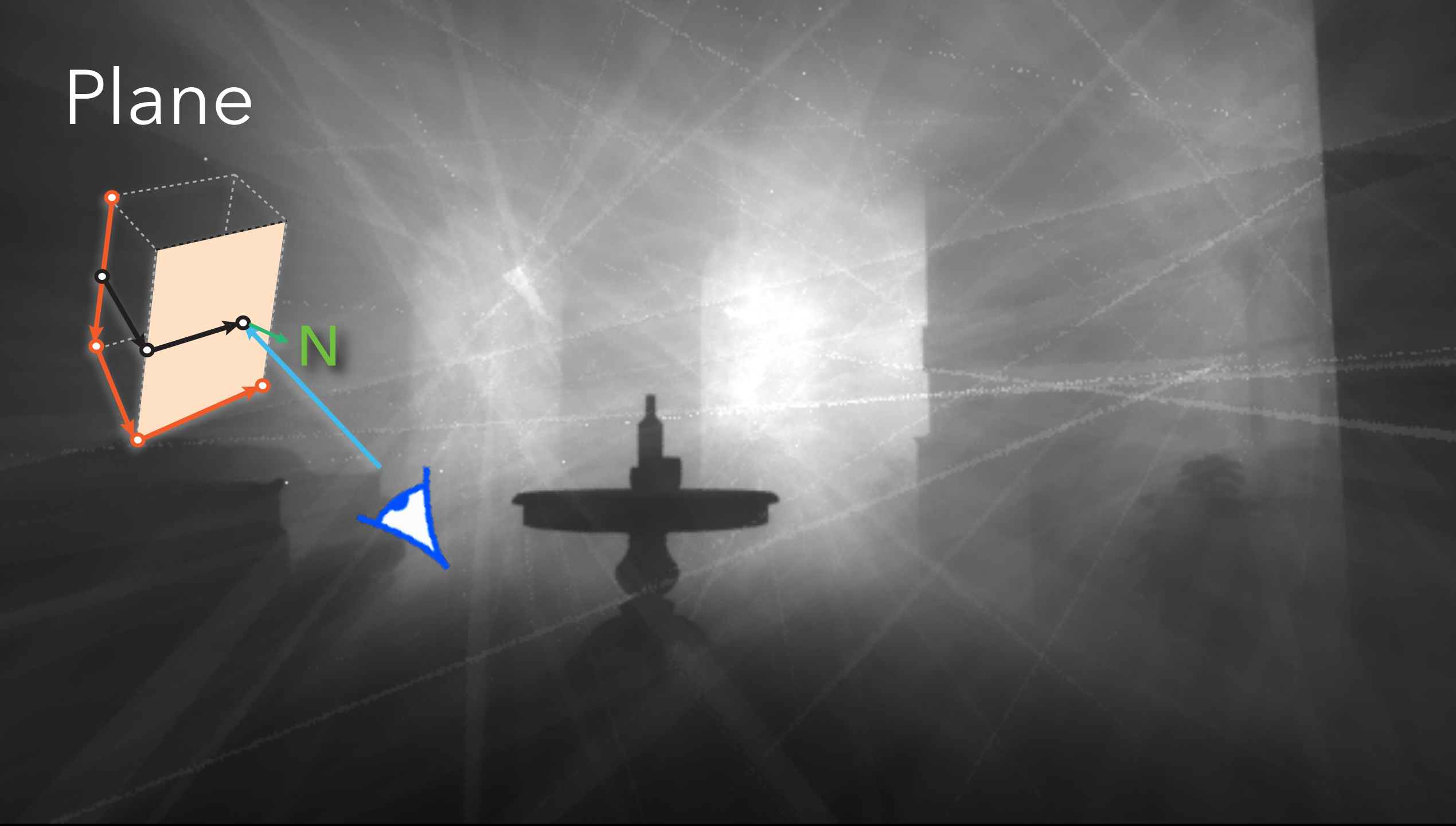
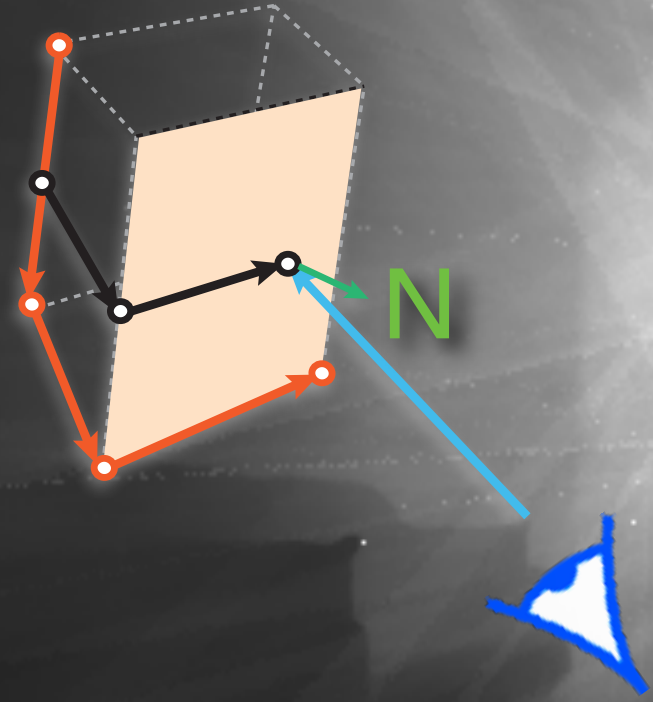


# How we render with photon surfaces

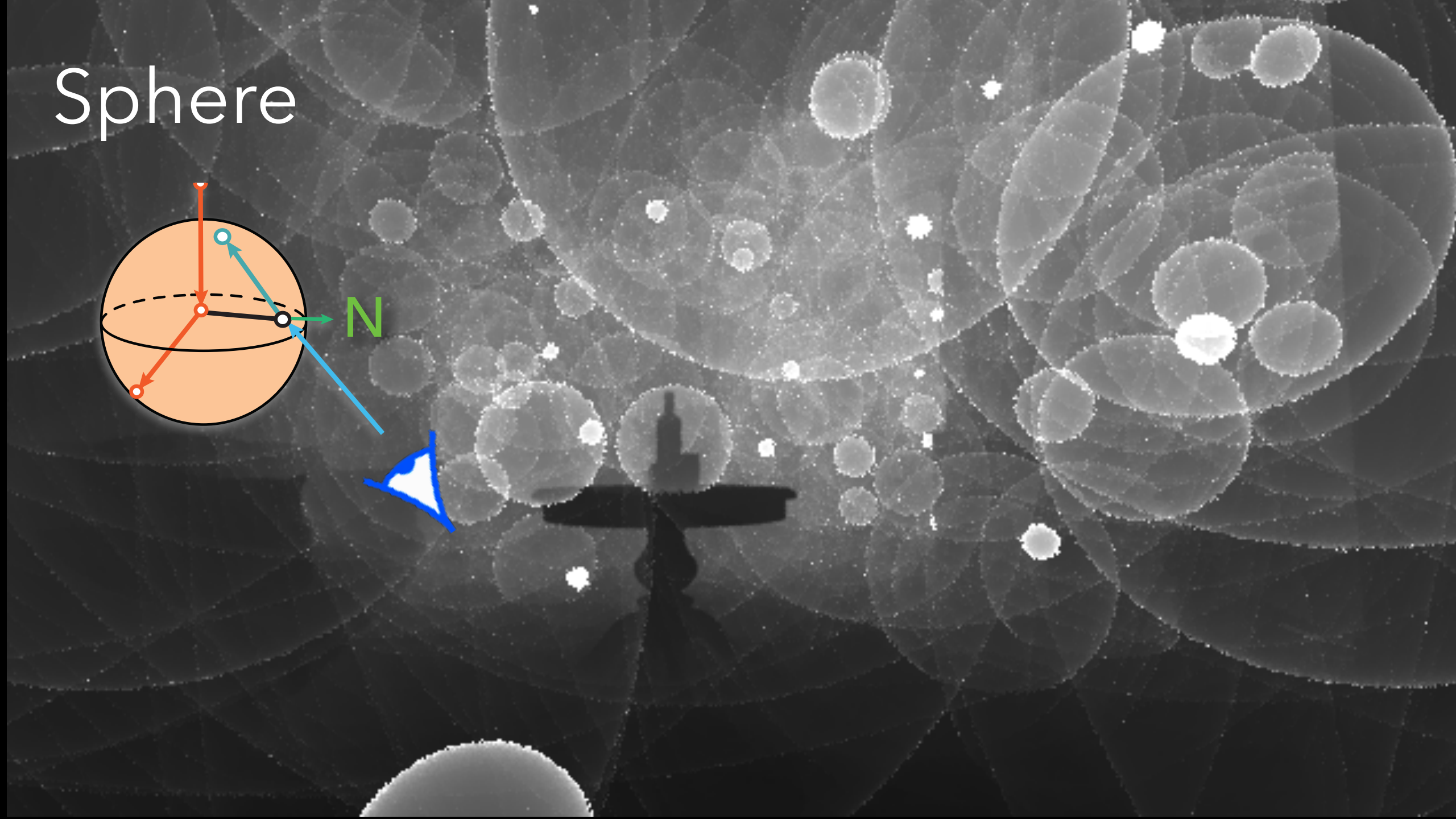
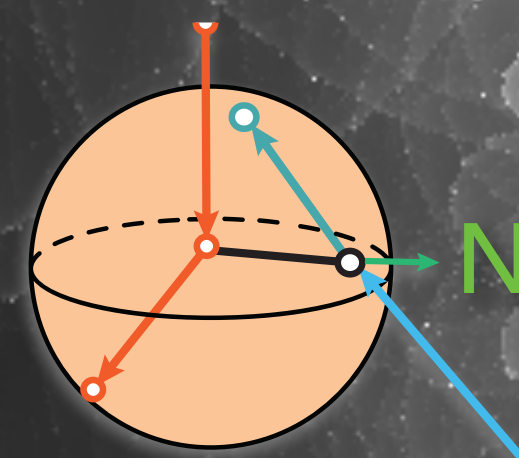
- 1: Trace
- 2: Create
- 3: Intersect
- 4: Evaluate



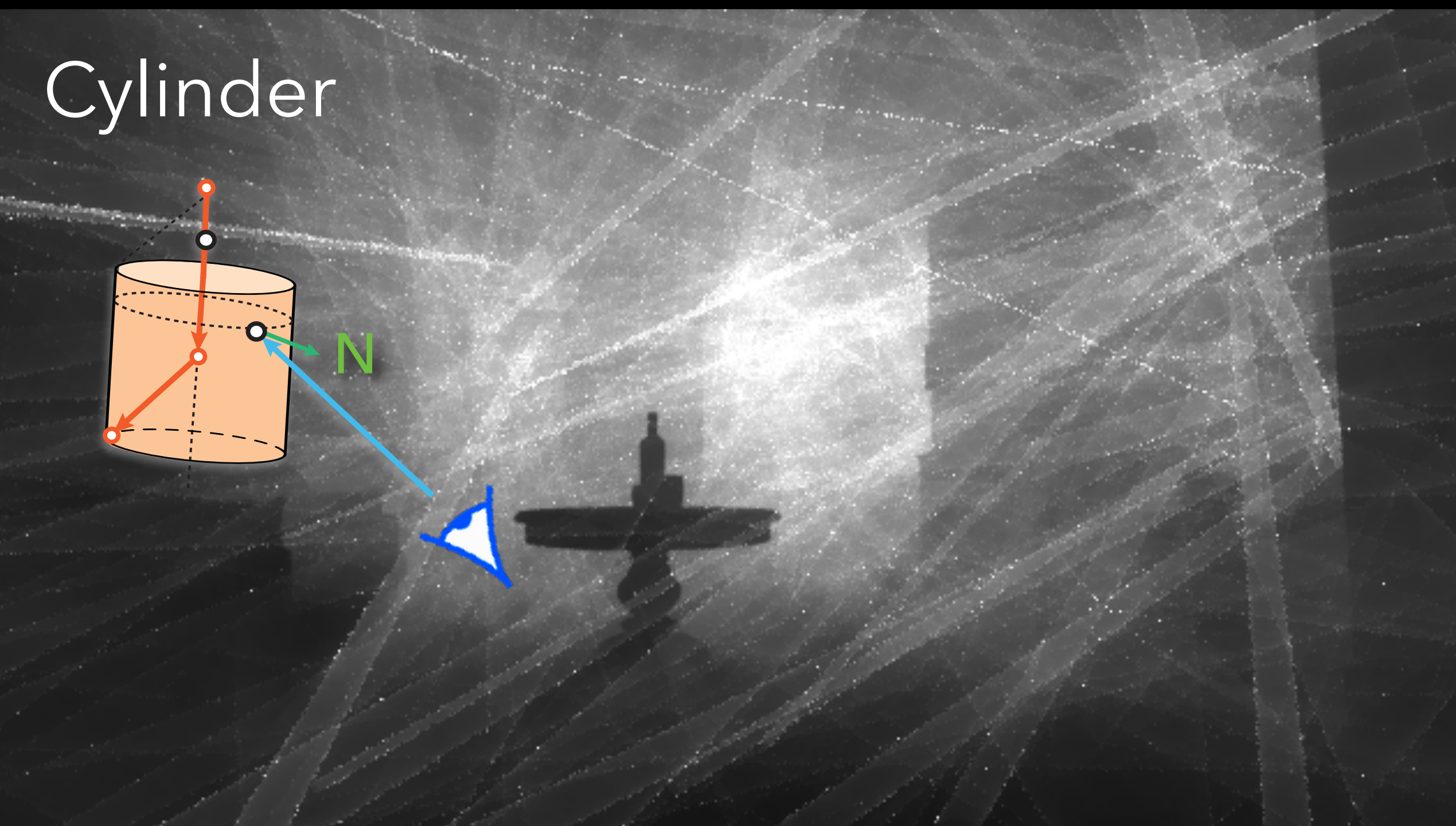
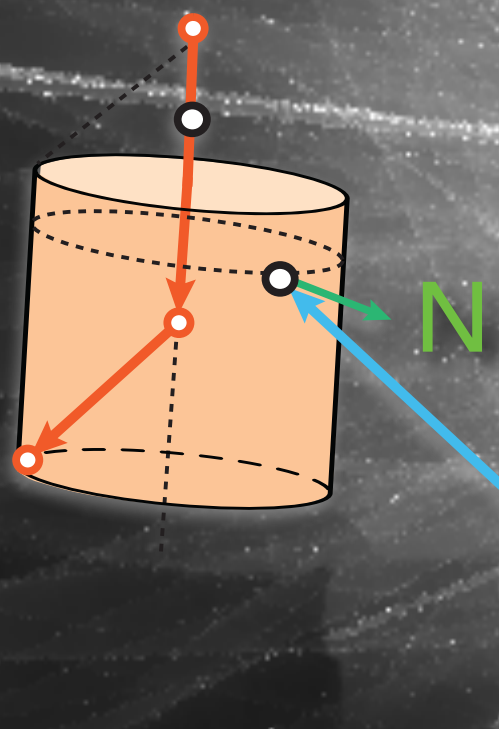
Plane



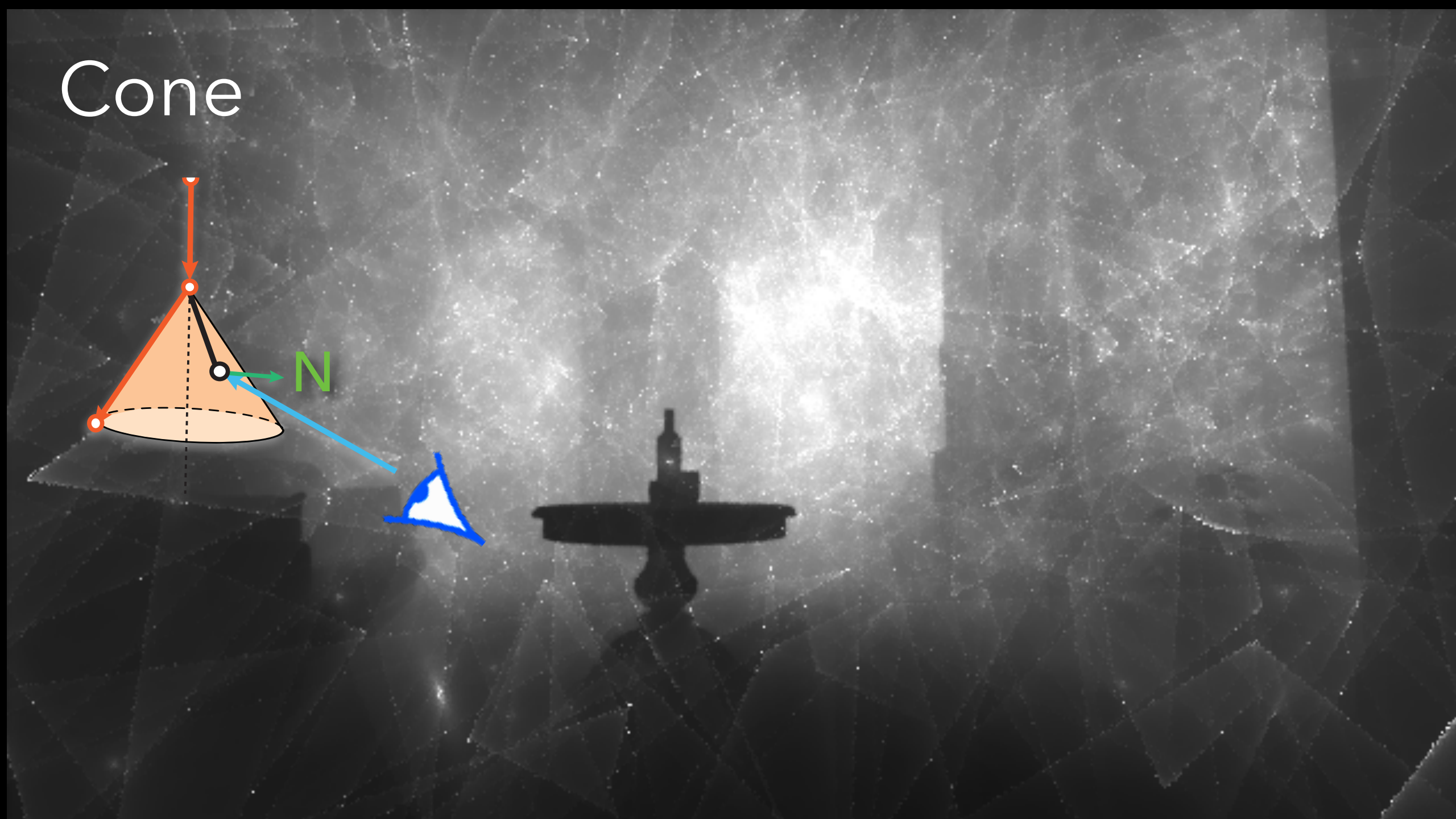
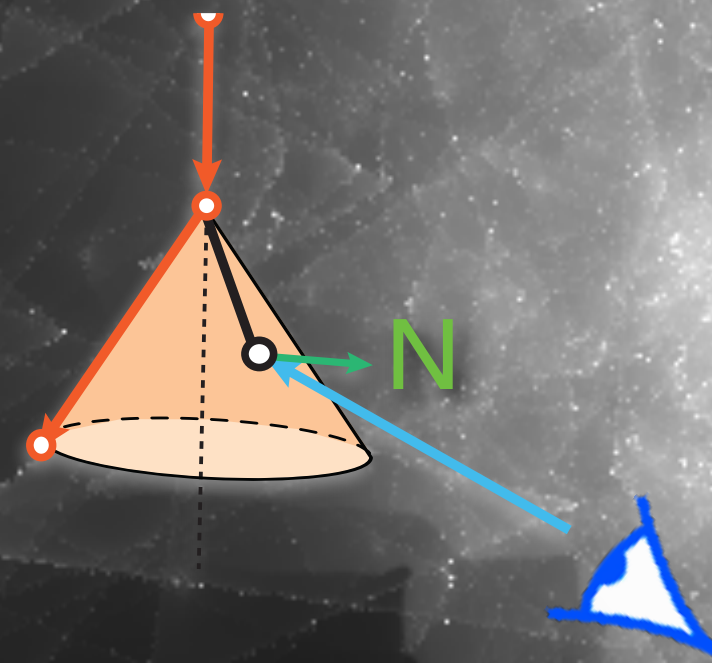
Sphere



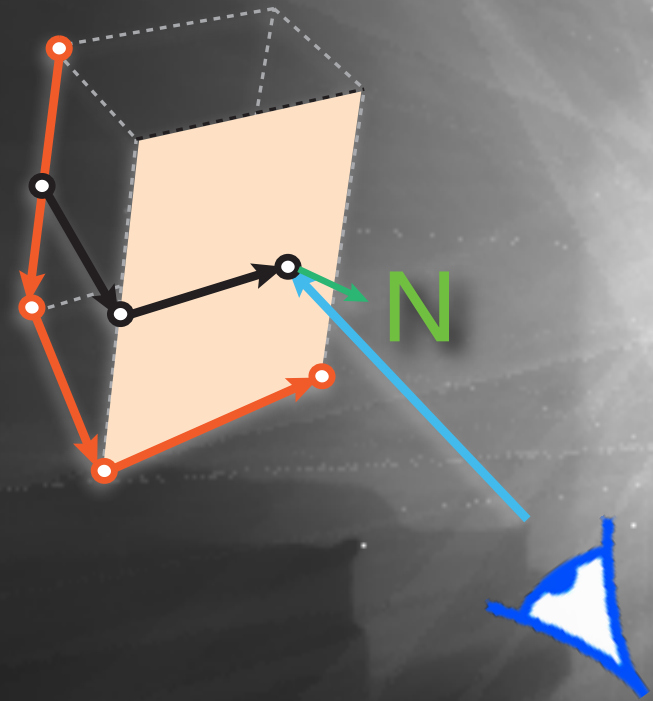
Cylinder



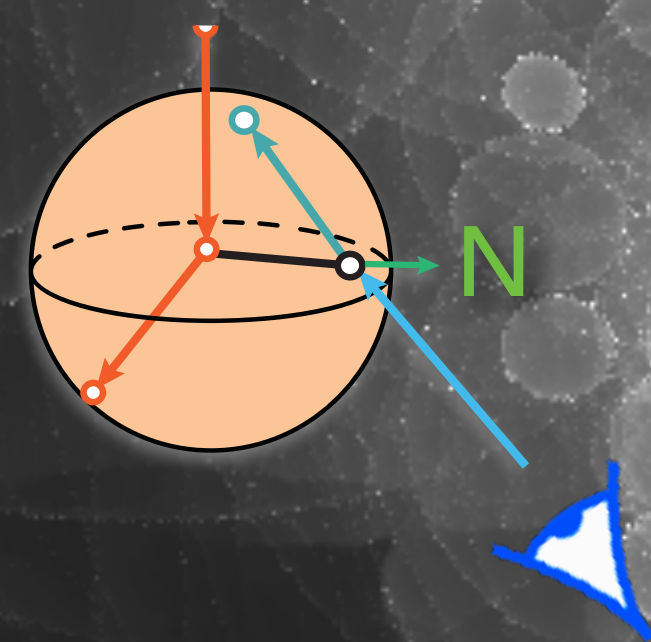
Cone



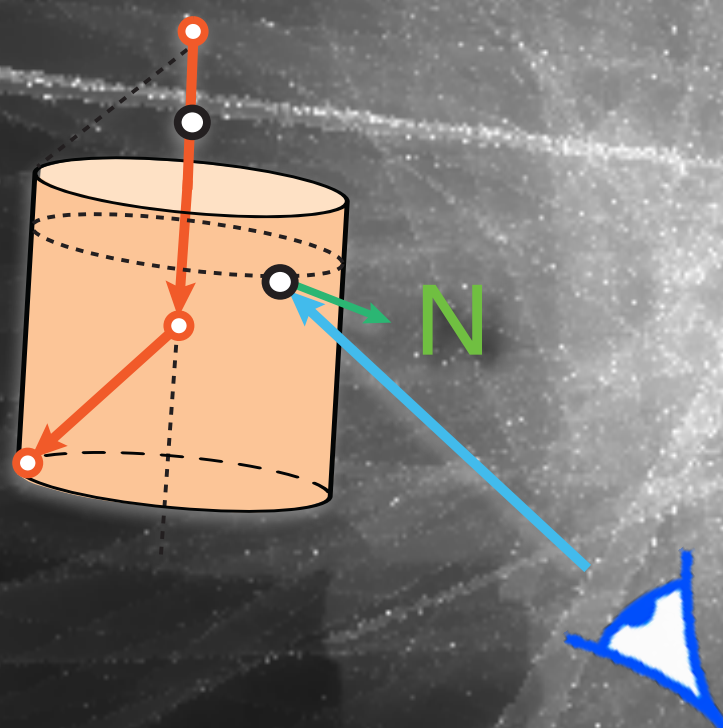
Plane



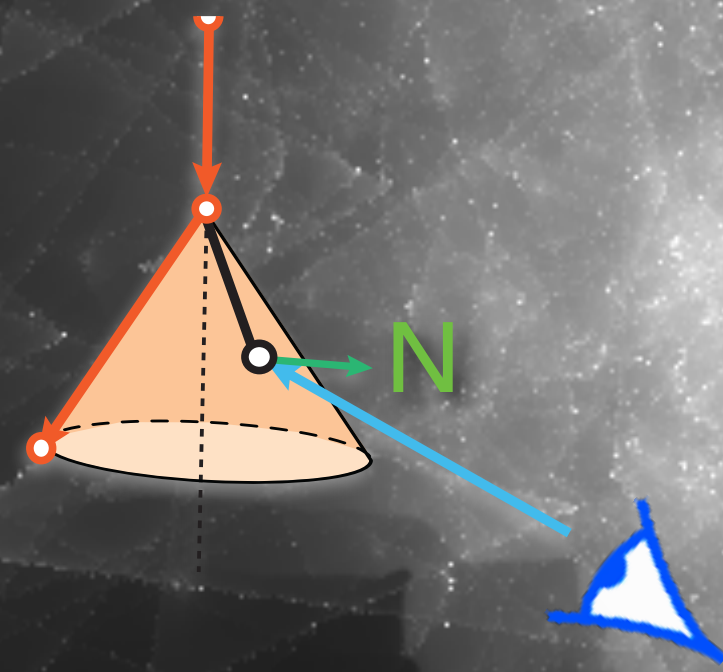
Sphere



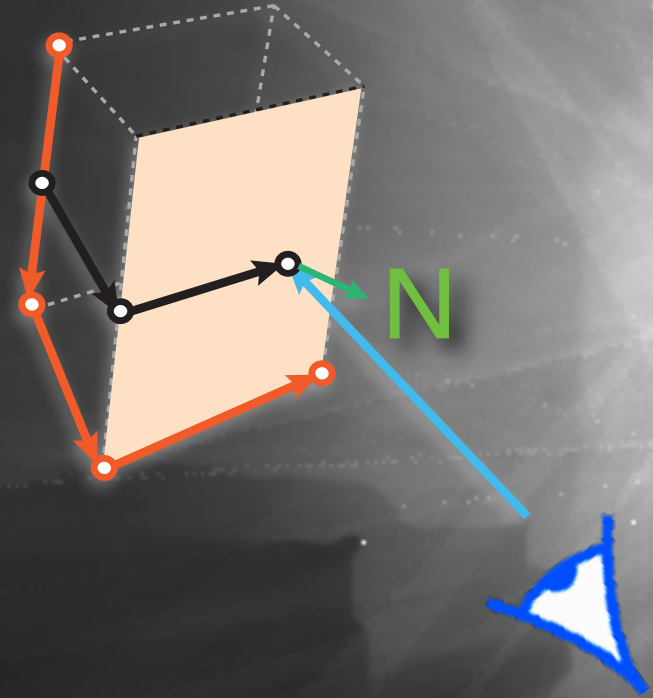
Cylinder



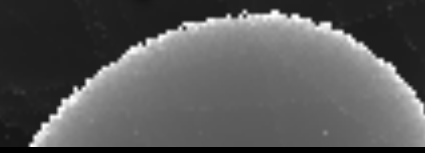
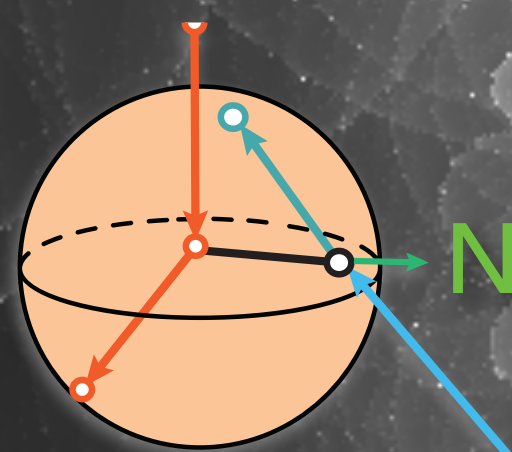
Cone



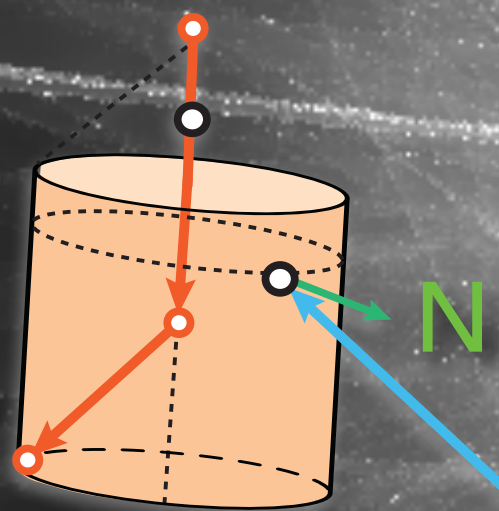
Plane



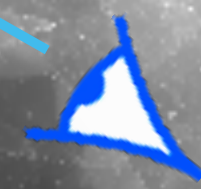
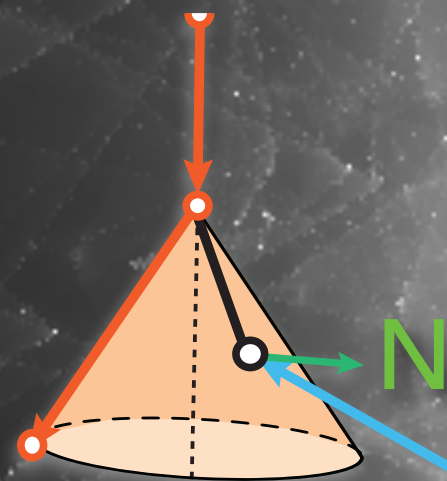
Sphere



Cylinder



Cone



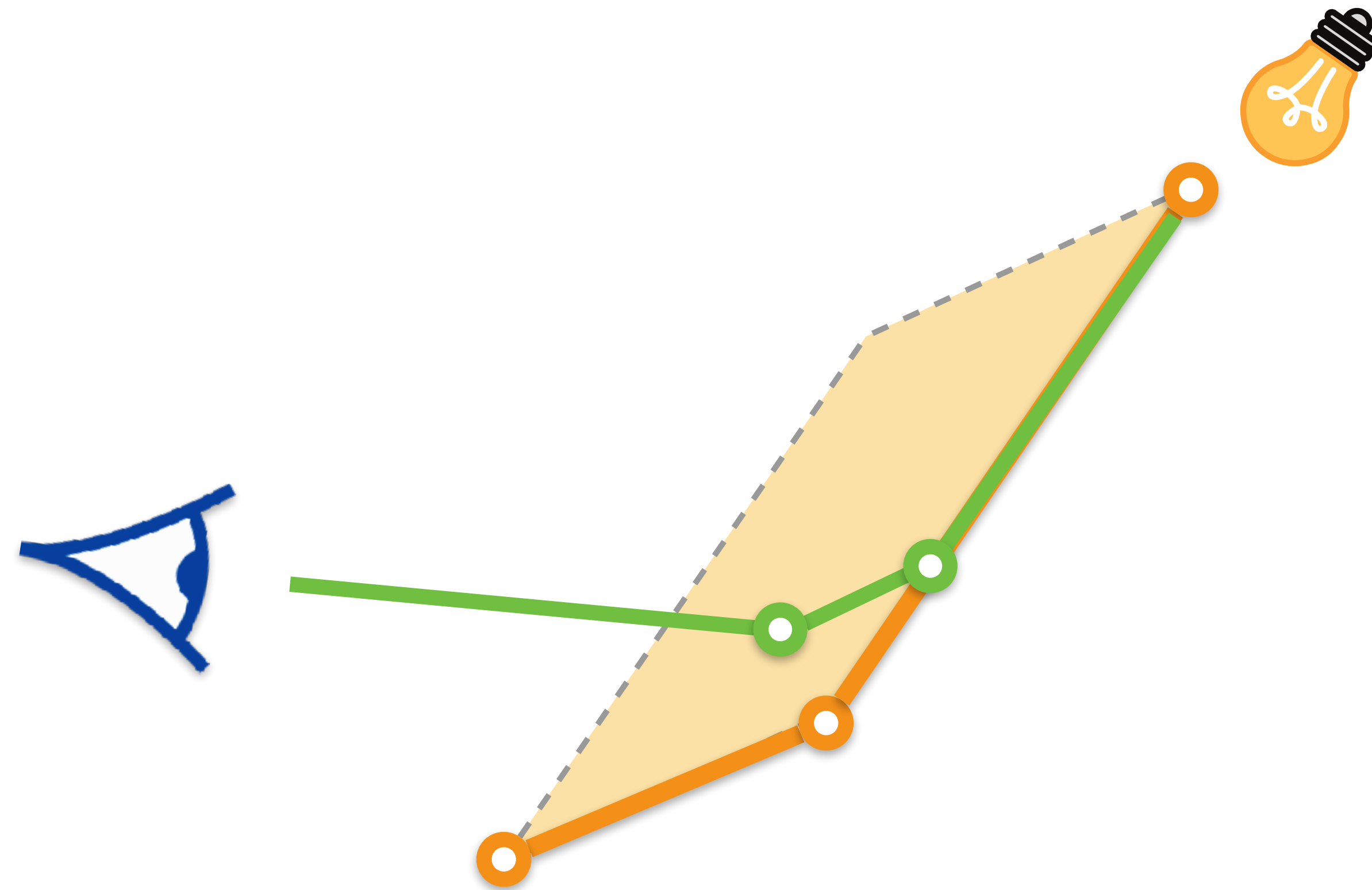
# Photon surfaces as sampling strategies

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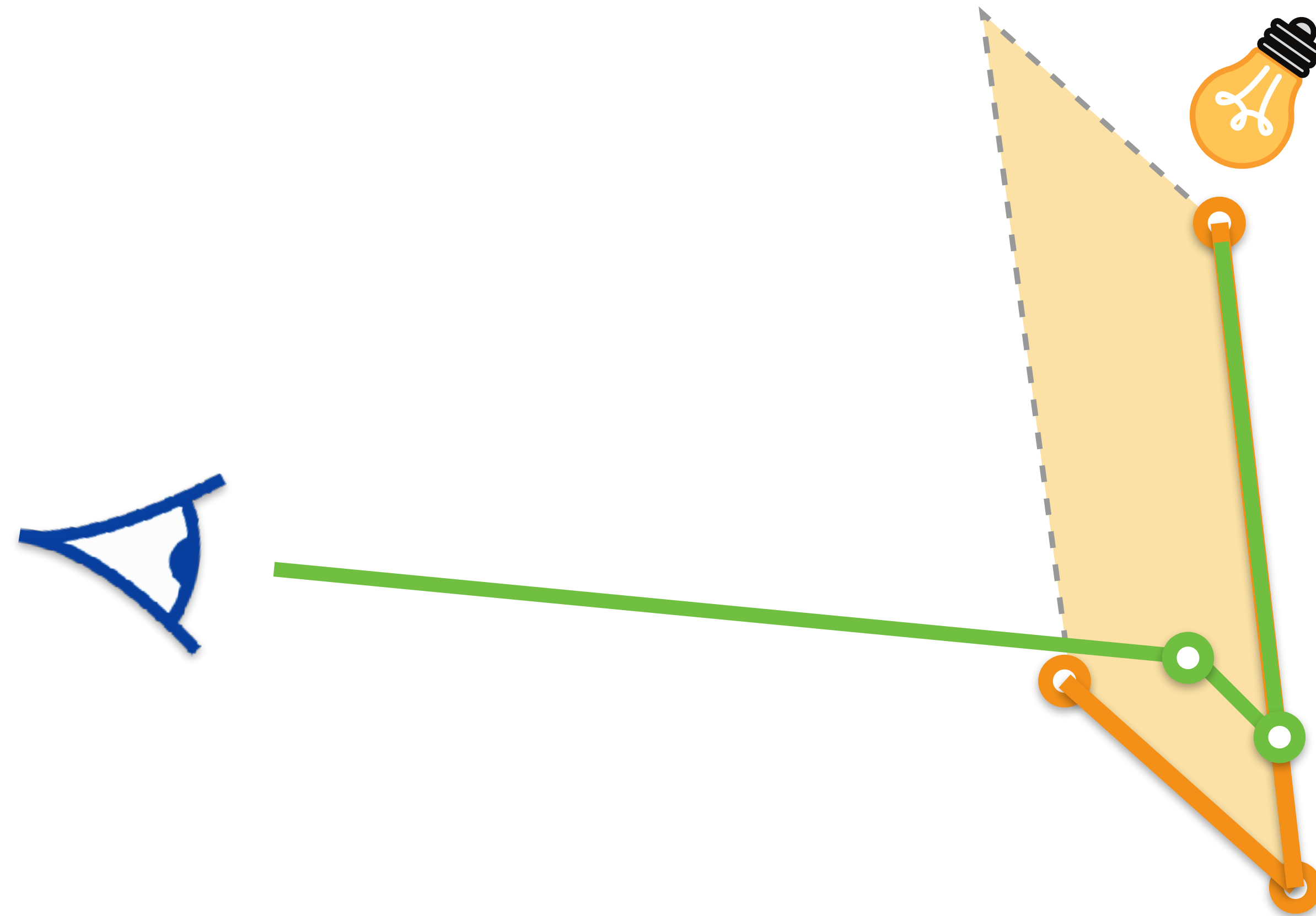




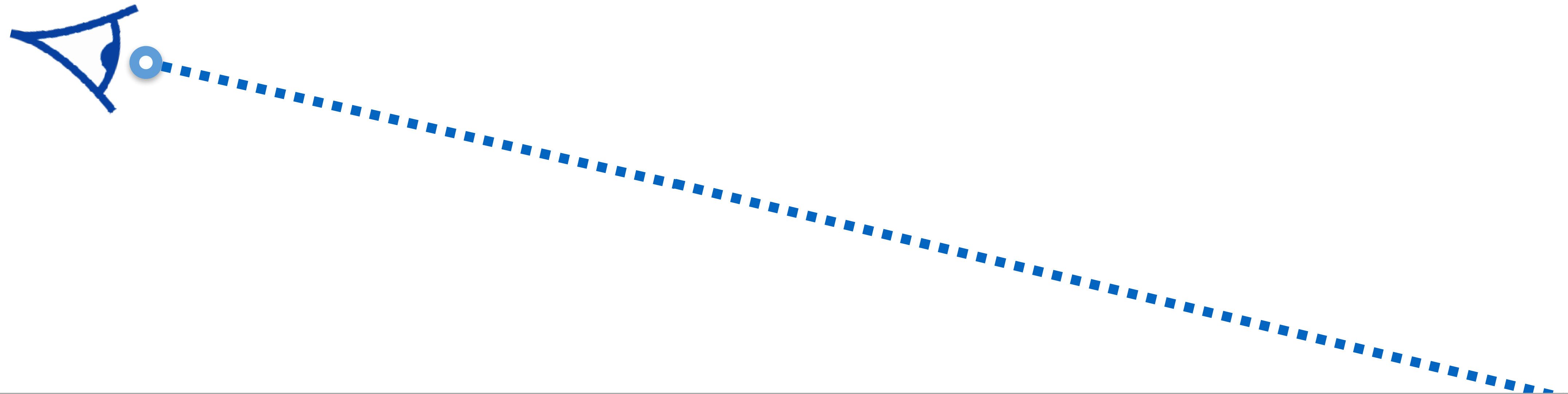
# Photon surfaces as sampling strategies



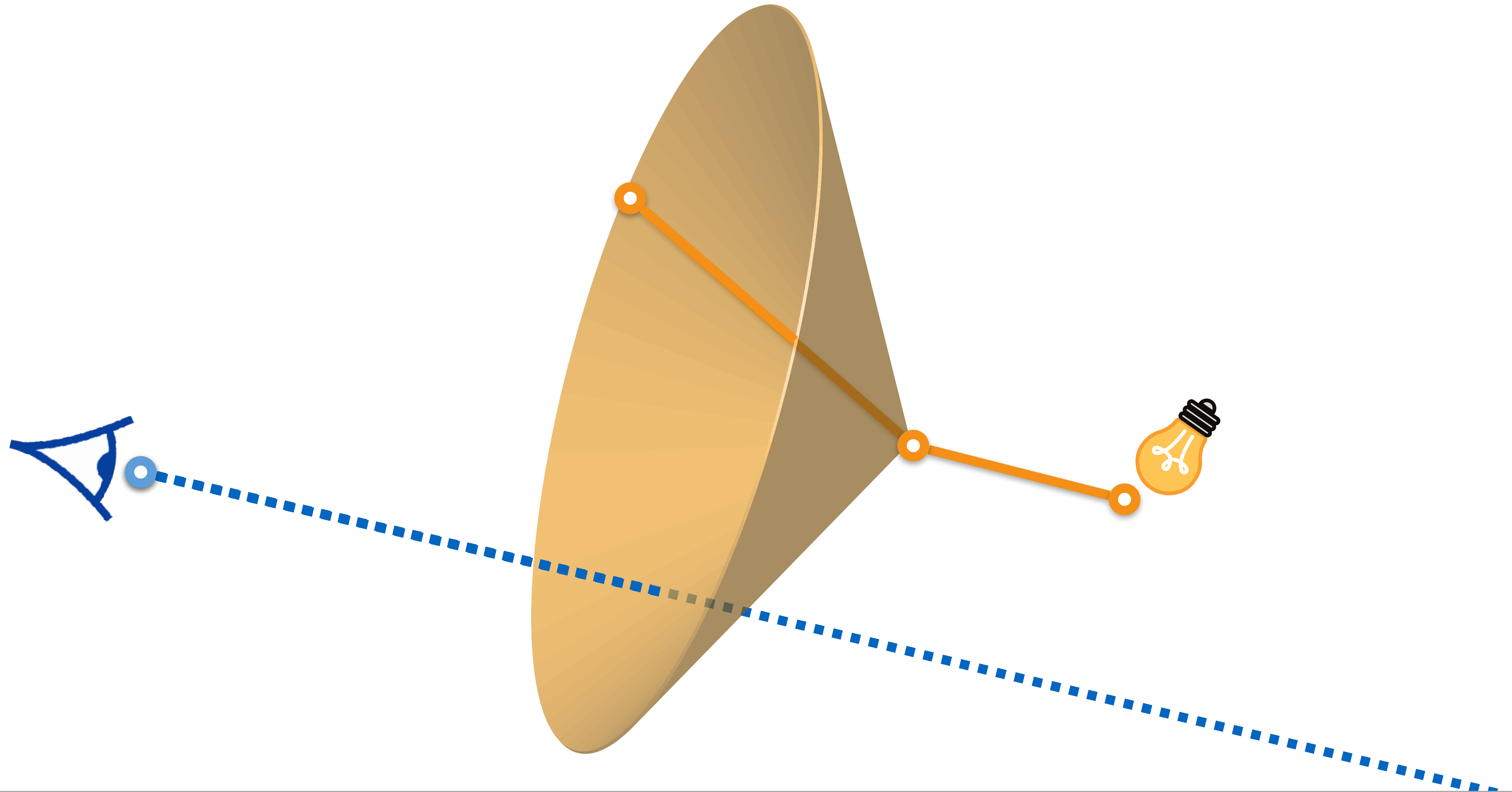
# Photon surfaces as sampling strategies



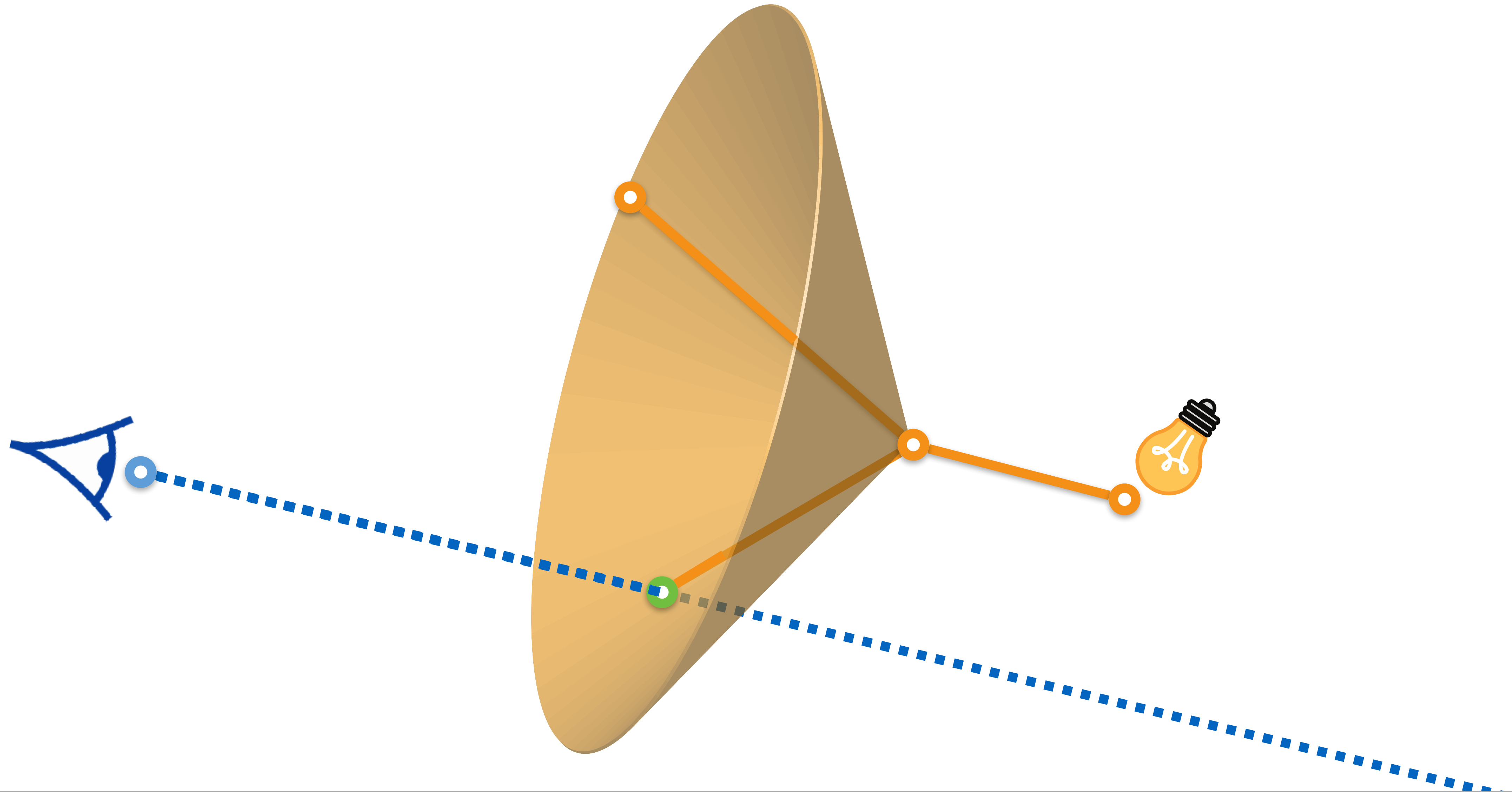
# Photon surfaces as sampling strategies



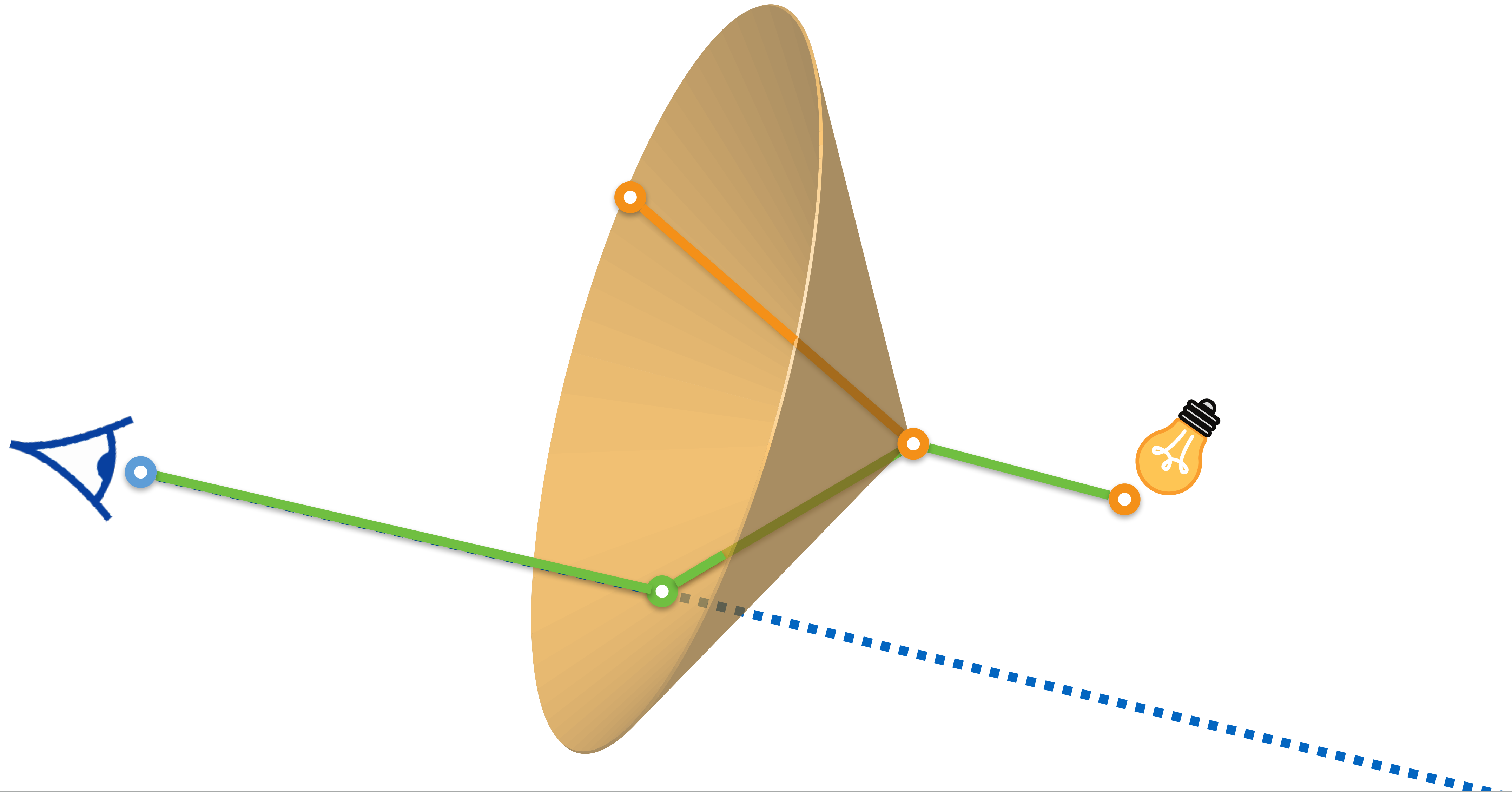
# Photon surfaces as sampling strategies



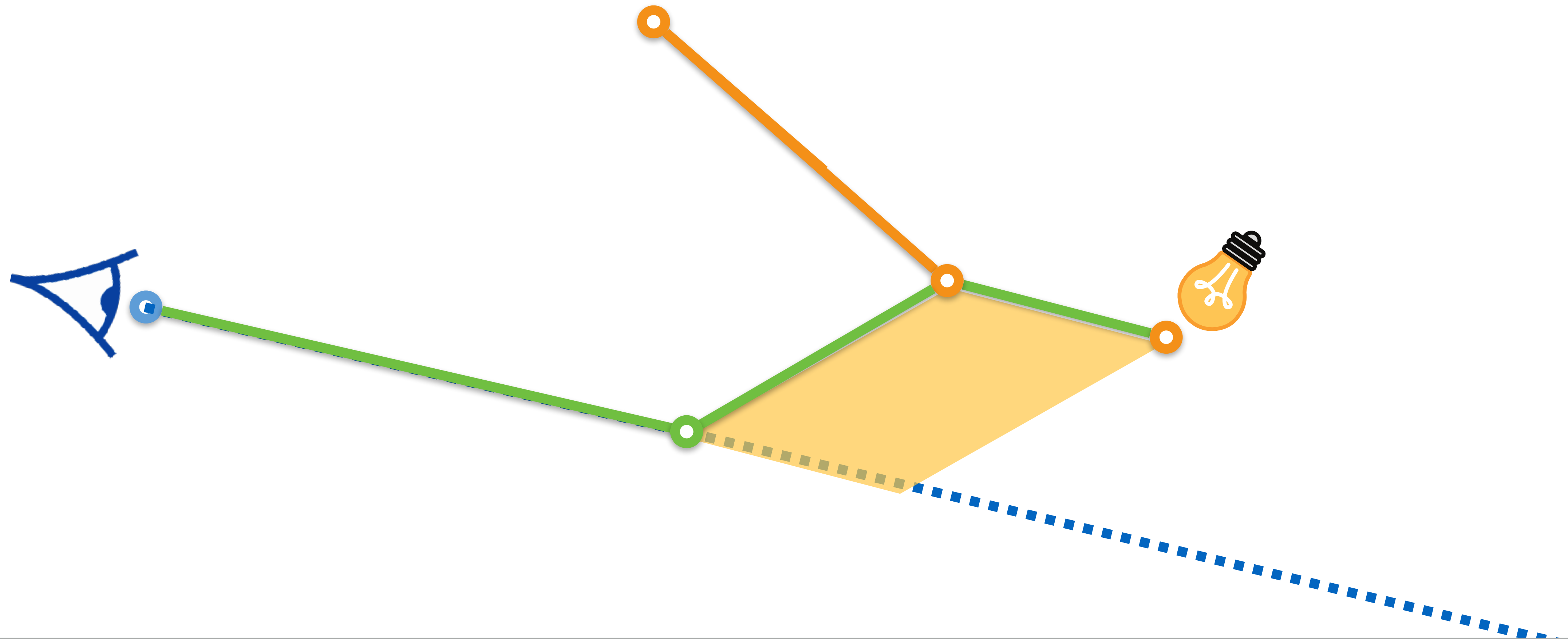
# Photon surfaces as sampling strategies



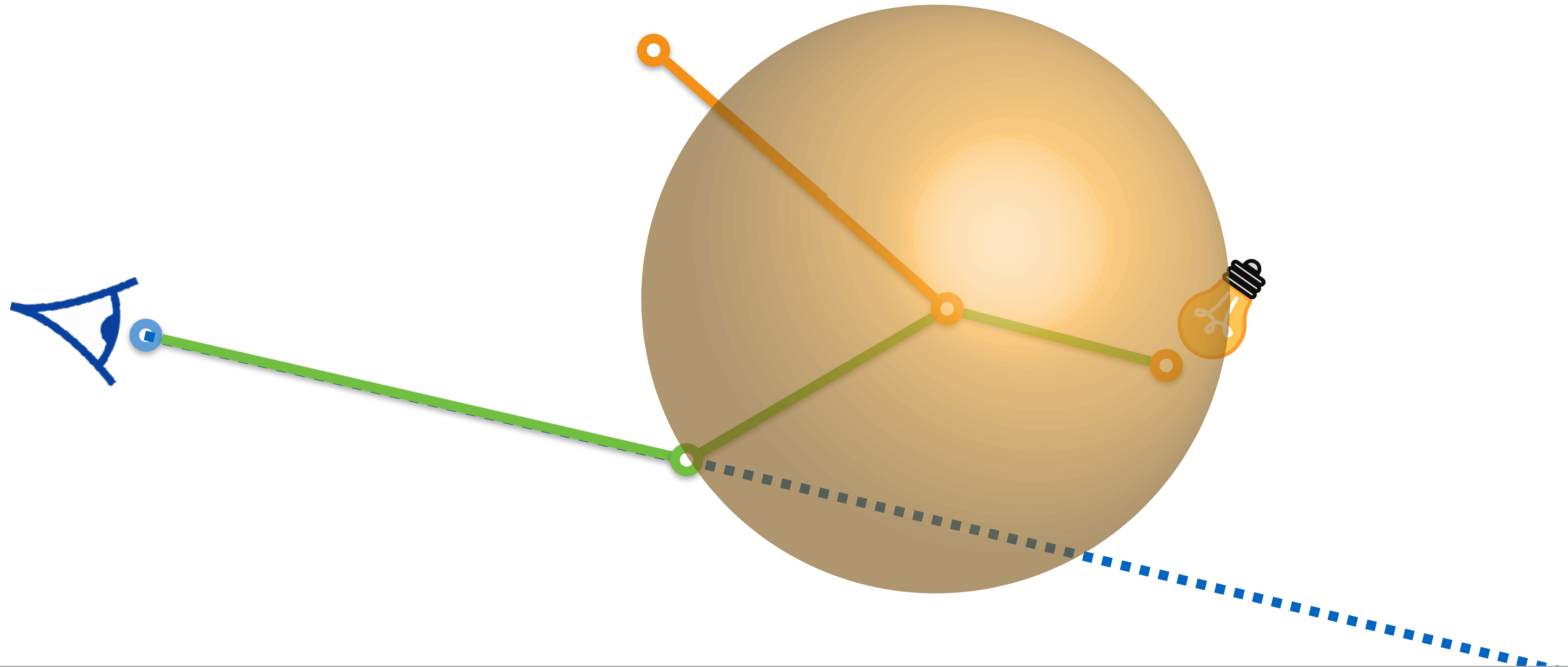
# Photon surfaces as sampling strategies



# Photon surfaces as sampling strategies



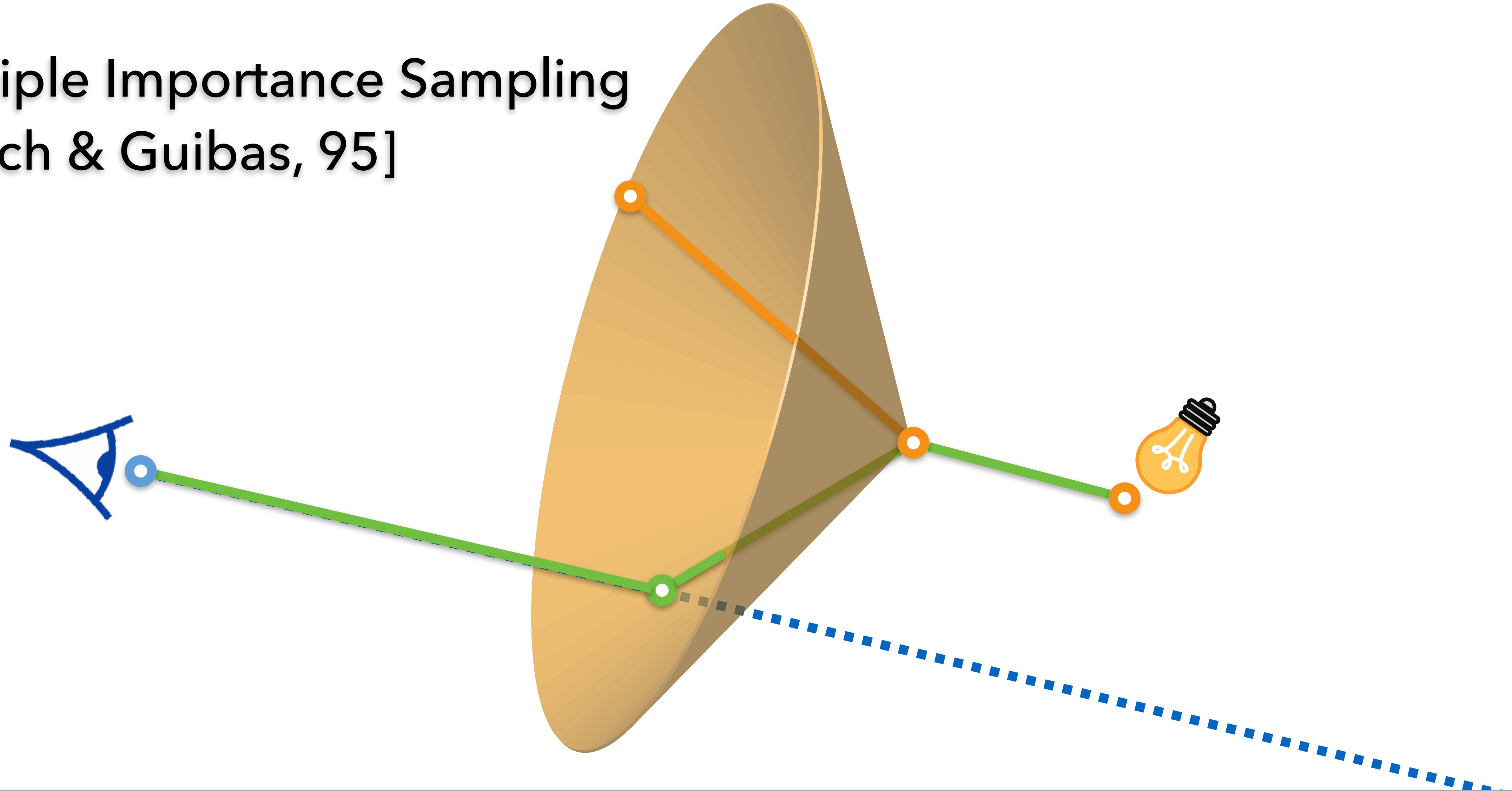
# Photon surfaces as sampling strategies



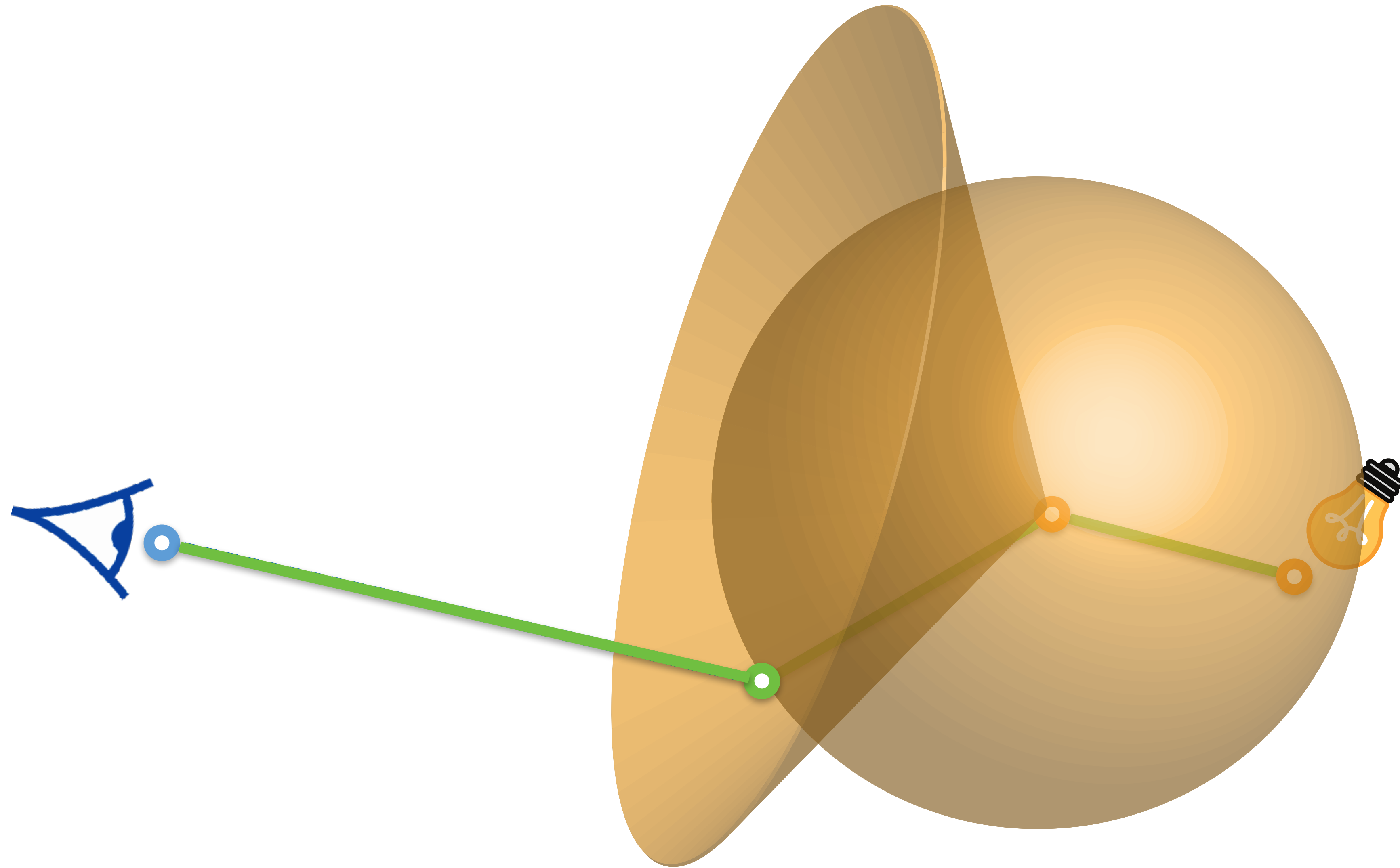


# Photon surfaces as sampling strategies

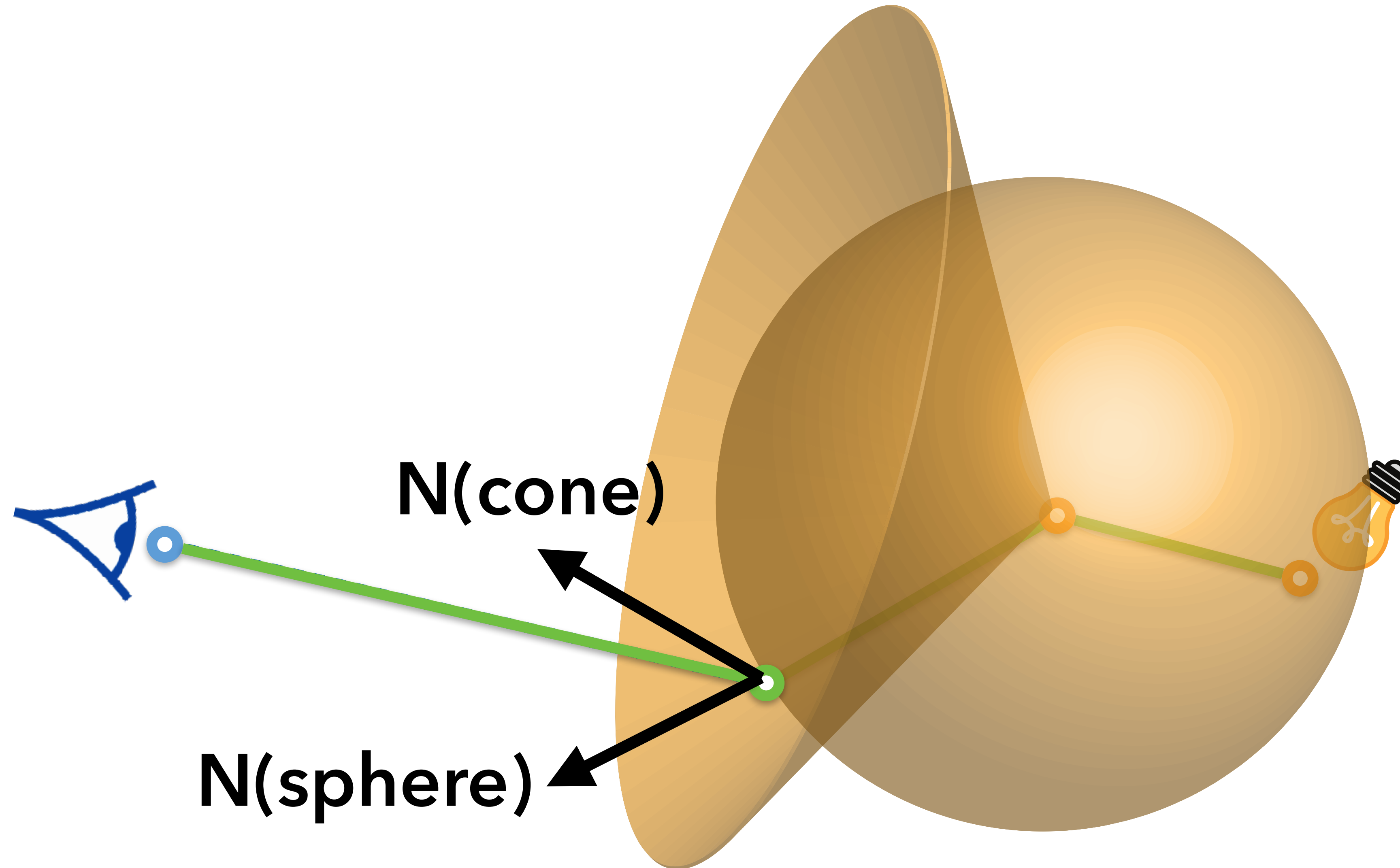
Multiple Importance Sampling  
[Veach & Guibas, 95]



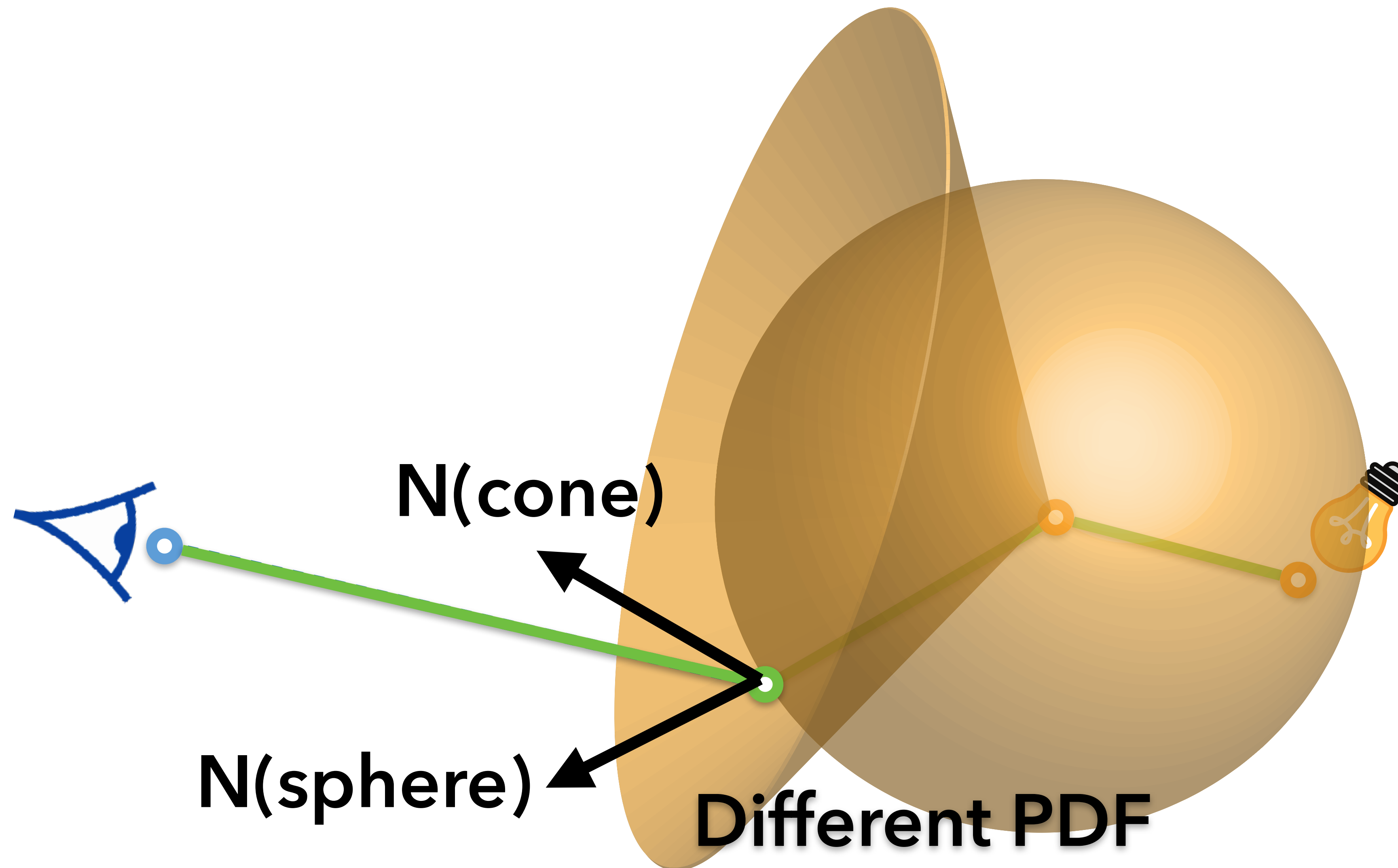
# Photon surfaces as sampling strategies

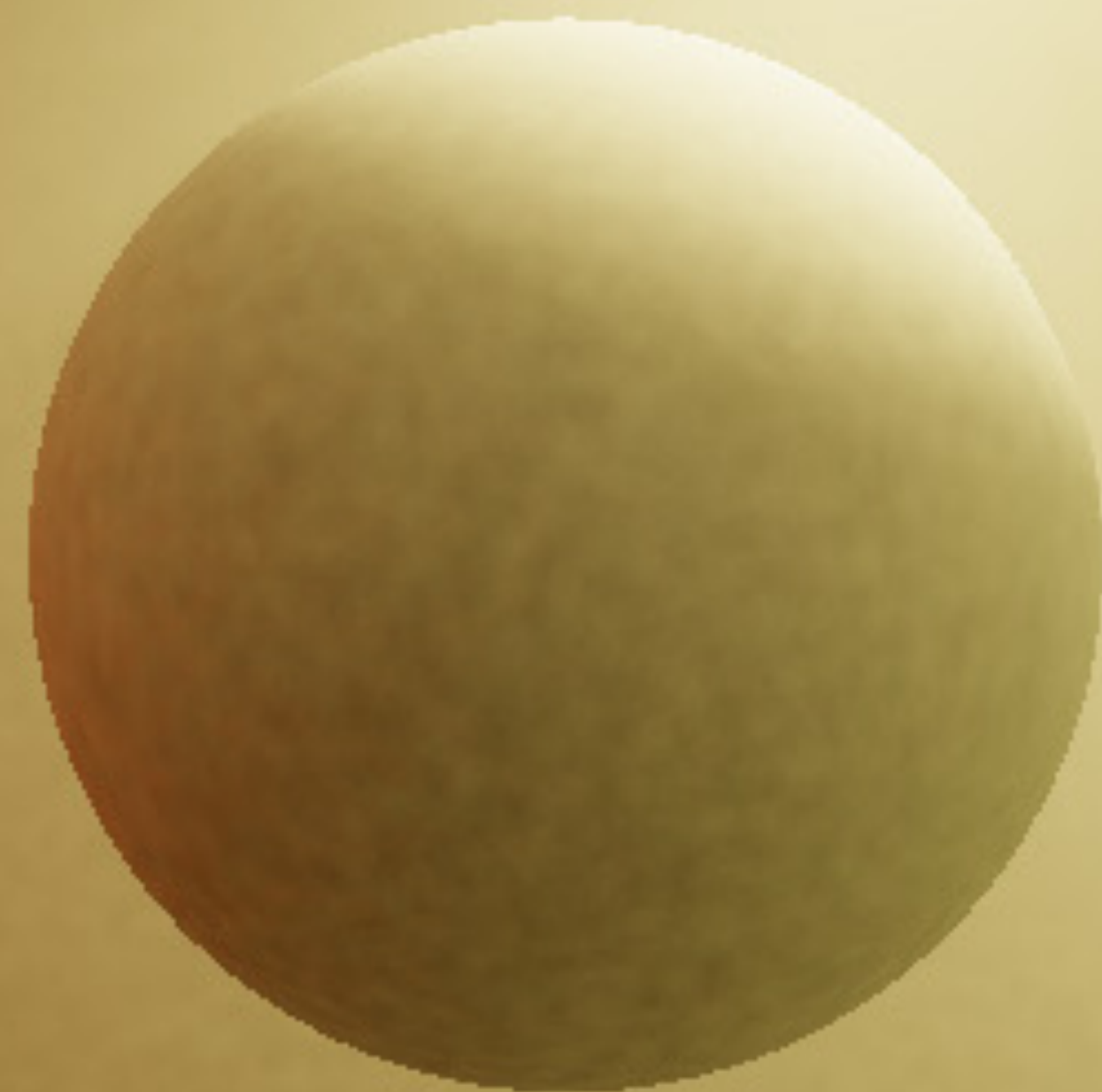


# Photon surfaces as sampling strategies



# Photon surfaces as sampling strategies





Full light transport



**2nd bounce**

The background features a complex arrangement of overlapping, semi-transparent shapes in shades of olive green and brown. These shapes include circles, rectangles, and irregular polygons, some of which are oriented vertically. Several thin, light-colored lines crisscross the composition, adding to the layered, geometric aesthetic. The overall effect is one of depth and intricate pattern.

Unweighted

The background features a complex arrangement of overlapping, semi-transparent shapes in shades of olive green and brown. These shapes include circles, triangles, and irregular polygons, some of which are oriented diagonally. Thin, light-colored lines crisscross the composition, connecting various points and creating a sense of movement and structure. The overall effect is layered and textured, with the darker colors of the shapes appearing more prominent where they overlap.

**Weighted**



# PROCEDURE (WITH MIS)

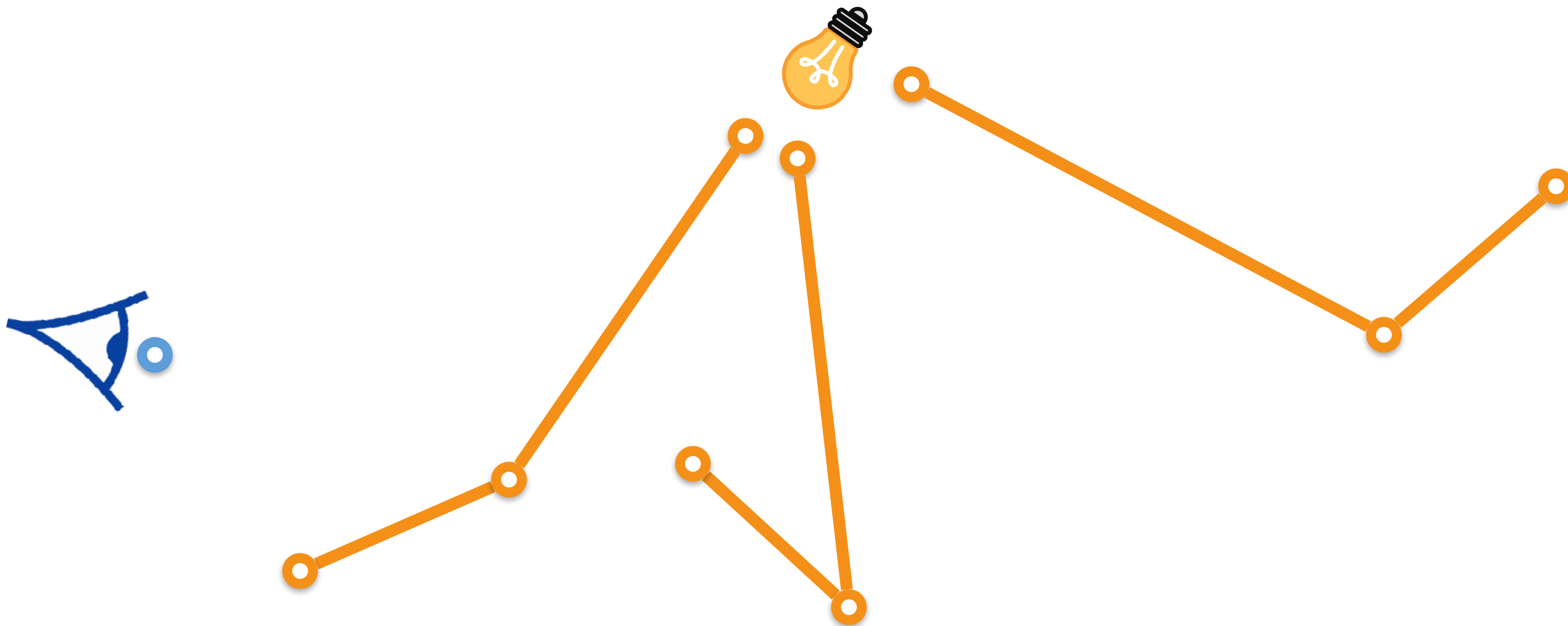
# Combining different estimators

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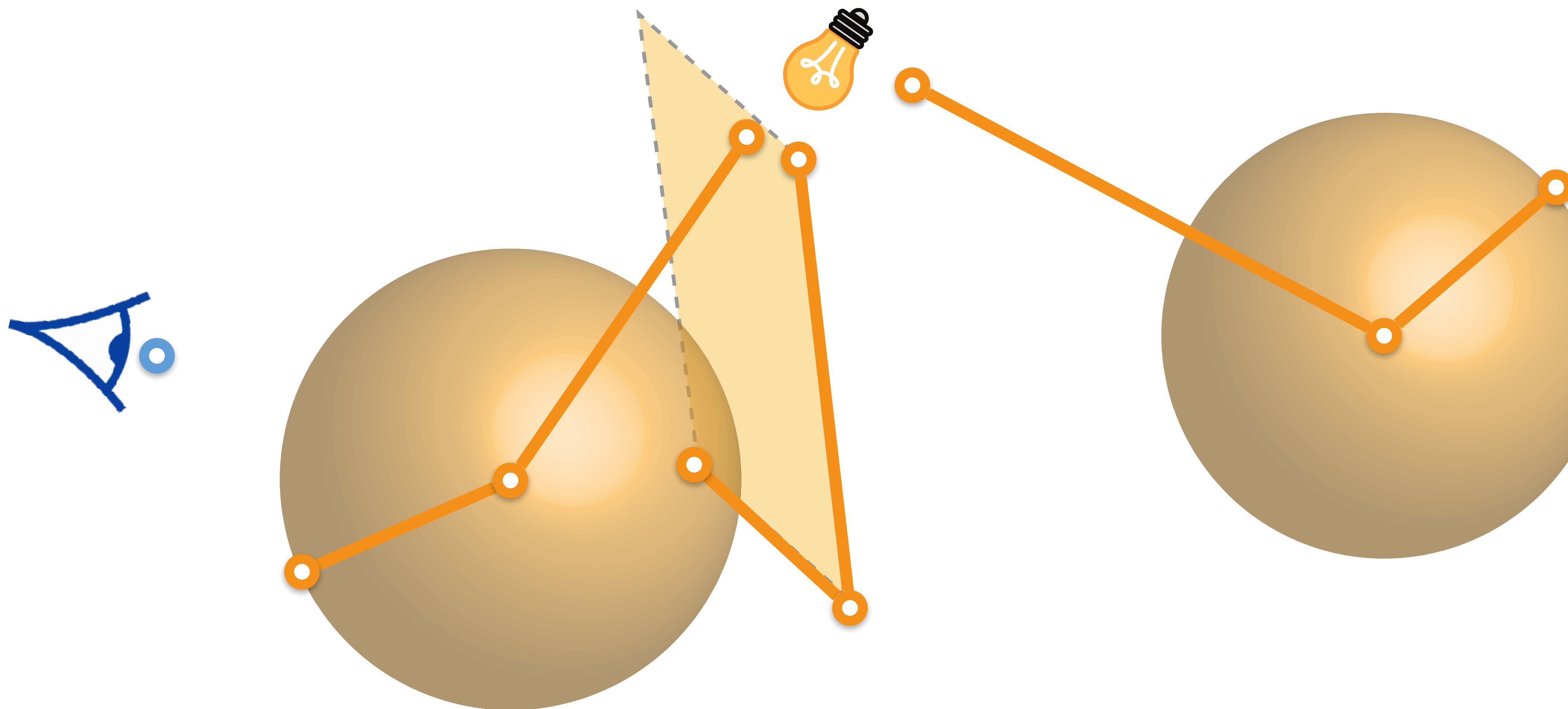
# Combining different estimators

1: Trace



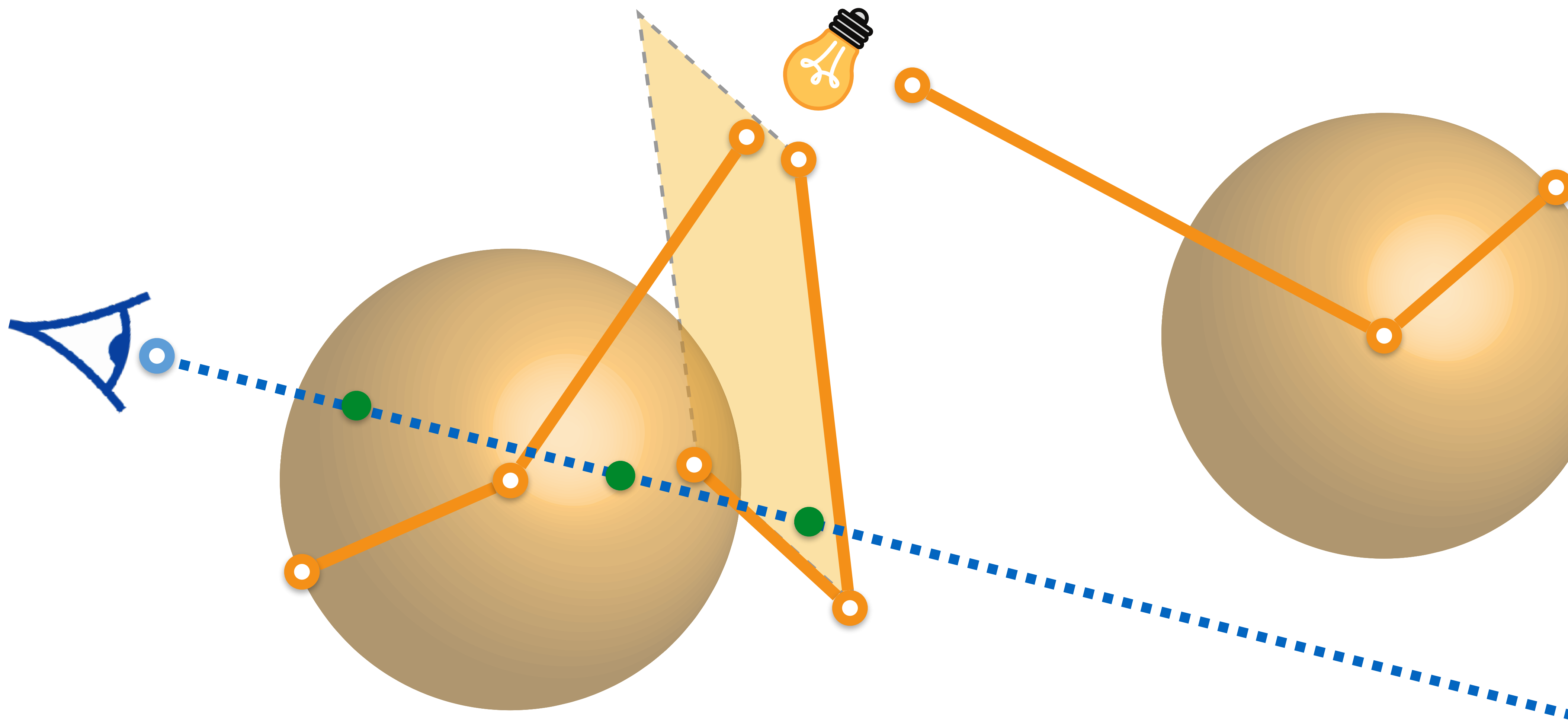
# Combining different estimators

- 1: Trace
- 2: Create



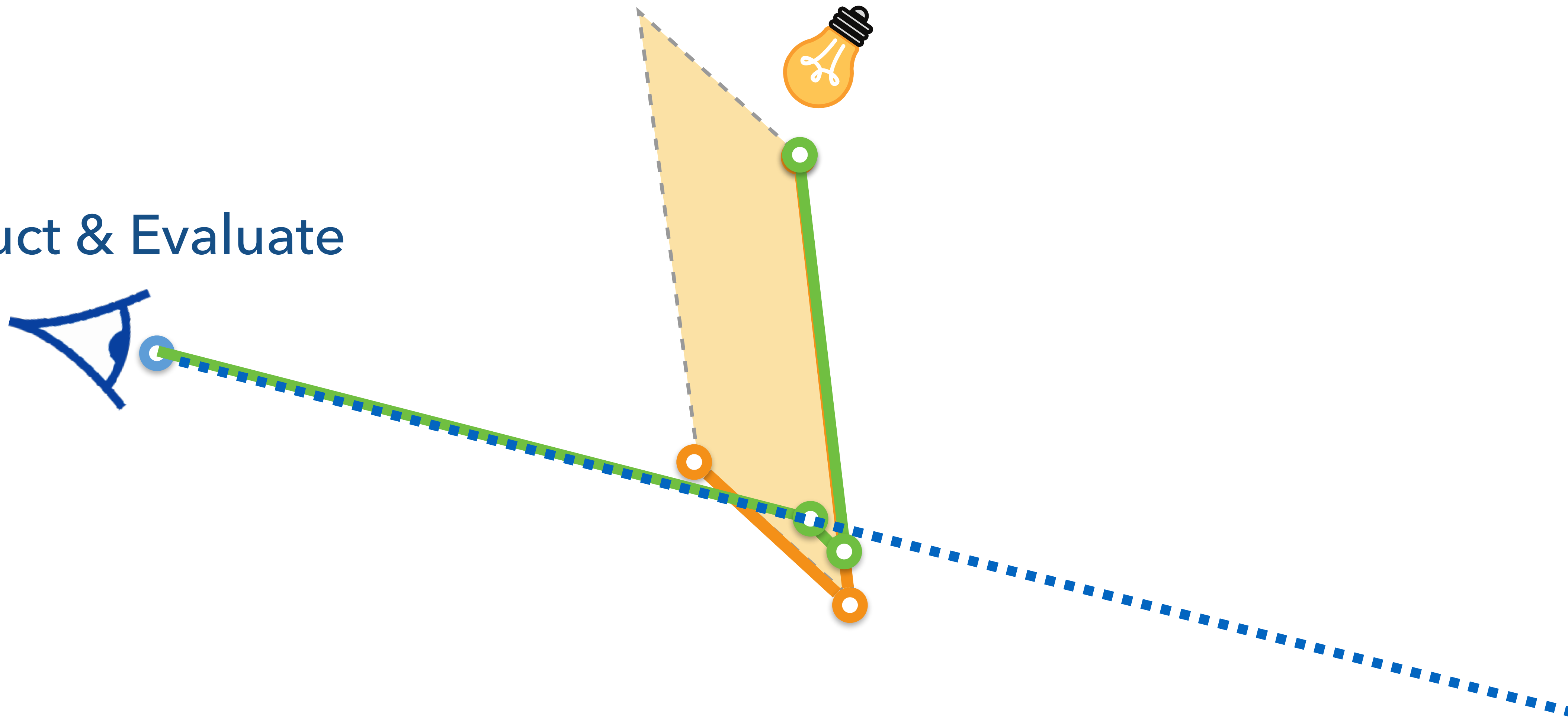
# Combining different estimators

- 1: Trace
- 2: Create
- 3: Intersect



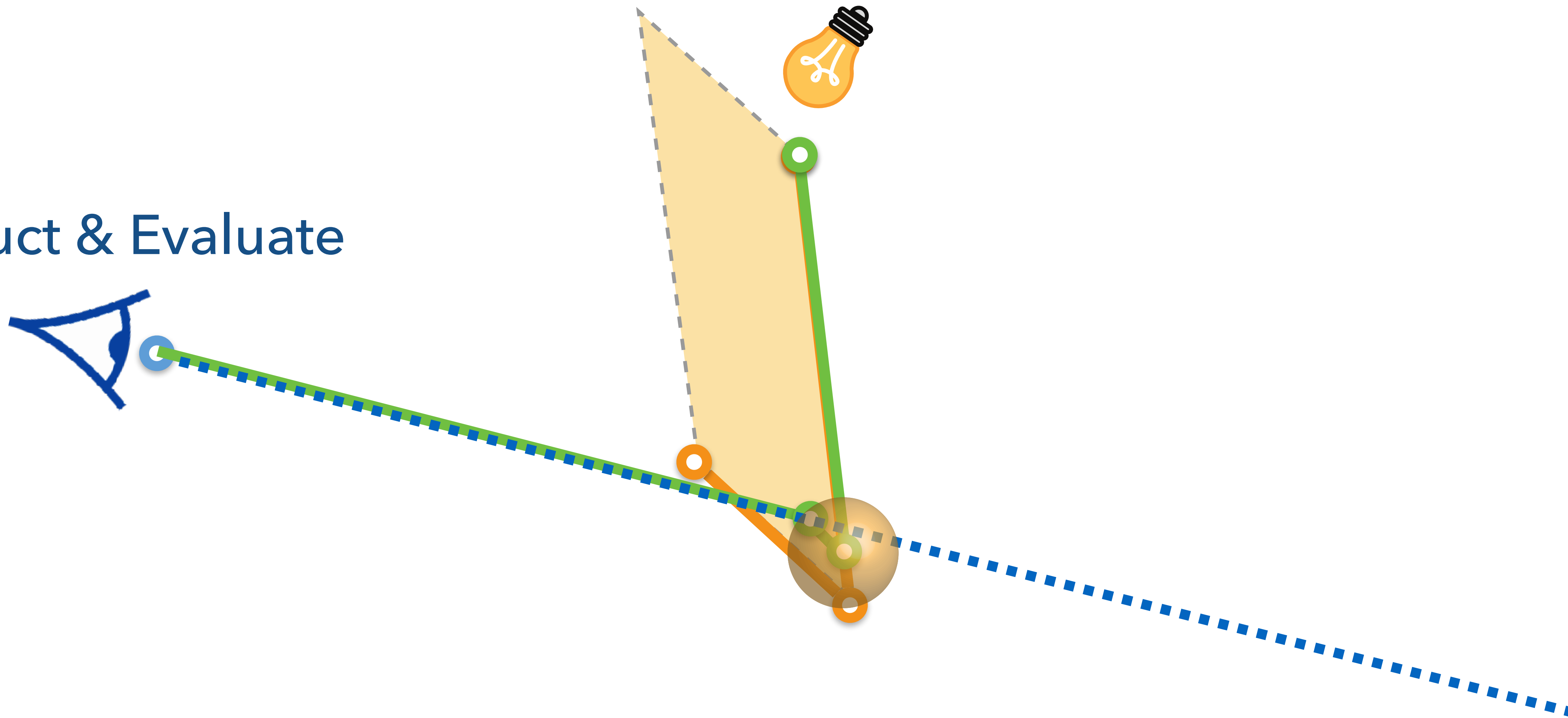
# Combining different estimators

- 1: Trace
- 2: Create
- 3: Intersect
- 4: Reconstruct & Evaluate



# Combining different estimators

- 1: Trace
- 2: Create
- 3: Intersect
- 4: Reconstruct & Evaluate



# RESULTS



# PERFORMANCE ANALYSIS

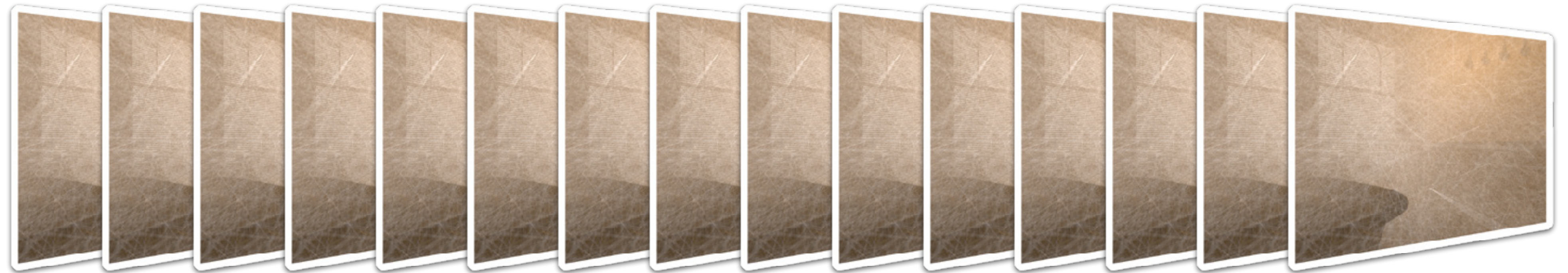
# Quantitative performance analysis

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5min / image

# Quantitative performance analysis

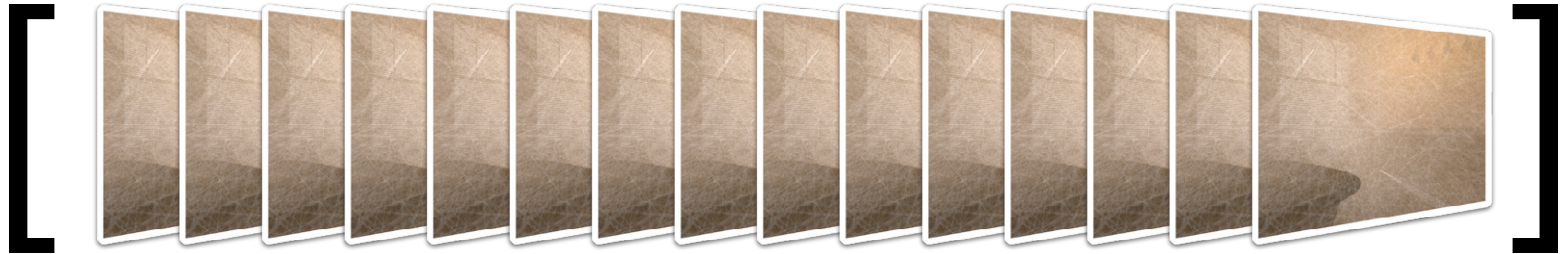
5min / image



# Quantitative performance analysis

5min / image

Var



# Scene 1



Medium only



# Photon beam



Var:1x

(Single-scattering)

Equal time

**Ours, MIS**

**(3 single-scattering planes, cone, sphere)**

**Var:0.129x**

**(Single-scattering)**

**Equal time**



# Photon plane

Var:1x

(Multi-scattering)

Equal time

# Ours, MIS (3 planes + cone + cylinder)

Var:0.229x

(Multi-scattering)

Equal time

# Scene 2



# Photon beam



Var: 1x

(Single-scattering)

Equal time

# Ours (3 single-scattering planes)

Var:0.389x

(Single-scattering)

Equal time

# Scene 3



# Photon plane



Var:1x

(Multi-scattering)

Equal time

# Ours, MIS (3 planes + cone + cylinder)

Var:0.483x

(Multi-scattering)

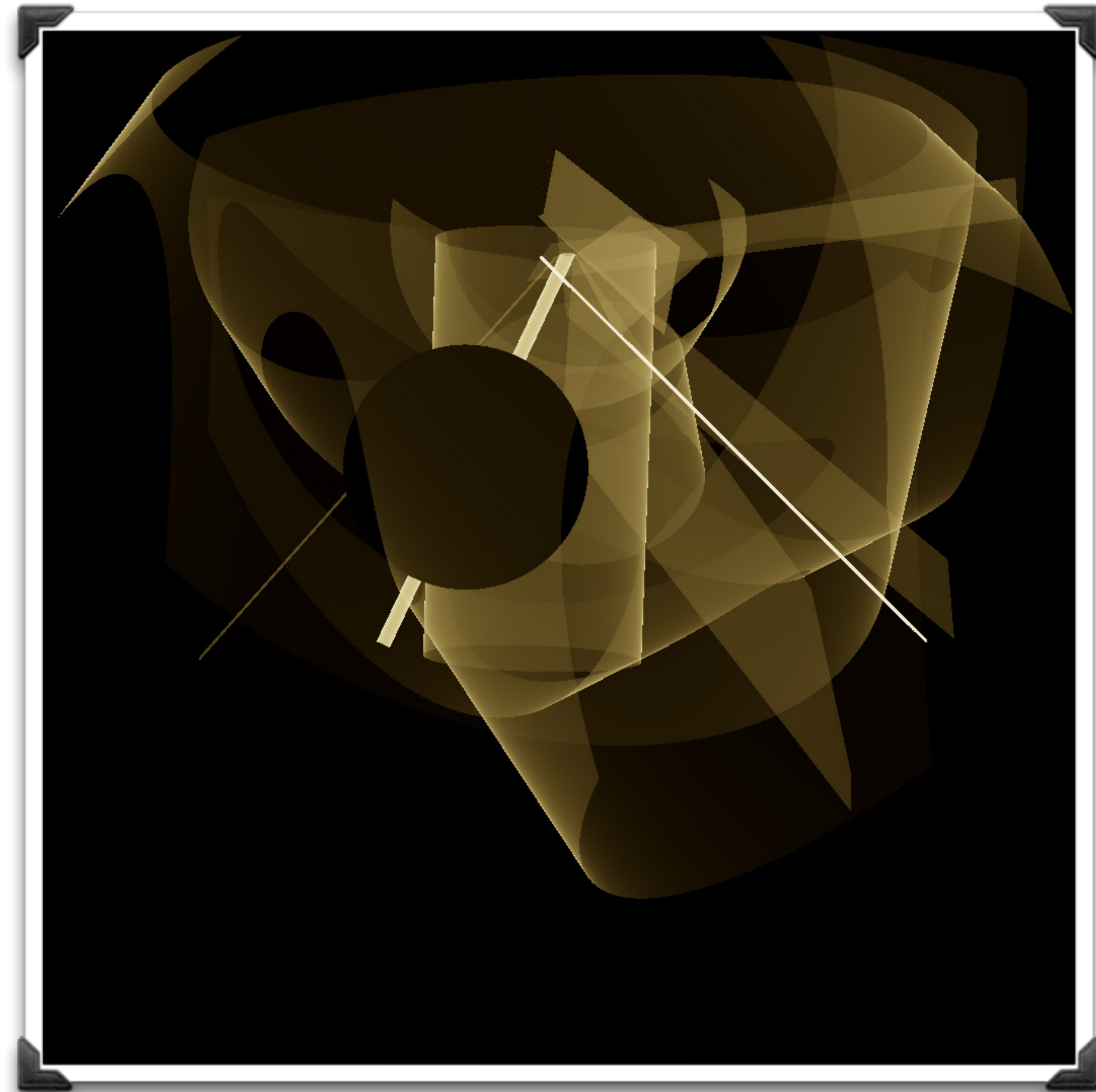
Equal time



# Conclusions

We've improved **unbiased** photon density estimation:

Photon surface



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Photon surface

Combination using MIS



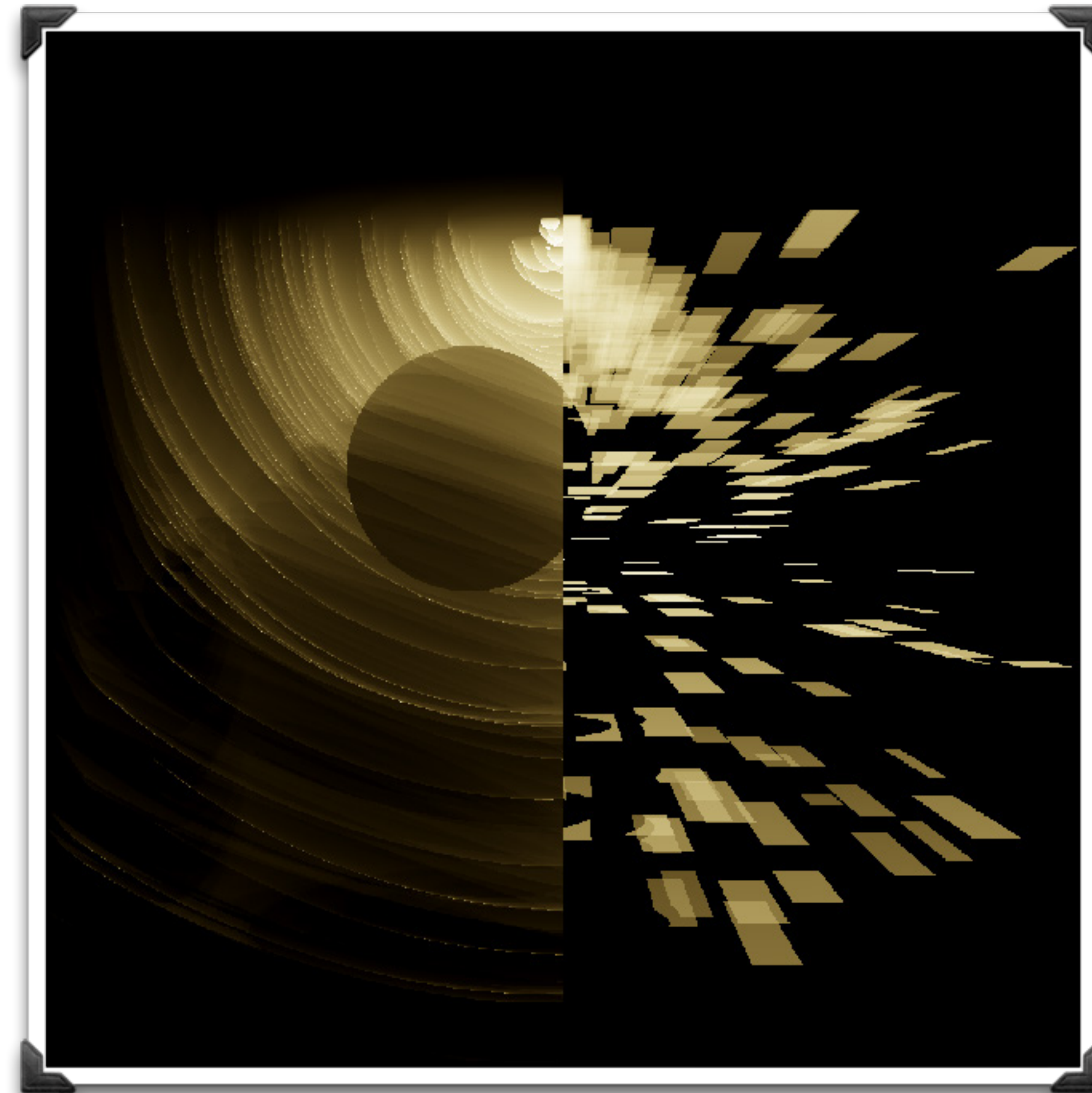
# Conclusions

We've improved **unbiased** photon density estimation:

Photon surface

Combination using MIS

Single-scattering



# Limitations

**Heterogeneity** is still too slow to be practical.



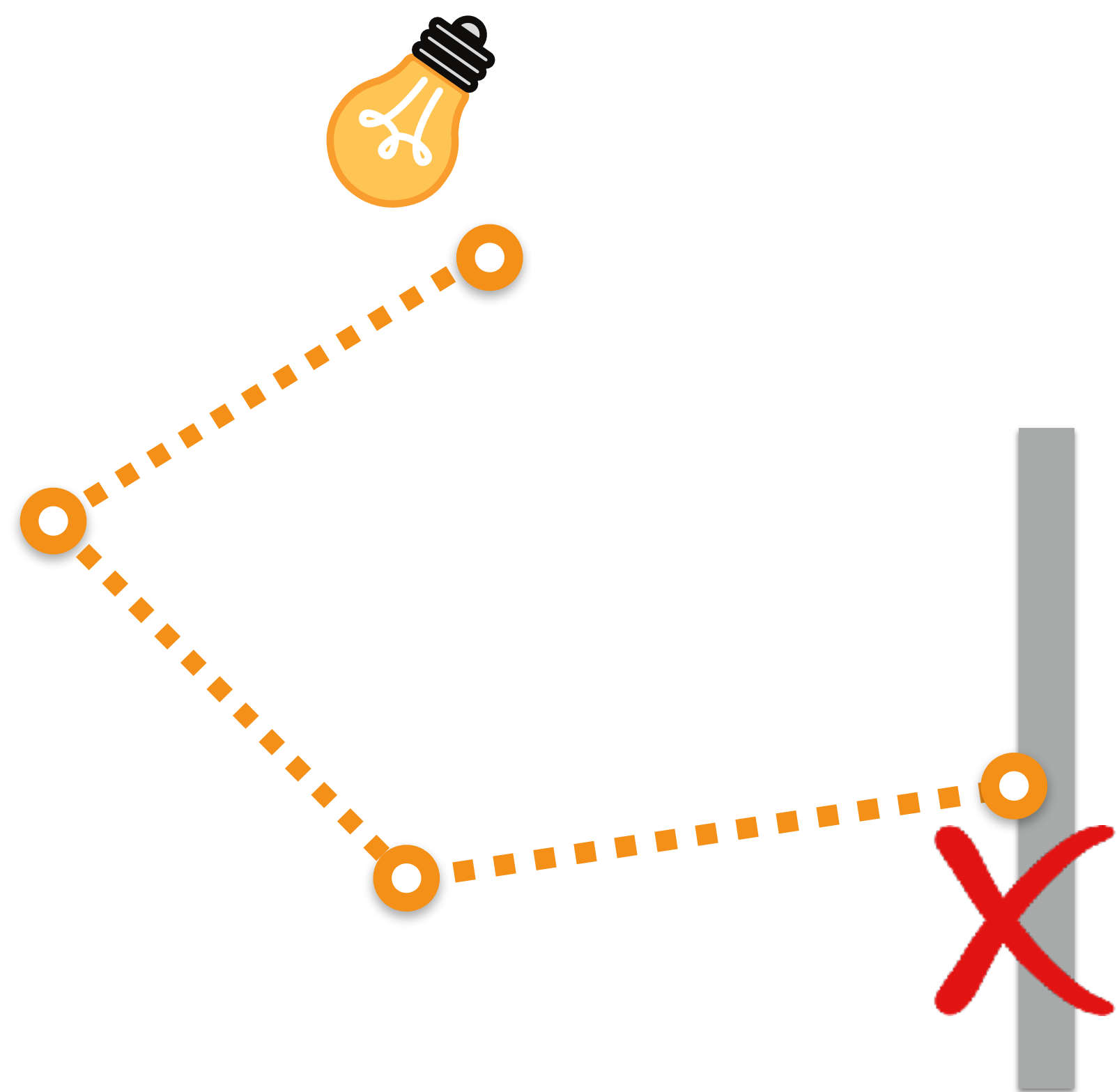
# Limitations

Some types of **light transport** cannot be handled efficiently

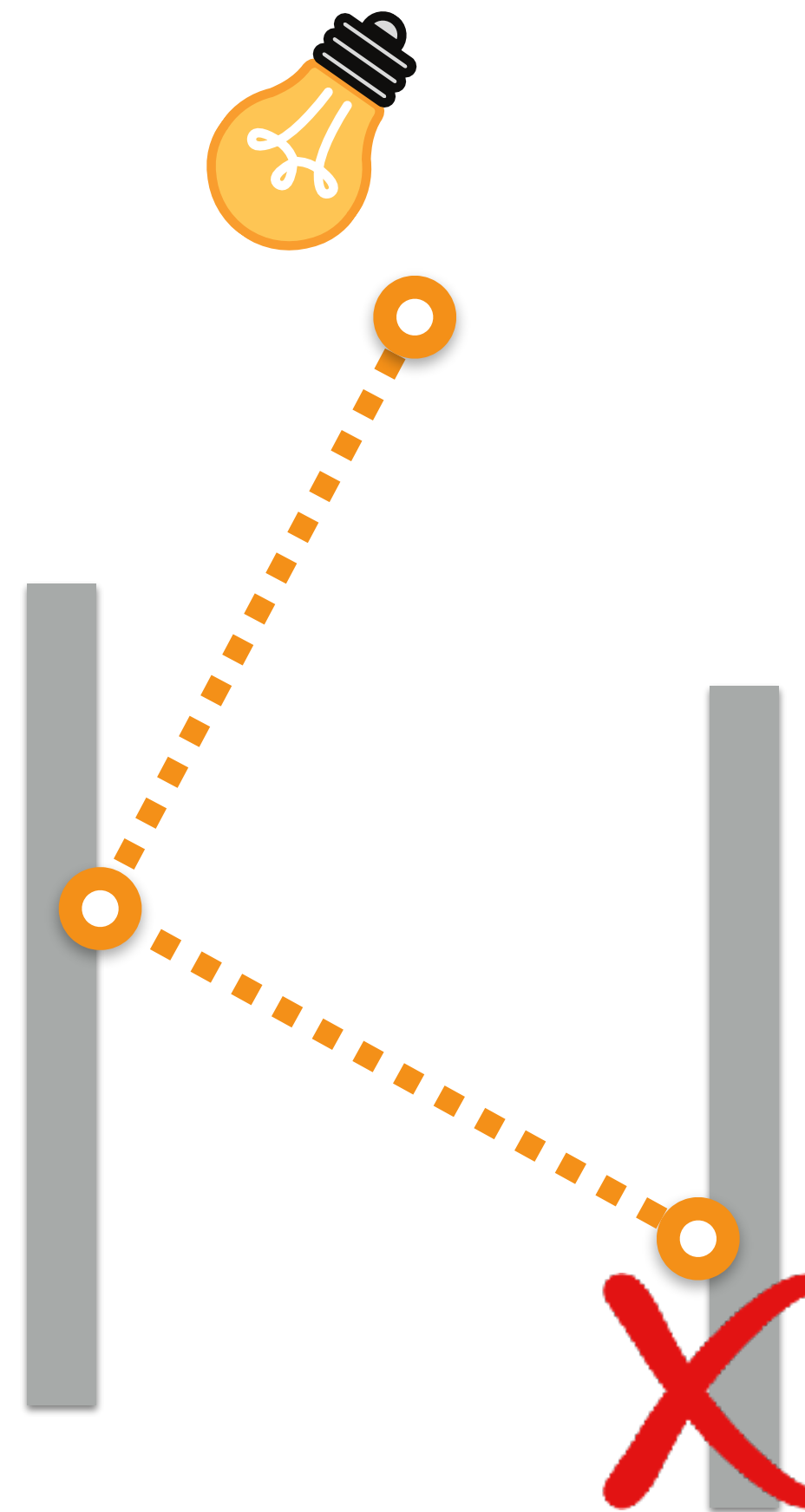


# Future Work

## Medium-to-surface



## Surface-to-surface



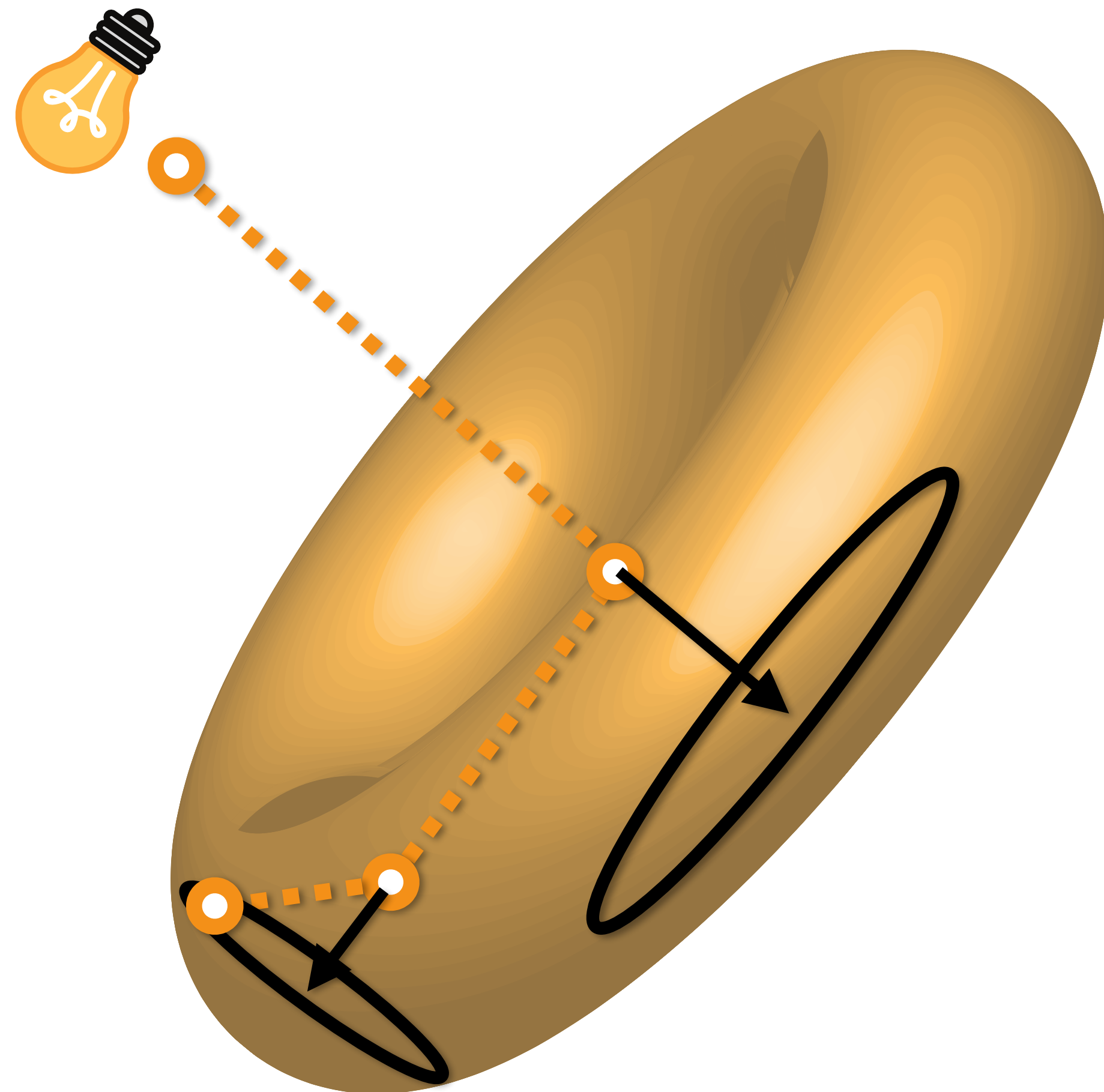
# Future Work

---

More possible **base estimators**

# Future Work

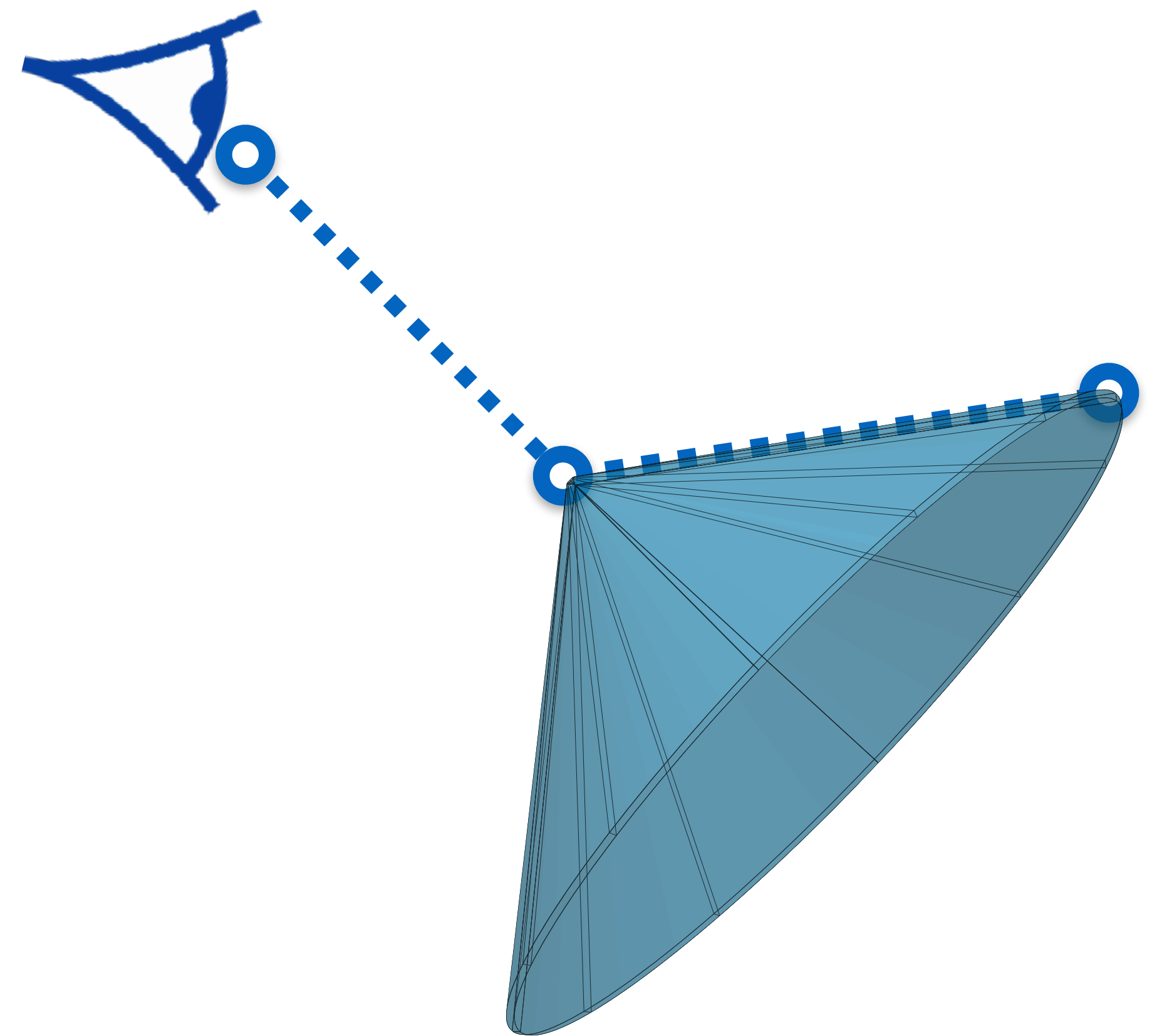
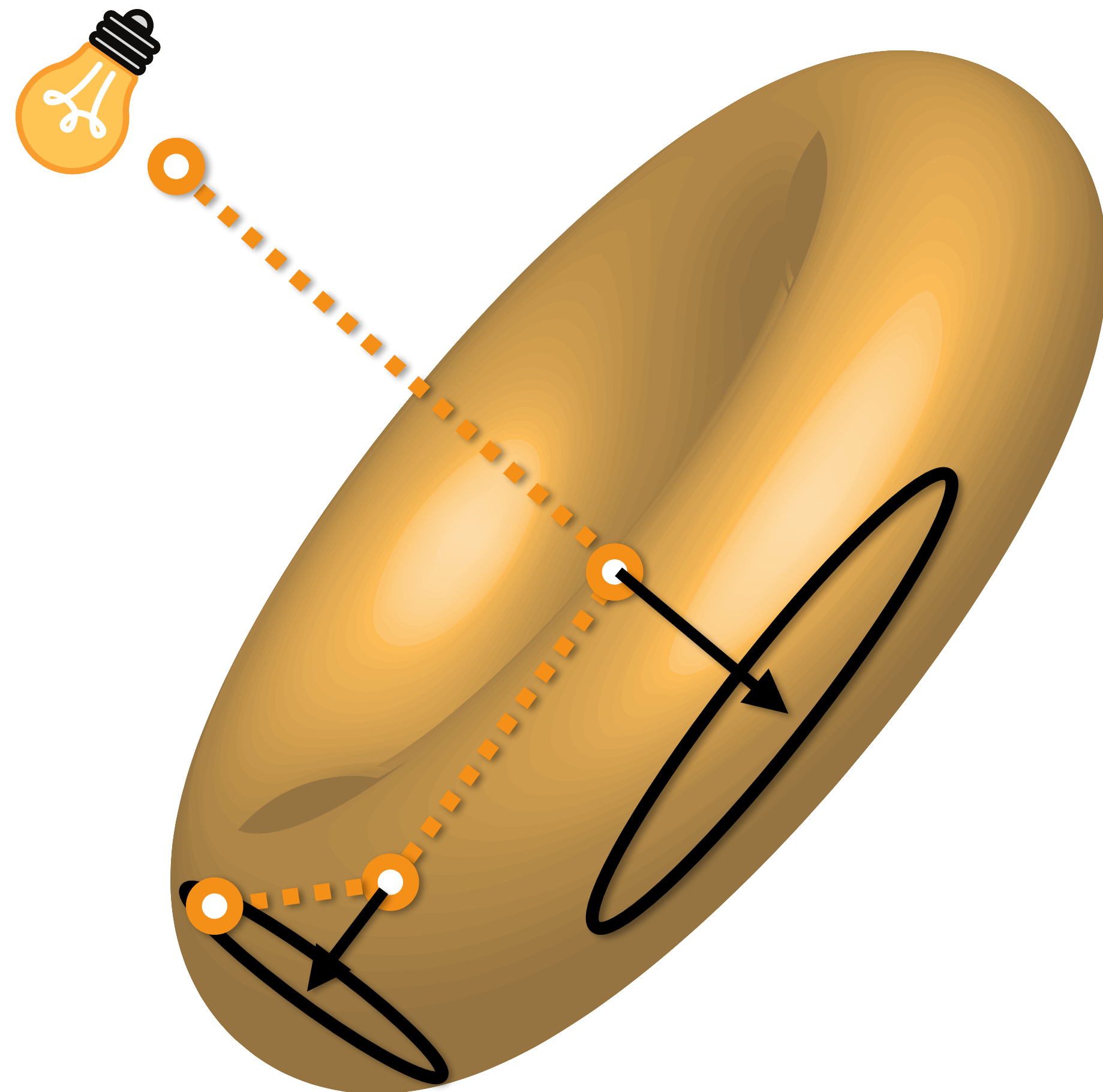
More possible **base estimators**





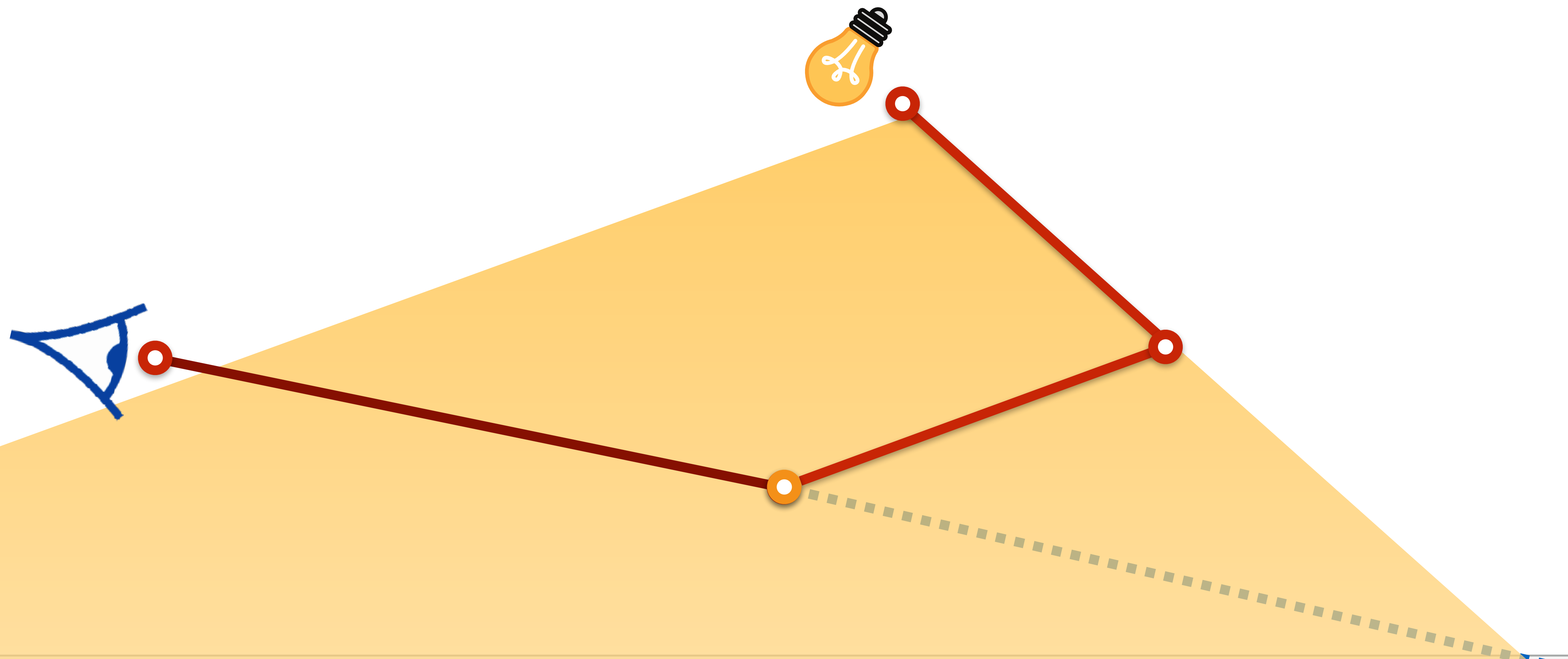
# Future Work

## More possible base estimators



# Future Work

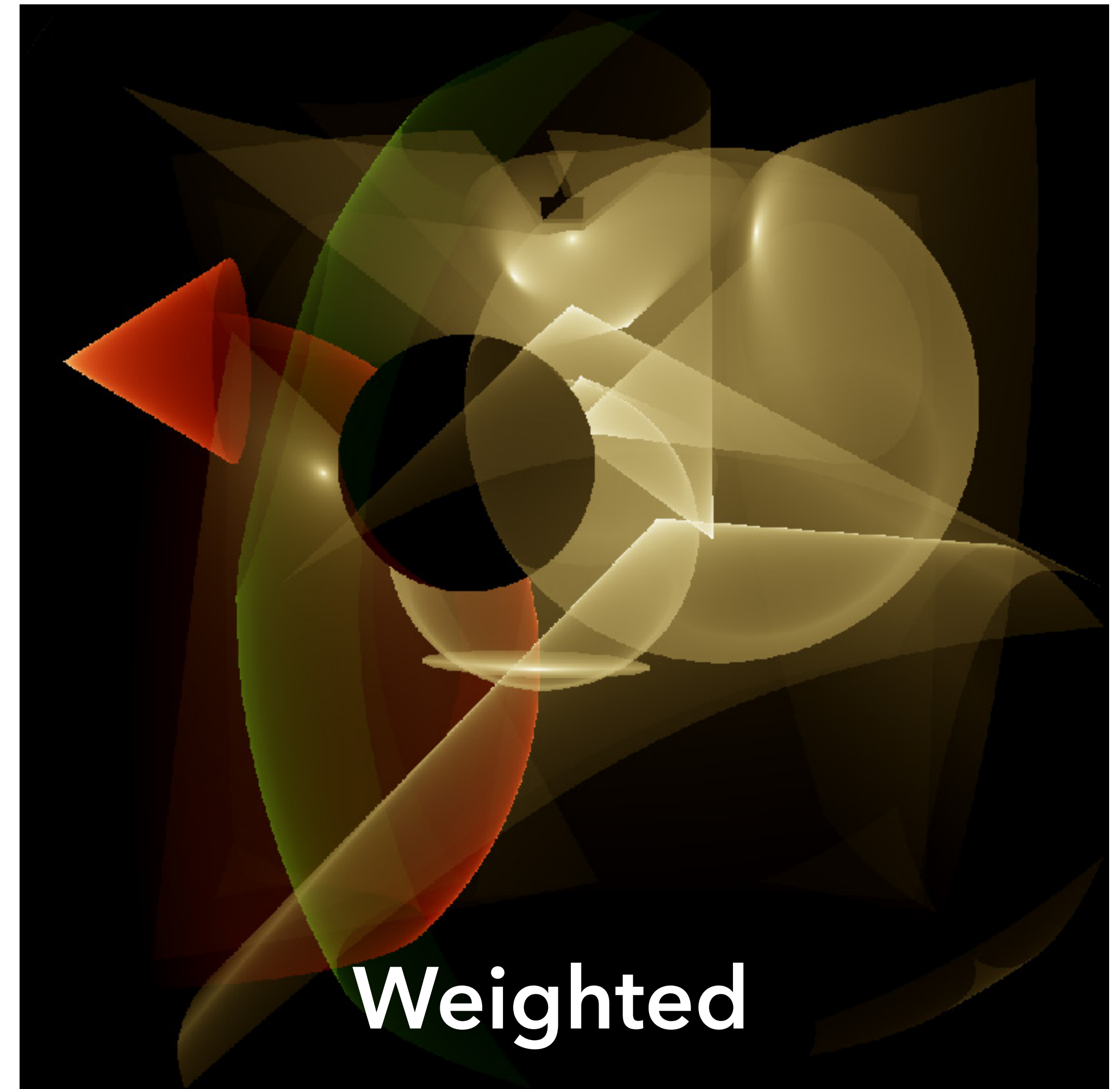
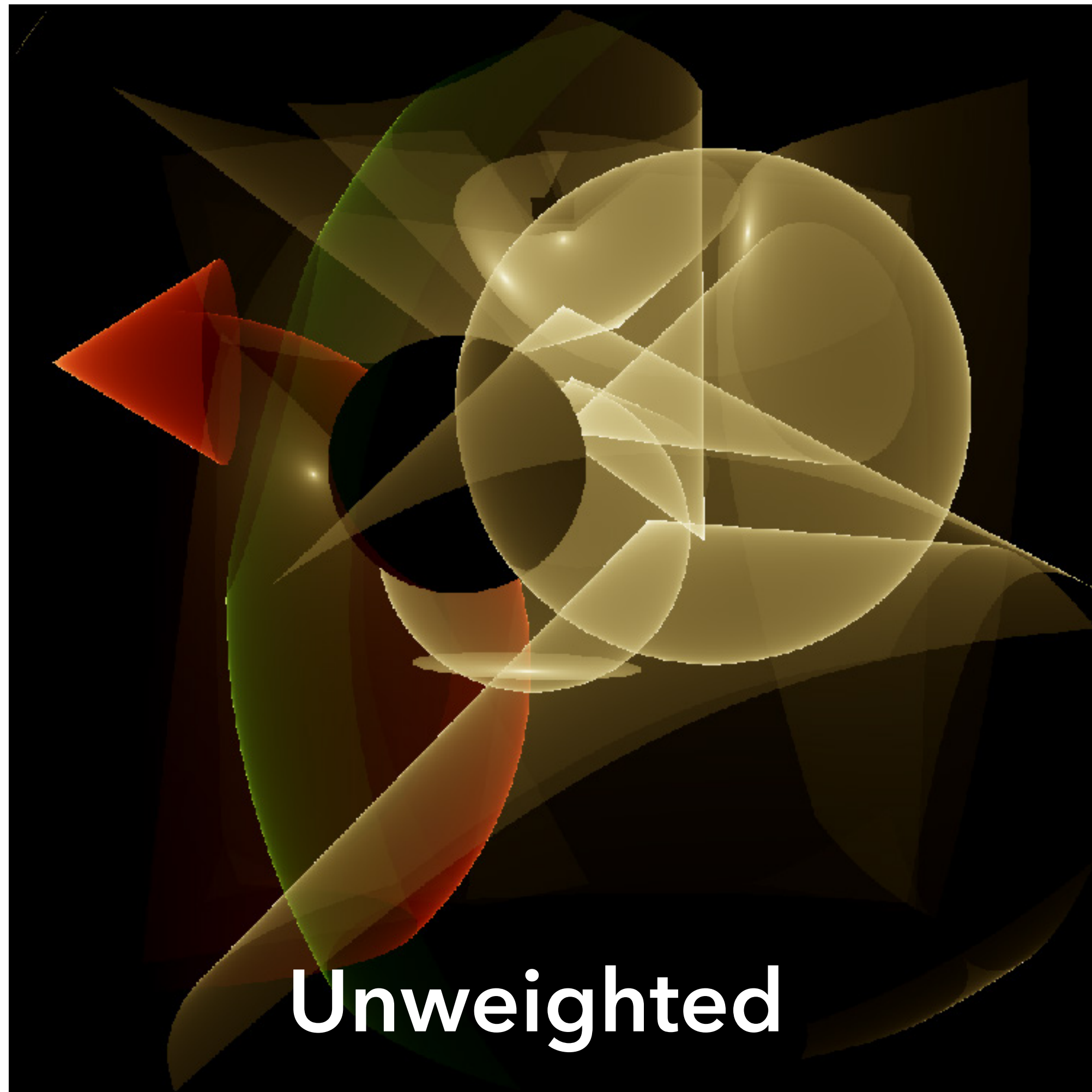
Implementing in a **path tracer**



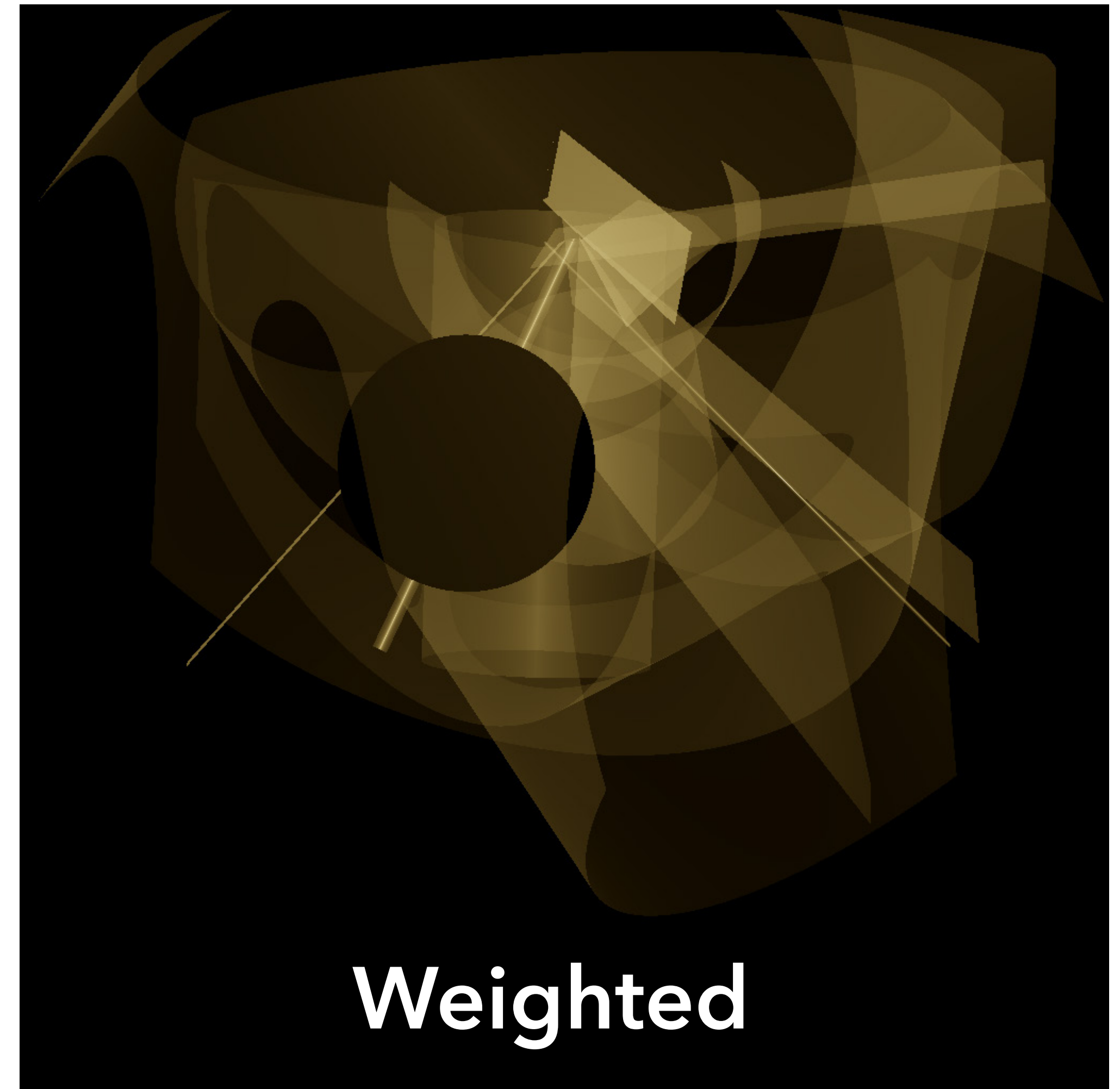
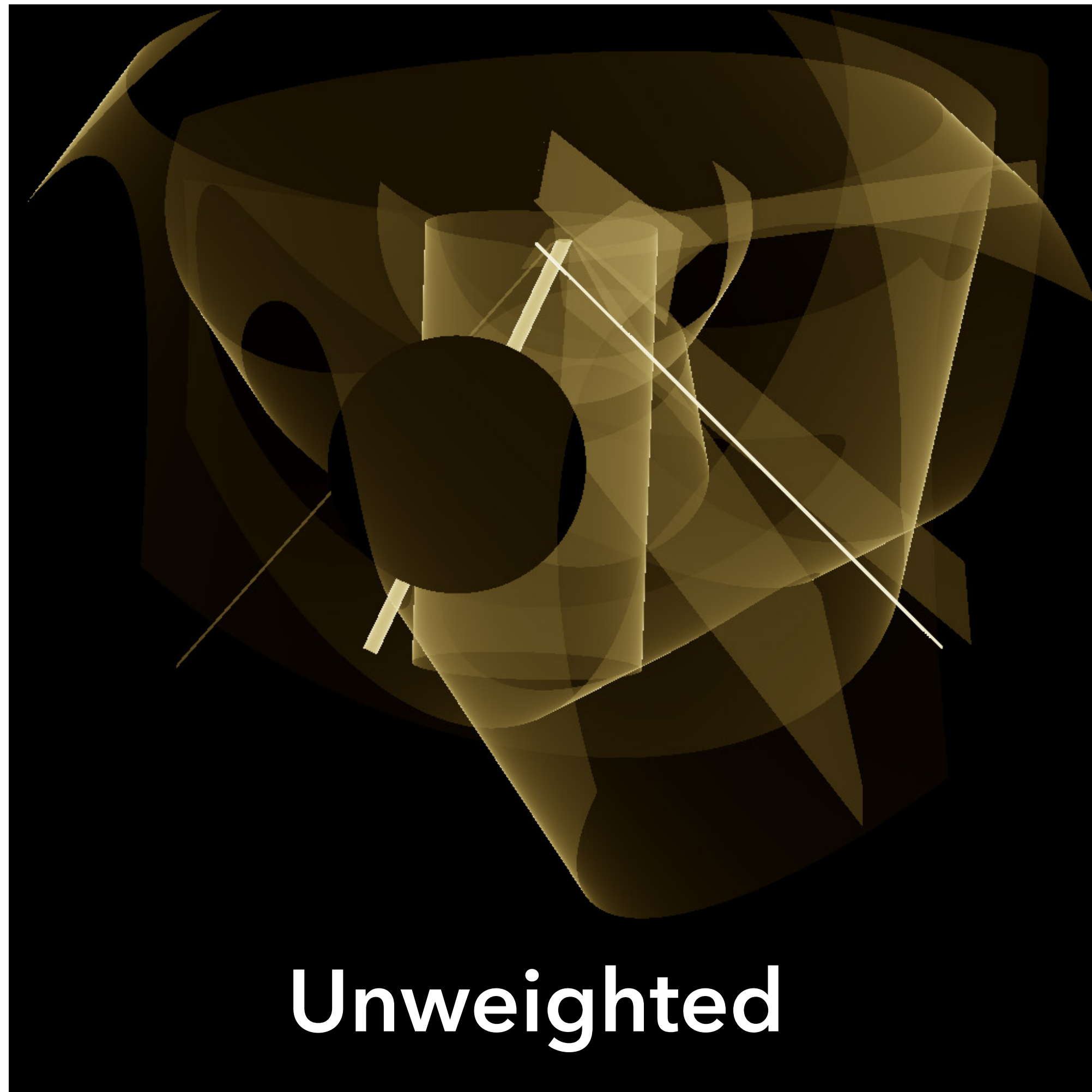
# Thank you!



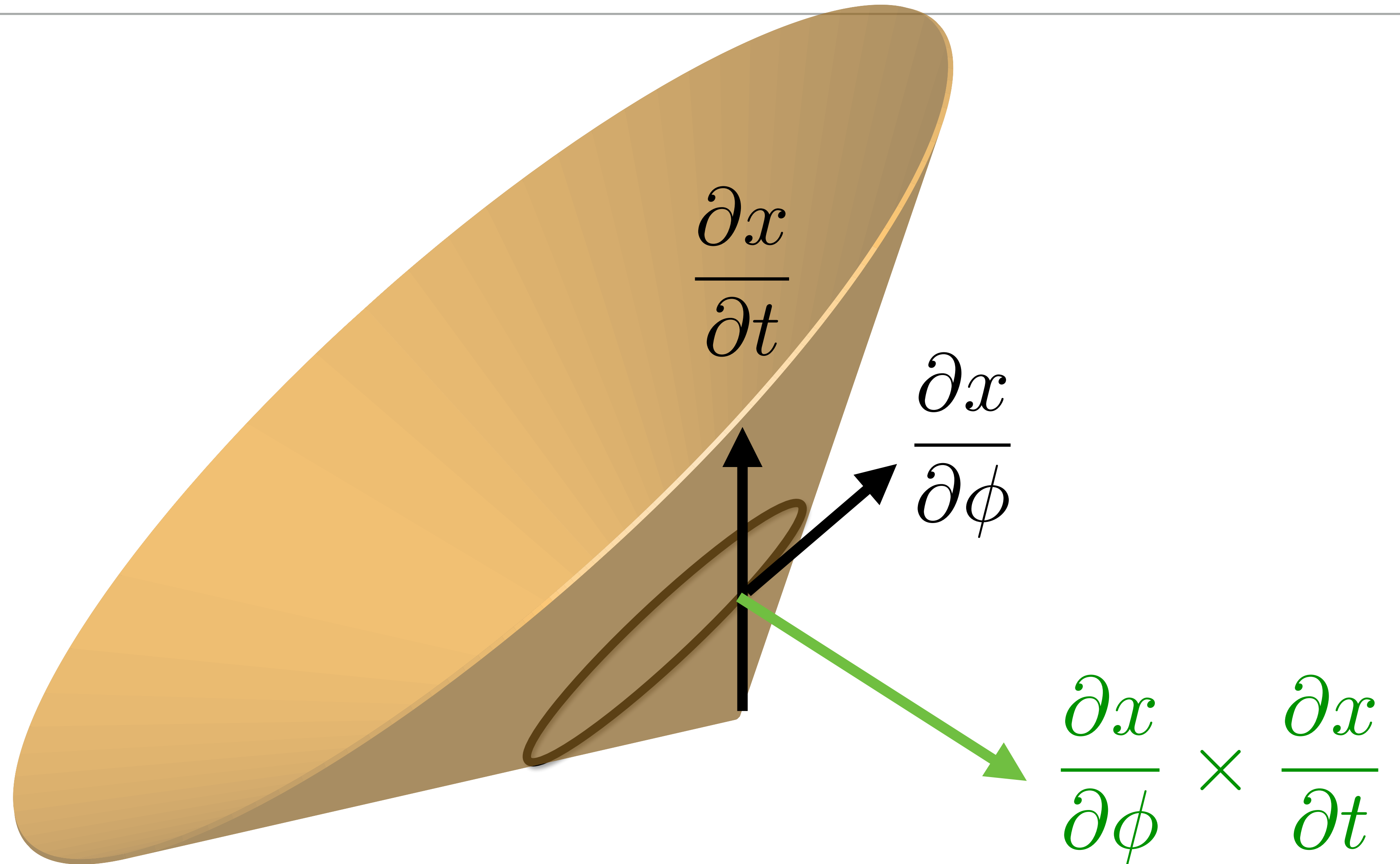
# Some combinations are much worse



# Some combinations are much better

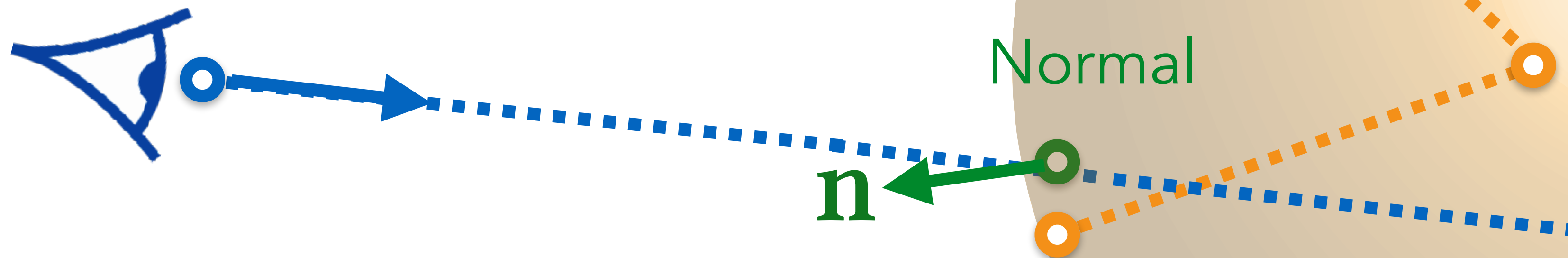


# Surface differential



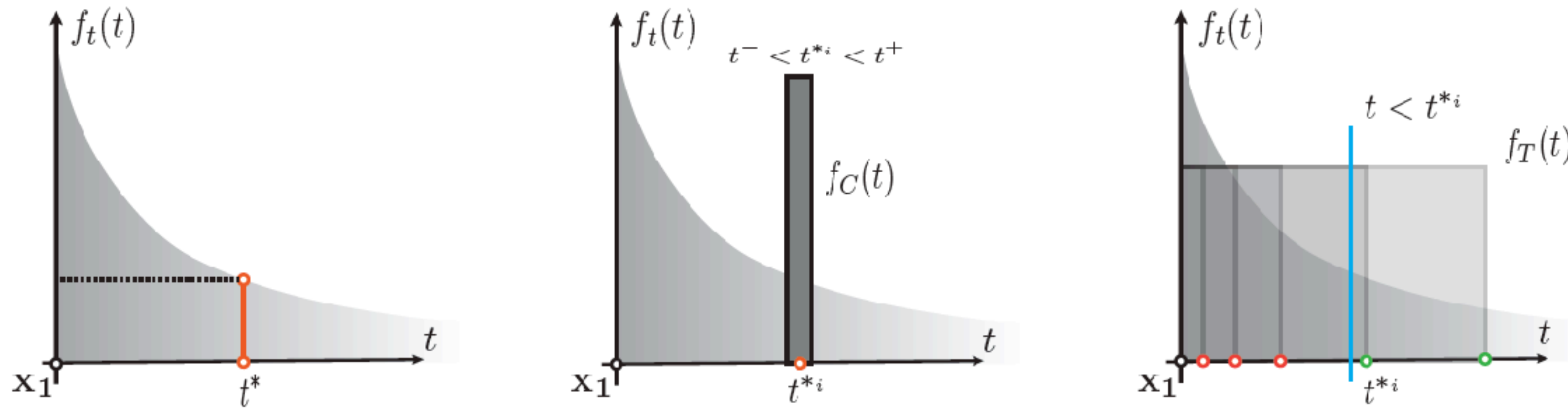
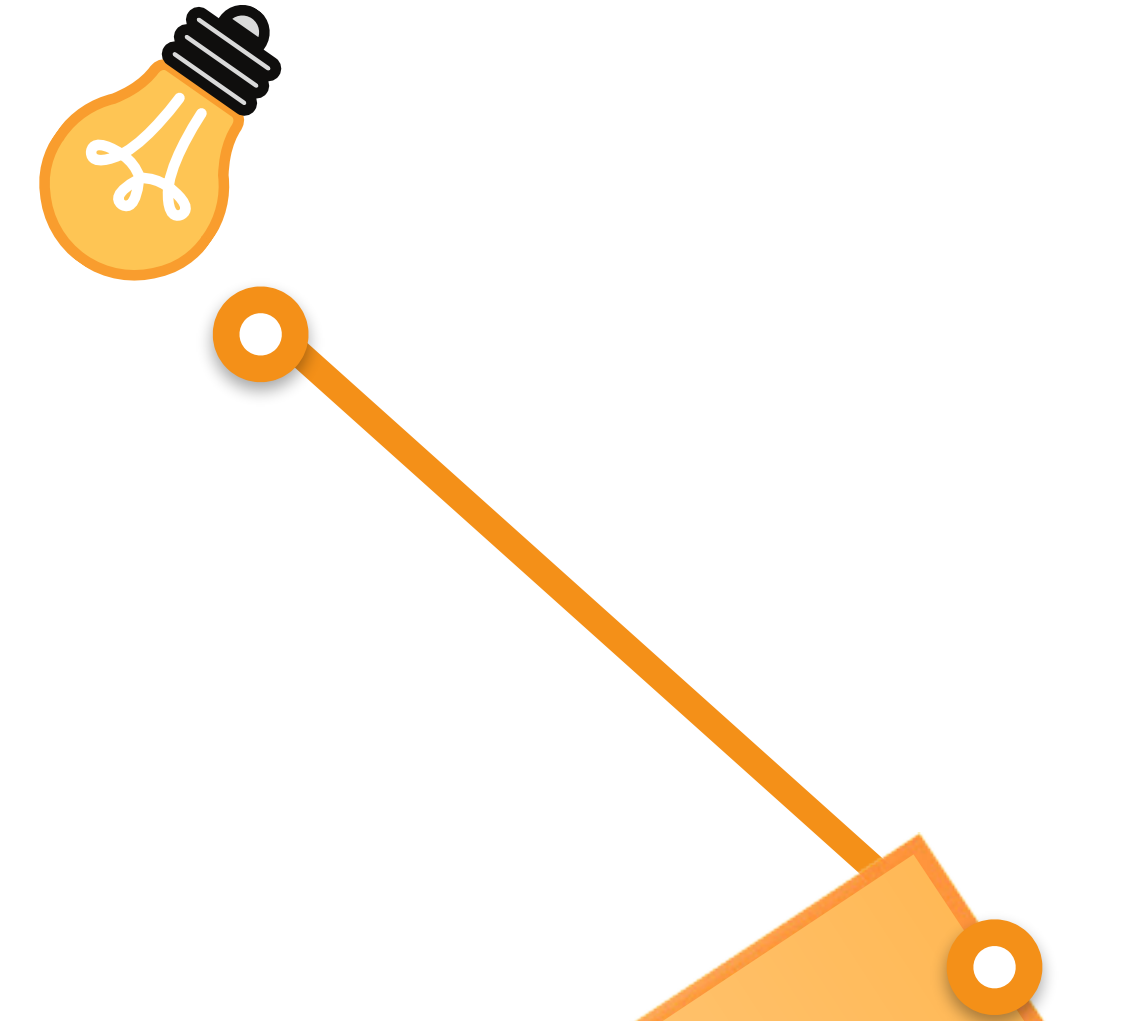
# Surface differential

$$\begin{aligned} |\omega \cdot n| &= \left| \det \left[ \frac{\partial(\mathbf{y} - \mathbf{x})}{\partial\theta_2}, \frac{\partial(\mathbf{y} - \mathbf{x})}{\partial\phi_2}, \frac{\partial(\mathbf{y} - \mathbf{x})}{\partial s_1} \right] \right| \\ &= \left| \det \left[ \frac{\partial\mathbf{x}(\theta_2, \phi_2)}{\partial\theta_2}, \frac{\partial\mathbf{x}(\theta_2, \phi_2)}{\partial\phi_2}, \frac{\partial(\mathbf{y}(s_1))}{\partial s_1} \right] \right| \end{aligned}$$



Variables:  $\{t_1, t_2, \theta_1, \theta_2, \phi_1, \phi_2, s_1\}$

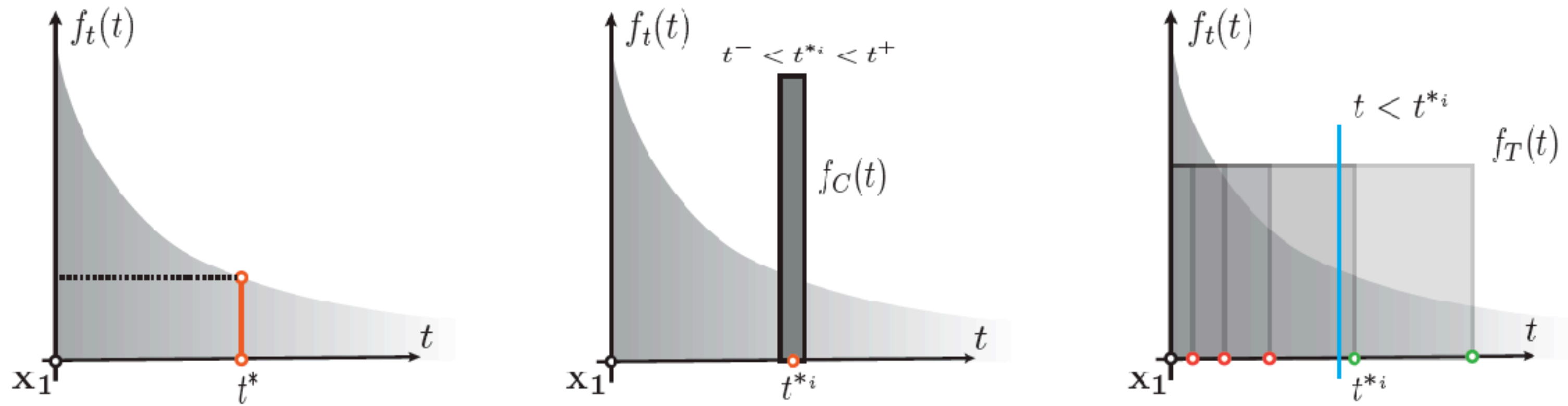
# Long Beam and Short Beam



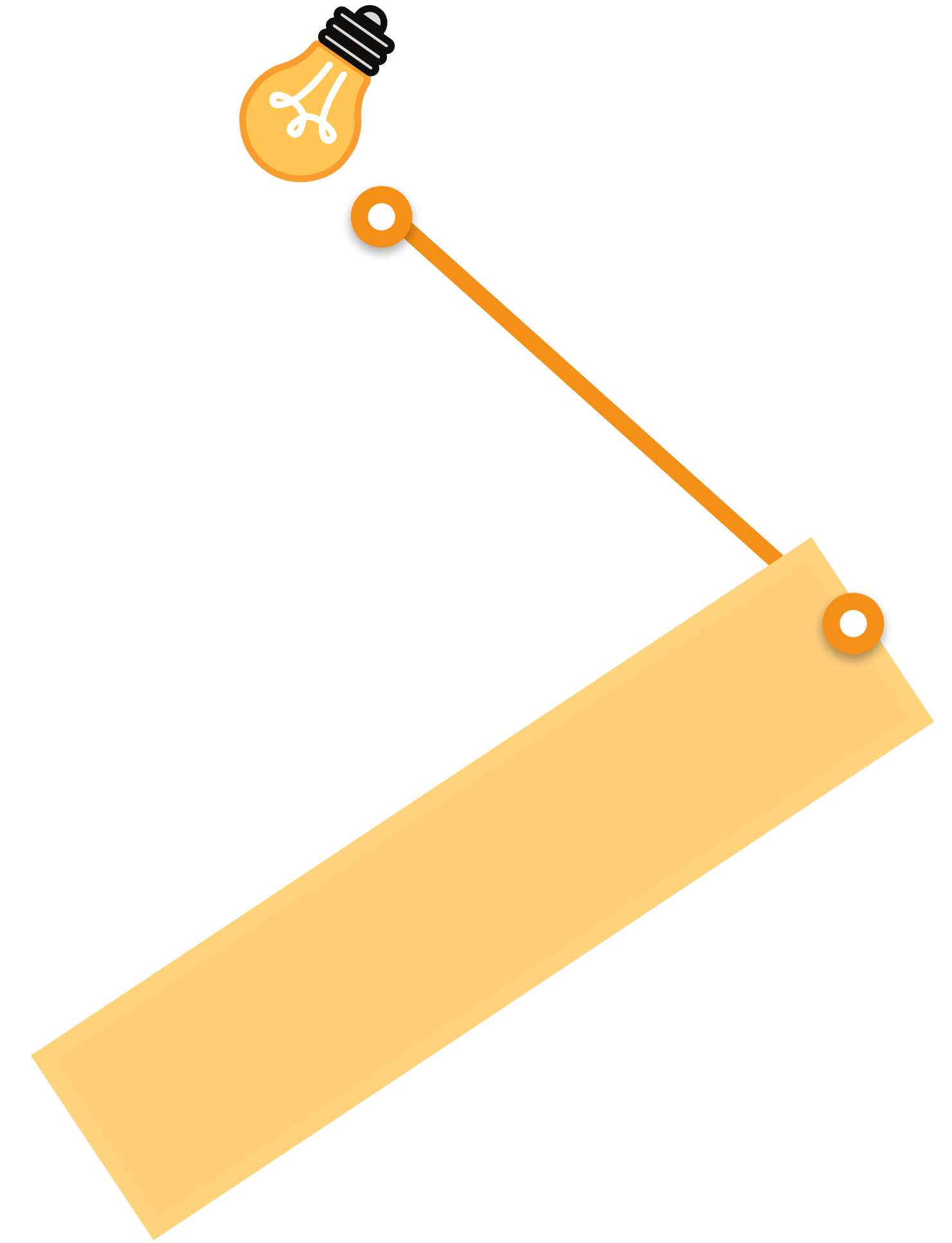
**Figure 5.2:** Expected value (right), collision estimator (middle) and tracklength estimator (left): Expected value directly compute the transmittance; collision estimator check if the sample  $t^{*i}$  falls in the interval  $[t^-, t^+]$ ; track-length estimator check if the sample go beyond some distance  $t$ .



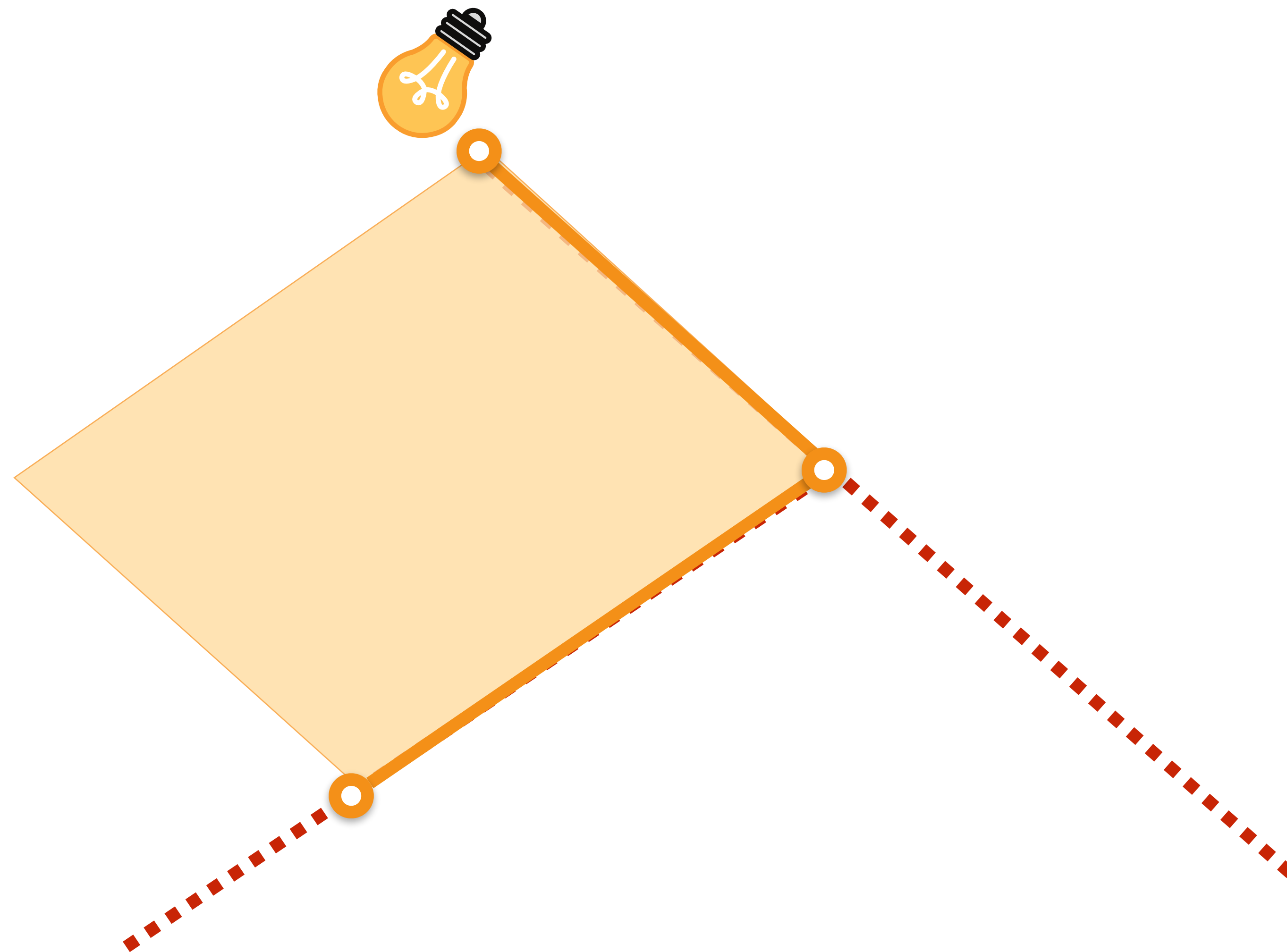
# Long Beam and Short Beam



**Figure 5.2:** Expected value (right), collision estimator (middle) and tracklength estimator (left): Expected value directly compute the transmittance; collision estimator check if the sample  $t^{*i}$  falls in the interval  $[t^-, t^+]$ ; track-length estimator check if the sample go beyond some distance  $t$ .



# Long & short surfaces



# Expense of each estimator

