Classical and Improved Diffusion Theory for Subsurface Scattering

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Abstract

This technical report provides a description and comparison of classical and improved diffusion theory as used for rendering subsurface scattering in computer graphics. We discuss the searchlight problem and describe the diffusion equation and Green's functions of each diffusion theory. We discuss the commonly used single-depth approximation leading to the diffusion dipole and elaborate on its connection to the desired extended-source function.

1. Introduction

This technical report provides an overview of classical and improved diffusion theory as introduced in physics and related fields, and utilized for rendering subsurface scattering in computer graphics.

When introducing the quantized diffusion method for subsurface scattering, d'Eon and Irving [dI11] presented several improvements to diffusion *theory* in tandem with their proposed *solution* (using quantized diffusion and Gaussians). In this technical report we explicitly separate these concepts to make potential connections with alternate solution methods more clear.

Our Monte Carlo solution method [HCJ13] builds on the same modified diffusion theory as quantized diffusion, but due to space limitations we could not provide an adequate exposition of the diffusion theory in that paper. We have therefore chosen to write this tech report to make the classical and improved diffusion theory, and the built-in assumptions, as clear as possible.

2. Light Transport in Scattering Media

Light transport in a participating medium can be described by the *radiance transport equation* (RTE) [Cha60]:

$$(\vec{\omega}\cdot\vec{\nabla})L(\vec{x},\vec{\omega}) = -\sigma_t L(\vec{x},\vec{\omega}) + Q(\vec{x},\vec{\omega}) + \sigma_s \int_{4\pi} L(\vec{x},\vec{\omega})\, f_s(\vec{\omega},\vec{\omega}')\, \mathrm{d}\vec{\omega}', \quad (1)$$

which states that the change in radiance L in direction $\vec{\omega}$ is the sum of three terms: a decrease in radiance dictated by the extinction coefficient ($\sigma_t = \sigma_s + \sigma_a$), and an increase due to the source function Q and the in-scattering integral on the second line. We summarize our notation in Table 1.

Table 1: *Notation*

Symbol	Description	Units
$L(\vec{x},\vec{\omega})$	Radiance at position $\vec{x}$ from direction $\vec{\omega}$	$[\mathrm{W\,m^{-2}\,sr^{-1}}]$
S	The BSSRDF	$[\mathrm{m^{-2}\,sr^{-1}}]$
$R_d(\vec{x})$	Diffuse reflectance profile value at $\vec{x}$	$[\mathrm{m^{-2}}]$
$f_s(\vec{\omega},\vec{\omega}')$	Phase function (normalized to $1/4\pi$)	$[\mathrm{sr^{-1}}]$
g	Average cosine of scattering	–
σ_s	Scattering coefficient	$[\mathrm{m^{-1}}]$
σ_s'	Reduced scattering coefficient: $(1-g)\,\sigma_s$	$[\mathrm{m^{-1}}]$
σ_a	Absorption coefficient	$[\mathrm{m^{-1}}]$
σ_t	Extinction coefficient: $\sigma_s+\sigma_a$	$[\mathrm{m^{-1}}]$
σ_t'	Reduced extinction coefficient: $\sigma_s'+\sigma_a$	$[\mathrm{m^{-1}}]$
η	Relative index of refraction	–
α	Scattering albedo: σ_s/σ_t	–
α'	Reduced scattering albedo: σ_s'/σ_t'	–
$\phi(\vec{x})$	Fluence at $\vec{x}$	$[\mathrm{W\,m^{-2}}]$
$\phi^m(\vec{x})$	Fluence at $\vec{x}$ due to a monopole source	$[\mathrm{W\,m^{-2}}]$
$\phi^d(\vec{x})$	Fluence at $\vec{x}$ due to a dipole source	$[\mathrm{W\,m^{-2}}]$
$\phi^b(\vec{x})$	Fluence at $\vec{x}$ due to a beam source	$[\mathrm{W\,m^{-2}}]$
$\vec{E}(\vec{x})$	Vector flux (vector irradiance) at $\vec{x}$	$[\mathrm{W\,m^{-2}}]$
$Q(\vec{x})$	Source function at $\vec{x}$	$[\mathrm{W\,m^{-3}}]$
D	Diffusion coefficient	$[\mathrm{m}]$
σ_{tr}	Transport coefficient: $\sqrt{\sigma_a/D}$	$[\mathrm{m^{-1}}]$

In the most general form, simulating light transport in translucent materials requires solving the RTE (accounting for scattering and absorption within the medium) with suitable boundary conditions imposed by the enclosing surfaces.

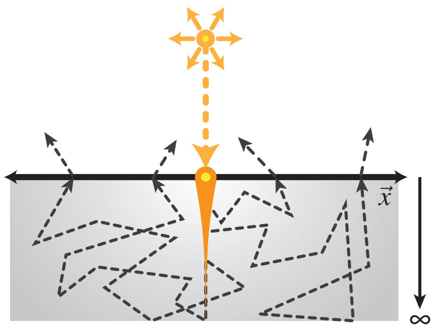

Figure 1: *The searchlight problem consisting of a semi-infinite planar homogeneous medium, a Fresnel boundary and a focused pencil beam of light entering the media. The pencil beam continues after the refraction as an exponentially decaying linear source. The light is absorbed and scattered and some of it exits the top boundary.*

2.1. The BSSRDF and the Searchlight Problem

When rendering translucent materials, it is often convenient to re-formulate this problem in analogy to the local surface reflection integral. This results in an integral equation which computes the outgoing radiance, L_o, at position and direction $(\vec{x}_o, \vec{\omega}_o)$ as a convolution of the incident illumination, L_i, and the BSSRDF, S, over all incident positions and directions $(\vec{x}_i, \vec{\omega}_i)$:

$$L_o(\vec{x}_o,\vec{\omega}_o) = \int_A \int_{2\pi} S(\vec{x}_i,\vec{\omega}_i;\vec{x}_o,\vec{\omega}_o) L_i(\vec{x}_i,\vec{\omega}_i)(\vec{n}\cdot\vec{\omega}_i)\, \mathrm{d}\vec{\omega}_i\, \mathrm{d}A(\vec{x}_i). \quad (2)$$

The most general method to solve this integral is brute-force Monte Carlo particle tracing. While this method is both general, intuitive, and simple to implement, unfortunately it gives noisy results and converges very slowly. Hence, researchers in fields such as medical physics, astrophysics, and recently computer graphics have developed alternative solution methods, and usually only use brute-force Monte Carlo particle tracing for validation.

For efficiency, S is often decomposed into reduced-radiance, single-scattering, and multi-scattering terms, $S = S^{(0)} + S^{(1)} + S_d$, so that each can be handled by specialized algorithms. Most previous techniques in graphics have approximated $S^{(1)}$ with the refractive single-scattering approximation proposed by Jensen et al. [JMLH01] and we propose an alternate diffuse single-scattering approximation [HCJ13]. Accurate brute-force simulation of the multi-scattering term S_d is as expensive as the general problem, so we rely on simplifications to the multi-scattering problem based on approximate solutions to the so-called "searchlight problem."

The searchlight problem, illustrated in Figure 1, considers a simplified setting where a focused pencil beam of unit flux is orthogonally incident at the origin on a semi-infinite planar homogeneous medium. Photons refract through a Fresnel boundary and travel in the downward direction until they are scattered or absorbed by the medium. The unabsorbed/unscattered photons form an exponentially decaying linear source inside the medium. The scattered photons can be scattered again and ultimately get absorbed or escape the medium. The sum of all photons exiting the upper boundary at each point $\vec{x}$ forms a spatial diffuse reflectance profile $R_d(\vec{x})$, which is radially symmetric (1D) for normally-incident light, $R_d(\vec{x}) = R_d(\|\vec{x}\|)$. Even though the searchlight problem is simpler than the fully general setting, exact solutions only exist for special cases—we still rely on brute-force Monte Carlo particle tracing for validation, just as for the general problem.

Most methods in graphics simplify S_d as a product of a typically 1D, diffuse reflectance profile R_d and a directional, energy-preserving Fresnel reshaping term [JMLH01, dI11]:

$$S_d(\vec{x}_i,\vec{\omega}_i;\vec{x}_o,\vec{\omega}_o) = \frac{1}{\pi} F_t(\vec{x}_i,\vec{\omega}_i,\eta) R_d(\vec{x}_o - \vec{x}_i) \frac{F_t(\vec{x}_o,\vec{\omega}_o,\eta)}{4C_\phi(1/\eta)}, \quad (3)$$

where the reflectance profile is now centered at the incident light position, $\vec{x}_i$, and $4C_\phi$ is a constant needed for normalization. To evaluate this expression efficiently, most methods have relied on diffusion theory to obtain an analytic approximation for R_d.

3. Classical Diffusion Theory

3.1. The Classical Diffusion Equation

The classical diffusion approximation solves Equation (1) by considering only a first-order spherical harmonic expansion of the radiance:

$$L(\vec{x},\vec{\omega}) \approx \frac{1}{4\pi}\phi(\vec{x}) + \frac{3}{4\pi}\vec{E}(\vec{x})\cdot\vec{\omega}, \quad (4)$$

where the fluence ϕ and vector flux $\vec{E}$ are the first two angular moments of the radiance distribution:

$$\phi(\vec{x}) = \int_{4\pi} L(\vec{x},\vec{\omega})\, \mathrm{d}\vec{\omega}, \qquad \vec{E}(\vec{x}) = \int_{4\pi} L(\vec{x},\vec{\omega})\, \vec{\omega}\, \mathrm{d}\vec{\omega}. \quad (5)$$

If we further assume that the source function Q is isotropic, the classical diffusion equation follows from this approximation:

$$-D\nabla^2\phi(\vec{x}) + \sigma_a\phi(\vec{x}) = Q(\vec{x}), \quad (6)$$

where $D = 1/3\sigma_t'$ is the diffusion coefficient and σ_t' is the reduced extinction coefficient obtained by applying similarity theory with $\sigma_t' = (1-g)\sigma_s + \sigma_a$. In the case of a unit power isotropic point source (a monopole) in an infinite homogeneous medium, the solution to Equation (6) is the classical diffusion Green's function:

$$\phi^m(\vec{x}) = \frac{1}{4\pi D}\frac{e^{-\sigma_{tr} d(\vec{x})}}{d(\vec{x})}, \quad (7)$$

where $\sigma_{tr} = \sqrt{\sigma_a/D}$ is the transport coefficient and $d(\vec{x})$ is the distance from $\vec{x}$ to the point source. We use the superscript on ϕ^m to indicate fluence from a monopole.

3.2. Obtaining the Diffusion Profile

The Source Function. To render subsurface scattering, a suitable source function needs to be defined. In reality, each ray of light incident on a smooth surface gives rise to a refracted ray within the medium with exponentially decreasing intensity. To avoid integrating from this extended beam source, Farrell et al. [FPW92] proposed concentrating all the light at the center of mass of the beam (a depth of one mean free path ($1/\sigma_t'$) inside the medium), effectively replacing the beam source with a single isotropic point source inside the medium. This single-depth approximation was later adopted by Jensen et al. [JMLH01].

Boundary Conditions. For semi-infinite configurations, the boundary condition can be handled in an approximate fashion with the "method of images" by placing a mirrored negative source outside the medium for every positive source inside the medium. The mirroring is performed above an extrapolated boundary such that the fluence is zero at a distance $z_b = 2AD$ above the surface. This offset takes into account the index of refraction mismatch at the boundary through the *reflection parameter* A. Jensen et al. [JMLH01] used $A(\eta) \approx \frac{1+2C_1(\eta)}{1-2C_1(\eta)}$ where η is the relative index of refraction at the surface and C_i is the i^{th} angular moment of the Fresnel function. Analytic solution to the angular moments are defined by Aronson [Aro95]. Approximations to the angular moments, including constant factors used in the calculations (2 for $C_1(\eta)$ and 3 for $C_2(\eta)$), are provided by d'Eon and Irving [dI11]:

$$2C_1(\eta) \approx \begin{cases} 0.919317 - 3.4793\eta + 6.75335\eta^2 \\ -7.80989\eta^3 + 4.98554\eta^4 - 1.36881\eta^5, & \eta < 1 \\ \\ -9.23372 + 22.2272\eta - 20.9292\eta^2 \\ +10.2291\eta^3 - 2.54396\eta^4 + 0.254913\eta^5, & \eta \geq 1 \end{cases} \tag{8}$$

$$3C_2(\eta) \approx \begin{cases} 0.828421 - 2.62051\eta + 3.36231\eta^2 \\ -1.95284\eta^3 + 0.236494\eta^4 + 0.145787\eta^5, & \eta < 1 \\ \\ -1641.1 + 135.926\eta^{-3} - 656.175\eta^{-2} \\ +1376.53\eta^{-1} + 1213.67\eta - 568.556\eta^2 \\ +164.798\eta^3 - 27.0181\eta^4 + 1.91826\eta^5, & \eta \geq 1. \end{cases} \tag{9}$$

The Dipole. The single-depth approximation, together with the method of images to satisfy the boundary conditions, forms a dipole consisting of a positive and negative monopole pair from Equation (7):

$$\phi^d(\vec{x}) = \frac{1}{4\pi D}\left(\frac{e^{-\sigma_{tr} d(\vec{x},\vec{x}_r)}}{d(\vec{x},\vec{x}_r)} - \frac{e^{-\sigma_{tr} d(\vec{x},\vec{x}_v)}}{d(\vec{x},\vec{x}_v)}\right) \tag{10}$$

where $d(\vec{x}_a,\vec{x}_b)$ is the distance between two points, the real source at $\vec{x}_r$ is at a depth $z_r = 1/\sigma_t'$ within the medium (regardless of incident light direction), and the mirrored virtual source at $\vec{x}_v = (-\vec{x}_r + 2z_b\vec{n})$ is placed at a height $z_v = z_r + 2z_b$ above the surface. We use the superscript on ϕ^d to indicate fluence due to a dipole. Figure 2 illustrates the resulting configuration.

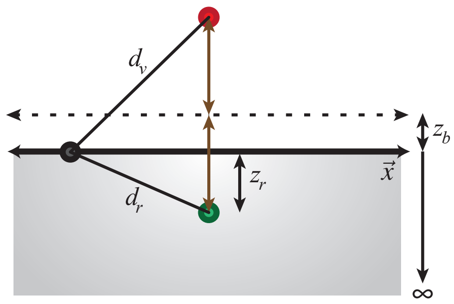

Figure 2: *The single-depth approximation replaces the extended beam source with a single isotropic point source (green) at a depth z_r inside the medium. This source is mirrored about an extrapolated boundary z_b above the surface, creating a negative point source (red) outside of the medium.*

Diffuse Reflectance Profile. The equations so far have expressed the fluence volumetrically within the medium. To render surfaces, however, we wish to compute the light leaving various points on the boundary.

To handle this, Jensen et al. [JMLH01] used Fick's law which states that (for isotropic sources) the vector flux is the gradient of the fluence:

$$\vec{E}(\vec{x}) = -D\vec{\nabla}\phi(\vec{x}). \tag{11}$$

Since the radiant exitance on the boundary is the dot product of the vector flux with the surface normal, we have†:

$$R_d(\vec{x}) = \vec{E}(\vec{x}) \cdot \vec{n} = -D(\vec{\nabla} \cdot \vec{n})\,\phi(\vec{x}). \tag{12}$$

Hence, to compute the diffusion profile $R_d(\vec{x})$ due to a dipole, we need to evaluate the directional derivative $(\vec{\nabla} \cdot \vec{n})$ of the fluence (10) in the direction of the normal. This gives us:

$$R_d(\vec{x}) = \frac{\alpha'}{4\pi}\left[\frac{z_r\,(1+\sigma_{tr} d_r)\,e^{-\sigma_{tr} d_r}}{d_r^3} + \frac{z_v\,(1+\sigma_{tr} d_v)\,e^{-\sigma_{tr} d_v}}{d_v^3}\right], \tag{13}$$

where we use the shorthand $d_r = d(\vec{x},\vec{x}_r)$ and $d_v = d(\vec{x},\vec{x}_v)$.

In classical diffusion theory, the diffuse reflectance profile $R_d(\vec{x})$ in Equation (12) depends only on the surface flux (the gradient of the fluence along the surface normal) and not the fluence $\phi(\vec{x})$ itself due to Fick's law. In more general

† Note that since the searchlight problem assumes incident light of unit flux (power), the diffuse reflectance profile $R_d(\vec{x})$ actually expresses the radiant exitance [W m^{-2}] response normalized per unit incident flux [W], resulting in units of [m^{-2}] for $R_d(\vec{x})$.

terms, this applies Neumann boundary conditions where only derivatives are specified. This can be formally expressed as:

$$R_d(\vec{x}) = C_\phi \phi(\vec{x}) + C_{\vec{E}}(-D(\vec{\nabla}\cdot\vec{n})\phi(\vec{x})). \tag{14}$$

with factors $C_\phi = 0$ and $C_{\vec{E}} = 1$ so only vector flux, and not fluence, contributes to the diffuse reflectance. These relative contributions change with other exitance calculations (other boundary conditions) as we will discuss in the next section.

4. Improved Diffusion Theory

D'Eon and Irving [dI11] introduced a wealth of improvements for diffusion theory from other fields such as medical physics and astrophysics to computer graphics, including a modified diffusion equation, a more accurate reflection parameter A, and a more accurate exitance calculation. We summarize these changes here, and in doing so we will introduce new terms for the aforementioned expressions with a subscript G.

4.1. Grosjean's Diffusion Equation

Grosjean [Gro56, Gro58] proposed a different approximation for the fluence due to an isotropic point source (monopole) in an infinite medium,

$$\phi^m_G(\vec{x}) = \frac{e^{-\sigma'_t d(\vec{x})}}{4\pi d^2(\vec{x})} + \frac{\alpha'}{4\pi D_G}\frac{e^{-\sigma_{tr,G} d(\vec{x})}}{d(\vec{x})}. \tag{15}$$

This is the sum of the exact single scattering and approximate multi scattering using Grosjean's modified diffusion coefficient

$$D_G = \frac{2\sigma_a + \sigma'_s}{3\sigma_t'^2} = \frac{1}{3\sigma'_t} + \frac{\sigma_a}{3\sigma_t'^2}, \tag{16}$$

and the transport coefficient is defined in the same way as in classical diffusion $\sigma_{tr,G} = \sqrt{\sigma_a/D_G}$, but using this modified diffusion coefficient D_G.

The single-scattering term is not considered part of the diffusion theory and is therefore handled separately. In contrast to classical diffusion theory, this explicitly excludes single scattering from the diffusion approximation. Single scattering can be calculated exactly using Monte Carlo methods [WZHB09, JC98, DJ07, JNSJ11], or it can be approximated [JMLH01, HCJ13].

Grosjean's approximation gives rise to a modified diffusion equation:

$$-D_G \nabla^2 \phi(\vec{x}) + \sigma_a \phi(\vec{x}) = \alpha' Q(\vec{x}). \tag{17}$$

Compared to Equation (6), this diffusion equation multiplies the source term by the reduced albedo α'. This multiplication is a result of the exclusion of single scattering, which provides more accurate solutions near the source and for low albedos. The Green's function of Equation (17) is the second term of Equation (15) which likewise has an additional albedo term compared to the classical diffusion Green's function (7).

Boundary Conditions. To handle the surface boundary, D'Eon and Irving still use the same boundary condition where fluence is forced to zero at a distance $z_b = 2A_G D_G$, but use Grosjean's D_G (16) and an improved reflection parameter, $A_G(\eta) \approx \frac{1+3C_2(\eta)}{1-2C_1(\eta)}$ [PG95].

Improved Dipole. With these changes, the fluence due to a dipole defined with the single-depth approximation and method of images is [d'E12]:

$$\phi^d_G(\vec{x},\vec{x}_r) = \frac{\alpha'}{4\pi D_G}\left(\frac{e^{-\sigma_{tr,G} d(\vec{x},\vec{x}_r)}}{d(\vec{x},\vec{x}_r)} - \frac{e^{-\sigma_{tr,G} d(\vec{x},\vec{x}_v)}}{d(\vec{x},\vec{x}_v)}\right), \tag{18}$$

where, compared to Equation (10), the additional α' takes the single-scattering separation into account. Please note that we call this the "improved dipole" rather than the "better dipole" [d'E12] for naming consistency.

Radiant Exitance. Instead of using Fick's Law, which relies solely on vector flux, d'Eon and Irving use a Robin boundary condition introduced by Kienle and Patterson [Aro95, KP97] which uses a linear combination of the fluence and its derivative, the vector flux. The result is a more accurate radiant exitance, defined as in Equation (14):

$$R_{d,G}(\vec{x}) = C_{\phi,G}\phi(\vec{x}) + C_{\vec{E},G}(-D(\vec{\nabla}\cdot\vec{n})\phi(\vec{x})), \tag{19}$$

but where $C_{\phi,G} = \frac{1}{4}(1-2C_1(\eta))$ and $C_{\vec{E},G} = \frac{1}{2}(1-3C_2(\eta))$. Evaluating the fluence contribution together with the vector flux gives us:

$$R_{d,G}(\vec{x}) = R^\phi_{d,G}(\vec{x}) + R^{\vec{E}}_{d,G}(\vec{x}) \tag{20}$$

with

$$R^\phi_{d,G}(\vec{x}) = C_{\phi,G}\frac{\alpha'^2}{4\pi D_G}\left(\frac{e^{-\sigma_{tr,G} d_r}}{d_r} - \frac{e^{-\sigma_{tr,G} d_v}}{d_v}\right), \quad \text{and} \tag{21}$$

$$R^{\vec{E}}_{d,G}(\vec{x}) = C_{\vec{E},G}\frac{\alpha'^2}{4\pi}\left[\frac{z_r\left(1+\sigma_{tr,G} d_r\right)e^{-\sigma_{tr,G} d_r}}{d_r^3} + \frac{(z_r+2z_b)\left(1+\sigma_{tr,G} d_v\right)e^{-\sigma_{tr,G} d_v}}{d_v^3}\right], \tag{22}$$

again using the shorthand $d_r = d(\vec{x},\vec{x}_r)$ and analogously for d_v, and where $z_r = 1/\sigma'_t$ is the fixed depth of the real source. ‡

The improved diffusion theory consistently delivers better results than the classical theory and should therefore be preferred for computer graphics purposes. Table 2 summarizes and compares important quantities from both diffusion theories. All other parameters such as the dipole depth z_b or the transport coefficient σ_{tr} follow from these definitions.

4.2. Extended Source

Accounting for the entire beam of photons entering the medium requires defining an extended source along the (re-

‡ Formulas corrected Sept. 2013

Table 2: *Comparison of classical and improved diffusion model*

Term	Classical Diffusion	Improved Diffusion
Diffusion Equation	$-D\nabla^2\phi(\vec{x}) + \sigma_a\phi(\vec{x}) = Q(\vec{x})$	$-D_G\nabla^2\phi(\vec{x}) + \sigma_a\phi(\vec{x}) = \alpha' Q(\vec{x})$
Green's Function Isotropic Source	$\phi^m(\vec{x}) = \frac{1}{4\pi D}\frac{e^{-\sigma_{tr}d(\vec{x})}}{d(\vec{x})}$	$\phi^m_G(\vec{x}) = \frac{\alpha'}{4\pi D_G}\frac{e^{-\sigma_{tr,G}d(\vec{x})}}{d(\vec{x})}$
Diffusion Coefficient	$D = \frac{1}{3\sigma'_t}$	$D_G = \frac{2\sigma_a+\sigma'_s}{3\sigma'^2_t} = \frac{1}{3\sigma'_t} + \frac{\sigma_a}{3\sigma'^2_t}$
Reflection Parameter	$A(\eta) = \frac{1+2C_1(\eta)}{1-2C_1(\eta)}$	$A_G(\eta) = \frac{1+3C_2(\eta)}{1-2C_1(\eta)}$
Exitance Parameter Fluence	$C_\phi = 0$	$C_{\phi,G} = \frac{1}{4}(1-2C_1(\eta))$
Exitance Parameter Flux	$C_{\vec{E}} = 1$	$C_{\vec{E},G} = \frac{1}{2}(1-3C_2(\eta))$

fracted) beam, with an exponential falloff formed by all first-scatter events inside the medium [FPW92]:

$$Q(t) = \alpha'\sigma'_t e^{-\sigma'_t t}, \tag{23}$$

where t is the distance along the source beam inside the medium.

Radiant Exitance from Extended Source. Obtaining the fluence due to the extended source in a semi-infinite medium requires sweeping the Green's function along the beam:

$$\phi^b(\vec{x},\vec{\omega}) = \int_0^\infty \phi^d(\vec{x},\vec{x}_r(t))\, Q(t)\, \mathrm{d}t, \tag{24}$$

where $\phi^d(\vec{x}_r(t))$ is the dipole Green's function (18) with the previously constant position of the real source $\vec{x}_r$ now varying along the beam $\vec{x}_r(t) = t\vec{\omega}$. Note that this creates a positive as well as a mirrored negative extended source due to the method of images for handling the semi-infinite medium.

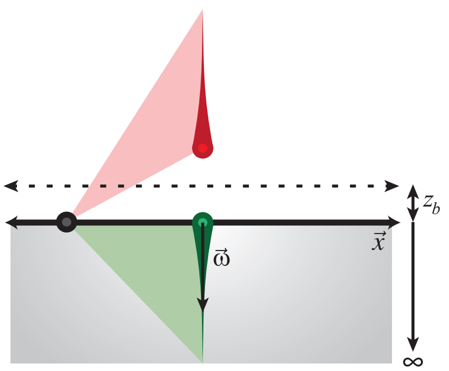

Figure 3: *The extended source is created by sweeping the dipole Green's function along the beam direction $\vec{\omega}$. Due to the method of images, a corresponding negative extended source is created.*

We use the superscript on ϕ^b to denote fluence from a beam source. The resulting configuration is depicted in Figure 3. This step is independent of the underlying diffusion theory and can be performed with any Green's function and exitance calculations, but due to its superiority, we only show the derivation for the improved diffusion theory.

To obtain the diffuse reflectance profile on the surface we need to instead integrate Equation (19) over the beam. This results in the integral

$$R_{d,G}(\vec{x},\vec{\omega}) = \int_0^\infty r(\vec{x},\vec{x}_r(t))\, Q(t)\, \mathrm{d}t, \tag{25}$$

where $r(\vec{x},\vec{x}_r(t)) = R^\phi_{d,G}(\vec{x},t) + R^{\vec{E}}_{d,G}(\vec{x},t)$ with:

$$R^\phi_{d,G}(\vec{x},t) = C_{\phi,G}\frac{\alpha'}{4\pi D_G}\left(\frac{e^{-\sigma_{tr,G}d_r(t)}}{d_r(t)} - \frac{e^{-\sigma_{tr,G}d_v(t)}}{d_v(t)}\right), \text{ and} \tag{26}$$

$$R^{\vec{E}}_{d,G}(\vec{x},t) = C_{\vec{E},G}\frac{\alpha'}{4\pi}\left[\frac{z_r(t)\left(1+\sigma_{tr,G}d_r(t)\right)e^{-\sigma_{tr,G}d_r(t)}}{d_r^3(t)} + \frac{(z_r(t)+2z_b)\left(1+\sigma_{tr,G}d_v(t)\right)e^{-\sigma_{tr,G}d_v(t)}}{d_v^3(t)}\right], \tag{27}$$

using the shorthand $d_r(t) = d(\vec{x},\vec{x}_r(t))$ and analogously for d_v, and where $z_r(t) = \vec{x}_r(t)\cdot\vec{n}$ is the depth of the real source. This redefines $R^\phi_{d,G}(\vec{x})$ and $R^{\vec{E}}_{d,G}(\vec{x})$ from the single-depth approximation (21–22) to be dependent on the depth of a point t on the extended source.

5. Solving the Extended Source

D'Eon and Irving [dI11] noted that Equation (25) has no closed-form solution, and proposed to approximate it using a sum of Gaussians:

$$R_d(\vec{x}) \approx \alpha' \sum_{i=0}^{k-1} (\mathrm{w}_R(i)\,\mathrm{w}_i)\; G_{2D}(v_i, x), \tag{28}$$

where $x = \|\vec{x}\|$, $(\mathrm{w}_R(i)\,\mathrm{w}_i)$ are the weights, and v_i are the variances of normalized 2D Gaussians G_{2D}. The equations necessary to obtain the weights and variances of the Gaussians are themselves fairly complex summations of integrals depending on further weights $w_{\phi R}(v,i), w_{\vec{E}R}(v,i)$.

In our recent article [HCJ13], we showed that it is efficient to numerically approximate Equation (25) using Monte Carlo integration with exponential and equiangular importance sampling [KF12, NNDJ12]. We further increased the accuracy of the improved diffusion model using an empirical correction factor.

6. Conclusion

This technical report has provided an overview of classical and improved diffusion theory, independent of any solution methods. So far, two solution methods have been used: quantized diffusion [dI11] and Monte Carlo integration [HCJ13], and it is our hope that this overview of diffusion theory might inspire other solution methods in the future.

References

[Aro95] ARONSON R.: Boundary conditions for diffusion of light. *Journal of the Optical Society of America A 12*, 11 (1995), 2532–2539. 3, 4

[Cha60] CHANDRASEKAR S.: *Radiative Transfer*. Dover, 1960. 1

[d'E12] D'EON E.: *A better dipole*. Tech. rep., http://www.eugenedeon.com, 2012. 4

[dI11] D'EON E., IRVING G.: A quantized-diffusion model for rendering translucent materials. *ACM Transactions on Graphics (Proc. SIGGRAPH) 30*, 4 (2011), 56:1–56:14. 1, 2, 3, 4, 5, 6

[DJ07] DONNER C., JENSEN H. W.: Rendering translucent materials using photon diffusion. *Rendering Techniques (Proc. Eurographics Symposium on Rendering)* (2007), 243–252. 4

[FPW92] FARRELL T. J., PATTERSON M. S., WILSON B.: A diffusion theory model of spatially resolved, steady-state diffuse reflectance for the noninvasive determination of tissue optical properties in vivo. *Medical Physics 19*, 4 (1992), 879–888. 3, 5

[Gro56] GROSJEAN C. C.: A high accuracy approximation for solving multiple scattering problems in infinite homogeneous media. *Il Nuovo Cimento (1955-1965) 3* (1956), 1262–1275. 4

[Gro58] GROSJEAN C. C.: Multiple isotropic scattering in convex homogeneous media bounded by vacuum. In *Proc. Second International Conference on the Peaceful Uses of Atomic Energy* (1958), vol. 413. 4

[HCJ13] HABEL R., CHRISTENSEN P. H., JAROSZ W.: Photon beam diffusion: A hybrid Monte Carlo method for subsurface scattering. *Computer Graphics Forum (Proc. Eurographics Symposium on Rendering) 32*, 4 (June 2013). 1, 2, 4, 6

[JC98] JENSEN H. W., CHRISTENSEN P. H.: Efficient simulation of light transport in scenes with participating media using photon maps. *Computer Graphics (Proc. SIGGRAPH) 32* (1998), 311–320. 4

[JMLH01] JENSEN H. W., MARSCHNER S. R., LEVOY M., HANRAHAN P.: A practical model for subsurface light transport. *Computer Graphics (Proc. SIGGRAPH) 35* (2001), 511–518. 2, 3, 4

[JNSJ11] JAROSZ W., NOWROUZEZAHRAI D., SADEGHI I., JENSEN H. W.: A comprehensive theory of volumetric radiance estimation using photon points and beams. *ACM Transactions on Graphics 30*, 1 (2011), 5:1–5:19. 4

[KF12] KULLA C., FAJARDO M.: Importance sampling techniques for path tracing in participating media. *Computer Graphics Forum (Proc. Eurographics Symposium on Rendering) 31*, 4 (2012), 1519–1528. 6

[KP97] KIENLE A., PATTERSON M. S.: Improved solutions of the steady-state and the time-resolved diffusion equations for reflectance from a semi-infinite turbid medium. *Journal of the Optical Society of America A 14*, 1 (1997), 246–254. 4

[NNDJ12] NOVÁK J., NOWROUZEZAHRAI D., DACHSBACHER C., JAROSZ W.: Virtual ray lights for rendering scenes with participating media. *ACM Transactions on Graphics (Proc. SIGGRAPH) 31*, 4 (2012), 60:1–60:11. 6

[PG95] POMRANING G. C., GANAPOL B. D.: Asymptotically consistent reflection boundary conditions for diffusion theory. *Annals of Nuclear Energy 22*, 12 (1995), 787–817. 4

[WZHB09] WALTER B., ZHAO S., HOLZSCHUCH N., BALA K.: Single scattering in refractive media with triangle mesh boundaries. *ACM Transactions on Graphics (Proc. SIGGRAPH) 28*, 3 (2009). 4