Progressive null-tracking for volumetric rendering supplemental

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ACM Reference Format:
Zackary Misso, Yining Karl Li, Brent Burley, Daniel Teece, and Wojciech Jarosz. 2023. Progressive null-tracking for volumetric rendering supplemental. In *Special Interest Group on Computer Graphics and Interactive Techniques Conference Conference Proceedings (SIGGRAPH '23 Conference Proceedings), August 6–10, 2023, Los Angeles, CA, USA.* ACM, New York, NY, USA, 2 pages. https://doi.org/10.1145/3588432.3591557

1 PROOF OF CONSISTENCY

In this section we prove the consistency of our progressive formulation,

$$\langle I \rangle = \frac{1}{n} \sum_{k=1}^{n} \left\langle I\left(\mu_t^{(k)}\right)\right\rangle, \tag{1}$$

where I represents a fully converged image containing some participating medium, $\langle I \rangle$ is an estimate for that image, and $\langle I(\mu_t^{(k)})\rangle$ corresponds to a single pixel sample evaluation using a specified combined extinction $\mu_t^{(k)}$. We choose to clamp the real density whenever the combined extinction $\mu_t^{(k)}$ is non-bounding, and we update the combined extinctions used for subsequent iterations based on the occurrences of non-bounding events as described in the main paper.

Clamping results in bias, however, we make the assumption that after some finite number (j) of iterations where $j < n$, the subsequent combined extinctions $\mu_t^{(k)}$ for $k > j$ will all become bounding. Meaning, for all indices $k > j$, $\langle I(\mu_t^{(k)})\rangle$ will be an unbiased estimate for the true solution I.

Based on this assumption, let us first write the problem in terms of expected values instead of estimators,

$$I \approx \frac{1}{n} \sum_{k=1}^{n} I\left(\mu_t^{(k)}\right), \tag{2}$$

and then split the sum by the first j biased iterations and $n - j - 1$ unbiased iterations,

$$I \approx \frac{1}{n}\left(\left[\sum_{k=1}^{j} I\left(\mu_t^{(k)}\right)\right] + \left[\sum_{k=j+1}^{n} I\left(\mu_t^{(k)}\right)\right]\right). \tag{3}$$

Let us denote β_k to be equal to the bias for iteration k. Then our modified expected value becomes,

$$I \approx \frac{1}{n}\left(\left[\sum_{k=1}^{j} I + \beta_k\right] + \left[\sum_{k=j+1}^{n} I\right]\right). \tag{4}$$

Combining like terms we arrive at,

$$I \approx \frac{1}{n}\left(nI + \sum_{k=1}^{j} \beta_k\right), \tag{5}$$

which simplifies to,

$$I \approx I + \frac{1}{n} \sum_{k=1}^{j} \beta_k. \tag{6}$$

For simplicity we will denote the total bias as the constant $B = \sum_{k=1}^{j} \beta_k$,

$$I \approx I + \frac{B}{n}. \tag{7}$$

Since B is a constant it should be obvious that our progressive formulation, and any progressive formulation where only the first j finite iterations are biased, remain consistent since,

$$\lim_{n\to\infty} \frac{B}{n} = 0, \tag{8}$$

the bias disappears in the limit.

2 ASYMPTOTIC ANALYSIS OF MSE CONVERGENCE

Since our goal is to propose a technique for production rendering, one major consideration is how our progressive formulation and the introduction of bias will impact the performance in production scenes. While we do not yet have an extensive practical suite of scenes showcasing our method, we can perform some asymptotic analysis to convey how we would expect our method to generally perform.

We choose to quantify performance by using the mean squared error. Mean squared error can be defined as the sum of the variance

SIGGRAPH '23 Conference Proceedings, August 6–10, 2023, Los Angeles, CA, USA

ACM ISBN 979-8-4007-0159-7/23/08…$15.00
https://doi.org/10.1145/3588432.3591557

and squared bias,

$$MSE[\langle I\rangle] = Var[\langle I\rangle] + Bias[\langle I\rangle]^2, \tag{9}$$

of an estimator. Most prior null-scattering techniques are unbiased, meaning their mean squared error would only be dependent upon their variance. Since our progressive technique introduces bias we will first analyze the asymptotic behavior of its bias and variance independently.

2.1 Asymptotic behaviour of bias

Since we are analyzing the asymptotic behaviour, we choose to only consider the cases where we have already reached the point where all future iterations $j < n$ will be unbiased, meaning we have already converged to bounding combined extinctions. We make this assumption, because once $n > j$, the total bias $Bias[\langle I\rangle] = \frac{B}{n}$, will behave in a predictable manner,

$$Bias[\langle I\rangle] = O(n^{-1}), \tag{10}$$

as we showed in Eq. (7). This means that,

$$Bias[\langle I\rangle]^2 = O(n^{-2}), \tag{11}$$

the squared bias disappear at a rate of $O(n^{-2})$ as more pixel samples are evaluated.

2.2 Asymptotic behaviour of variance

The variance of our progressive estimator Eq. (1),

$$Var[\langle I\rangle] = Var\left[\frac{1}{n}\sum_{k=1}^{n}\left\langle I\left(\mu_t^{(k)}\right)\right\rangle\right] \tag{12}$$

$$Var[\langle I\rangle] = \frac{1}{n^2}\sum_{k=1}^{n} Var\left[\left\langle I\left(\mu_t^{(k)}\right)\right\rangle\right], \tag{13}$$

will simplify to the sum of the iteration variances assuming each iterations is evaluated independently from one another. However, the variances of the iterations will not all be the same since the first j iterations might use different majorants, while the variances of the last $(n - j - 1)$ iterations will be identical,

$$Var[\langle I\rangle] = \frac{1}{n^2}\left((n-j-1)Var\left[\left\langle I\left(\mu_t^{(n)}\right)\right\rangle\right] + \sum_{k=1}^{j} Var\left[\left\langle I\left(\mu_t^{(k)}\right)\right\rangle\right]\right). \tag{14}$$

So let us define V^+ such that,

$$V^+ := max\left\{Var\left[\left\langle I\left(\mu_t^{(k)}\right)\right\rangle\right] : k = 1...n\right\}. \tag{15}$$

Assuming that the variance of every iteration is finite then,

$$Var[\langle I\rangle] < \frac{1}{n^2}\left(nV^+\right) \tag{16}$$

$$Var[\langle I\rangle] < \frac{V^+}{n} \tag{17}$$

$$Var[\langle I\rangle] \approx O(n^{-1}), \tag{18}$$

which is the typical Monte Carlo convergence rate for variance.

2.3 Asymptotic behaviour of MSE

Given both the asymptotic convergence rates for the variance Eq. (18) and squared bias Eq. (11), we can compute the asymptotic convergence for the MSE,

$$MSE[\langle I\rangle] = Var[\langle I\rangle] + Bias[\langle I\rangle]^2, \tag{19}$$

$$MSE[\langle I\rangle] \approx O(n^{-1}) + O(n^{-2}), \tag{20}$$

$$MSE[\langle I\rangle] \approx O(n^{-1}). \tag{21}$$

Thus, even though we introduce bias through the use of our progressive technique, the mean squared error will eventually be dominated by the variance instead of the bias. Once our majorants have converged to become bounding, the squared bias will converge at rate which is an order of magnitude faster than the variance.

This implies that in practice, it would be ideal to initialize the guesses for the bounding majorants and use updating strategies which learn bounding majorants quickly.