# Unbiased and consistent rendering using biased estimators

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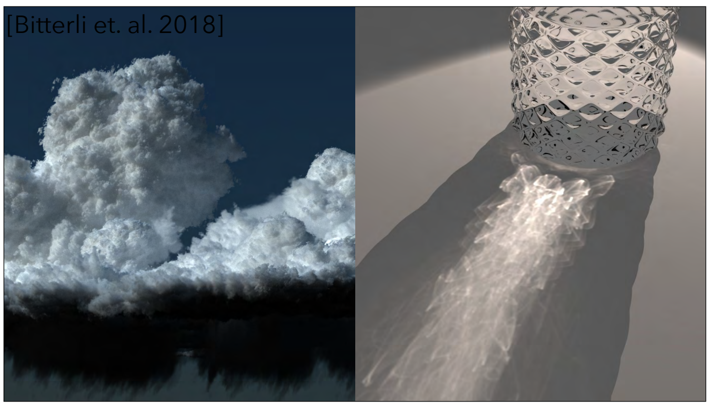



- In the field of light transport it is our goal to accurately simulate complicated visual phenomena such as volumetric media or caustics.

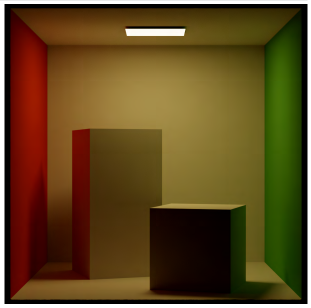
## Unbiased solutions



And ideally, we want solutions which are unbiased.

## Unbiased solutions

$$I$$



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If we denote a ground truth image as the quantity I,

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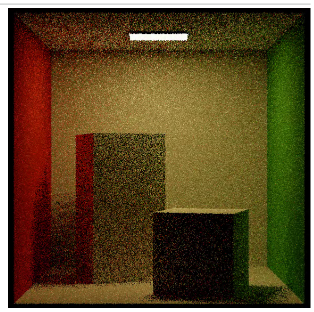
## Unbiased solutions

$$\langle I \rangle$$



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And then denote a stochastic estimator for I, as I surrounded by angle brackets.

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## Unbiased solutions

$$\mathrm{E}\left[\langle I \rangle\right] = I$$

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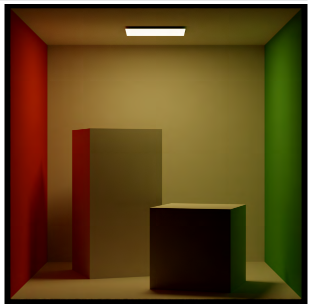
Then an unbiased solution is one whose expected value is always equal to the ground truth.

## Biased solutions

$$\mathrm{E}\left[\langle I \rangle\right] \neq I$$

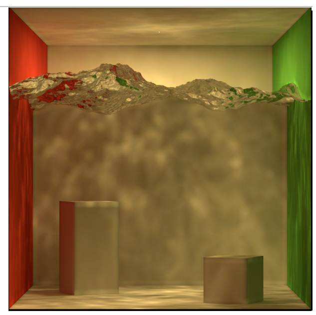




Unfortunately, there are situations in rendering where we do not have unbiased solutions and instead have to fallback on biased ones.

## Biased solutions

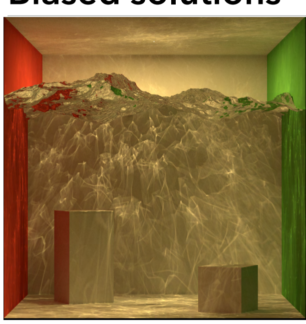
$$\mathrm{E}\left[\langle I \rangle\right] \neq I$$

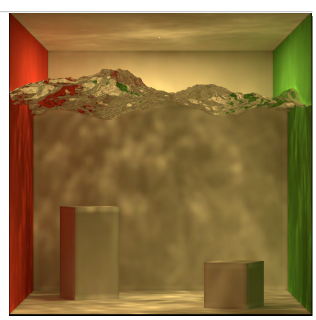




As an example of this consider photon mapping. It will give us a biased version

## Biased solutions







Of the actual ground truth.

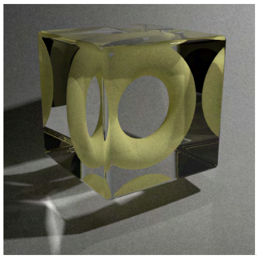
# Motivation

Speaking of photon mapping

## Progressive photon mapping

[Hachisuka et. al. 2008]

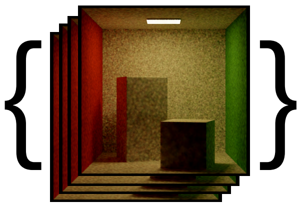
[Knaus et. al. 2011]





there has actually been prior work which eventually gets rid of this bias. The method is known as progressive photon mapping

## Progressive photon mapping

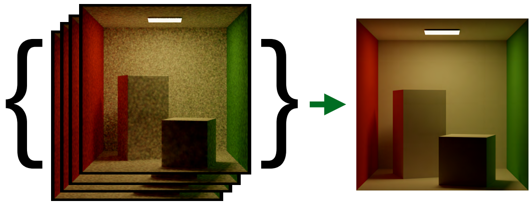


[Knaus et. al. 2011]



It works by using an iterative process which runs many different instances of photon mapping. While each individual instance is biased,

## Progressive photon mapping



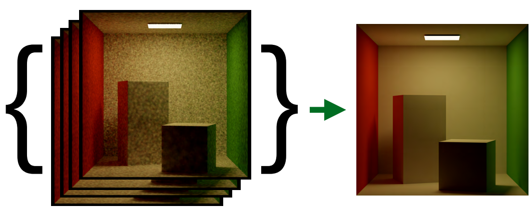
[Knaus et. al. 2011]



When all of them are combined you will eventually get the ground truth.

## Progressive photon mapping

$$\lim_{k \to \infty}$$

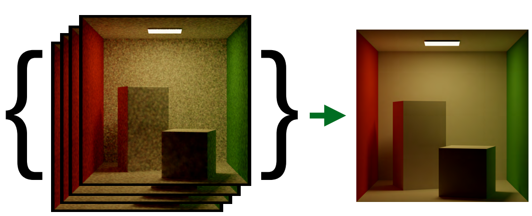


[Knaus et. al. 2011]



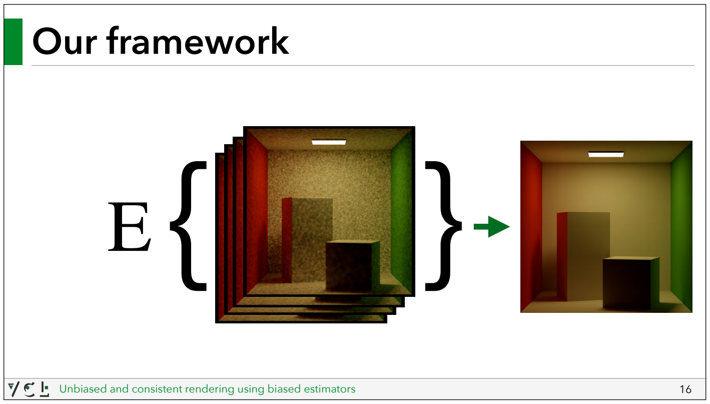
However, we only get the ground truth in the limit of infinite work.

## Our framework

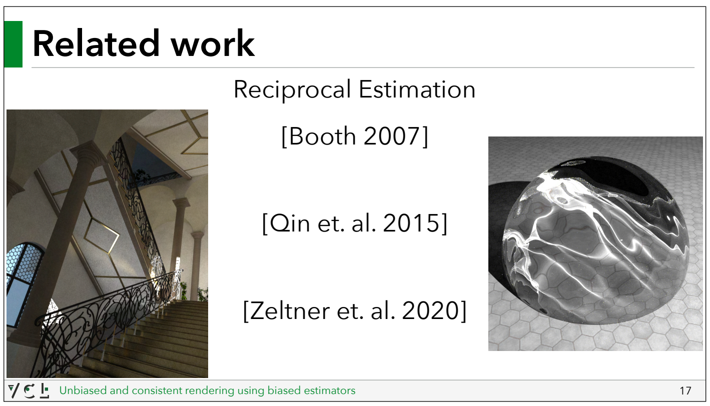




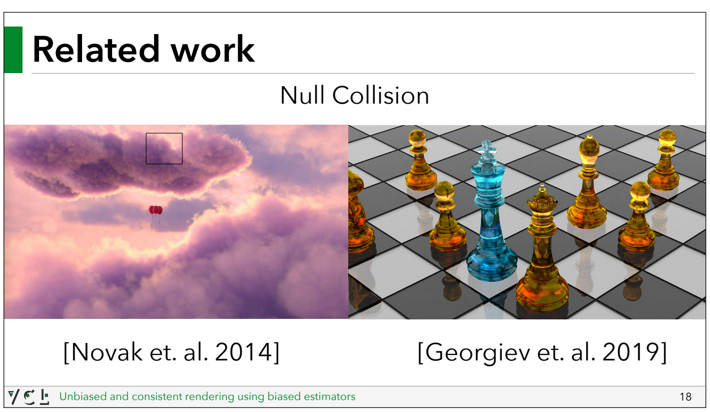
Inspired by this process, we instead propose a framework that takes the results of a bunch of biased instances, and




combines them in such a way that our algorithm is unbiased and always has the correct expected value. In essence we propose a framework for debiasing biased solutions. Our framework is general and can be applied across many different problems in rendering.




One such problem is reciprocal estimation.




While another example is the null collision formulation for volumetric media

## Applicable problems



But, for our framework to be applicable to a problem,

## Applicable problems

$$I(k)$$



We first assume that we have a biased algorithm whose expected result is denoted as I(k). This algorithm has to have a controllable amount of bias, which

## Applicable problems

$$I(\boxed{k})$$



Is directly controlled by the parameter k.

## Applicable problems

$$\lim_{k \to \infty} I(k) = I$$

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As long as the bias vanishes in the limit, then our framework is applicable and we can derive unbiased estimators.

## Path Tracing - max path depth

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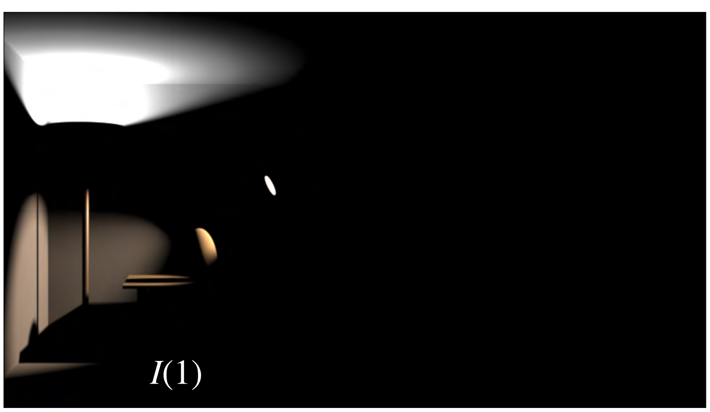
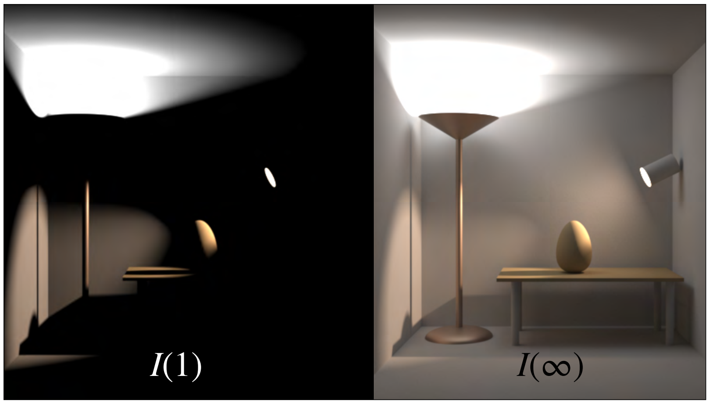
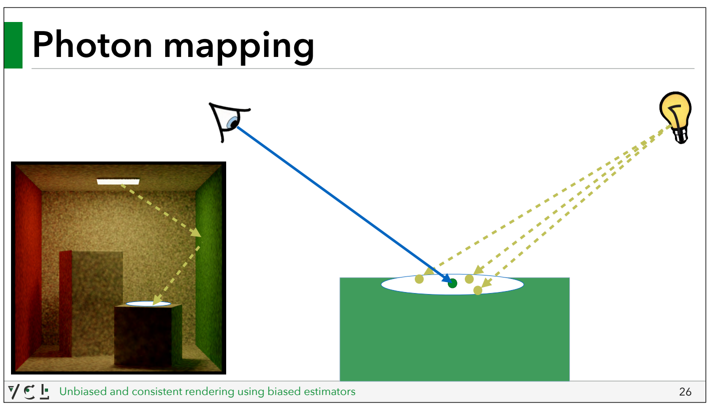
As a simple example of an applicable problem, consider hard-coding the maximum path-depth in a typical path-tracer.



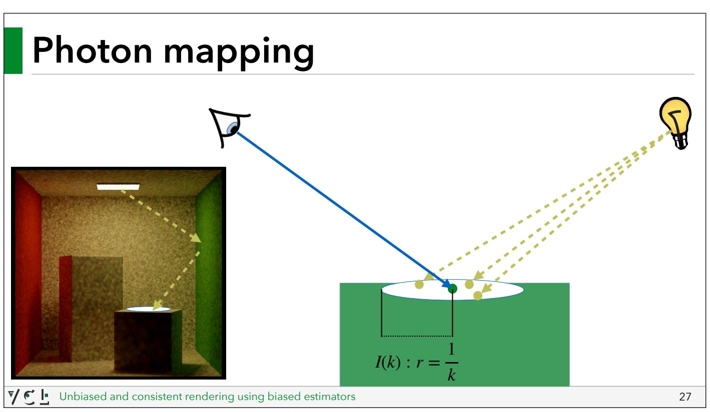

By setting the maximum path-depth equal to 1, we would only visualize direct illumination, which is biased.



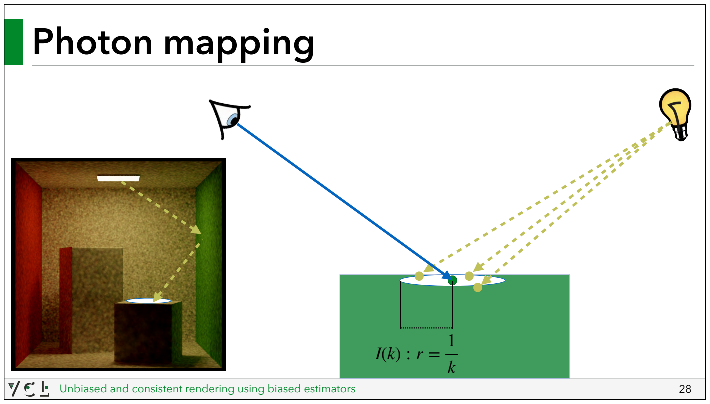

By setting no maximum path depth, we would render the full unbiased image.



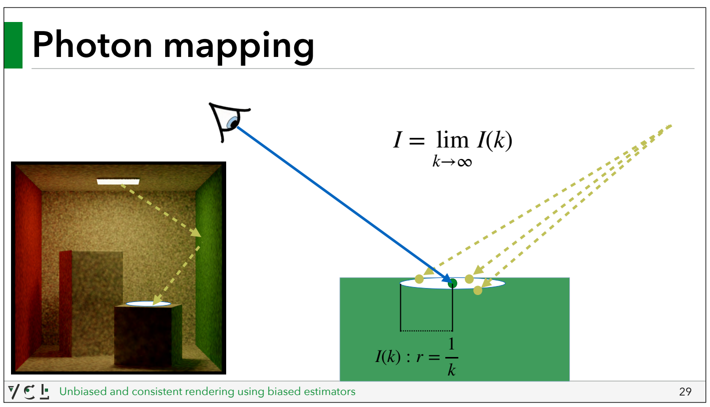

- As a more complicated example, consider photon mapping.



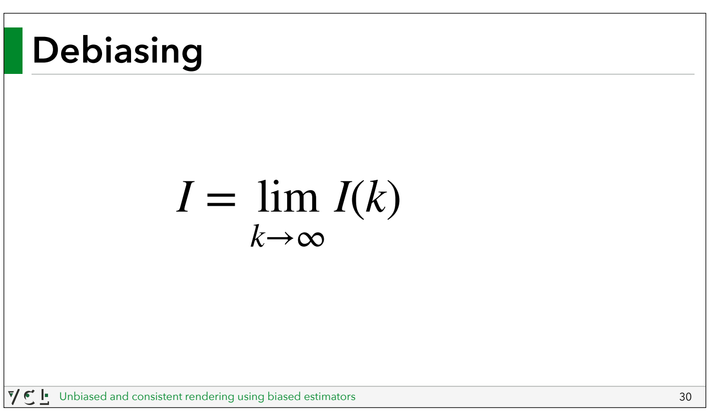

- If we choose to set the radius inversely proportional to k, the amount of bias in photon mapping will




- decrease as k increases.




- And in the limit, photon mapping will theoretically result in the true ground truth image.




- Now to formulate unbiased estimators we need a formulation for this limit that it is amenable to unbiased estimation.

## Debiasing

$$I(\infty) = \lim_{k \to \infty} I(k)$$



- Effectively, we want to solve for I of the limit itself.

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## Debiasing

$$I(\infty) = I(k) + \left[ I(\infty) - I(k) \right]$$



- And, to do this, we first rewrite the problem like so.

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## Debiasing

$$I(\infty) = \boxed{I(k)} + \left[ I(\infty) - \boxed{I(k)} \right]$$



- Trivially, this equation is true as the two instances of I(k) are going to cancel. We choose to rewrite this problem in this way,

## Debiasing

$$\overbrace{I(\infty)}^{\text{GT}} = I(k) + \big[I(\infty) - I(k)\big]$$



- because it redefines the ground truth,

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## Debiasing

$$\overbrace{I(\infty)}^{\text{GT}} = \overbrace{I(k)}^{\text{Biased}} + \big[I(\infty) - I(k)\big]$$



- In terms of the sum of a biased algorithm,

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## Debiasing

$$\overbrace{I(\infty)}^{\text{GT}} = \overbrace{I(k)}^{\text{Biased}} + \overbrace{\big[I(\infty) - I(k)\big]}^{\text{Bias Correction}}$$



- and a bias correction term since I at infinity minus I of k is exactly equal to the bias.

## Debiasing

GT | Biased | Bias Correction

$$I(\infty) = I(k) + \left[ I(\infty) - I(k) \right]$$



- Now, this form still can not be estimated because we assume we cannot directly evaluate I at infinity.

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## Debiasing

GT | Biased | Bias Correction

$$I(\infty) = I(k) + \left[ I(\infty) - I(k) \right]$$

$$I(\infty) = I(k) + \sum_{j=k}^{\infty} \left[ I(j+1) - I(j) \right]$$



- To get around this, we reformulate the bias correction term into an infinite telescoping series.

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## Debiasing

$$I(\infty) = I(k) + \sum_{j=k}^{\infty} \left[ I(j+1) - I(j) \right]$$



- With this infinite sum, we can choose to evaluate it using Monte Carlo.

## Debiasing

$$I(\infty) = I(k) + \frac{I(j+1) - I(j)}{p(j)}$$



- Which involves randomly choosing to evaluate a single term of the infinite sum based on some probability mass function.

## Debiasing

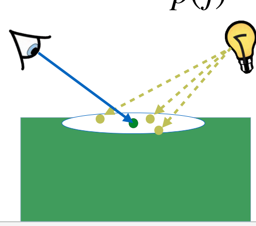
$$\langle I(\infty)\rangle = \langle I(k)\rangle + \frac{\langle I(j+1)\rangle - \langle I(j)\rangle}{p(j)}$$



- Finally, by replacing each instance of I with an estimator, we effectively arrive at an unbiased estimator for the ground truth which only evaluates biased estimates of the original problem.

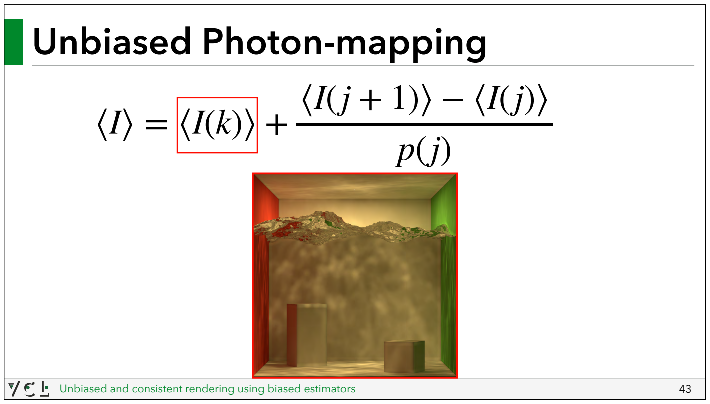
## Unbiased Photon-mapping

$$\langle I\rangle = \langle I(k)\rangle + \frac{\langle I(j+1)\rangle - \langle I(j)\rangle}{p(j)}$$

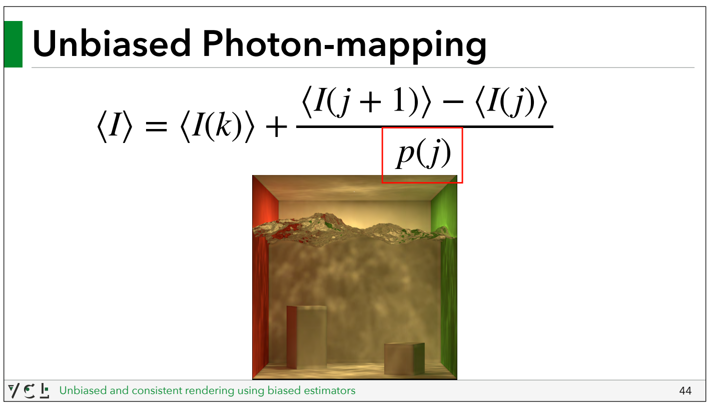




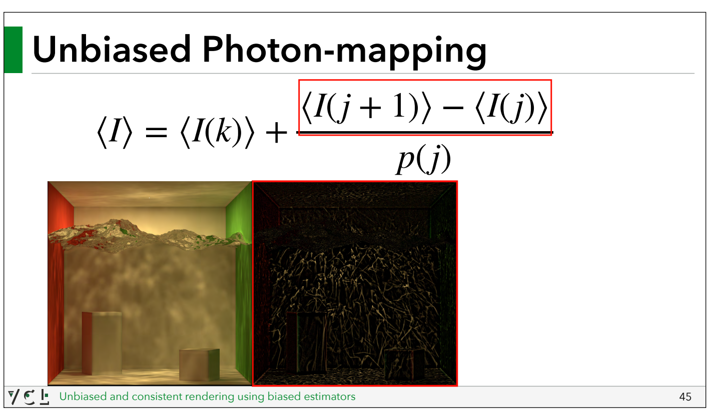
- As an example of how to apply this estimator in practice, lets walk through how an unbiased photon-mapping estimator would work,




- We first estimate I of k, by running photon mapping using some base radius with let's say 1 million photons.



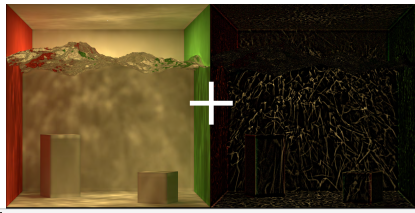

- Then we choose a term, j, according to some probability mass function.




- Next, we take the difference of two different instances of photon mapping which use two different radii. This will give us an unbiased estimate of the bias.

## Unbiased Photon-mapping

$$\langle I \rangle = \langle I(k) \rangle + \frac{\langle I(j+1) \rangle - \langle I(j) \rangle}{p(j)}$$

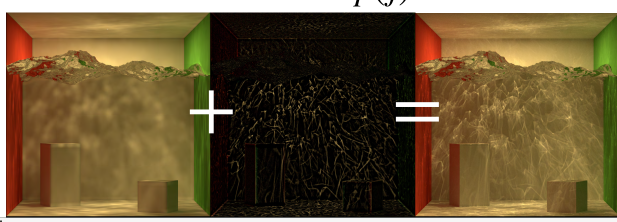




- By combining these two estimates

## Unbiased Photon-mapping

$$\langle I \rangle = \langle I(k) \rangle + \frac{\langle I(j+1) \rangle - \langle I(j) \rangle}{p(j)}$$

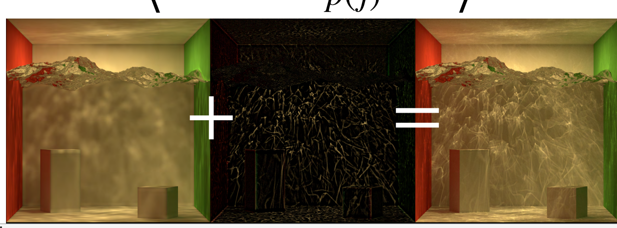




- We arrive at an unbiased result.

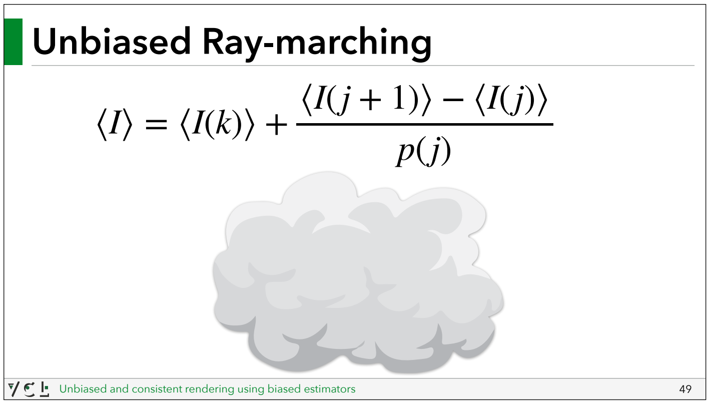
## Unbiased Photon-mapping

$$\langle I \rangle = \left\langle I(k) + \frac{I(j+1) - I(j)}{p(j)} \right\rangle$$

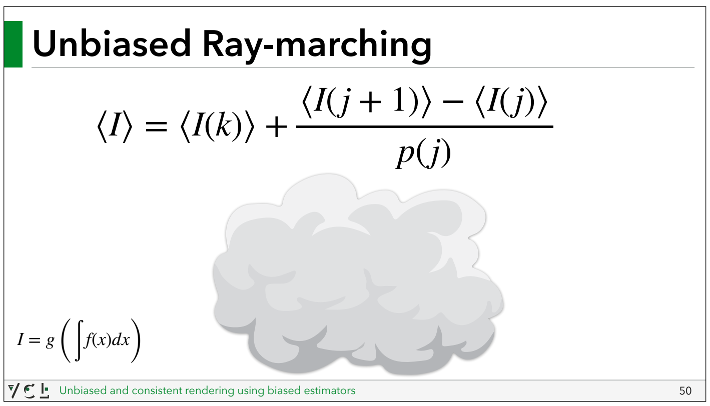




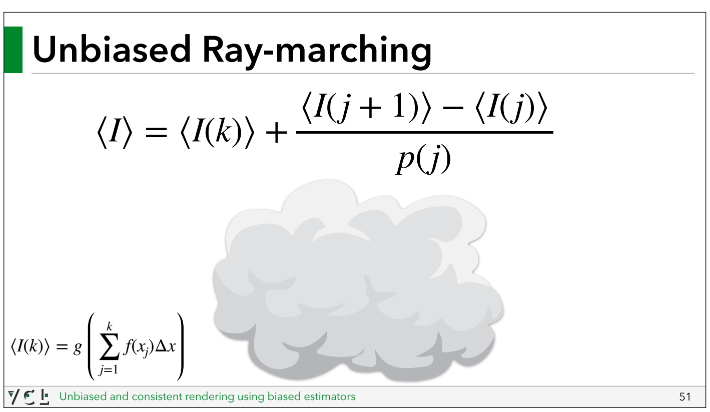
- To improve the convergence rate of this estimator, we can utilize correlations by sharing the same set of photons between all three photon mapping estimates.



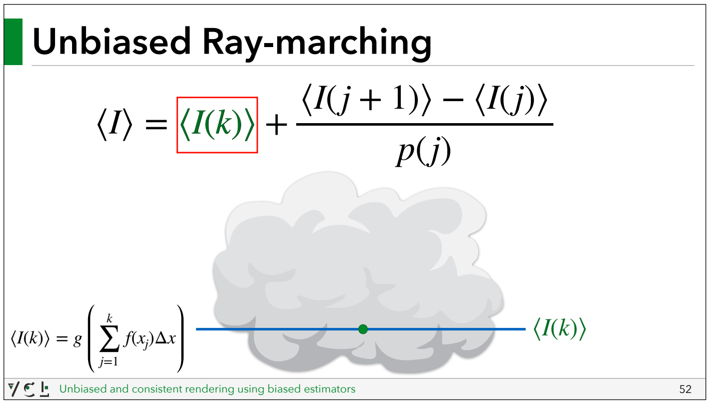

- We can also do something similar for estimating transmittance with ray-marching.



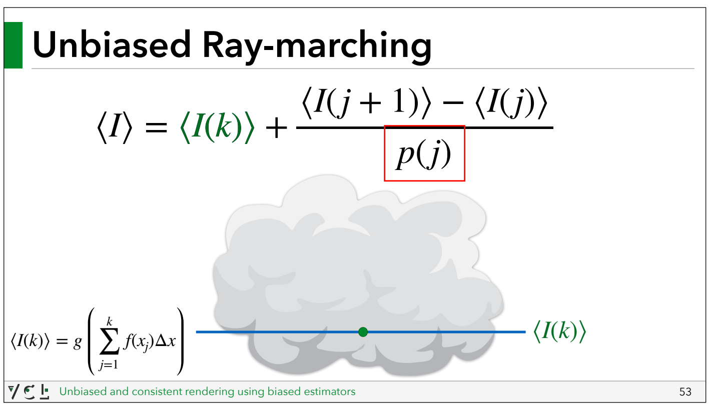

- Transmittance estimation involves estimating an integral over the medium density that is then modified by some non-linear function.



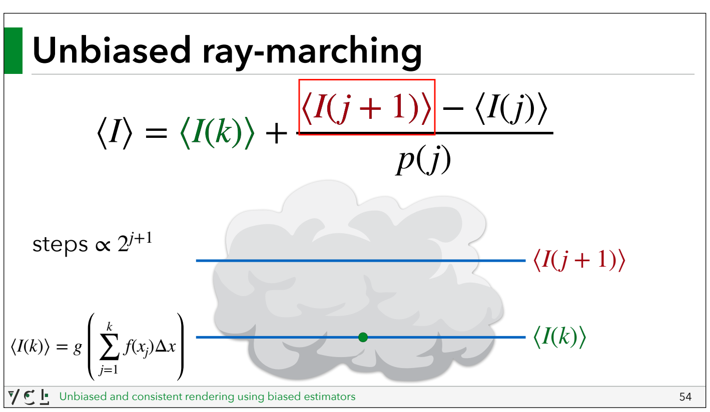

- Ray-marching effectively approximates transmittance by replacing the integral with a Riemann sum approximation. For any finite k, ray-marching will most likely be biased.



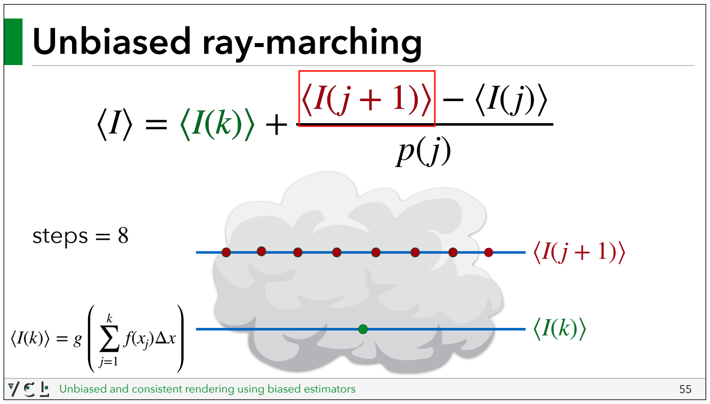

- A debiased version would first run ray-marching using a fairly large step size. The dot here represents where we evaluate the medium density.



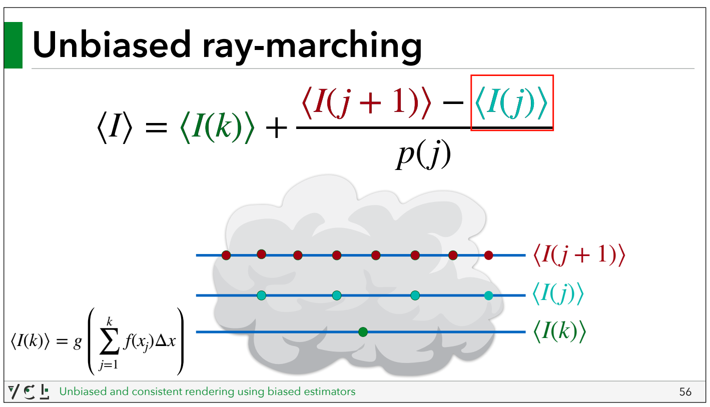

- Then we choose a term, j according to some probability mass function.




- Next we run ray marching again, except this time using a number of steps proportional to 2 raised to the j + 1 power.



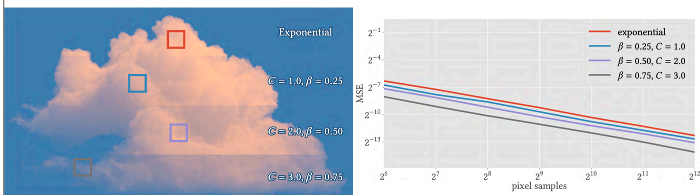

- If we choose j equal to 2, we take a total of eight steps.




- We then repeat the same process for I of j, using use half of the number of steps. Similar to photon mapping, you can also take advantage of correlations, however, for brevity we will skip that.


- Now on to some results

## Transmittance estimation




[Bitterli et. al. 2018]



- Our unbiased ray-marching algorithm is the first general unbiased transmittance estimator that supports any valid transmittance function. This is demonstrated by rendering a cloud using the non-exponential Davis model which was introduced to graphics in prior work. On the right we show convergence rates of our unbiased estimator for different transmittance parameters and also show that our estimator has the expected Monte Carlo convergence rates.

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## Probability mass function

$$\langle I \rangle = \langle I(k) \rangle + \frac{\langle I(j+1) \rangle - \langle I(j) \rangle}{p(j)}$$



- Up until now, we have haven't mentioned how to choose a probability mass function. While we refer you to the paper for a recipe on how to do this, it is worth mentioning that for any valid probability mass function,

## Probability mass function

$$\langle I \rangle = \langle I(k) \rangle + \frac{\langle I(j+1) \rangle - \langle I(j) \rangle}{p(j)}$$

$$\mathrm{E}\left[\langle I \rangle\right] = I$$



- The expected value will always be correct as long as bias vanishes in the limit.

## Probability mass function

$$\langle I \rangle = \langle I(k) \rangle + \frac{\langle I(j+1) \rangle - \langle I(j) \rangle}{p(j)}$$

$$\mathrm{E}\left[\langle I \rangle\right] = I \qquad \mathrm{V}\left[\langle I \rangle\right] = \infty$$

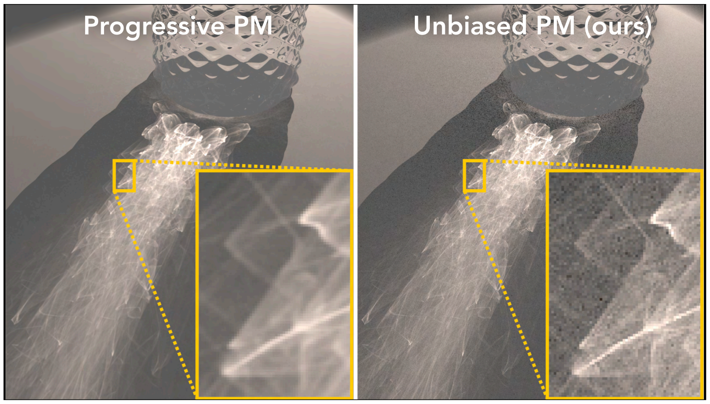


- However, variance can turn out to be infinite, but that does not stop us from having unbiased estimators. It only result in slower convergence rates.

## Photon mapping



- Photon mapping is an example of a problem which we prove in the paper will always have infinite variance. Despite this, we still derive an unbiased algorithm.




- In this scene we compare progressive photon mapping to our unbiased estimator for an equal number of photons. Our unbiased estimator effectively trades noise for bias in the form of blurring.

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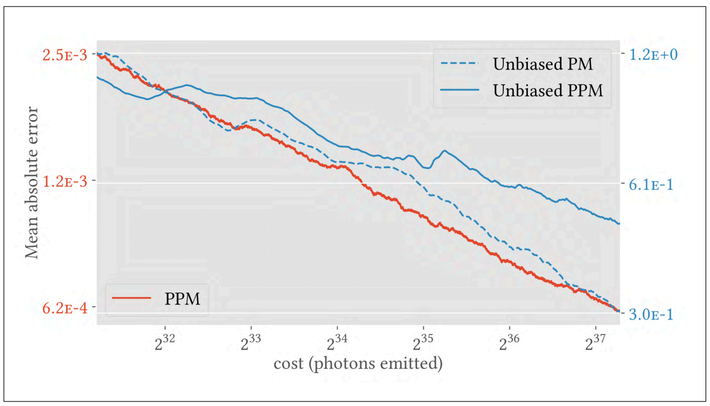


Here we plot the relative convergence rates for our unbiased methods versus progressive photon mapping. While the error of our methods is greater, if you look at the blue dashed line versus the red solid line you will see that the convergence rate, or slope, of our unbiased photon mapping algorithm is overall fairly identical to progressive photon mapping.

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## Additional Contributions

- Recipe
- Taylor series
- Infinite variance
- Finite differences



Additionally, In the paper (click) we outline a general recipe for deriving estimators for any applicable problem. (click) We also discuss another estimation technique which utilizes a Taylor series expansion. (click) We provide a more in depth discussion on the infinite variance case. (Click) And we also provide unbiased and progressive finite differences.

Thank you!

Thank you!