A BIDIRECTIONAL FORMULATION FOR WALK ON SPHERES

Yang Qi, Dario Seyb, Benedikt Bitterli, Wojciech Jarosz

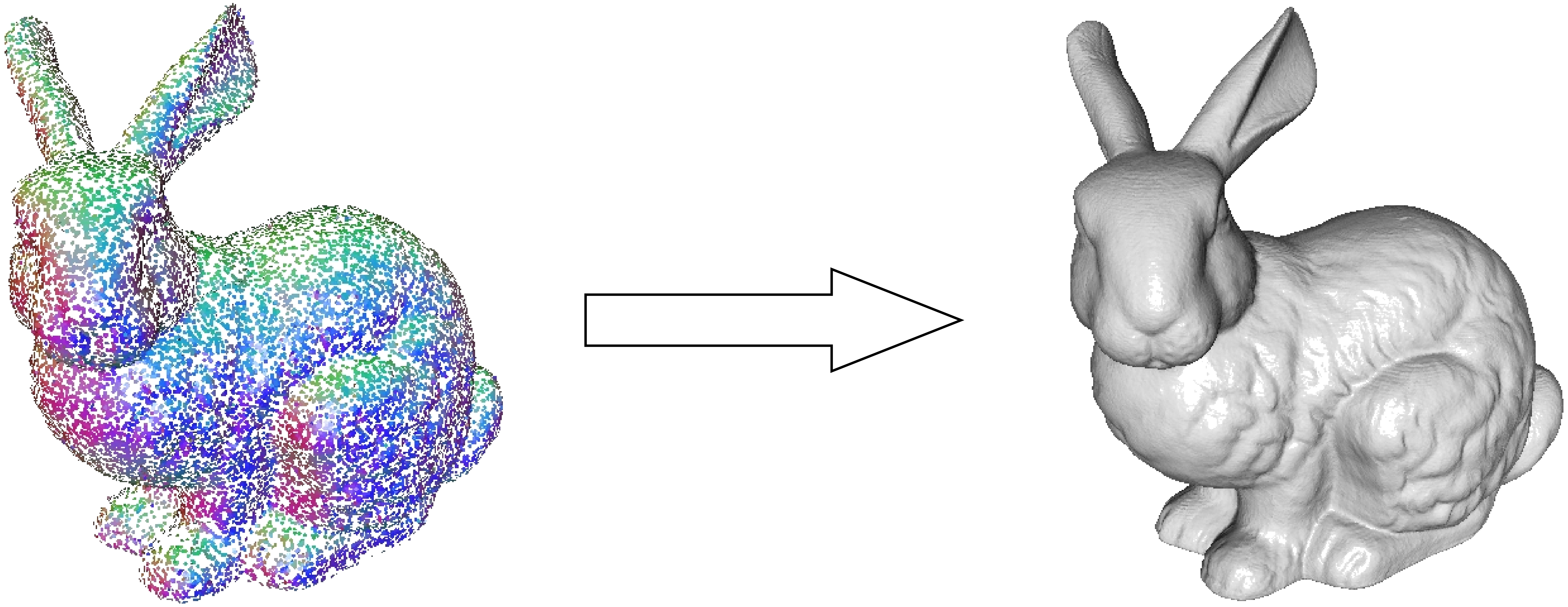
DARTMOUTH
VISUAL COMPUTING LAB

Diffusion curves

- Orzan, A. Bousseau, A. Winnemöller, H. Barla, P. Thollot, J. and Salesin, D. *"Diffusion curves: a vector representation for smooth-shaded images"* *(SIGGRAPH 2008)*
- Bowers, J. Leahey, J. and Wang, R. *"A Raytracing Approach to Diffusion Curve"* *(EGSR 2009)*
- Prévost, R. Jarosz, W. Sorkine-Hornung, O. *"A Vectorial Framework for Ray Traced Diffusion Curve"* (Computer Graphics Form 2015)
- …

Surface reconstruction

- Kazhdan, M. Bolith, M. and Hoppe, H.
  “Poisson Surface Reconstruction” (SGP 2006)
- Kazhdan, M. and Hoppe, H.
  “Screened Poisson surface reconstruction” (SIGGRAPH 2013)
- …



https://hhoppe.com/proj/screenedpoisson/

Subsurface Scattering

- Joe Stam *"Multiple scattering as a diffusion process" (EGSR 1995)*
- Christensen, N. and Jensen, H.
  "A Practical Guide to Global Illumination using Photon maps" (SIGGRAPH 2000)
- d'Eon, E. Irving, G.
  "A Quantized-Diffusion Model for Rendering Translucent Materials" (SIGGRAPH 2011)
- …

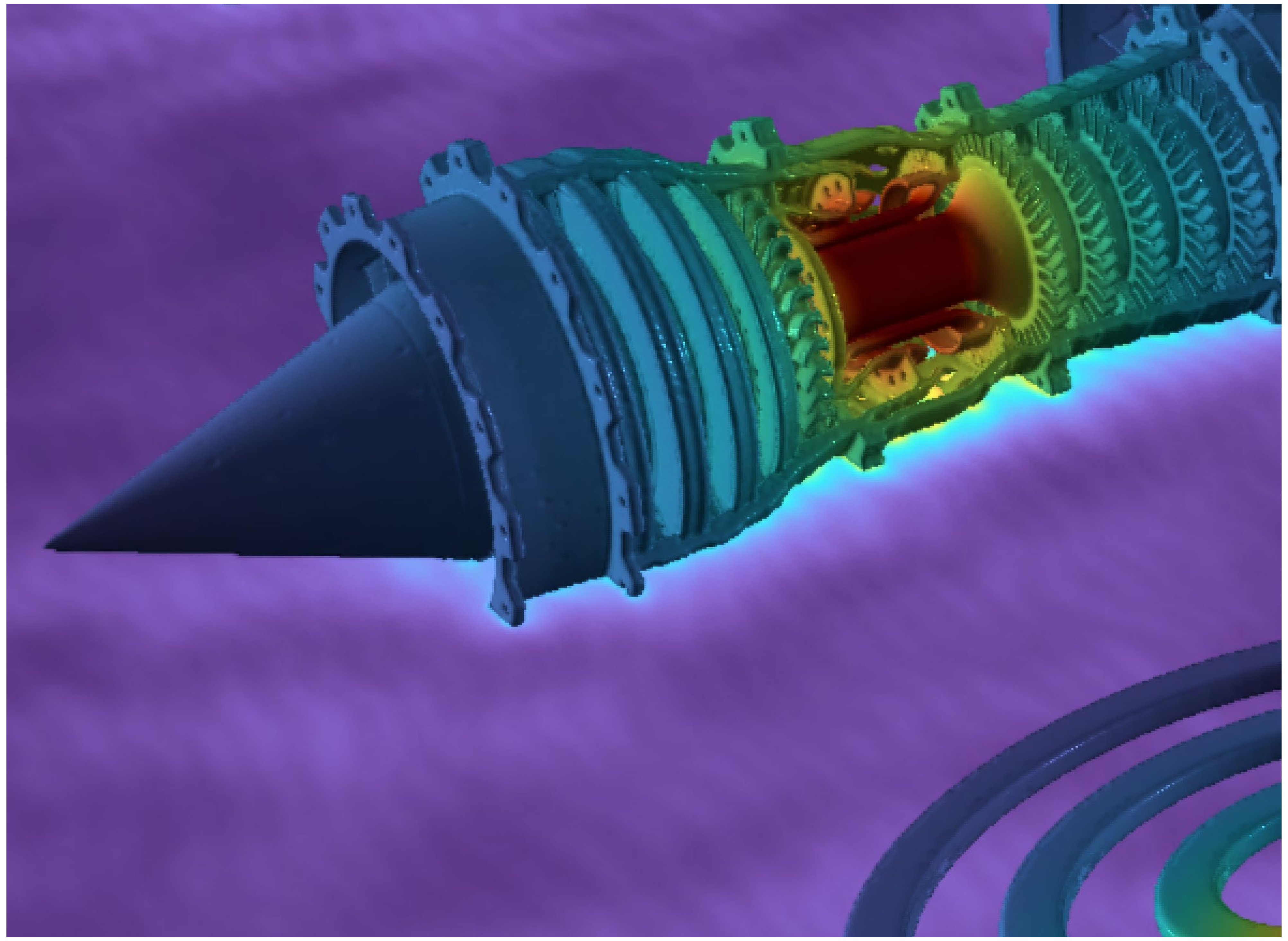
https://graphics.stanford.edu/courses/cs348b-00/course8.pdf

https://naml.us/paper/deon2011_subsurface.pdf

PDE Solver

Traditional PDE solver

- Finite element method
- Finite difference method



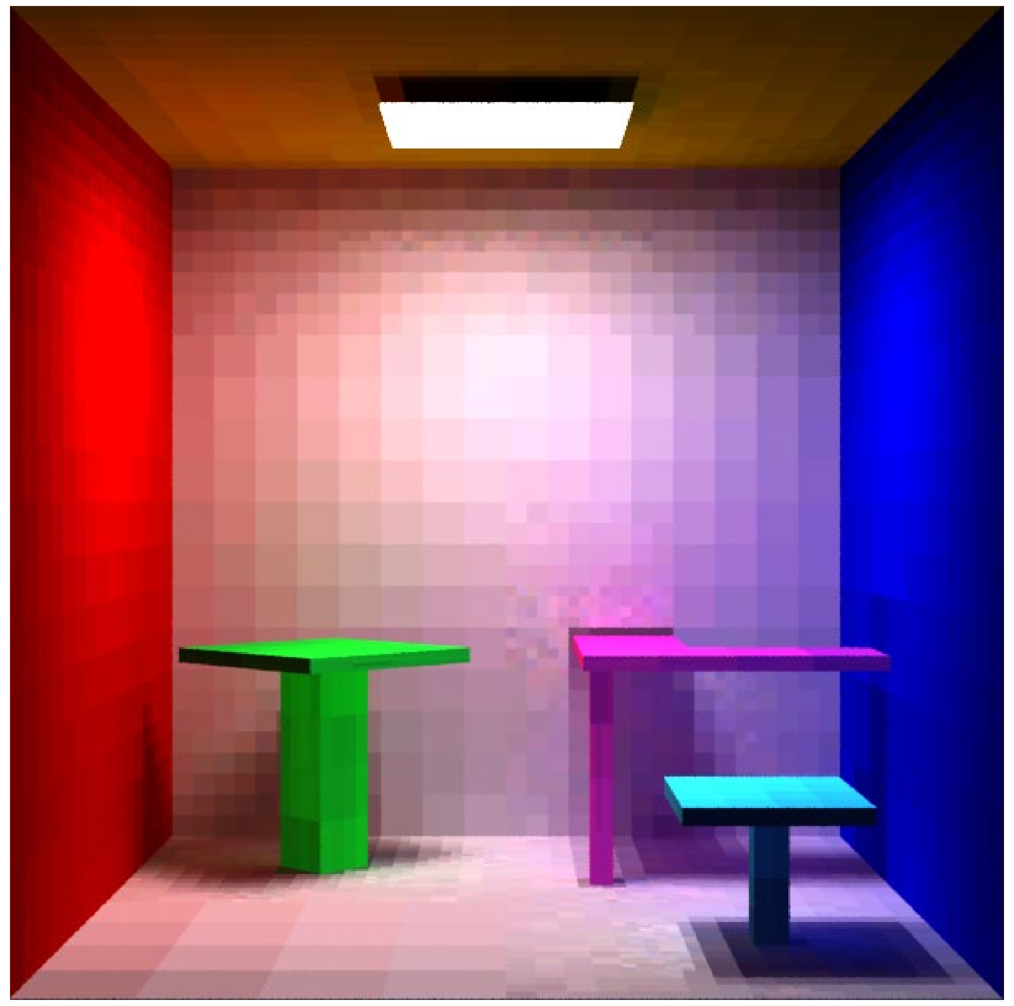
cs.dartmouth.edu/wjarosz/publications/sawhneyseyb22gridfree.html

Meshing is costly for complex shape

Methods in Rendering

Mesh based method

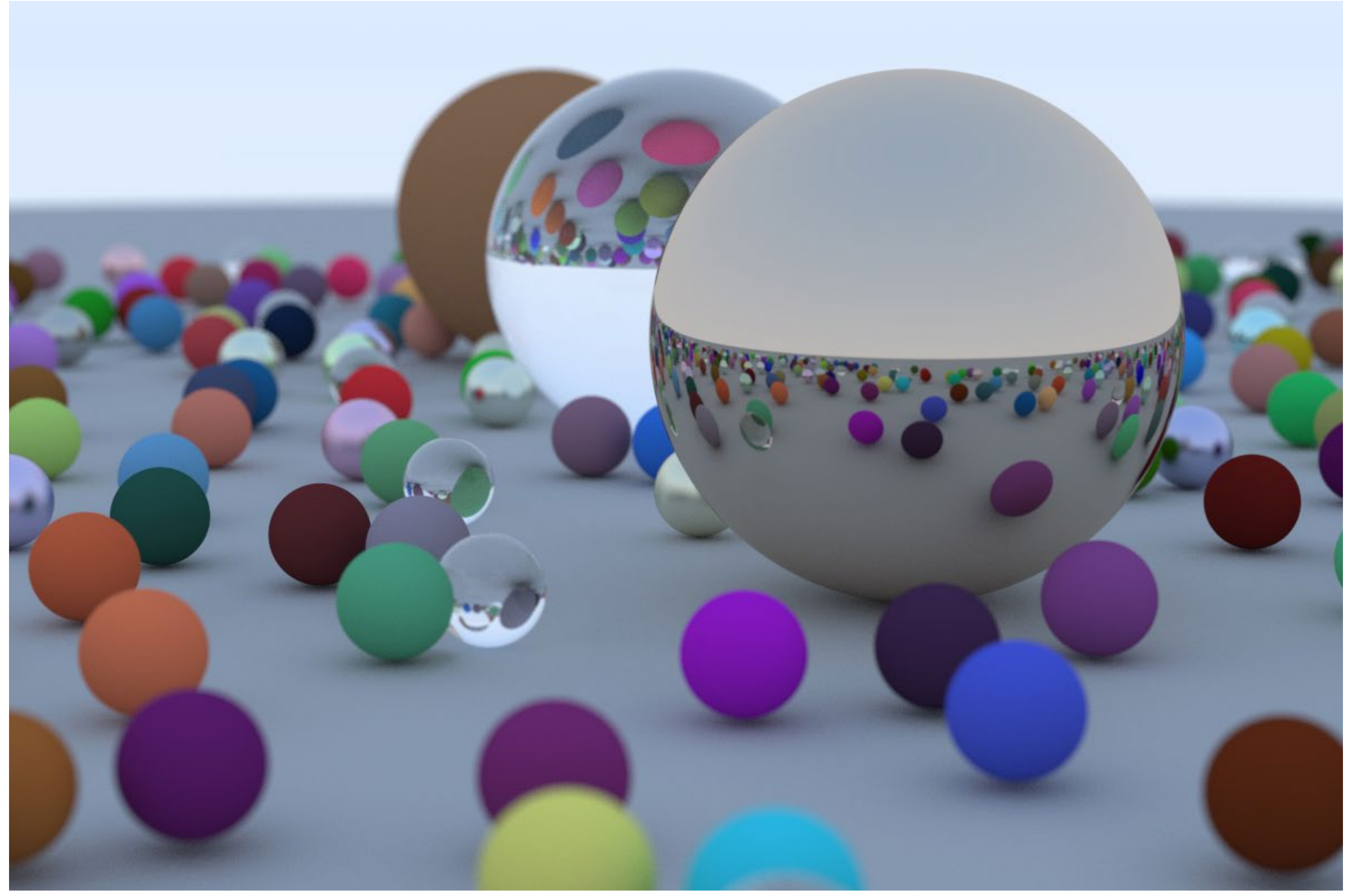
- *"Radiosity and Realistic Image Synthesis"*
  Wallace, J. and Cohen, M. (1993)
- …



www.cg.tuwien.ac.at/research/rendering/rays-radio/

Monte-Carlo method

- *"The Rendering Equation"*
  James T. Kajiya. (1986)
- …



raytracing.github.io/books/RayTracingInOneWeekend.html

Walk on Spheres (WoS)

- Mervin E. Muller *(Ann. Math. Statist. 1956)*
  "Some continuous monte carlo methods for the Dirichlet problem"

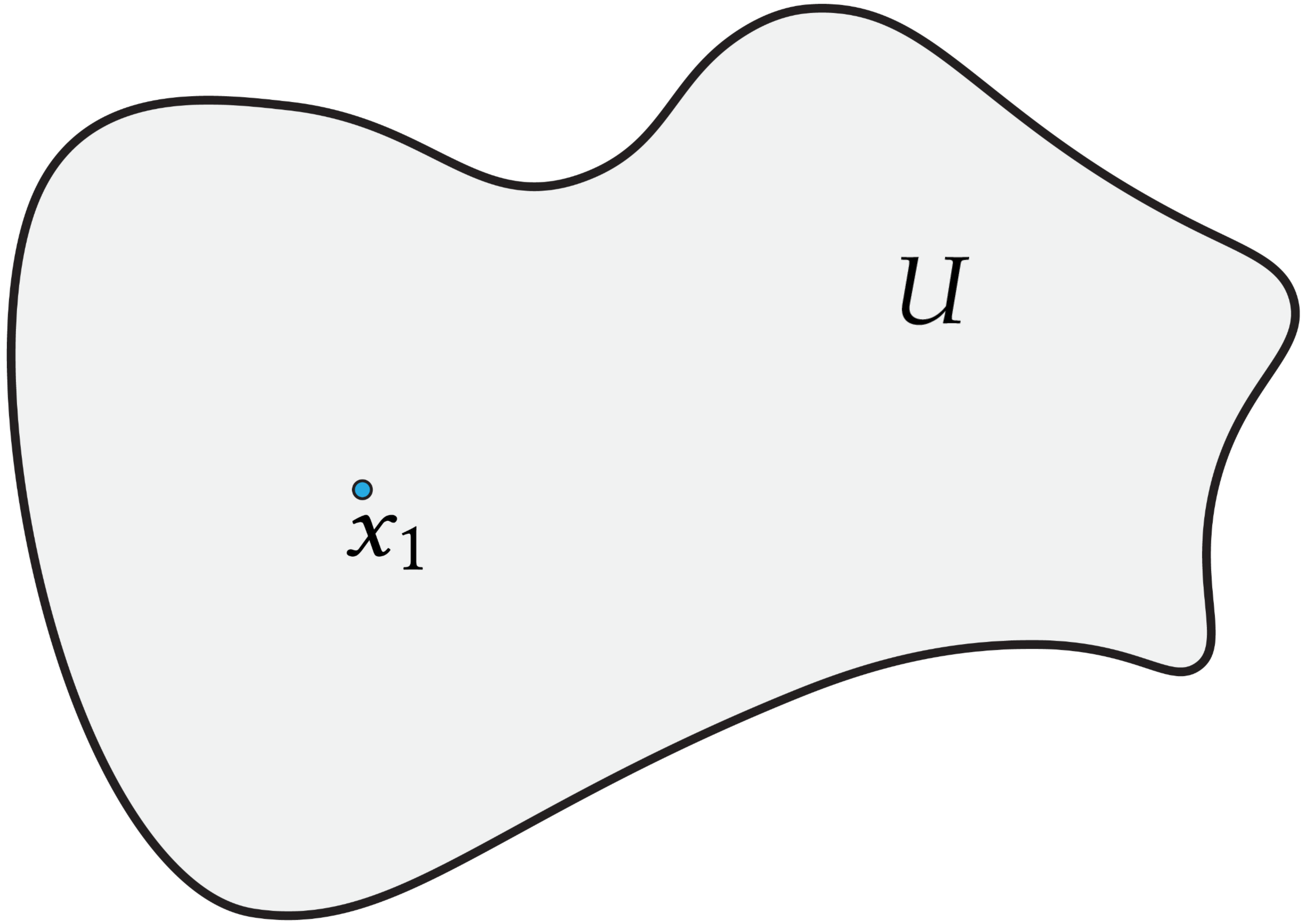
- Sawhney, R. and Crane, K. *(SIGGRAPH 2020)*
  "Monte Carlo geometry processing"

- Sawhney, R. Seyb, D. Jarosz, W. and Crane, K *(SIGGRAPH 2020)*
  "Grid free Monte Carlo for PDEs with spatially varying coefficients"

Walk on Spheres (WoS)

$$u(x) = g(x) \quad \text{if } x \in \partial U$$

How to solve u(x) inside U?



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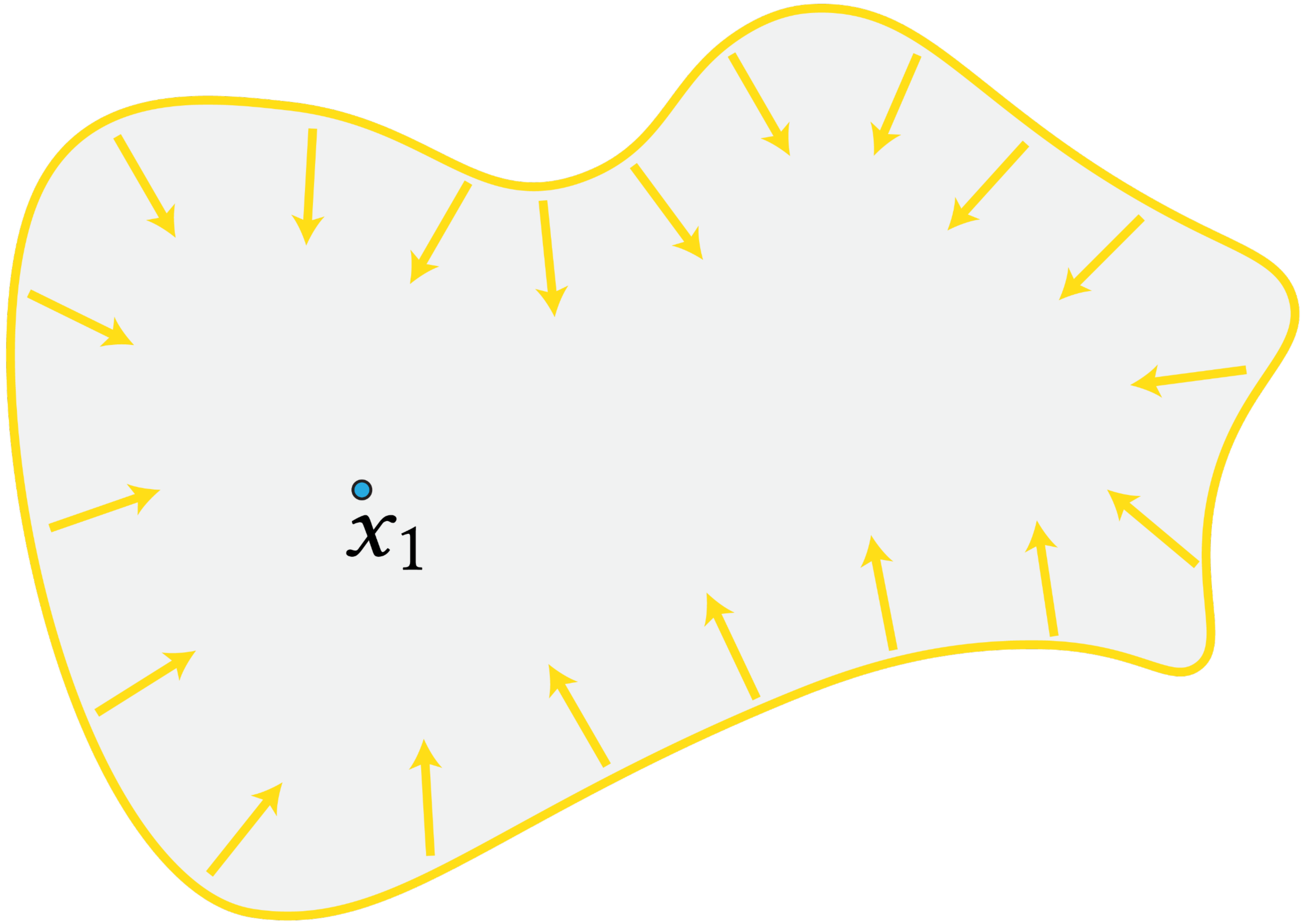
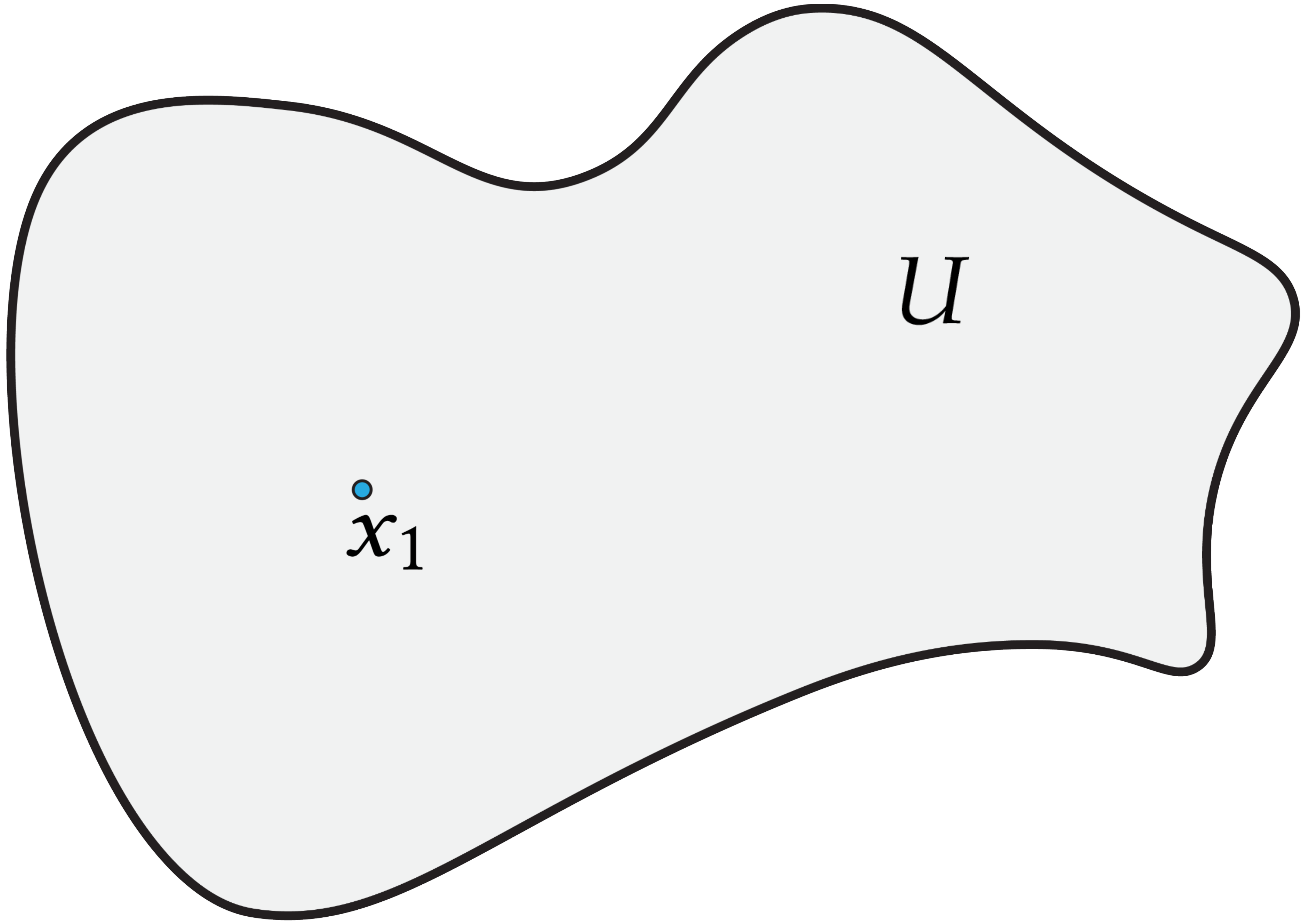


Walk on Spheres (WoS)

Mean value theorem:

$$u(x) = \frac{1}{|\partial B_x|} \int_{\partial B_x} u(x') \, \mathrm{d}x'$$



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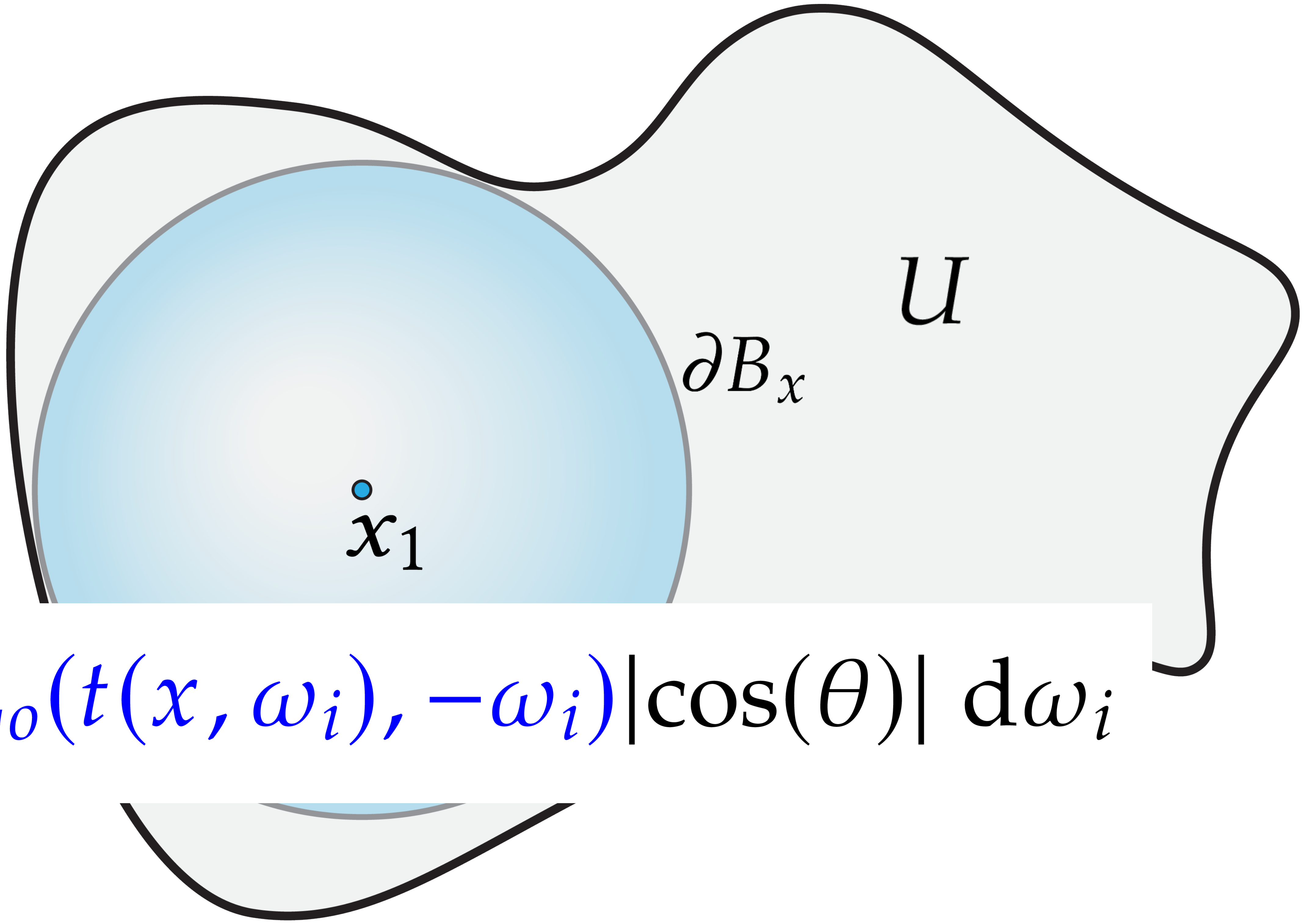


Rendering Equation[Kaj86]:

$$L_o(x, \omega_o) = L_e(x, \omega_o) + \int_{S^2} f(x, \omega_o, \omega_i) L_o(t(x, \omega_i), -\omega_i) |\cos(\theta)|\, \mathrm{d}\omega_i$$

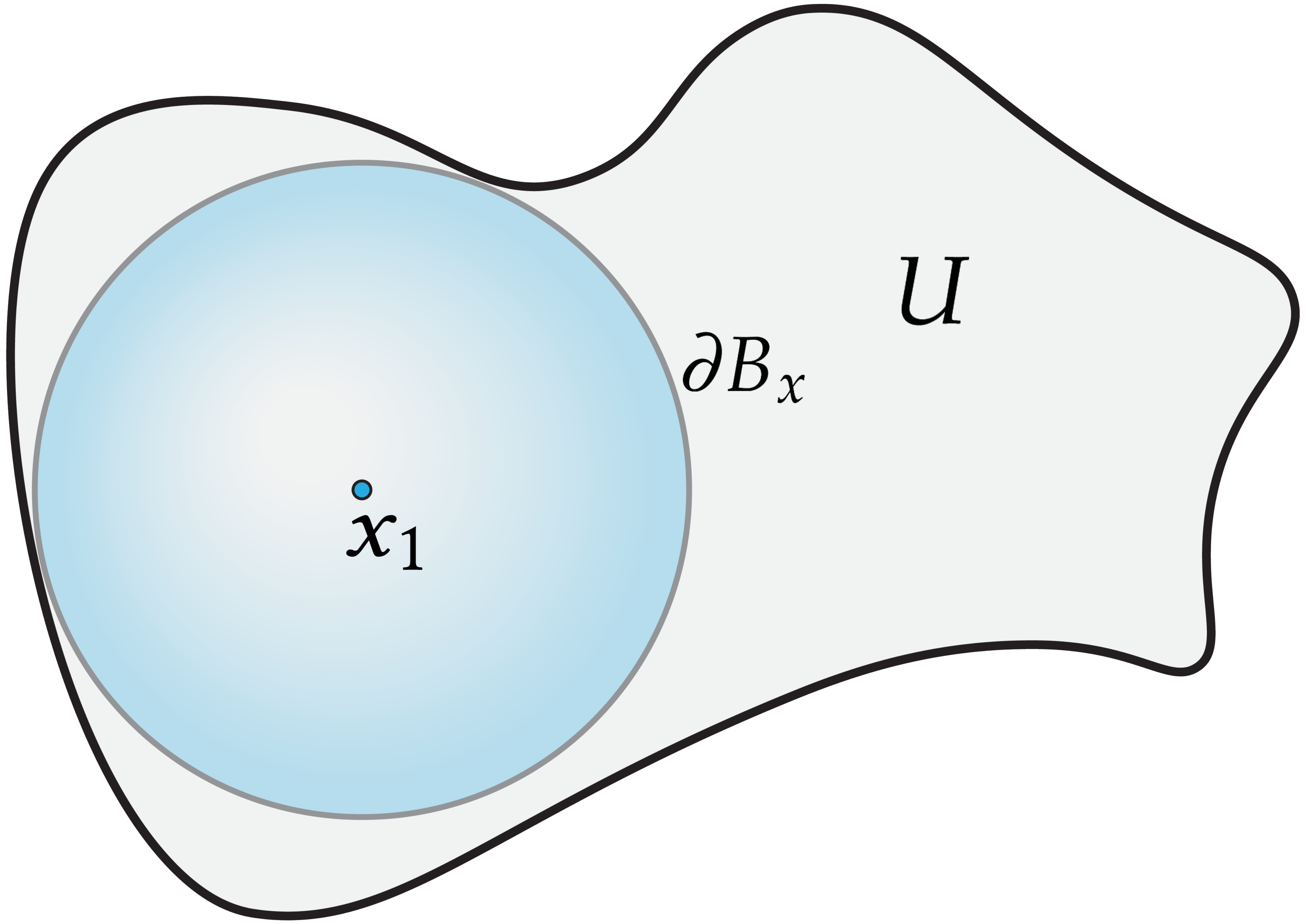
Walk on Spheres (WoS)

Mean value theorem:

$$u(x) = \frac{1}{|\partial B_x|} \int_{\partial B_x} u(x')\,\mathrm{d}x'$$

Estimator:

$$\langle u(x_i) \rangle = u(x_{i+1})$$



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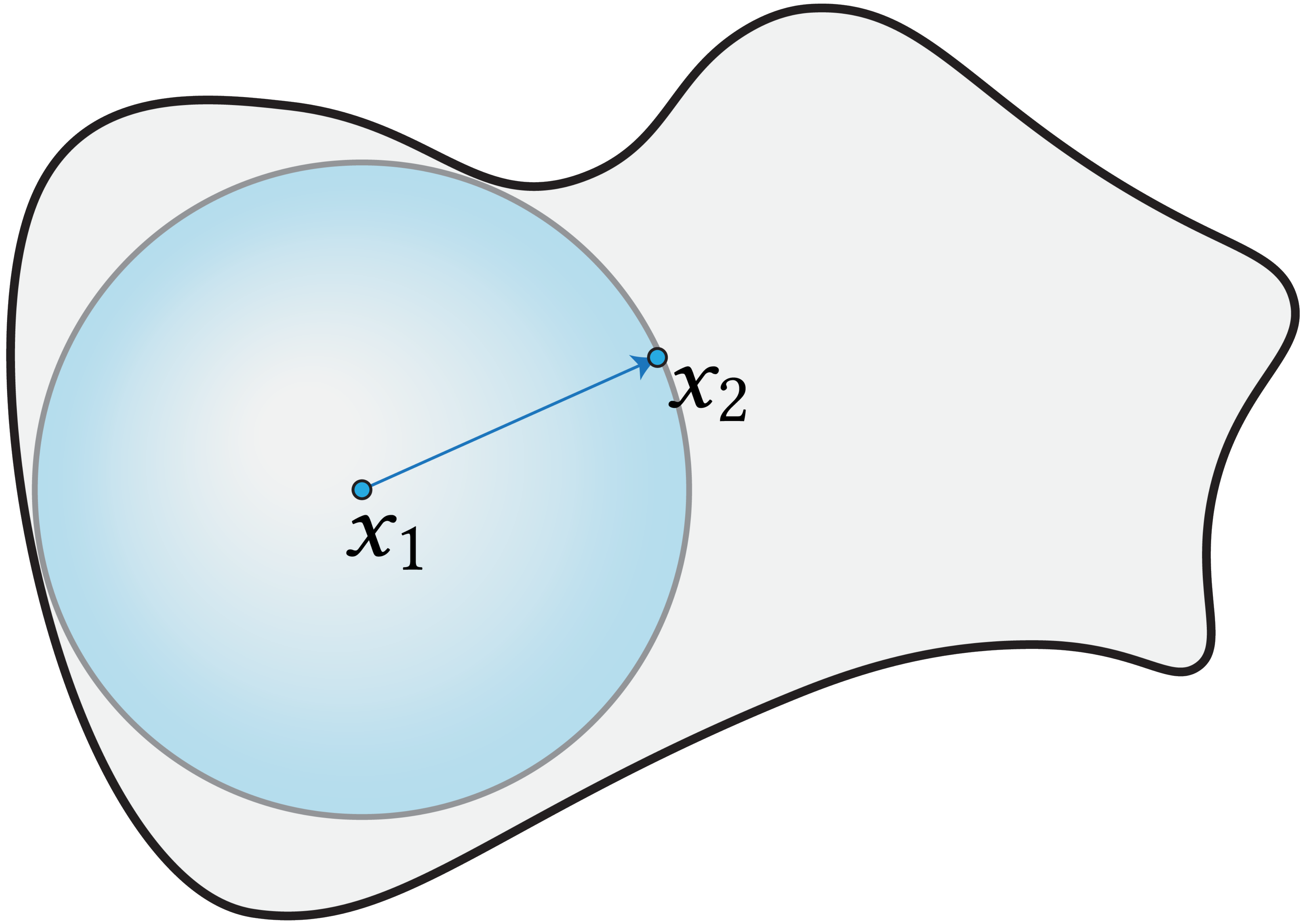

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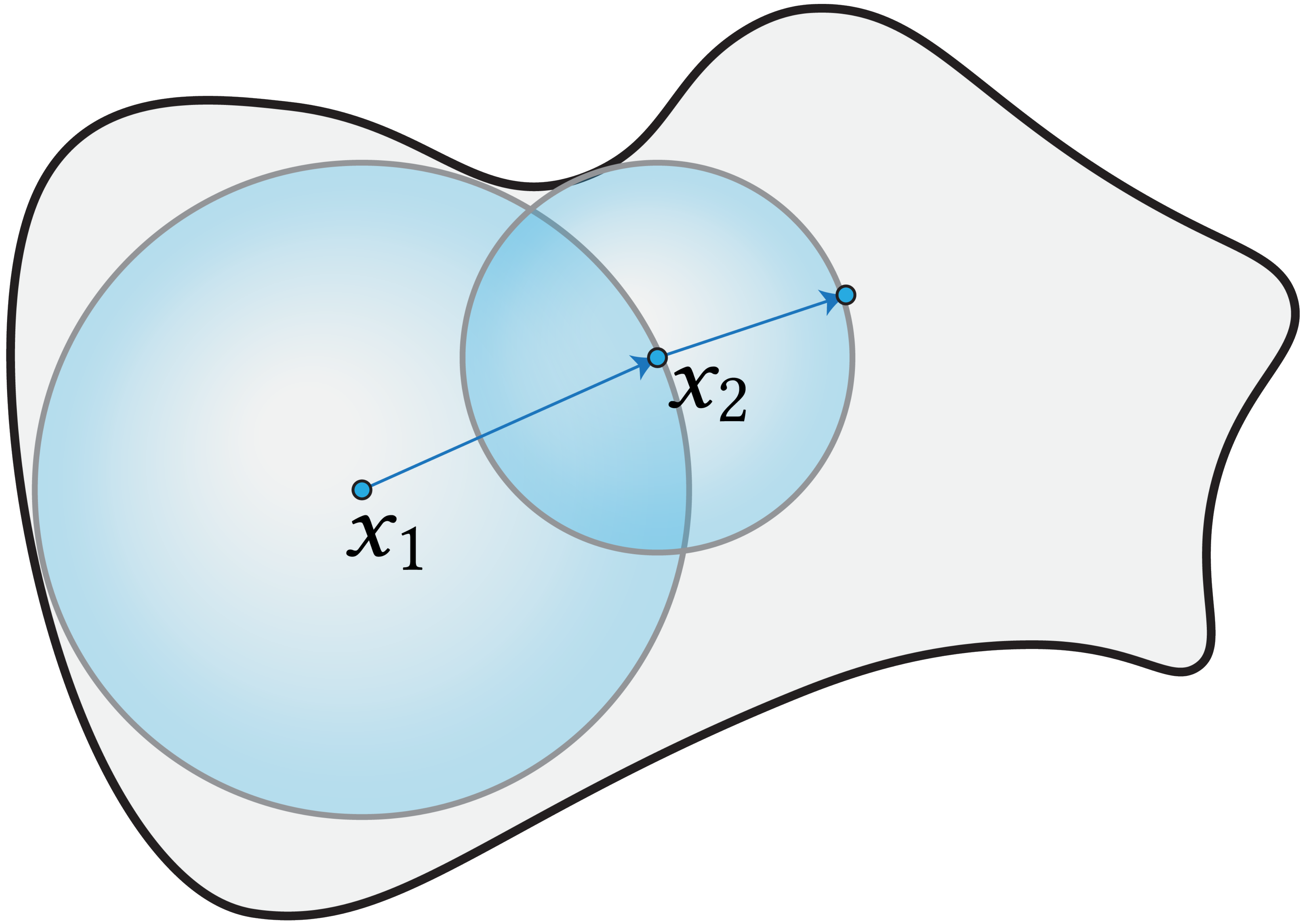

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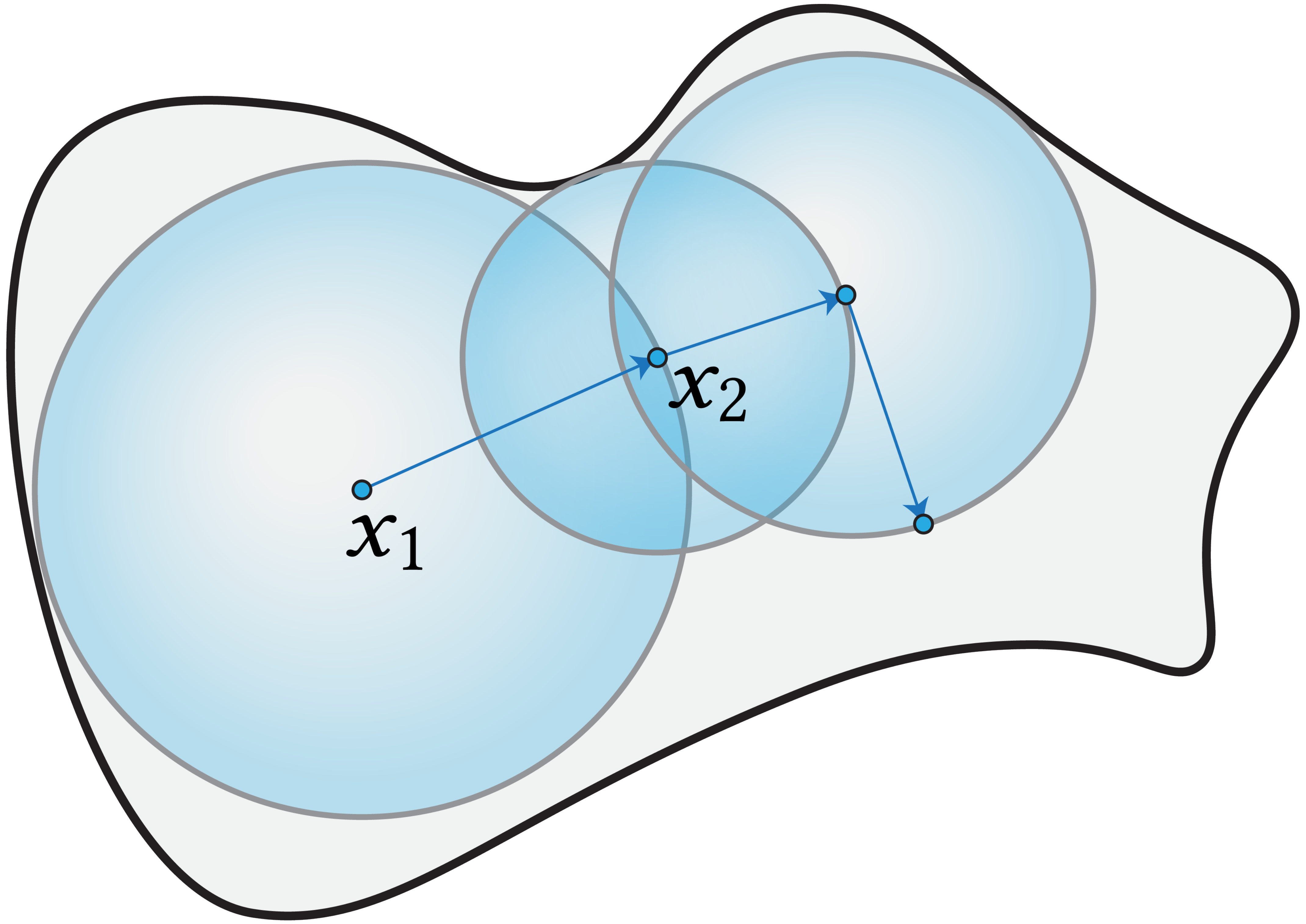

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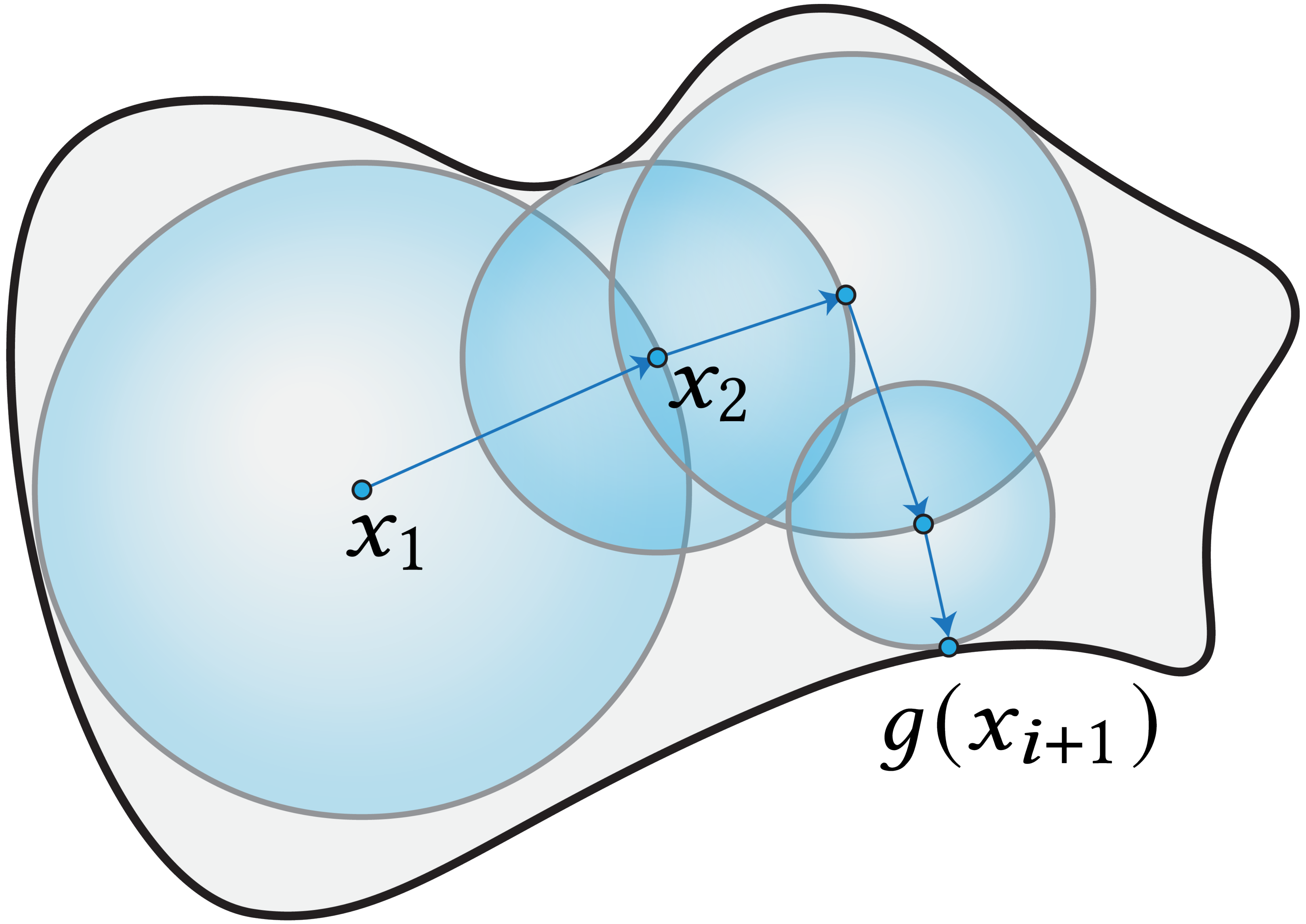
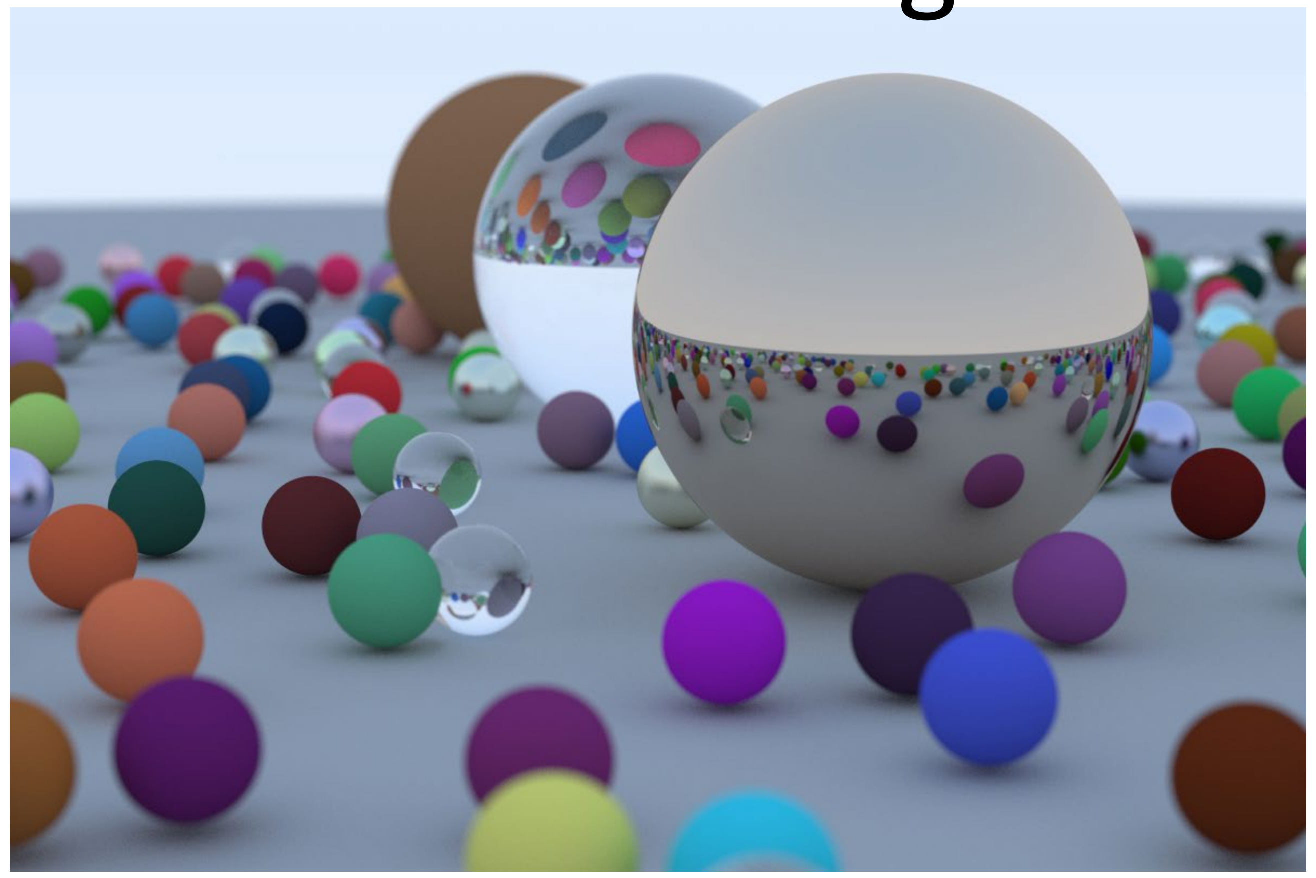
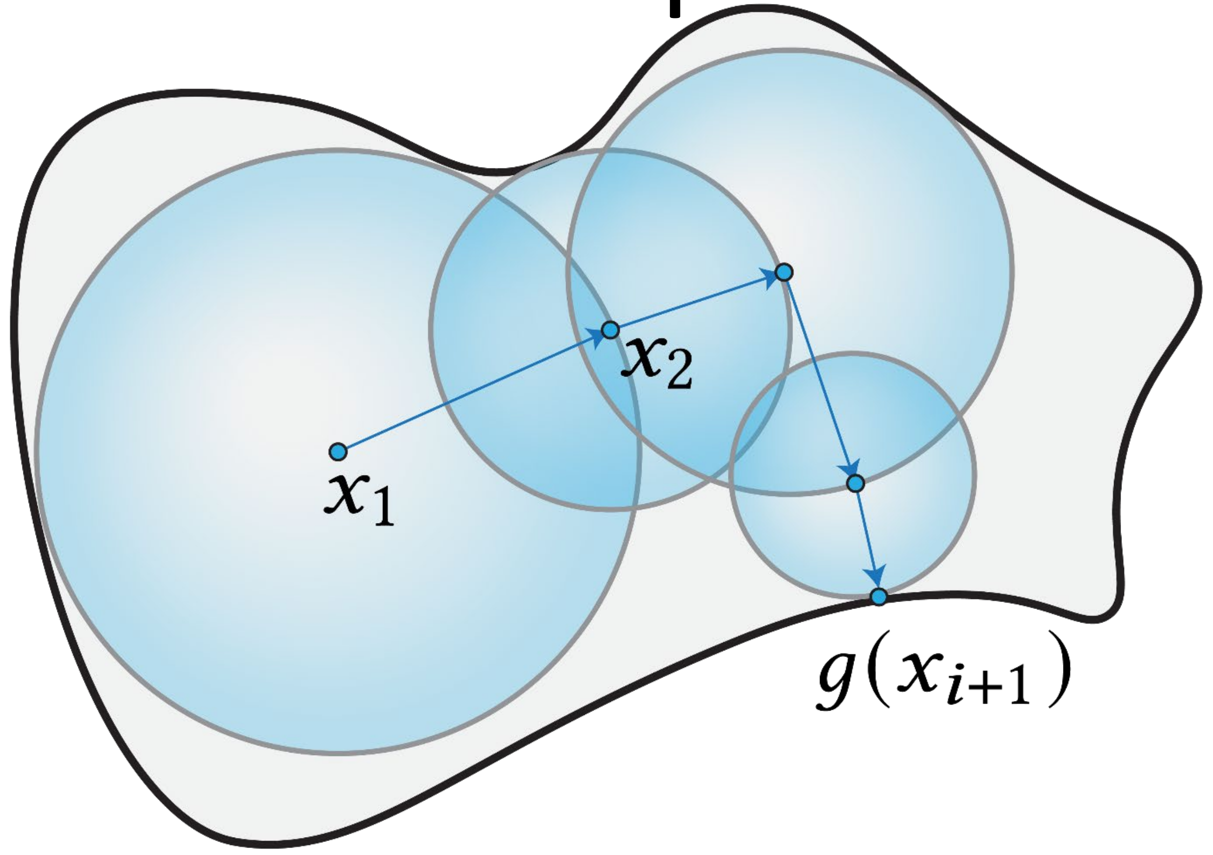
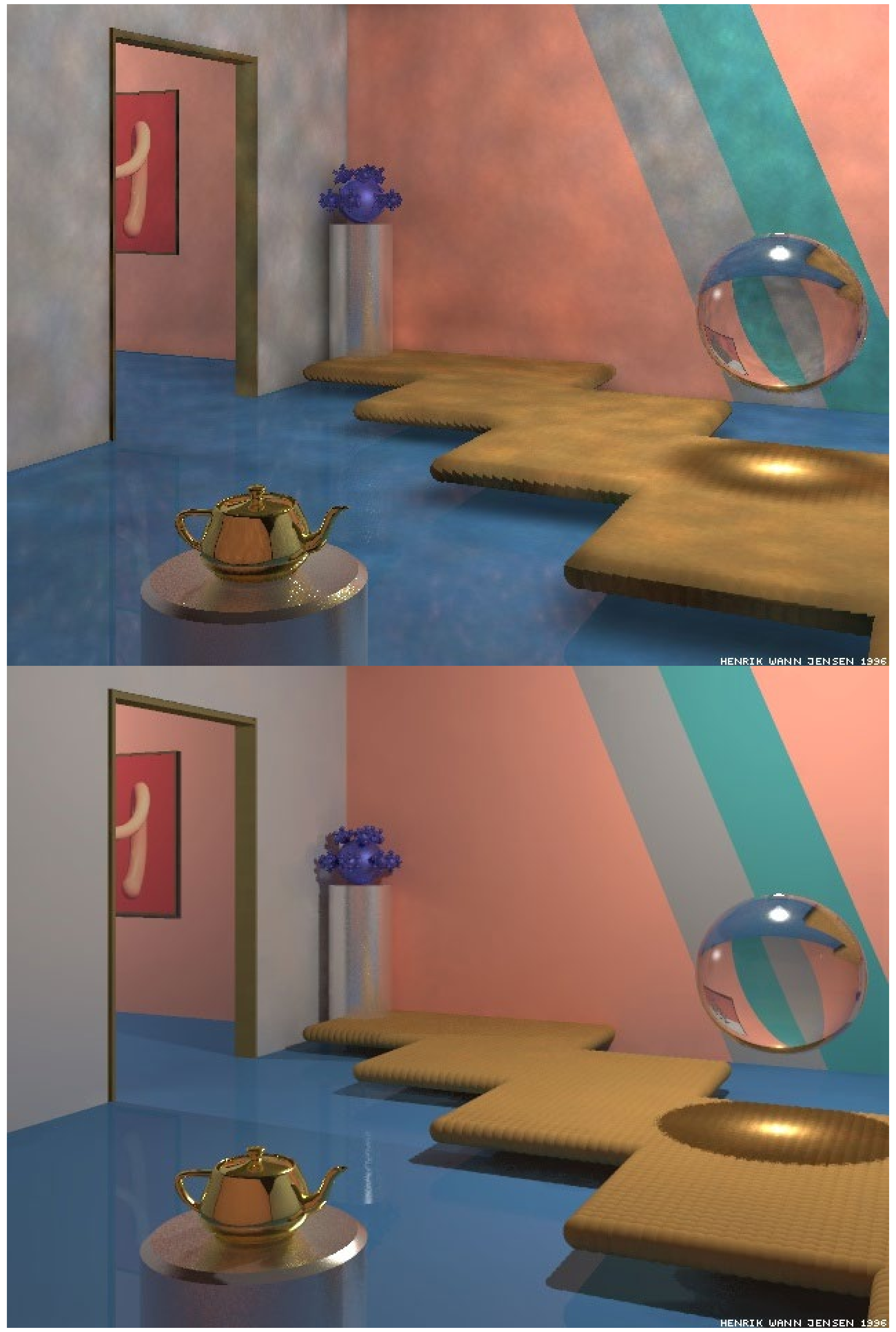


Rendering & PDE

	"Forward" Method	"Reverse" Method
Rendering:	Path Tracing  raytracing.github.io/books/RayTracingInOneWeekend.html	1. No reuse of paths 2. No global importance sampling
PDE:	Walk on Spheres 	

Photon mapping

- Henrik Wann Jensen
  “Global illumination using photon maps”
- Henrik Wann Jensen
  “Realistic Image Synthesis Using Photon Mapping”
- …

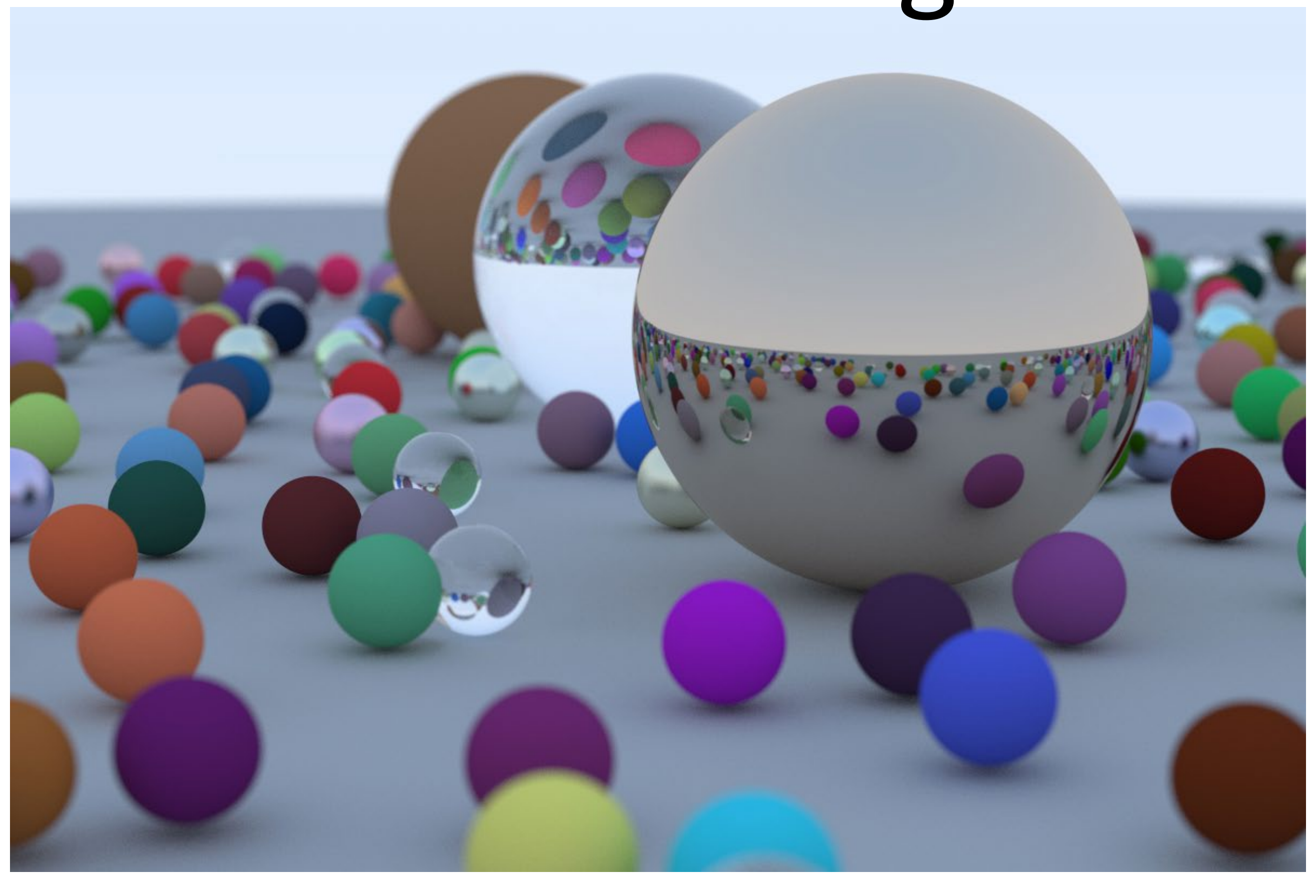
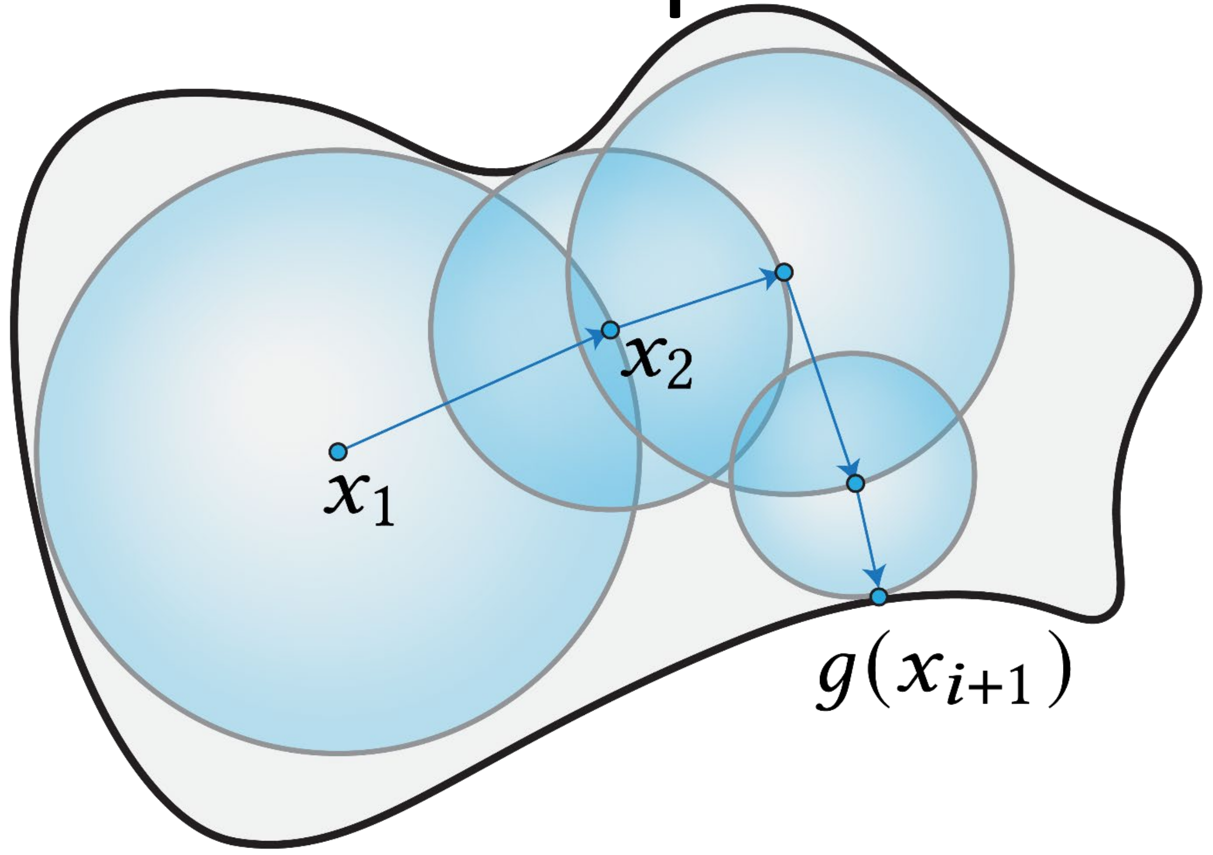


http://graphics.ucsd.edu/~henrik/papers/photon_map/

VPL / Many-light rendering

- Keller, A.
  "Instant Radiosity" (SIGGRAPH 1997)
- Walter, B. Fernadandez, S. Arbree, A. Bala, K. Donikian, M. and Greenberg, D. P.
  "Lightcuts: a scalable approach to illumination" *(SIGGRAPH 2005)*
- Walter, B., Arbree, A., Bala, K., and GREENBERG, D. P.
  "Multidimensional lightcuts" (SIGGRAPH 2006)
- Hašan, M., Pellacini, F., and Bala, K.
  "Matrix row-column sampling for the many-light problem" (SIGGRAPH 2007)
- …

Rendering & PDE

	"Forward" Method	"Reverse" Method
Rendering:	Path Tracing  raytracing.github.io/books/RayTracingInOneWeekend.html	Photon Mapping / VPLs  graphics.ucsd.edu/~henrik/papers/photon_map/
PDE:	Walk on Spheres 	"Reverse WoS"? ?

PDE

Laplace operator:

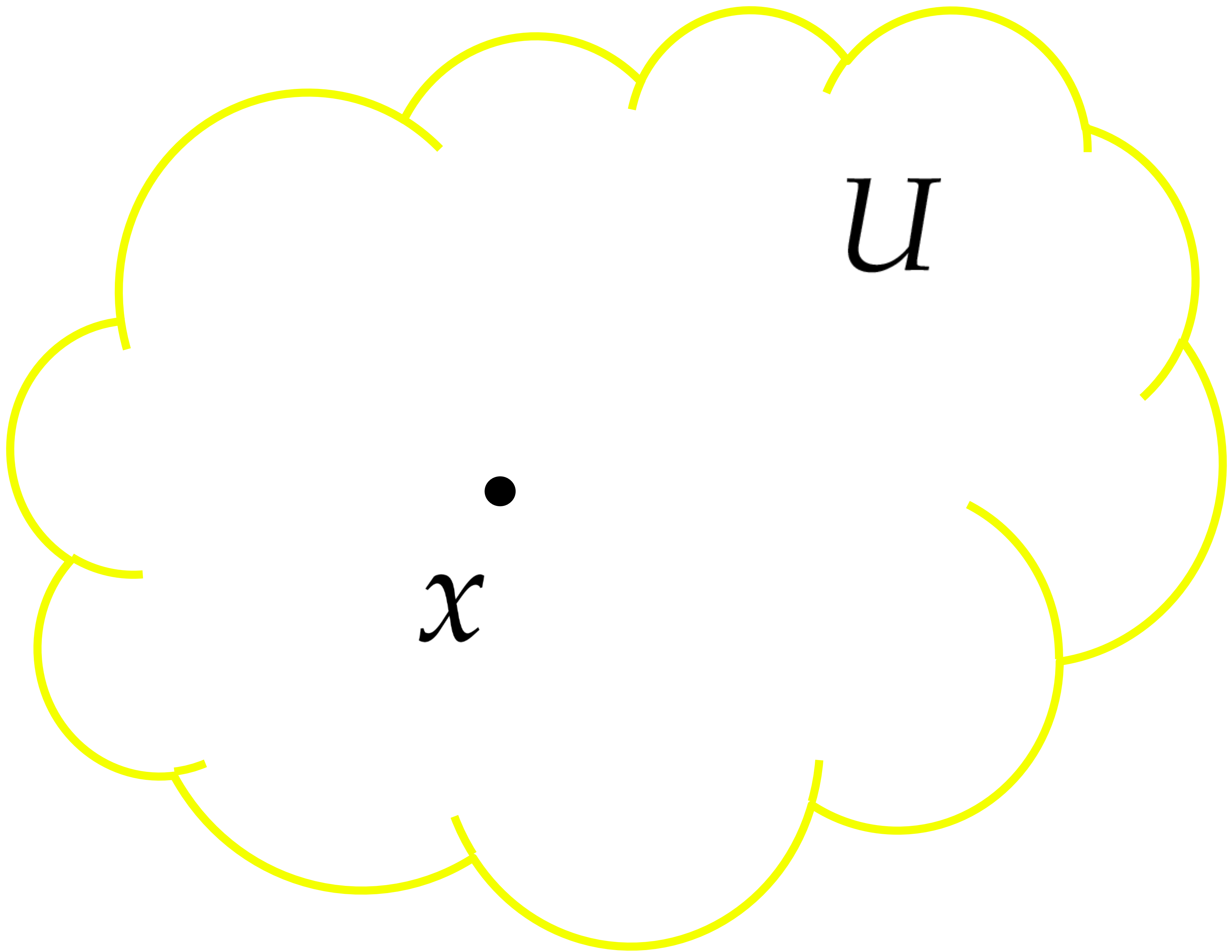
Sum of second derivatives

$$\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y)$$

PDE

Poisson's equation:

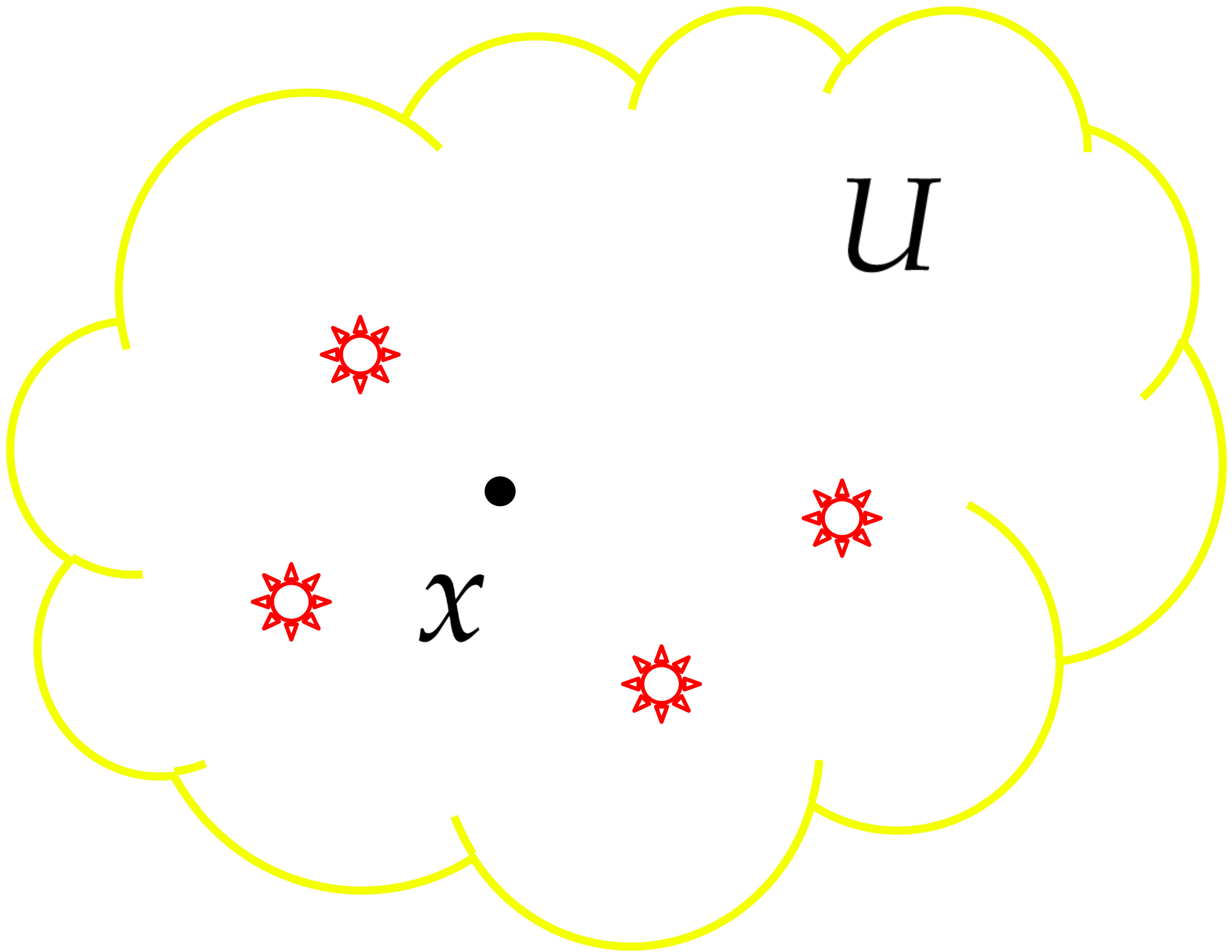
$$\Delta u(x) = 0 \qquad \text{if } x \in U,$$
$$u(x) = g(x) \qquad \text{if } x \in \partial U.$$



PDE

Poisson's equation (with sources):

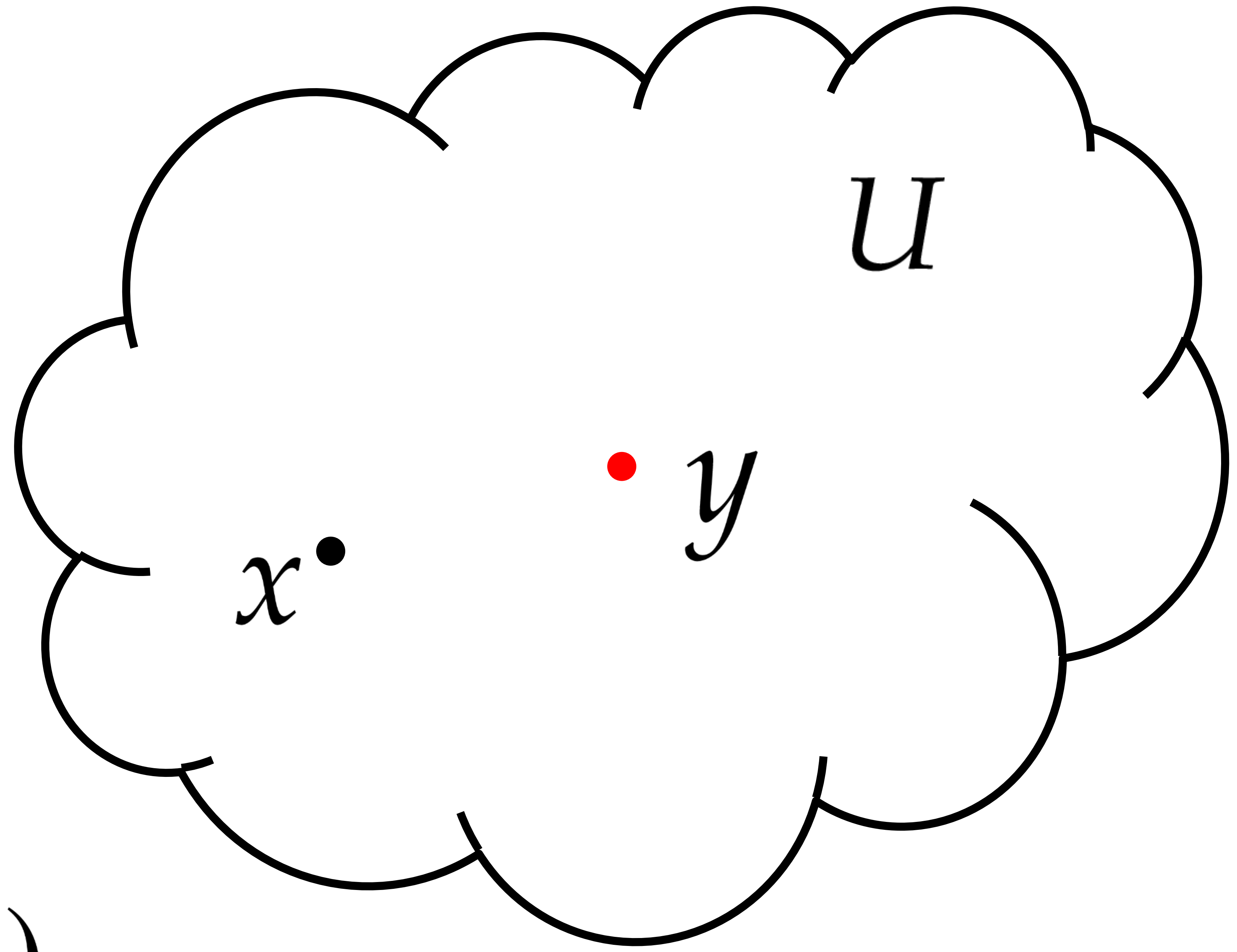
$$\Delta u(x) = f(x) \quad \text{if } x \in U,$$
$$u(x) = g(x) \quad \text{if } x \in \partial U.$$



Green's Function

Delta-Point source:

$$\Delta u(x) = \delta_y(x) \quad \text{if } x \in U$$
$$u(x) = 0 \quad \text{if } x \in \partial U$$



Green's function:

Solution to the delta point source PDE $\mathcal{G}(x, y)$

Green’s Function

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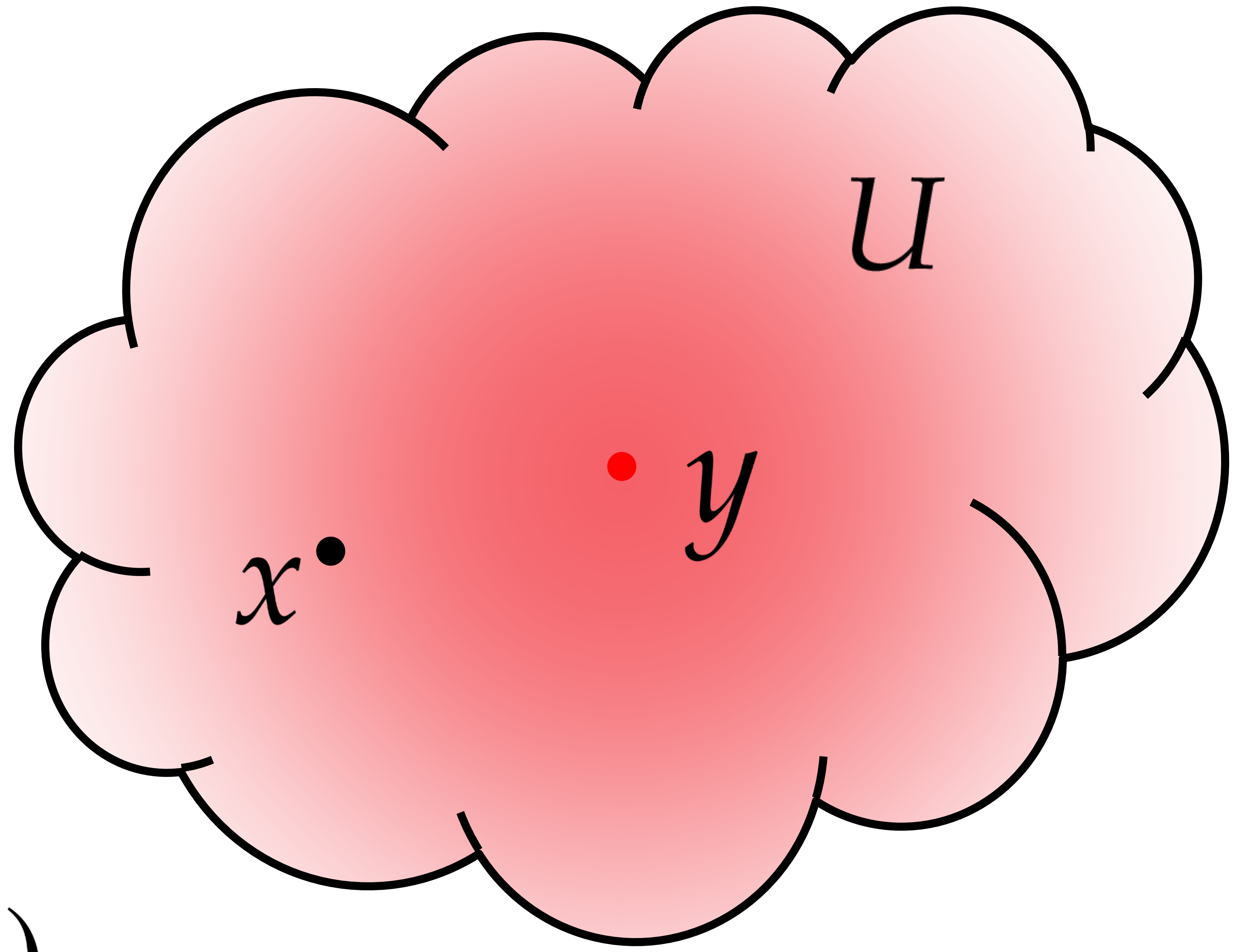


Green’s function:

Solution to the delta point source PDE $\mathcal{G}(x, y)$

Define $\mathcal{G}(x, y) = 0$ if x or y is not inside U

Green's Function

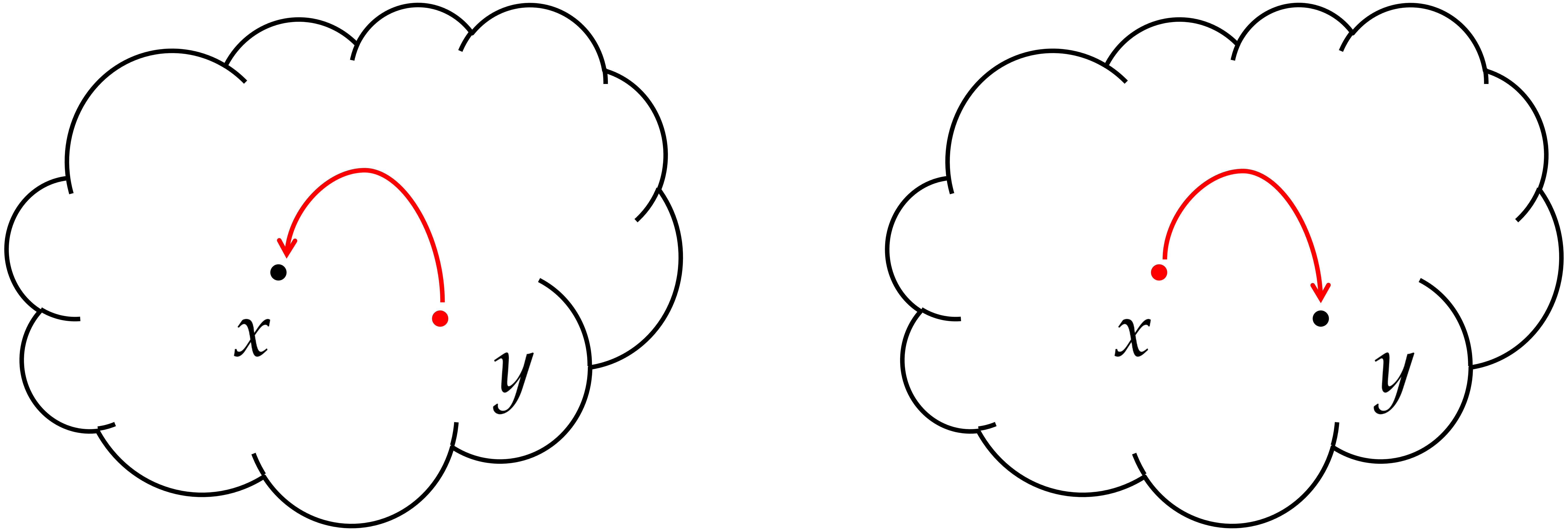
$\mathcal{G}(x, y)$ Green's function for the domain

(Usually hard to calculate)

$\mathcal{G}^{B_x}(x, y)$ Green's function for the ball

(Has a simple analytical form)

Symmetry property of Green's Function



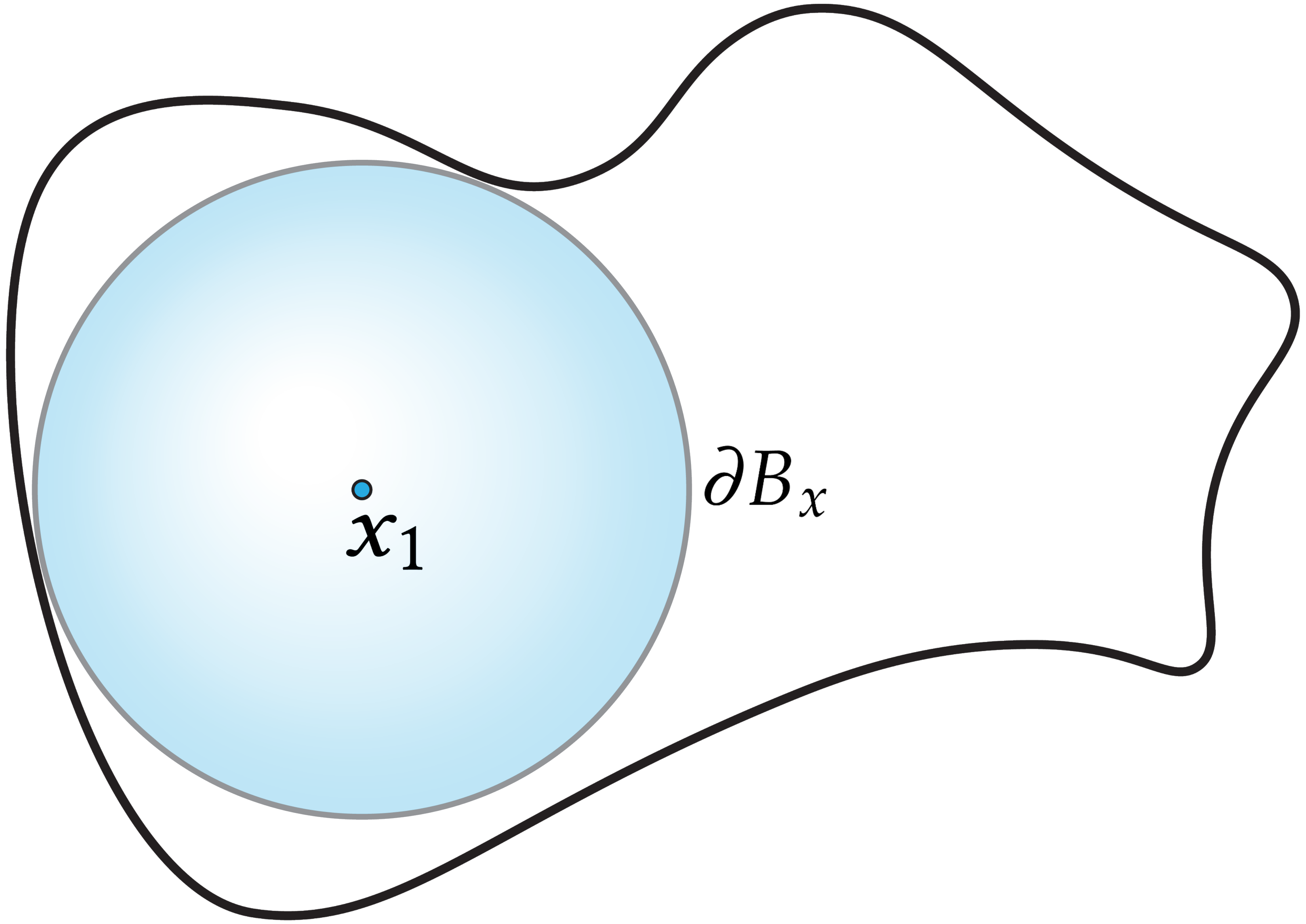
Use double side arrow to denote the symmetric property

$\mathcal{G}(x \leftrightarrow y) \coloneqq \mathcal{G}(x, y) = \mathcal{G}(y, x)$

Walk on Spheres with Sources

Mean Value Theorem:

$$u(x) = \frac{1}{|\partial B_x|} \int_{\partial B_x} u(x')\, \mathrm{d}x' + \int_{B_x} f(y) \mathcal{G}^{B_x}(x \leftrightarrow y)\, \mathrm{d}y$$



Walk on Spheres with Sources

Mean Value Theorem:

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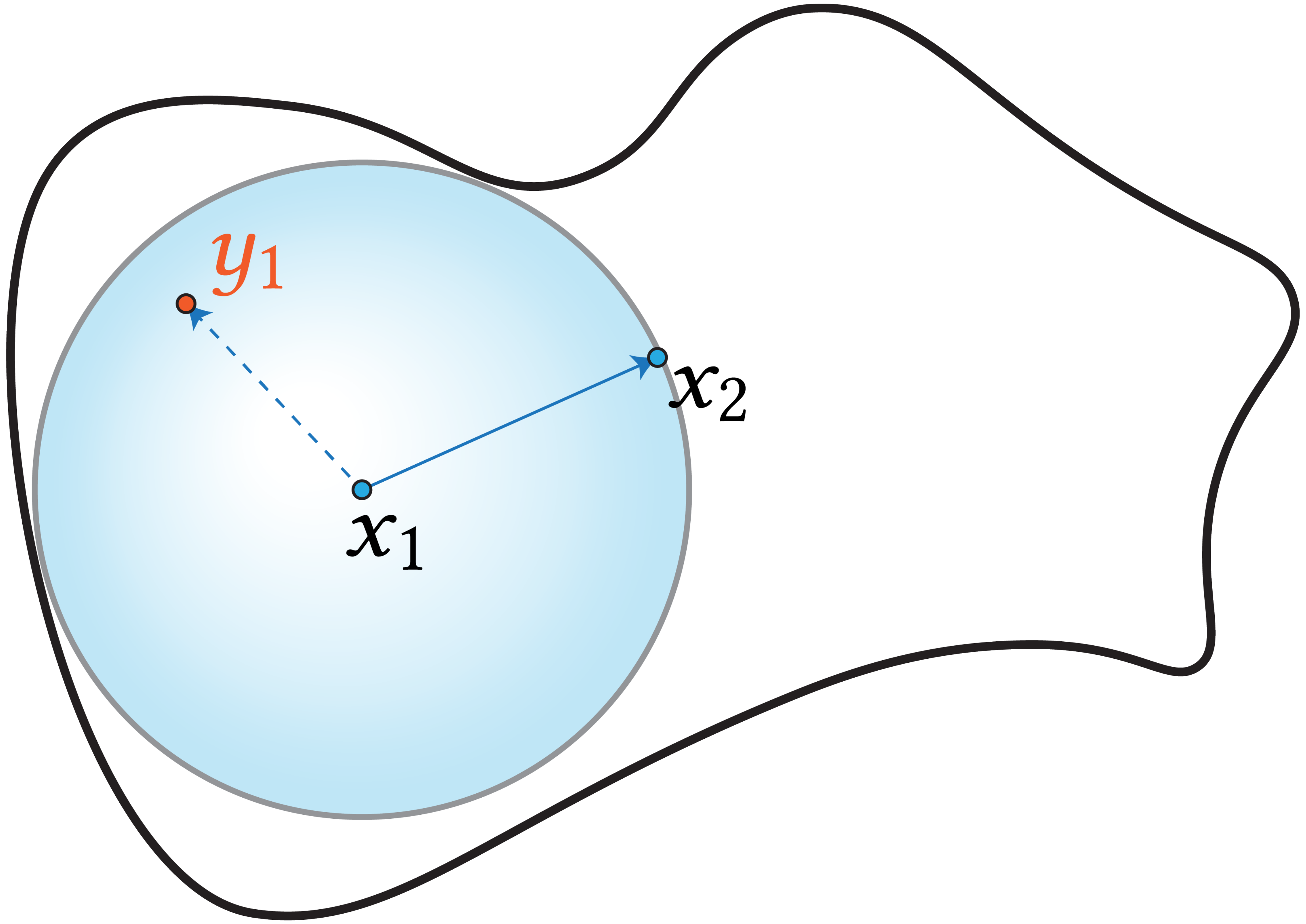
Estimator:

$$\langle u(x_i)\rangle = u(x_{i+1}) + \underbrace{\frac{f(y_i)\mathcal{G}^{B_{x_i}}(x_i \leftrightarrow y_i)}{p(y_i)}}$$

Estimate contribution from sources



Walk on Spheres with Sources

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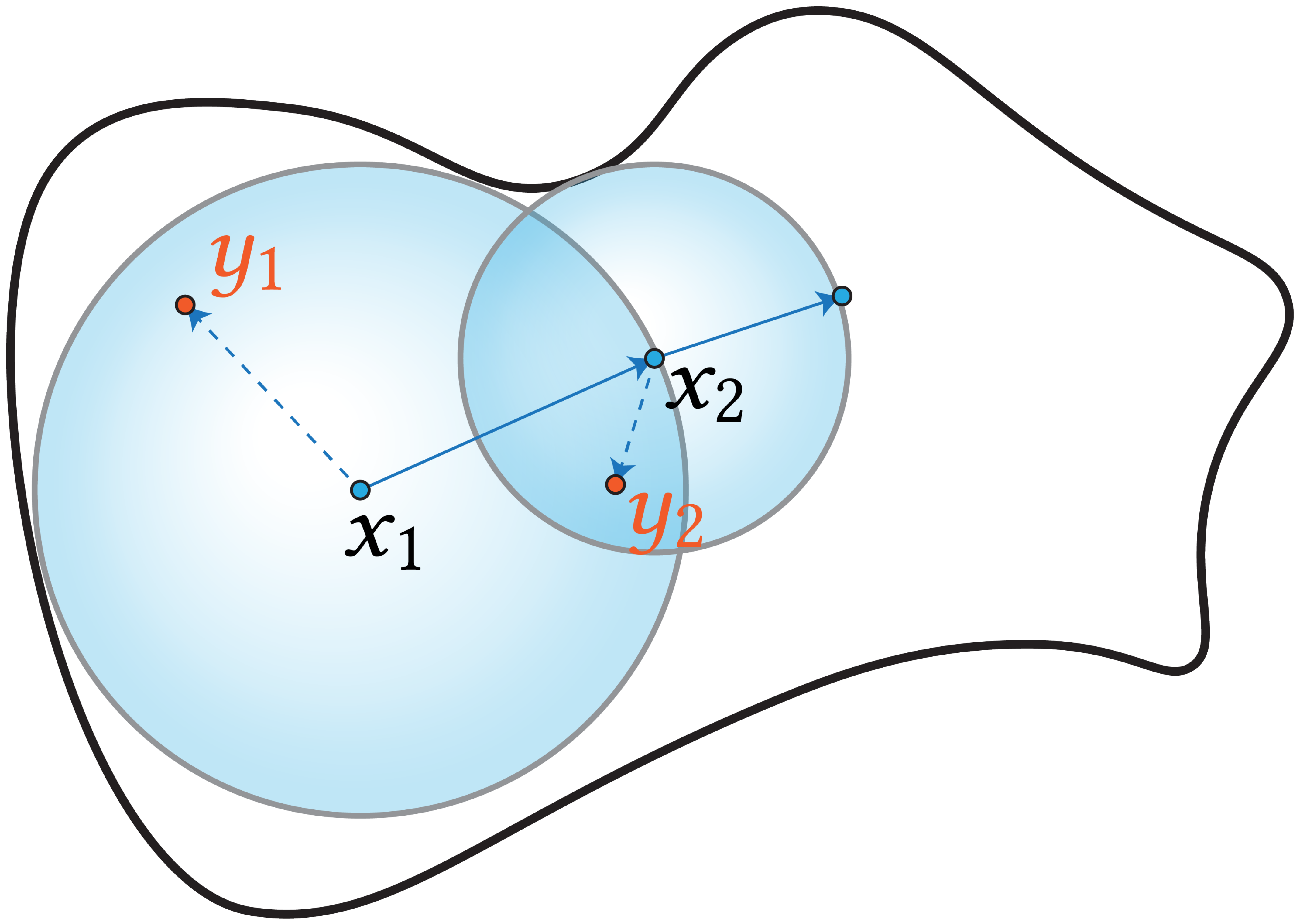

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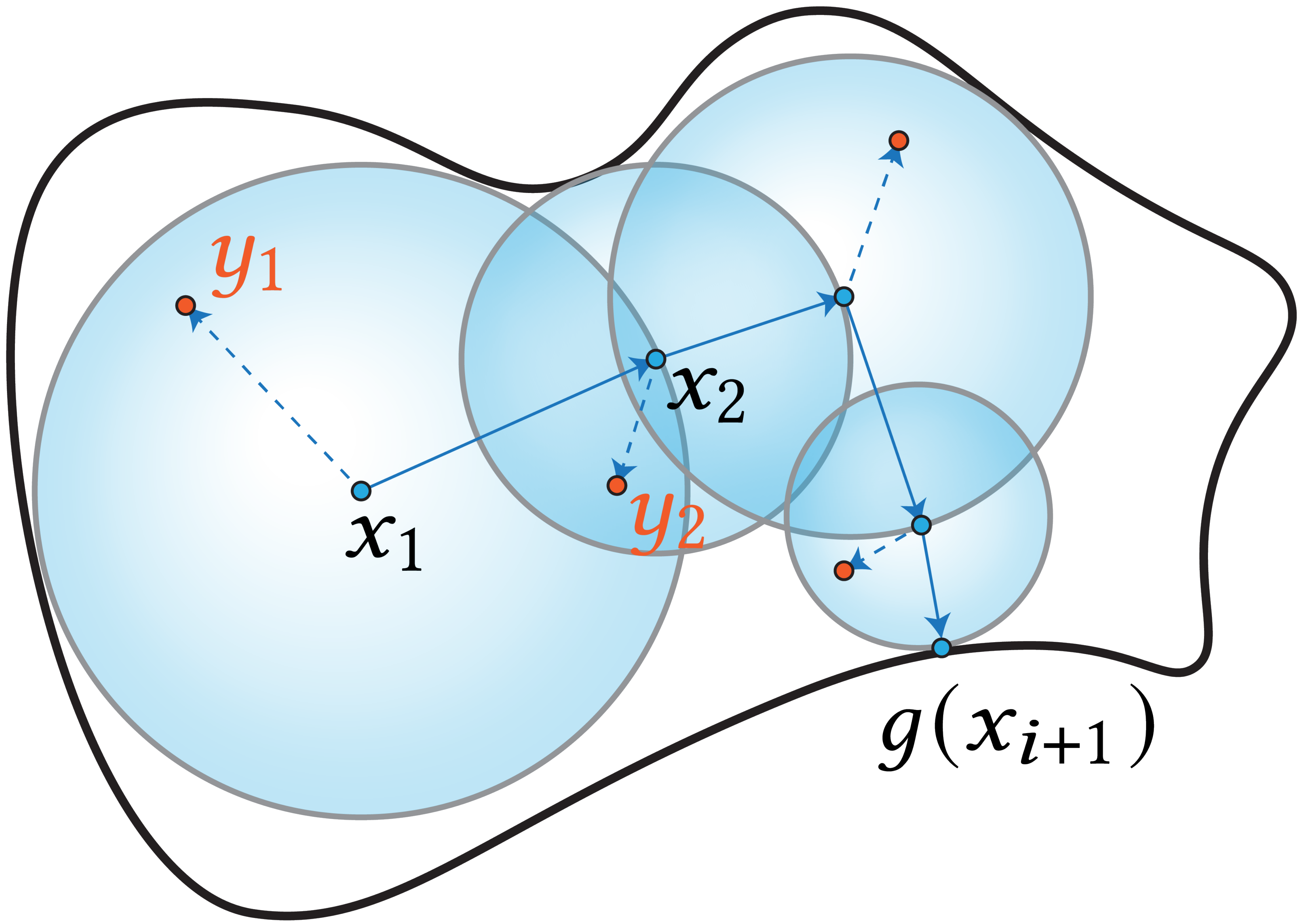


Our Approach

Mean Value Theorem:

$$u(x) = \frac{1}{|\partial B_x|} \int_{\partial B_x} u(z)\,\mathrm{d}z + \int_{B_x} f(y) \mathcal{G}^{B_x}(x \leftrightarrow y)\,\mathrm{d}y$$

Representation formula using Green's function:

$$u(x) = \int_{\partial U} g(z) \frac{\partial \mathcal{G}(x \leftrightarrow z)}{\partial z}\,\mathrm{d}z + \int_{U} f(y) \mathcal{G}(x \leftrightarrow y)\,\mathrm{d}y$$

Need an estimate for the Green's function!

MVT for Green's function

Delta point source PDE + Mean value theorem

= MVT for Green's function:

$$\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_x}(x \leftrightarrow y) + \frac{1}{|\partial B_x|} \int_{\partial B_x} \mathcal{G}(x' \leftrightarrow y) \, \mathrm{d}x' \, .$$

Swap x and y:

$$\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_y}(x \leftrightarrow y) + \frac{1}{|\partial B_y|} \int_{\partial B_y} \mathcal{G}(x \leftrightarrow y') \, \mathrm{d}y' \, .$$

MVT for Green's function

MVT for x:

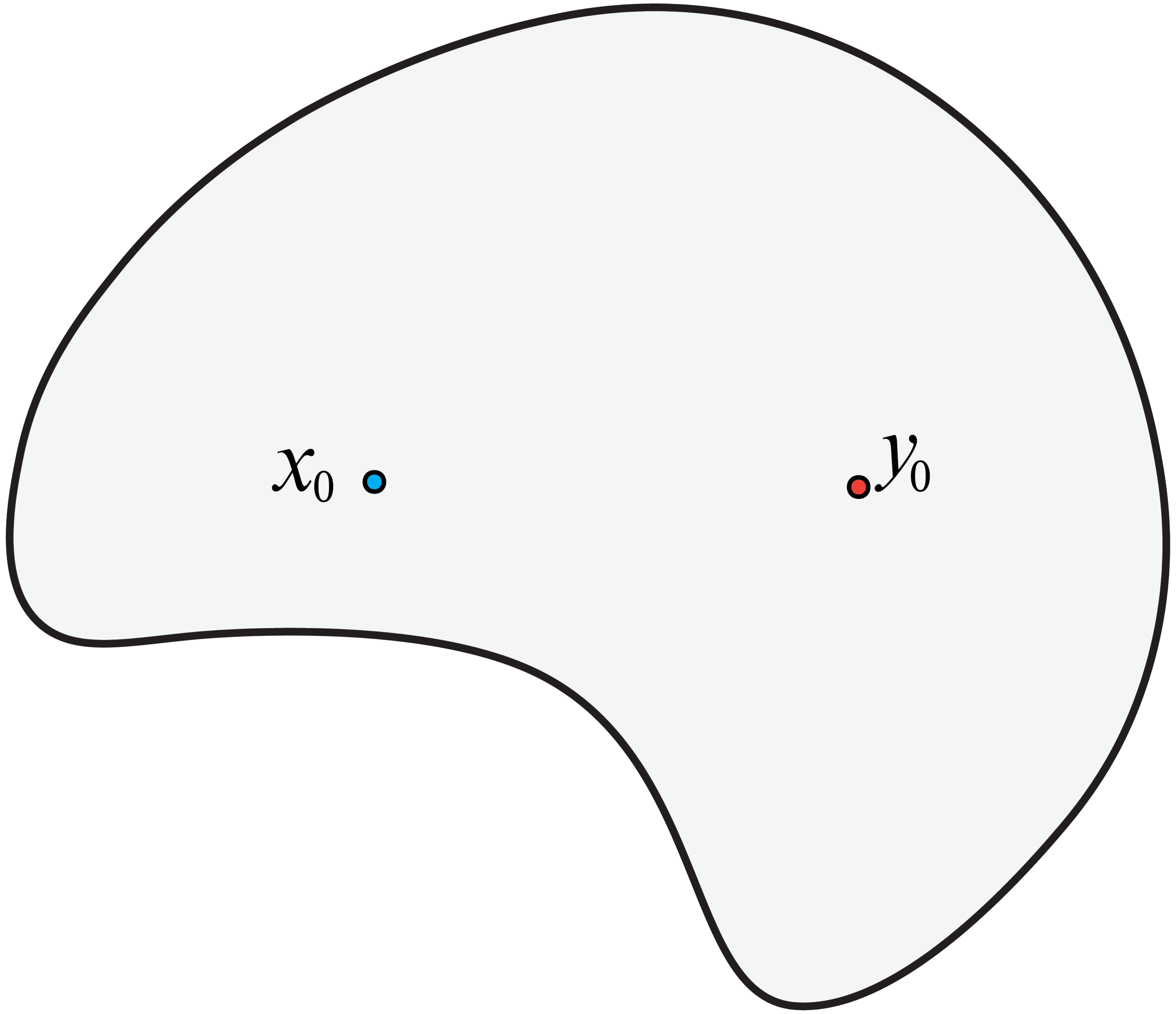
$$\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_x}(x \leftrightarrow y) + \frac{1}{|\partial B_x|} \int_{\partial B_x} \mathcal{G}(x' \leftrightarrow y) \, \mathrm{d}x'$$

MVT for y:

$$\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_y}(x \leftrightarrow y) + \frac{1}{|\partial B_y|} \int_{\partial B_y} \mathcal{G}(x \leftrightarrow y') \, \mathrm{d}y'$$

Forward Estimator

Forward: $$\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_x}(x \leftrightarrow y) + \frac{1}{|\partial B_x|} \int_{\partial B_x} \mathcal{G}(x' \leftrightarrow y)\, \mathrm{d}x' \,.$$



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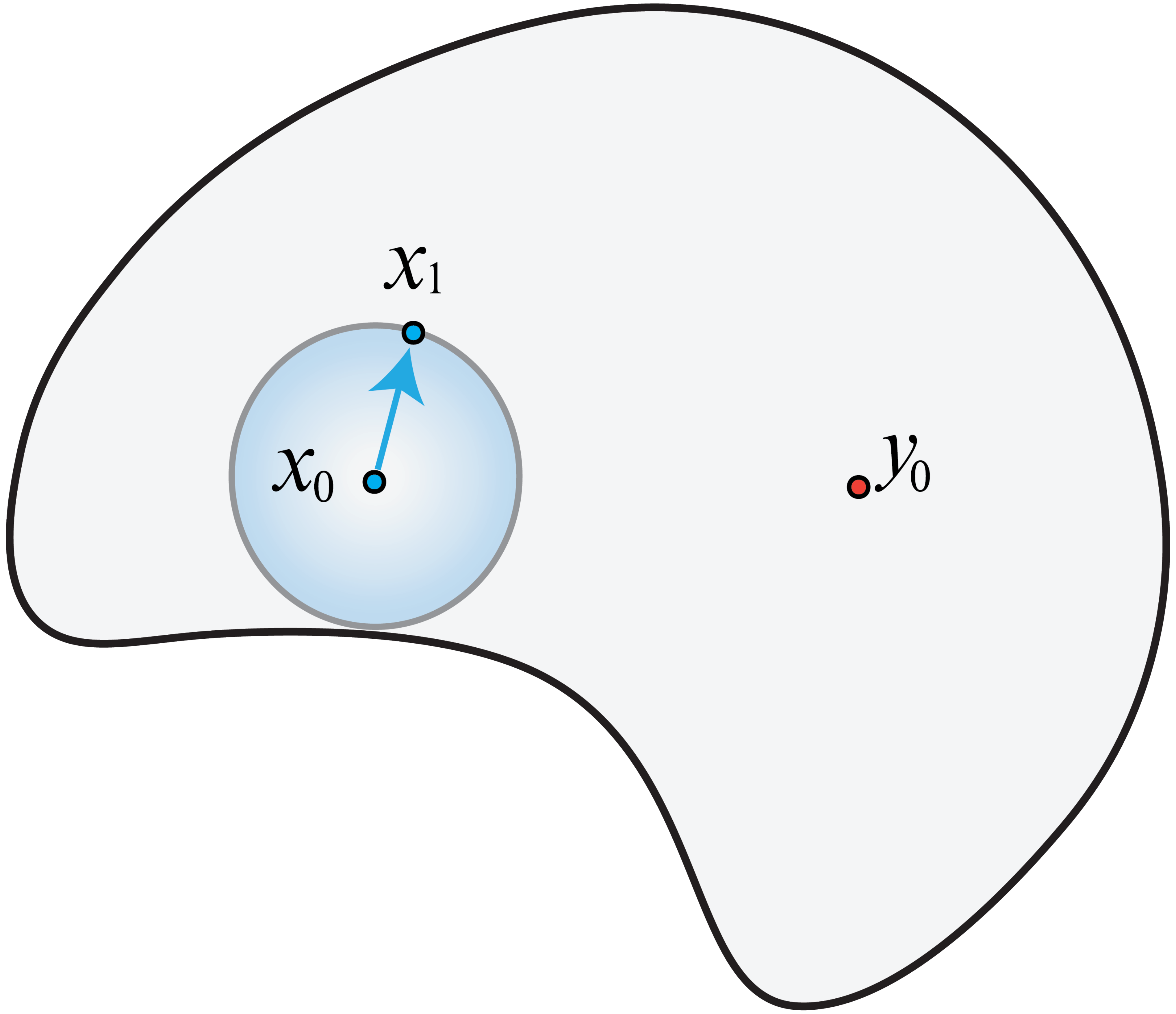

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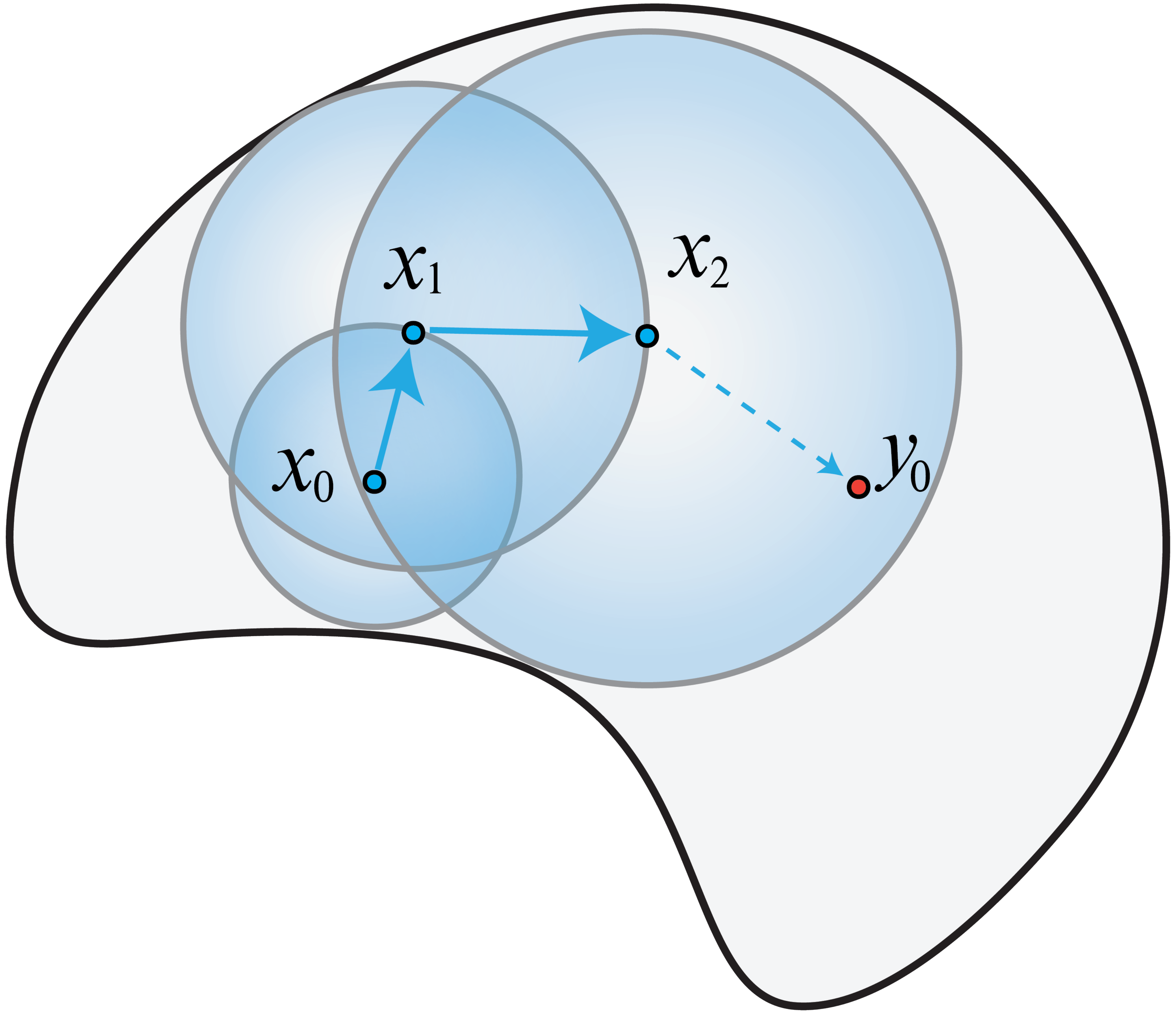

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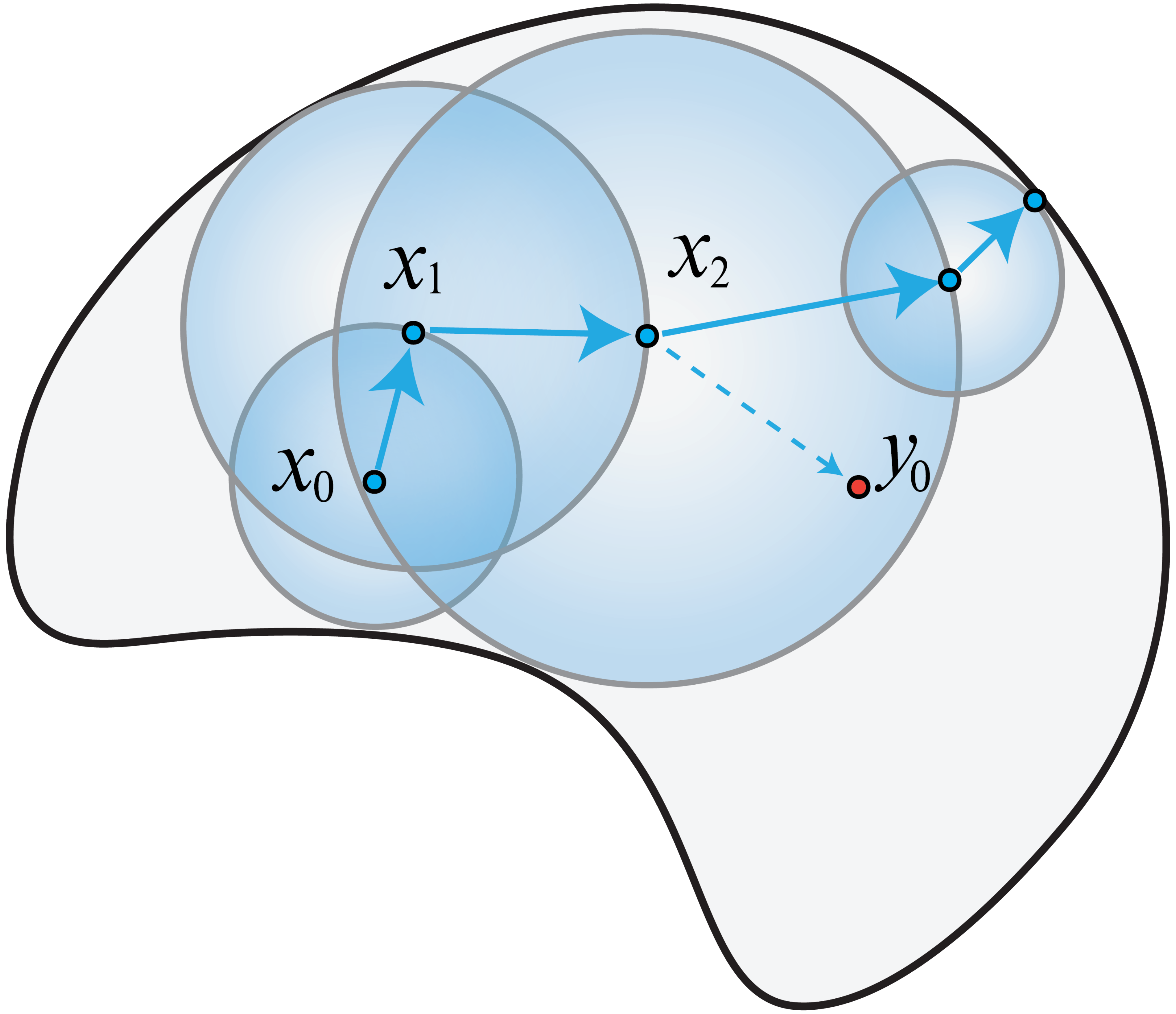

Forward Estimator:

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$$\langle \mathcal{G}(x \leftrightarrow y) \rangle = 0 \quad \text{if } x \in \partial U$$

Reverse Estimator

Reverse: $\mathcal{G}(x \leftrightarrow y) = \mathcal{G}^{B_y}(x \leftrightarrow y) + \frac{1}{|\partial B_y|} \int_{\partial B_y} \mathcal{G}(x \leftrightarrow y')\, \mathrm{d}y'\,.$

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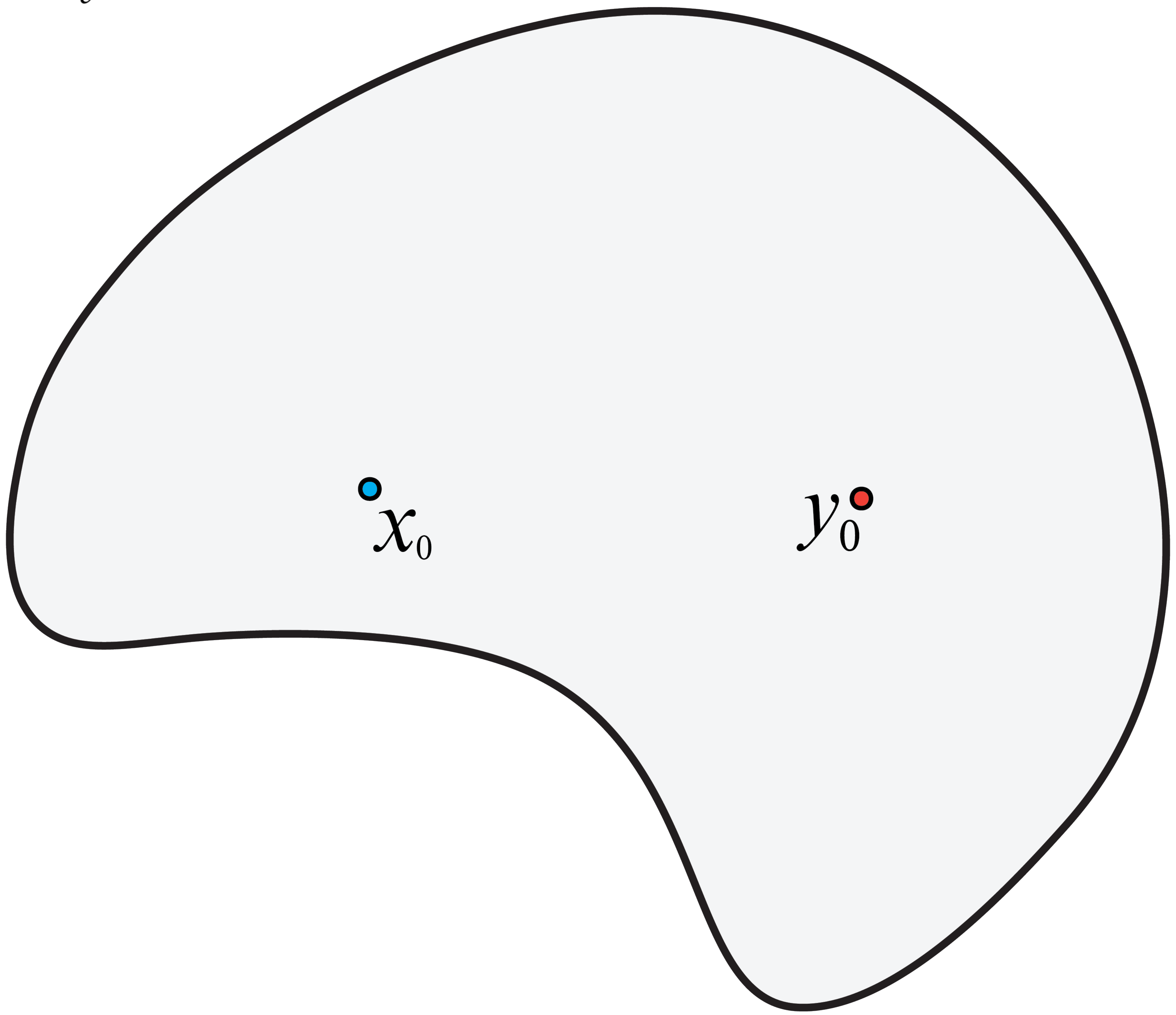

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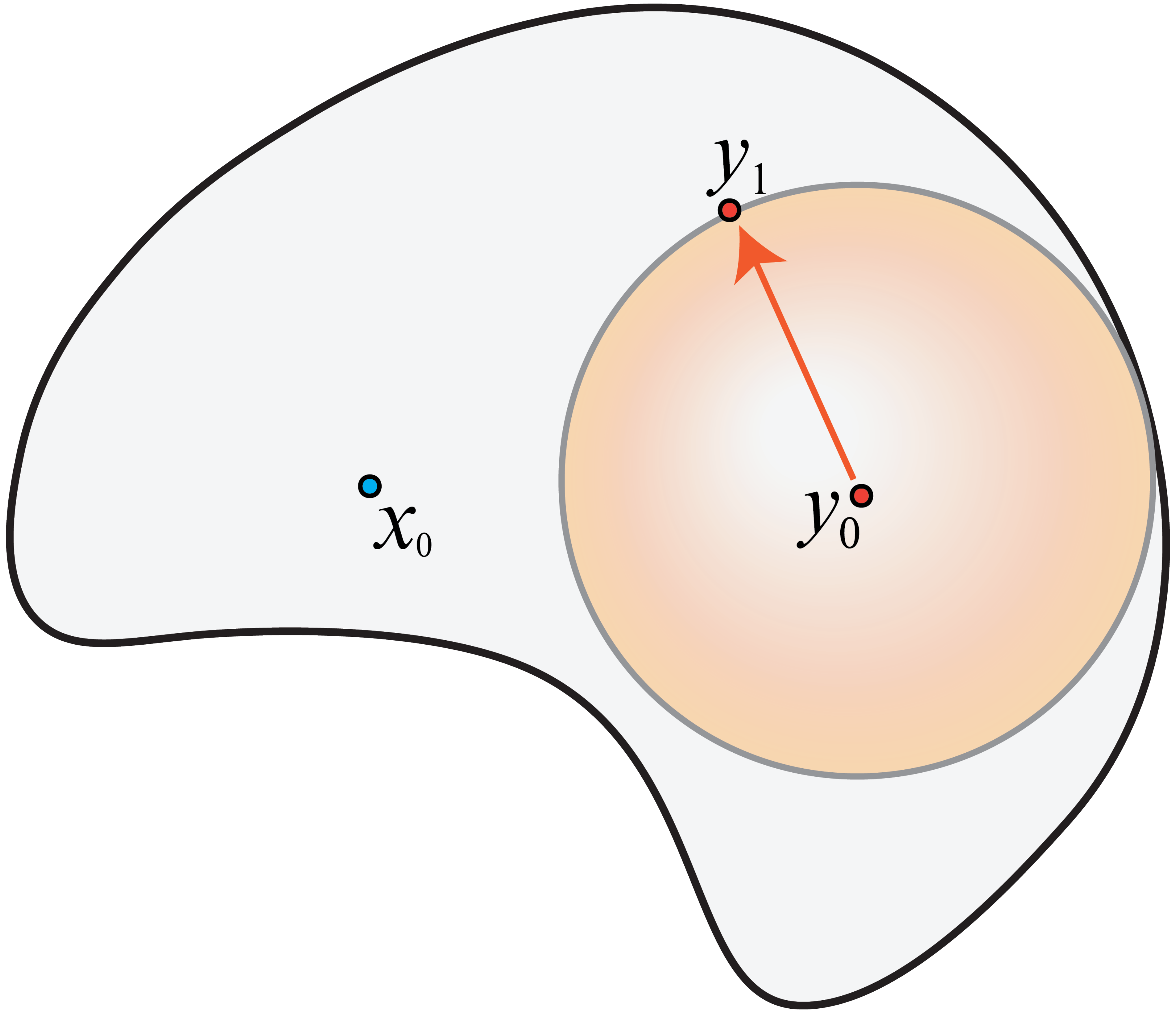

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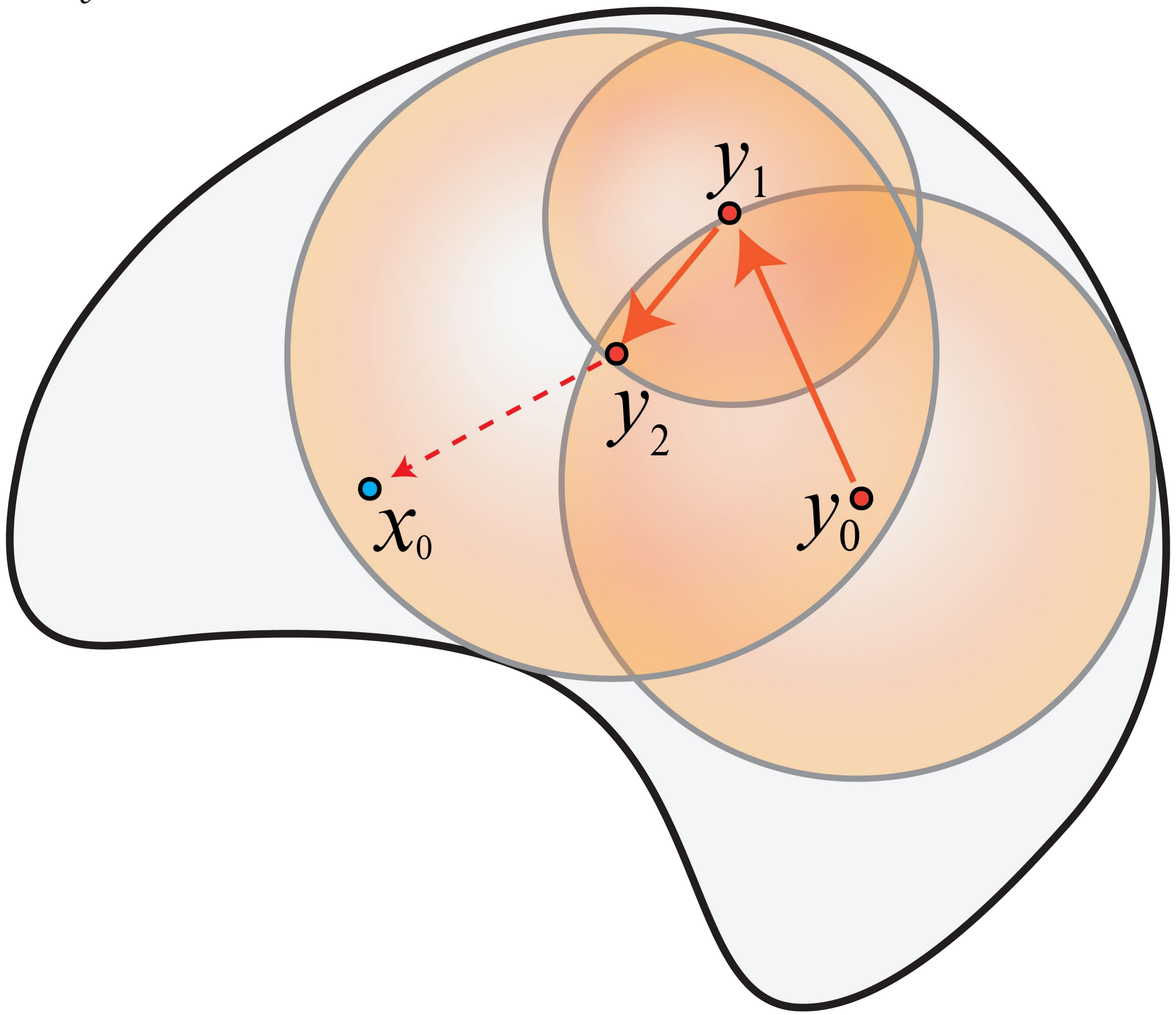

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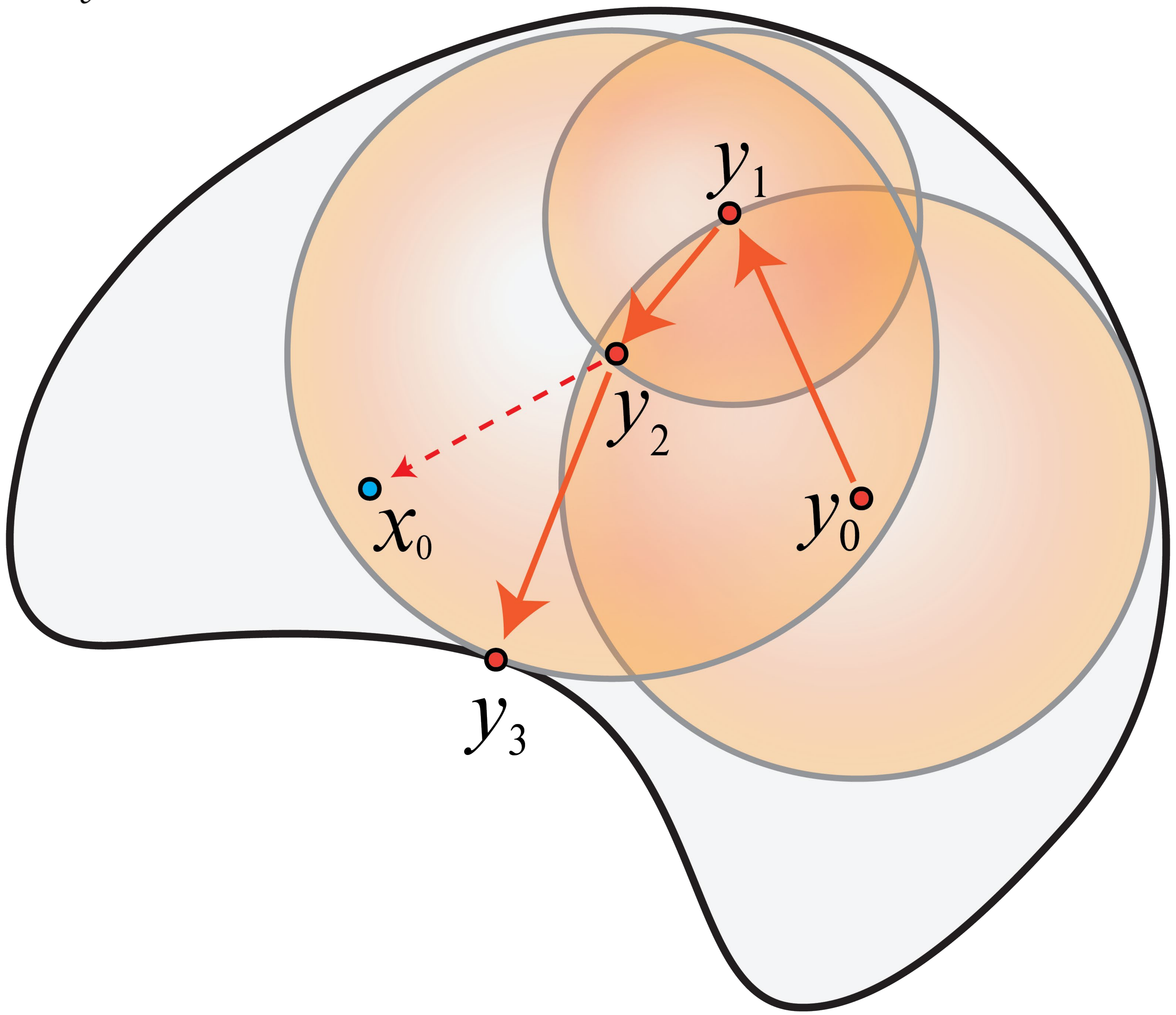
$$\langle \mathcal{G}(x \leftrightarrow y) \rangle = \mathcal{G}^{B_y}(x \leftrightarrow y) + \langle \mathcal{G}(x \leftrightarrow y') \rangle$$

$$\langle \mathcal{G}(x \leftrightarrow y) \rangle = 0 \quad \text{if } y \in \partial U$$



Reverse Estimator

- Start walks from sources
- Globally importance sampling
- Reuse walks

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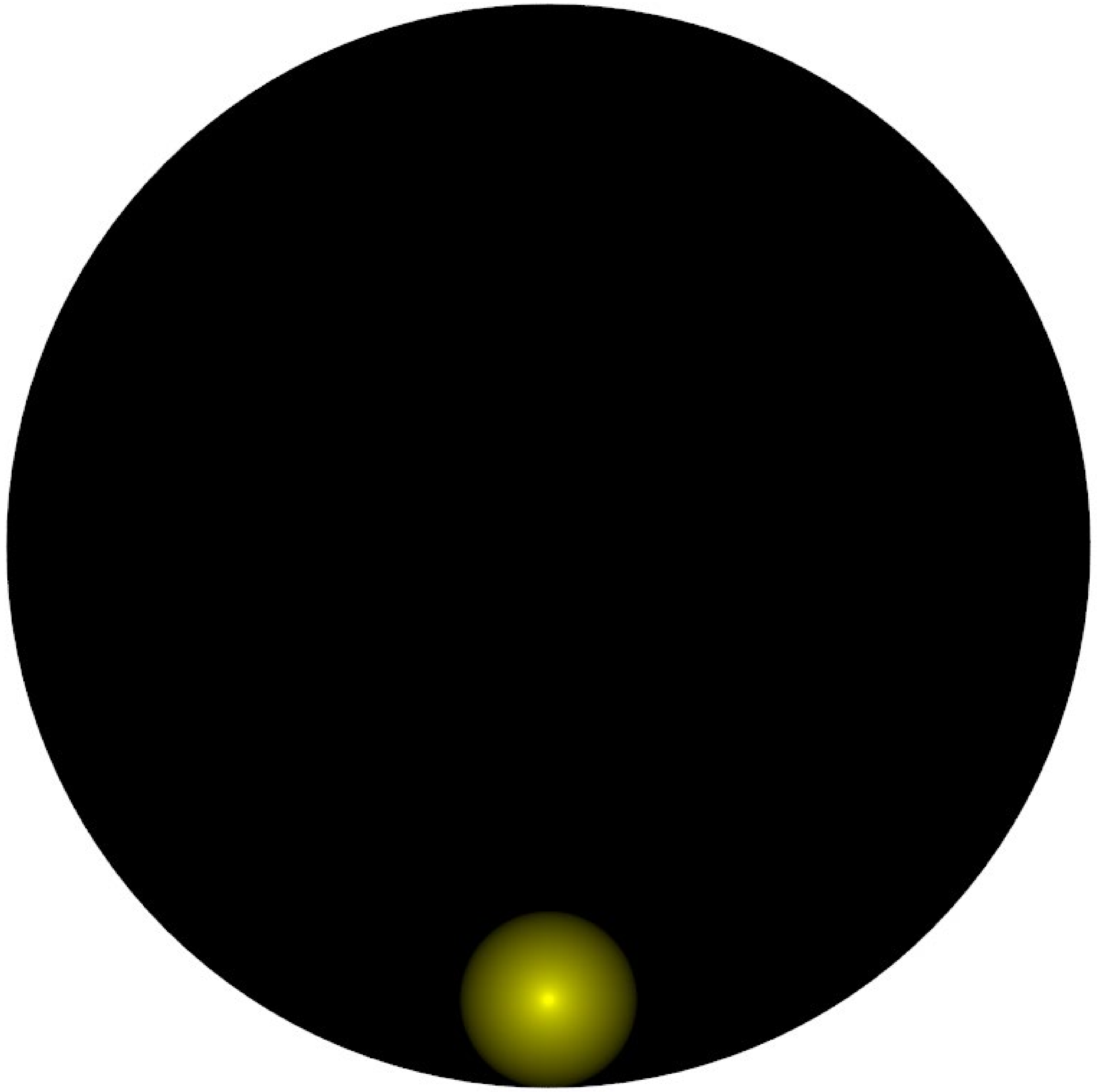
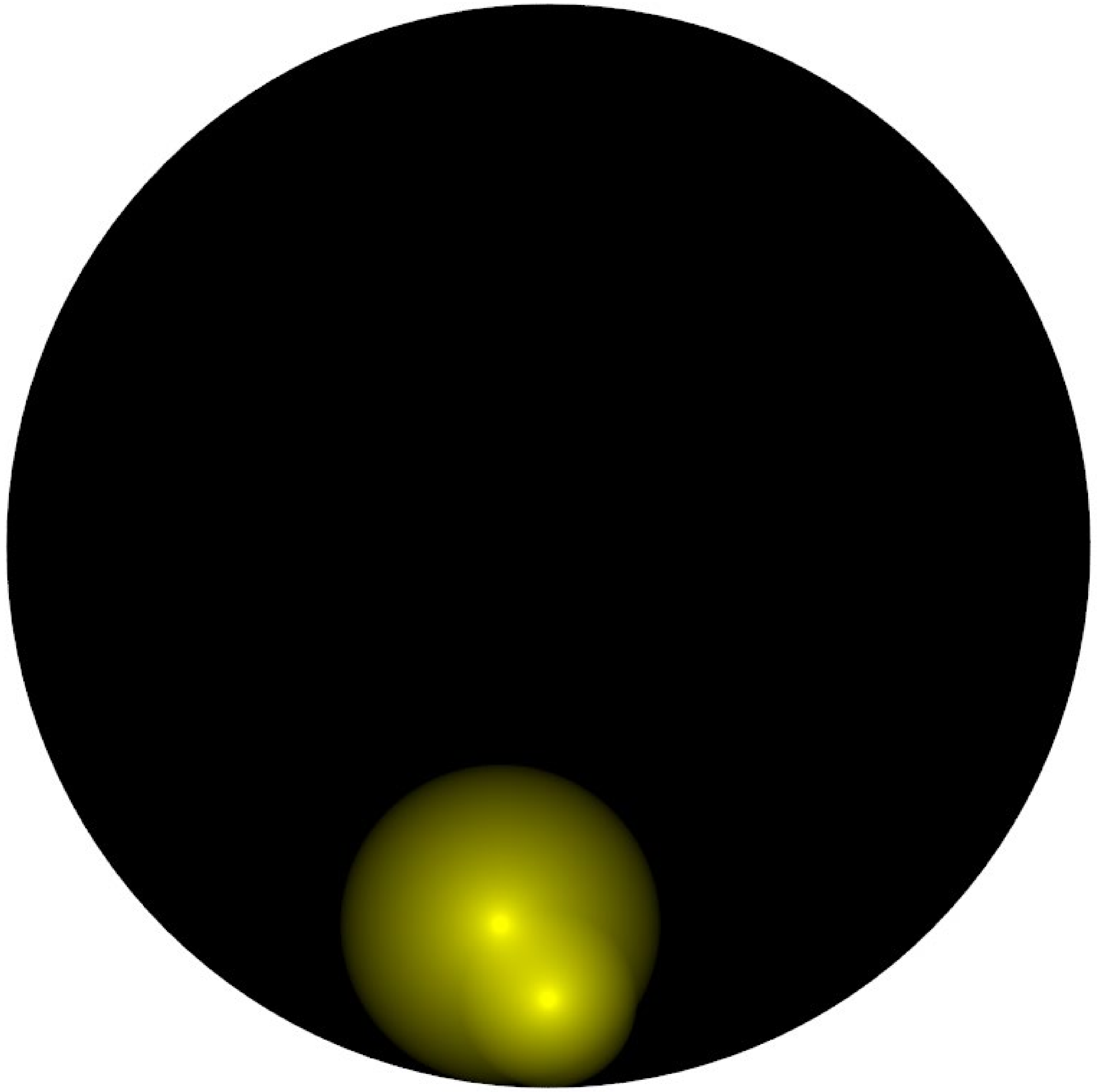


1 step

Reverse Estimator

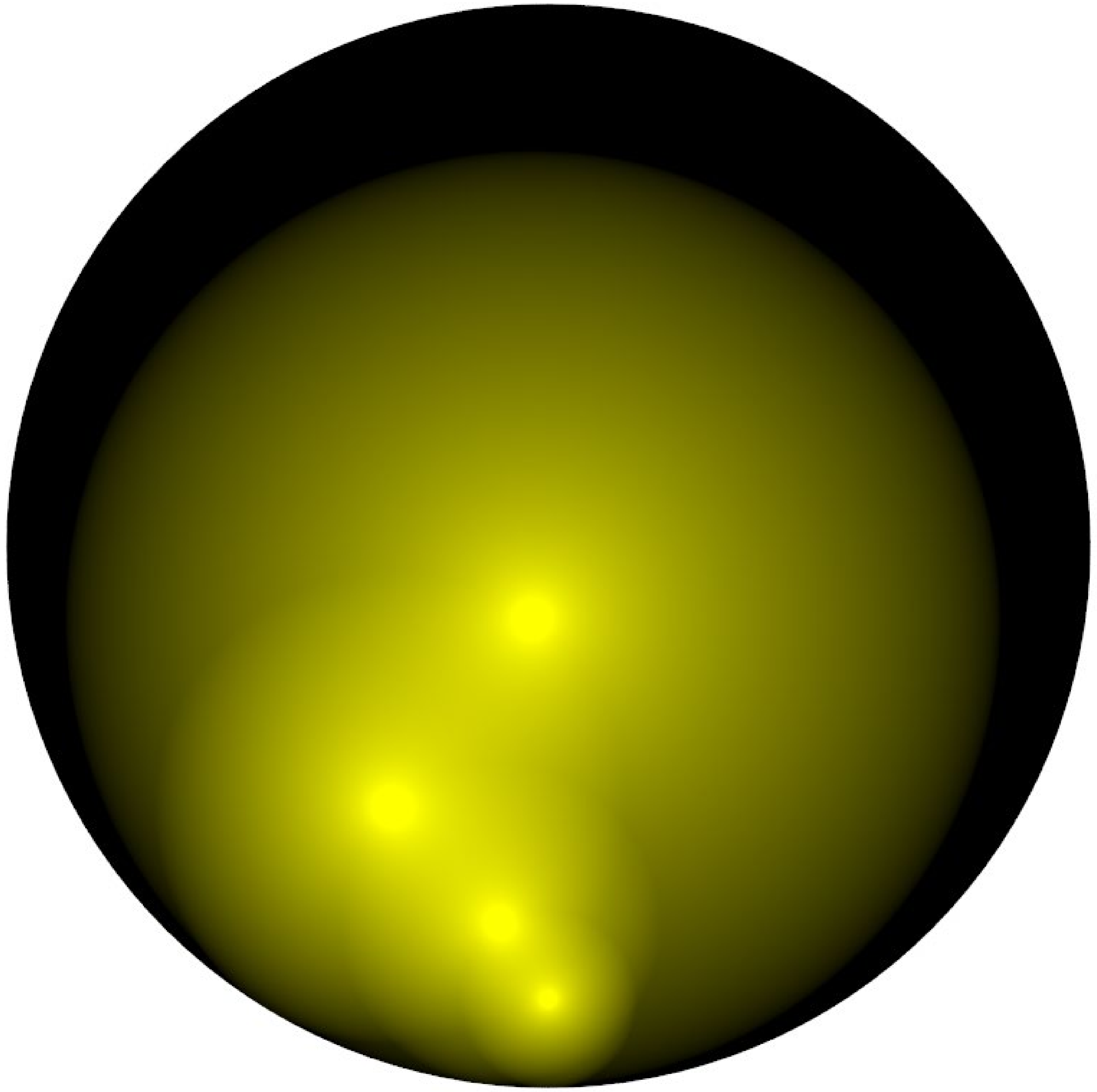
- Start walks from sources
- Globally importance sampling
- Reuse walks



2 steps

Reverse Estimator

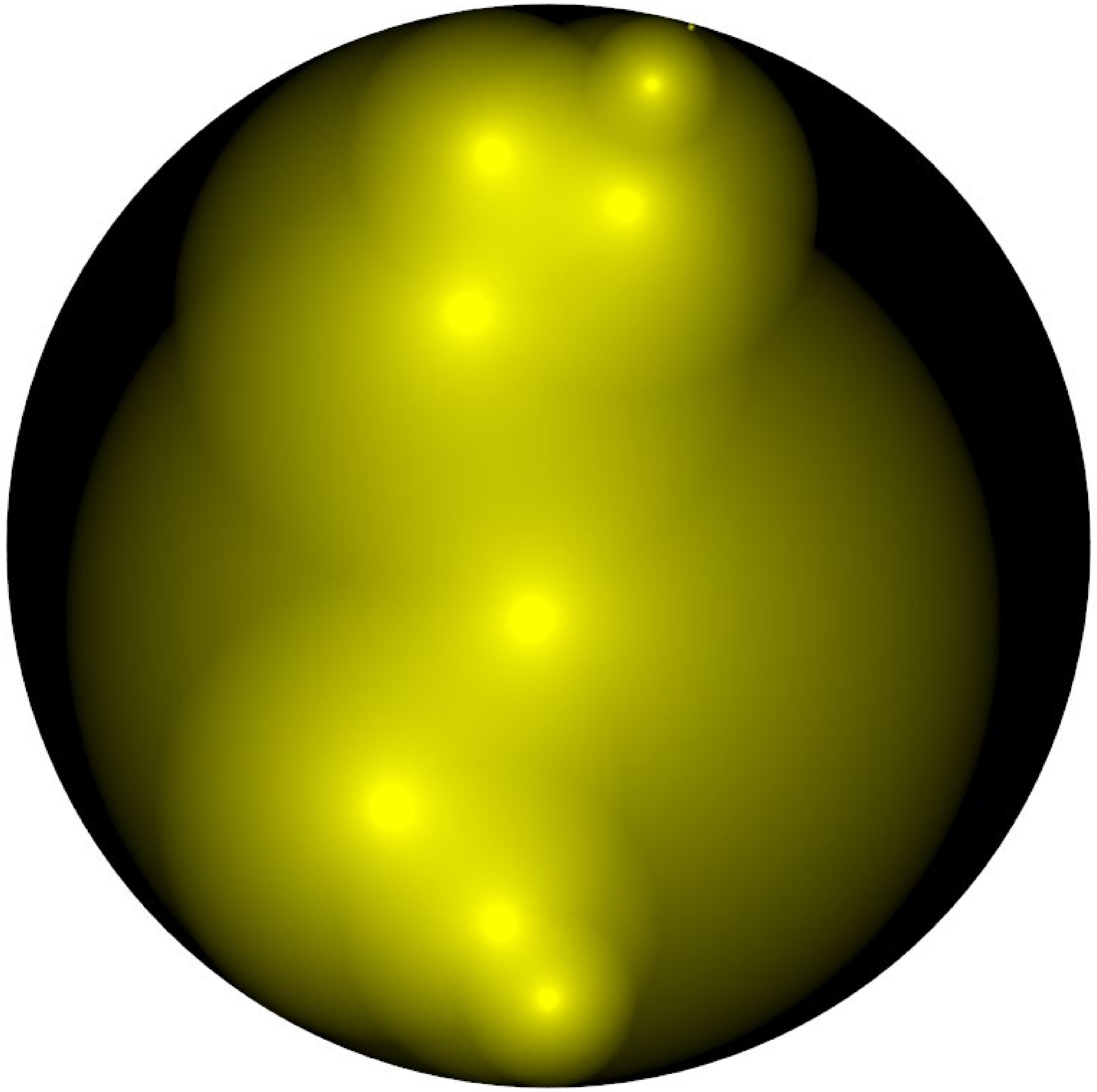
- Start walks from sources
- Globally importance sampling
- Reuse walks



4 steps

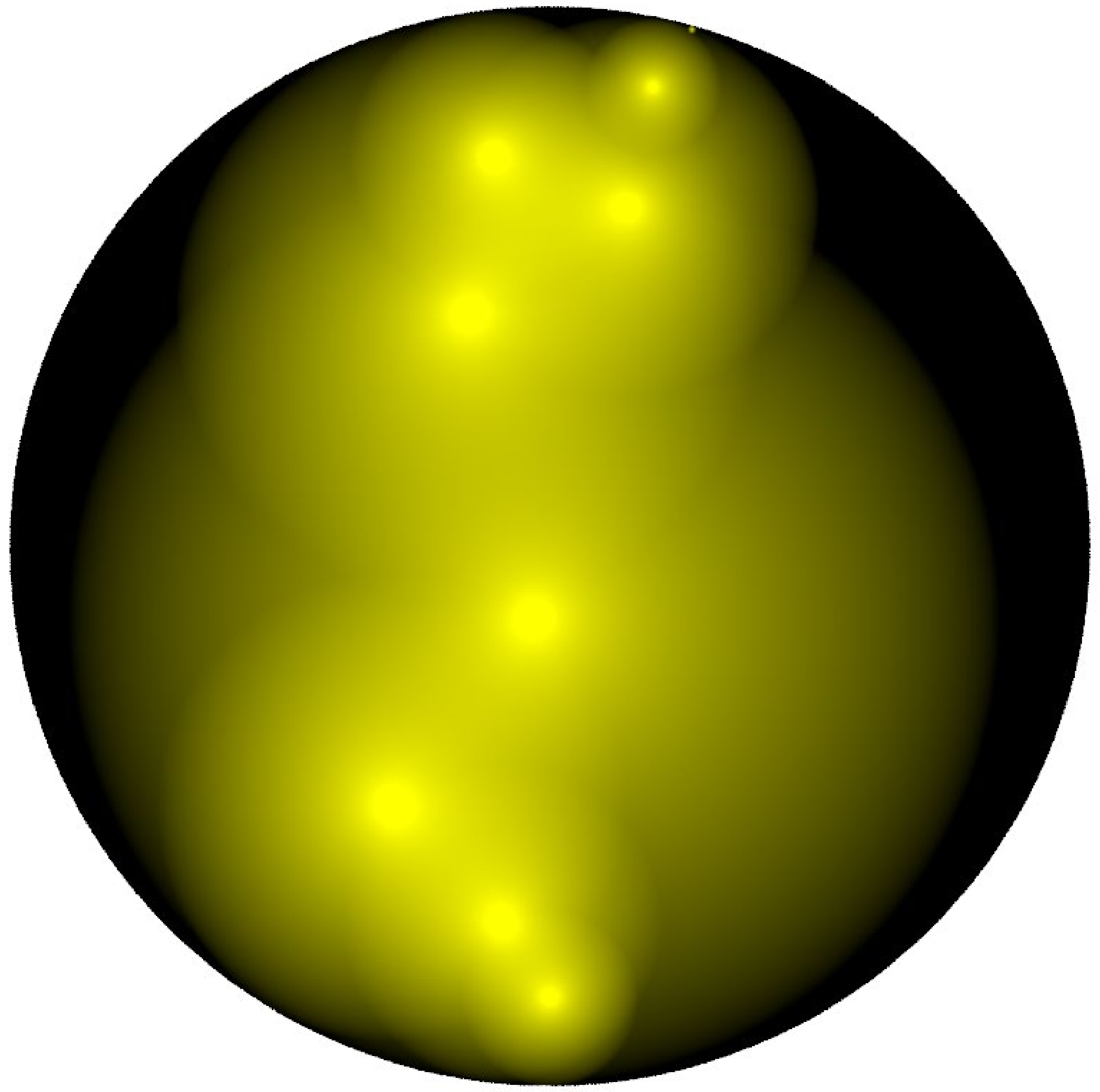
Reverse Estimator

- Start walks from sources
- Globally importance sampling
- Reuse walks

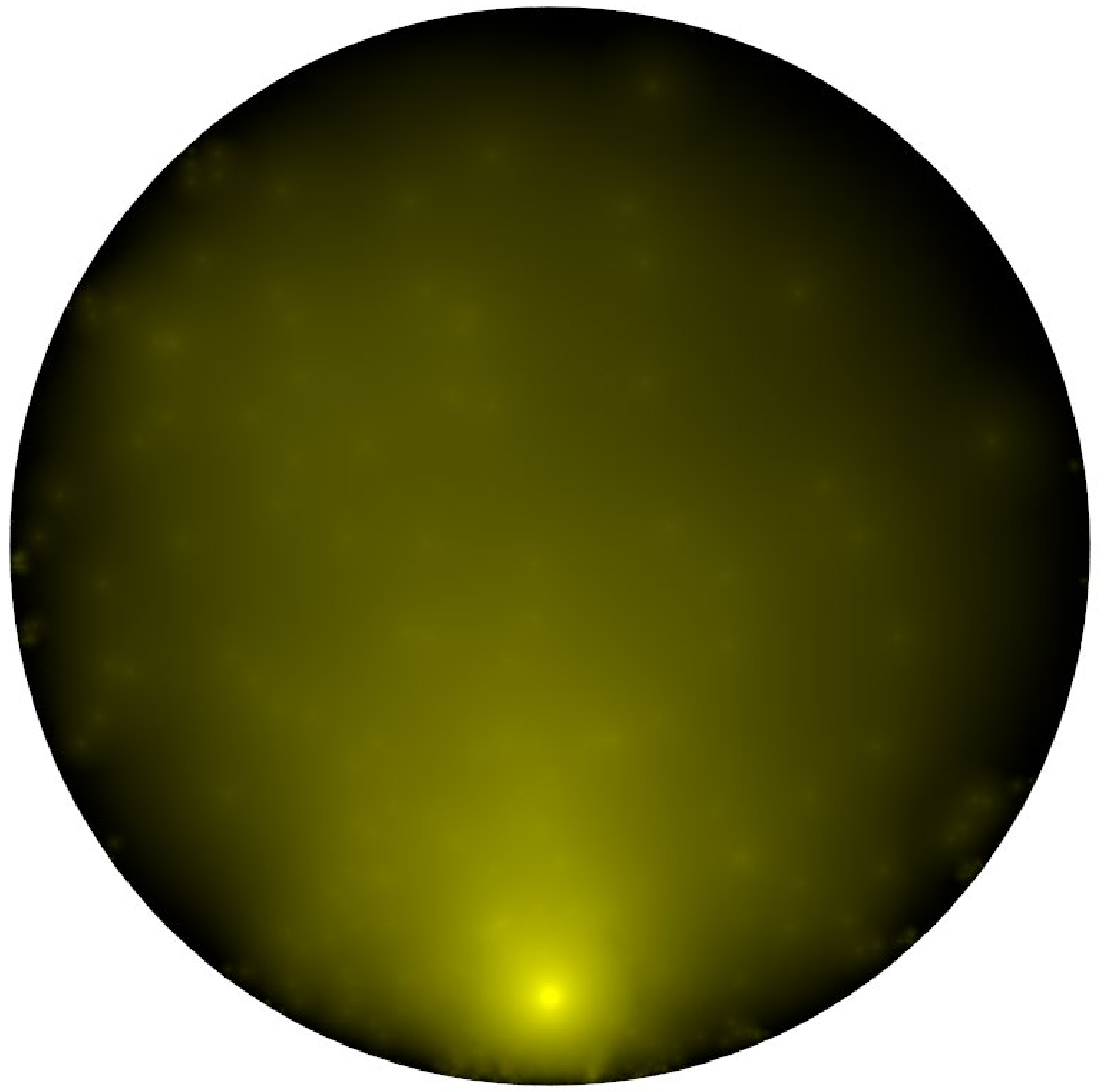


Hits boundary

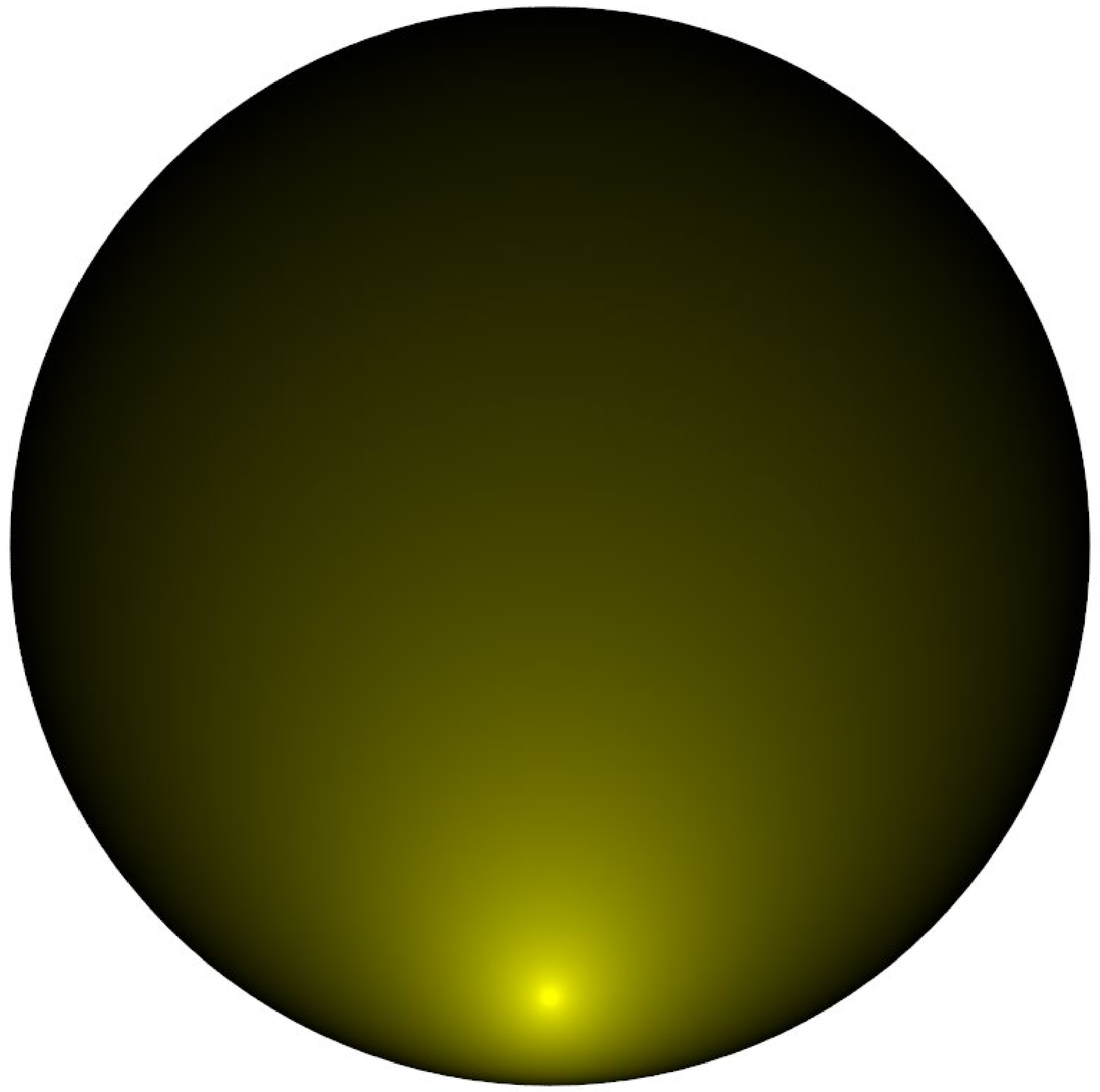
Reverse Estimator



1 sample



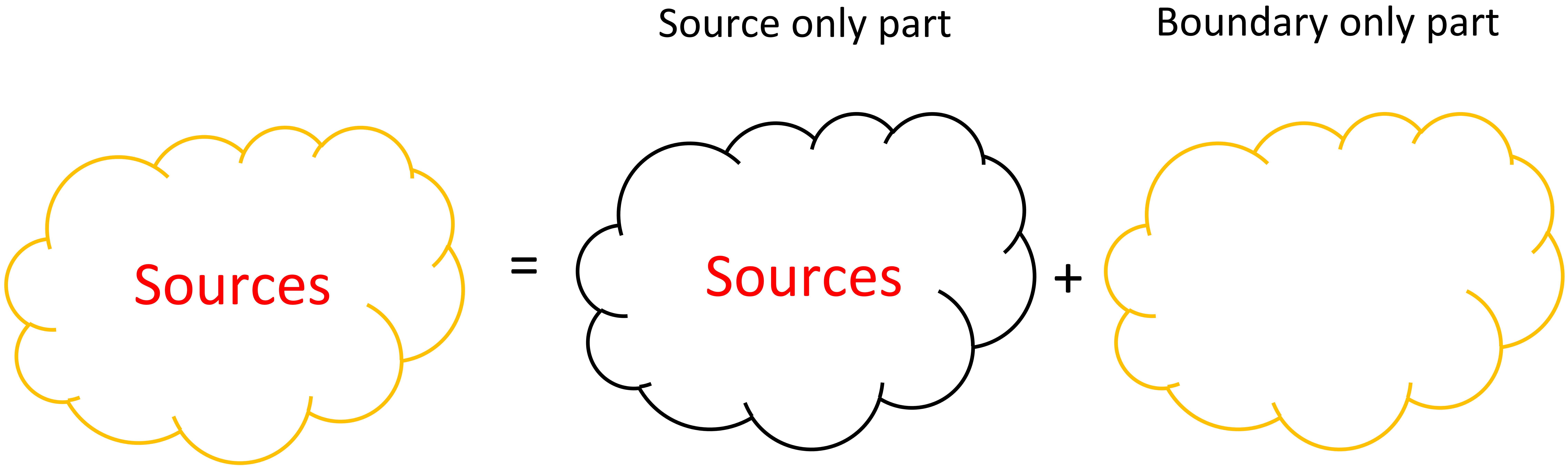
64 samples



Converged

Solve Poisson’s equation

Linear Property:



$$u(x) = \int_U f(y)\, G(x,y)\, \mathrm{d}y + \int_{\partial U} g(z)\, \frac{\partial G(x,z)}{\partial n(z)}\, \mathrm{d}z$$

Source Part

$$u(x) = \int_U f(y)\mathcal{G}(x \leftrightarrow y)\,\mathrm{d}y$$

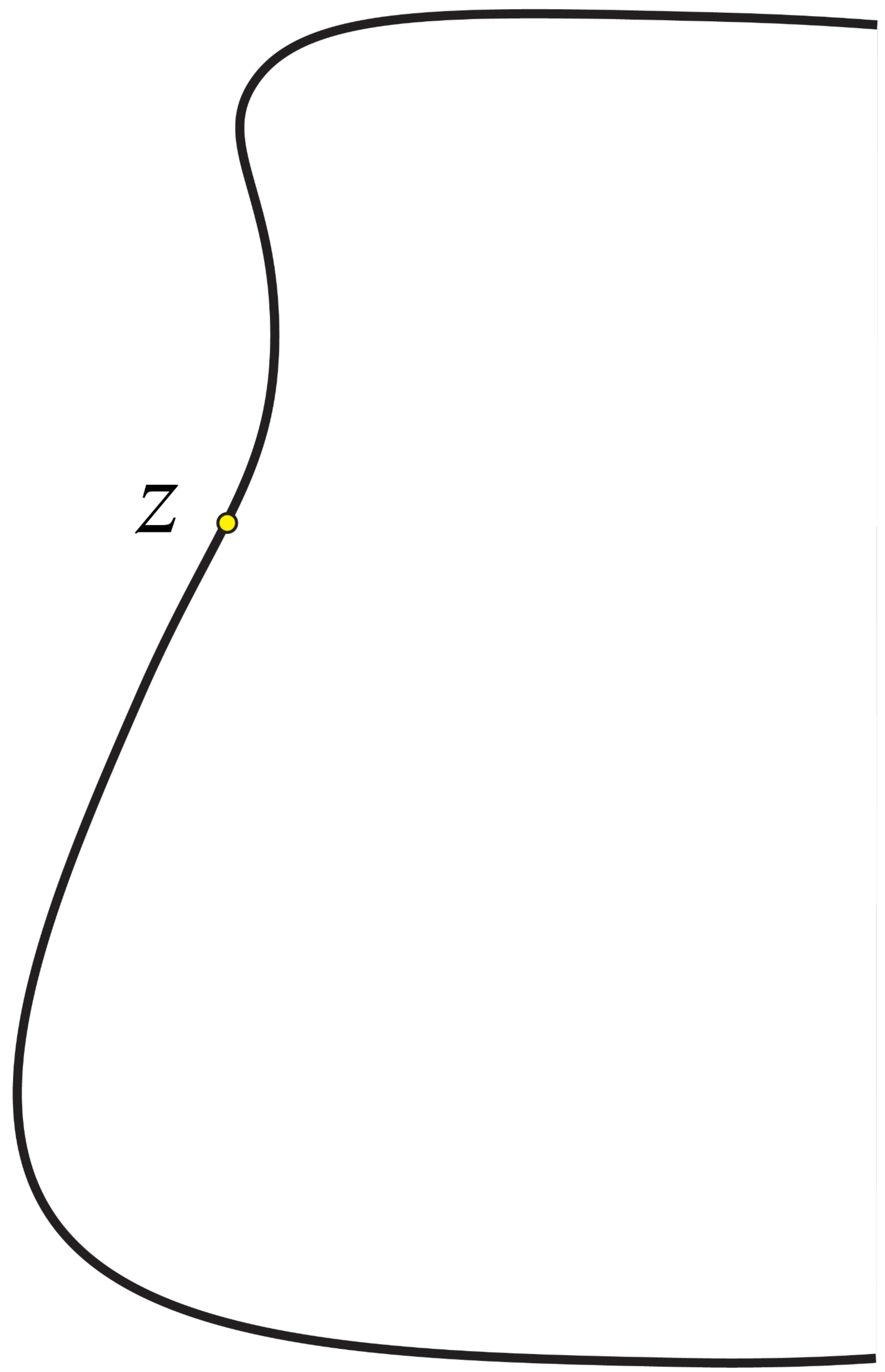
1. Sample $y \sim p^U(y)$
2. Estimate the solution using Green's function estimator $\langle u(x) \rangle = \frac{f(y)\langle \mathcal{G}(x \leftrightarrow y)\rangle}{p^U(y)}$

Boundary Part

$$u(x) = \int_{\partial U} g(z) \frac{\partial \mathcal{G}(x \leftrightarrow z)}{\partial z} \, \mathrm{d}z$$

Use finite difference method to estimate the derivative.

1. Sample $z \sim p^{\partial U}(z)$



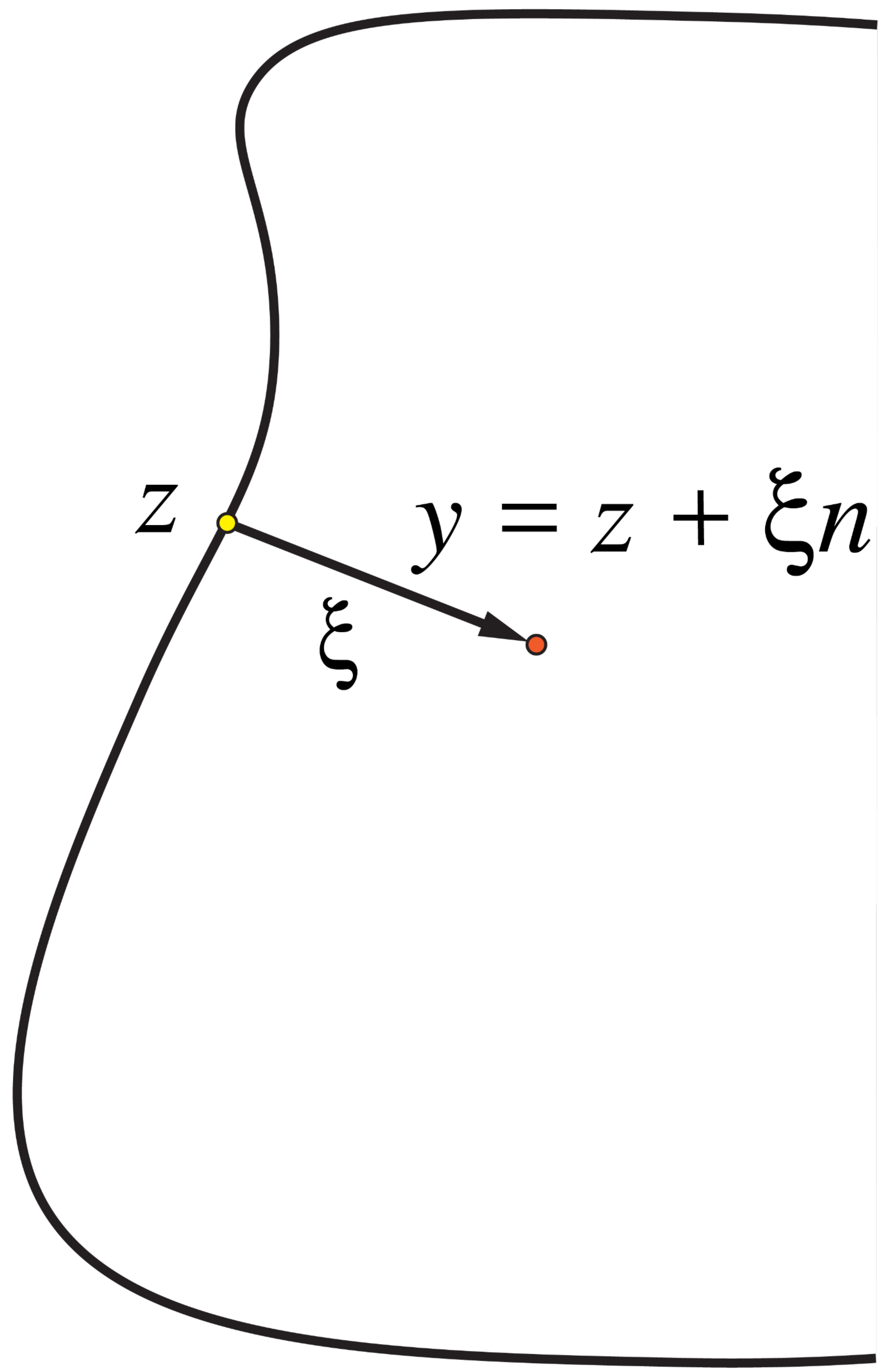
Boundary Part

$$u(x) = \int_{\partial U} g(z) \frac{\partial \mathcal{G}(x \leftrightarrow z)}{\partial z} \, \mathrm{d}z$$

Use finite difference method to estimate the derivative

1. Sample $z \sim p^{\partial U}(z)$

2. Push the point out by ξ



Boundary Part

$$u(x) = \int_{\partial U} g(z) \frac{\partial \mathcal{G}(x \leftrightarrow z)}{\partial z}\, \mathrm{d}z$$

Use finite difference method to estimate the derivative

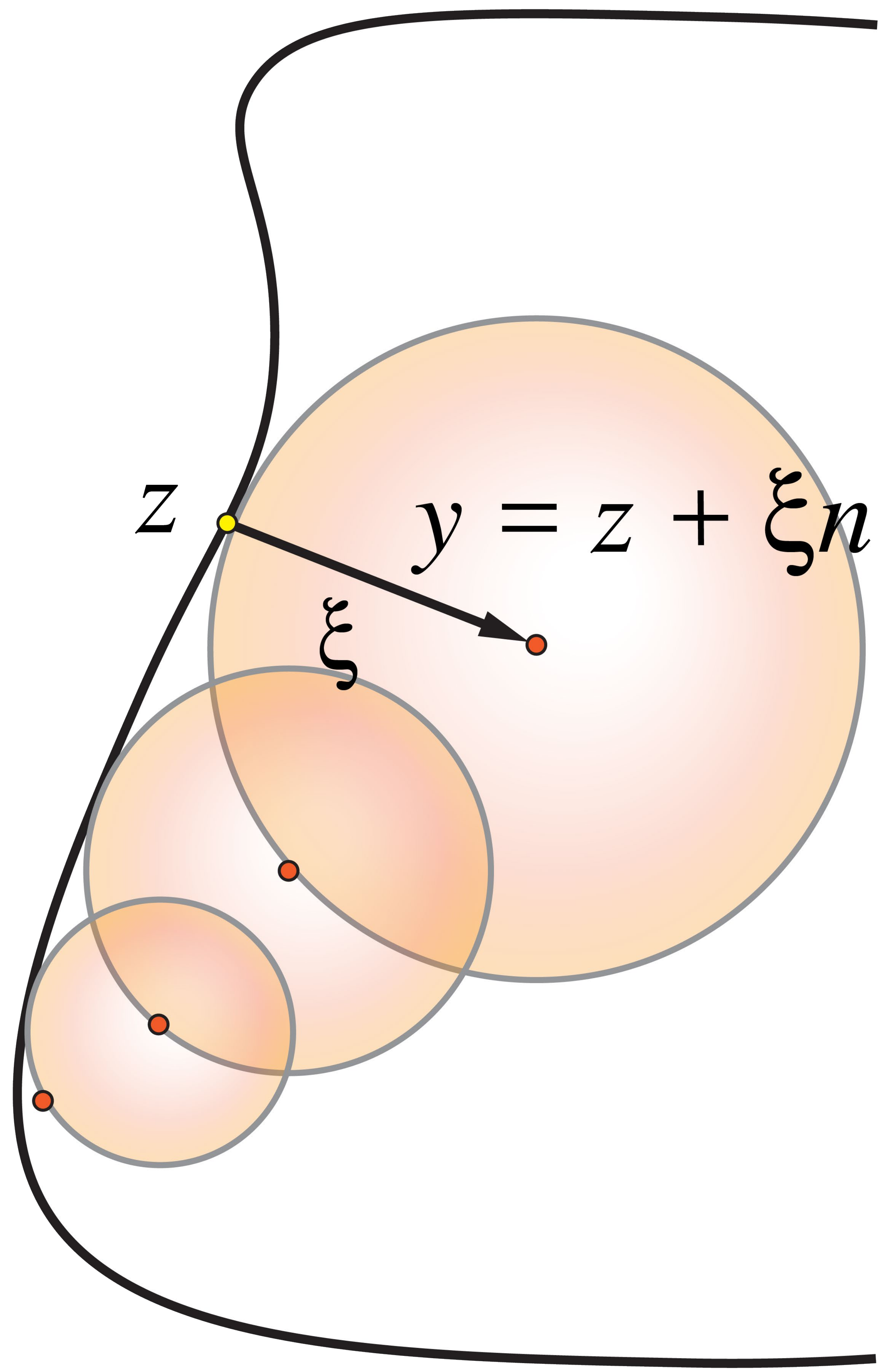
1. Sample $z \sim p^{\partial U}(z)$

2. Push the point out by ξ

3. Estimate the Green's function

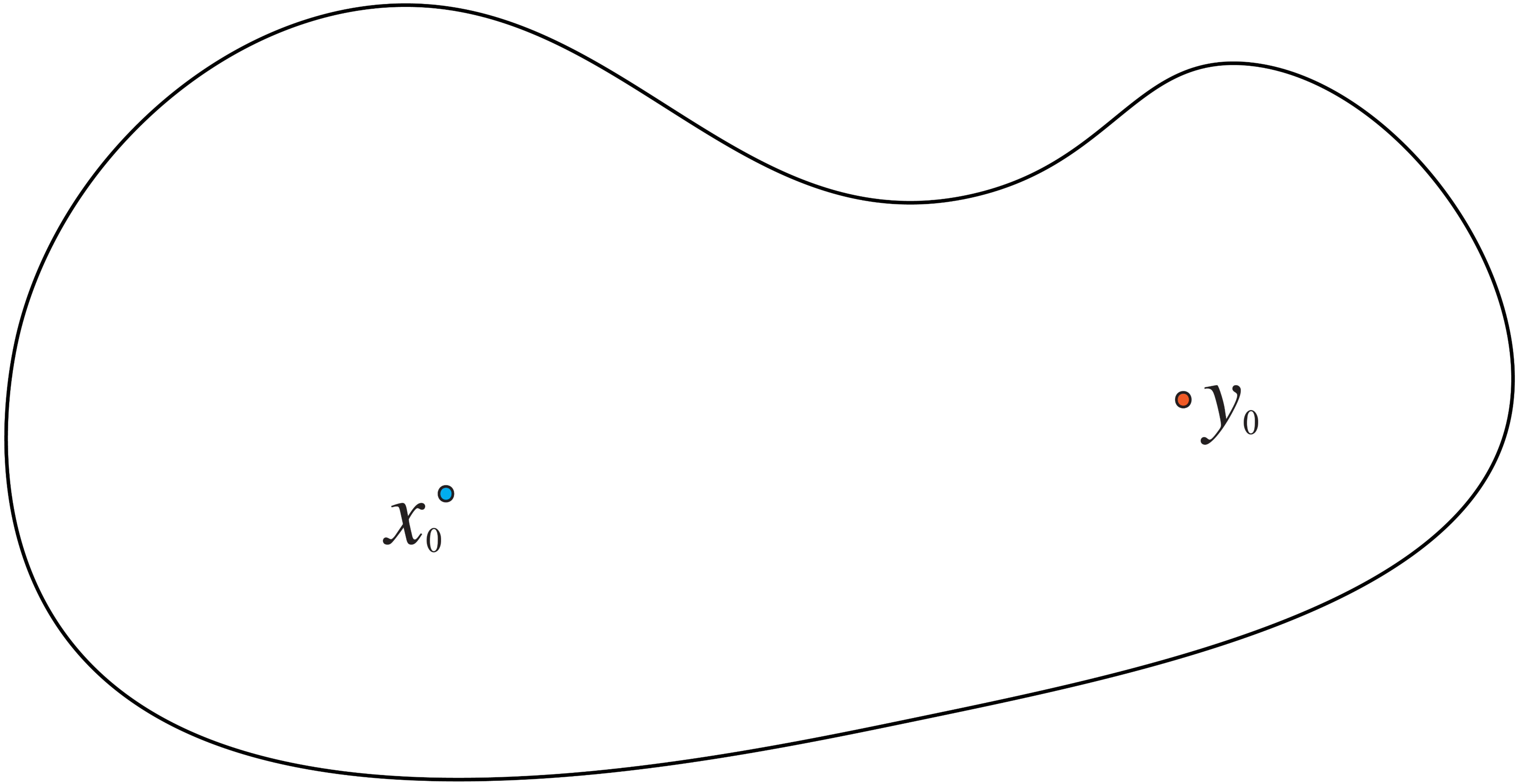
4. Estimate the solution through finite difference

$$\langle u(x) \rangle = \frac{g(z) \langle \mathcal{G}(x \leftrightarrow z) \rangle}{p^{\partial U}(z) \xi}$$



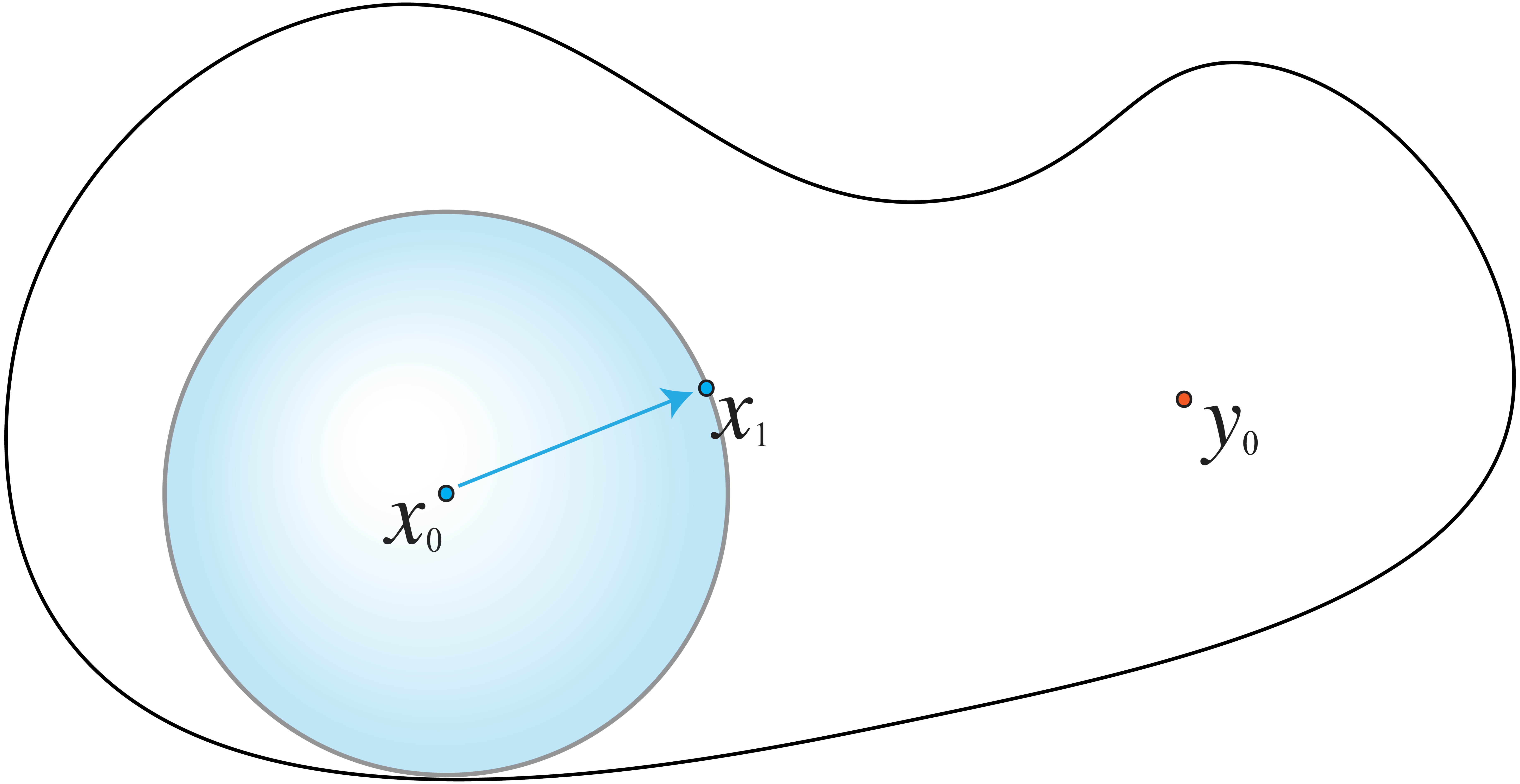
Combine Forward & Reverse

$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle =$



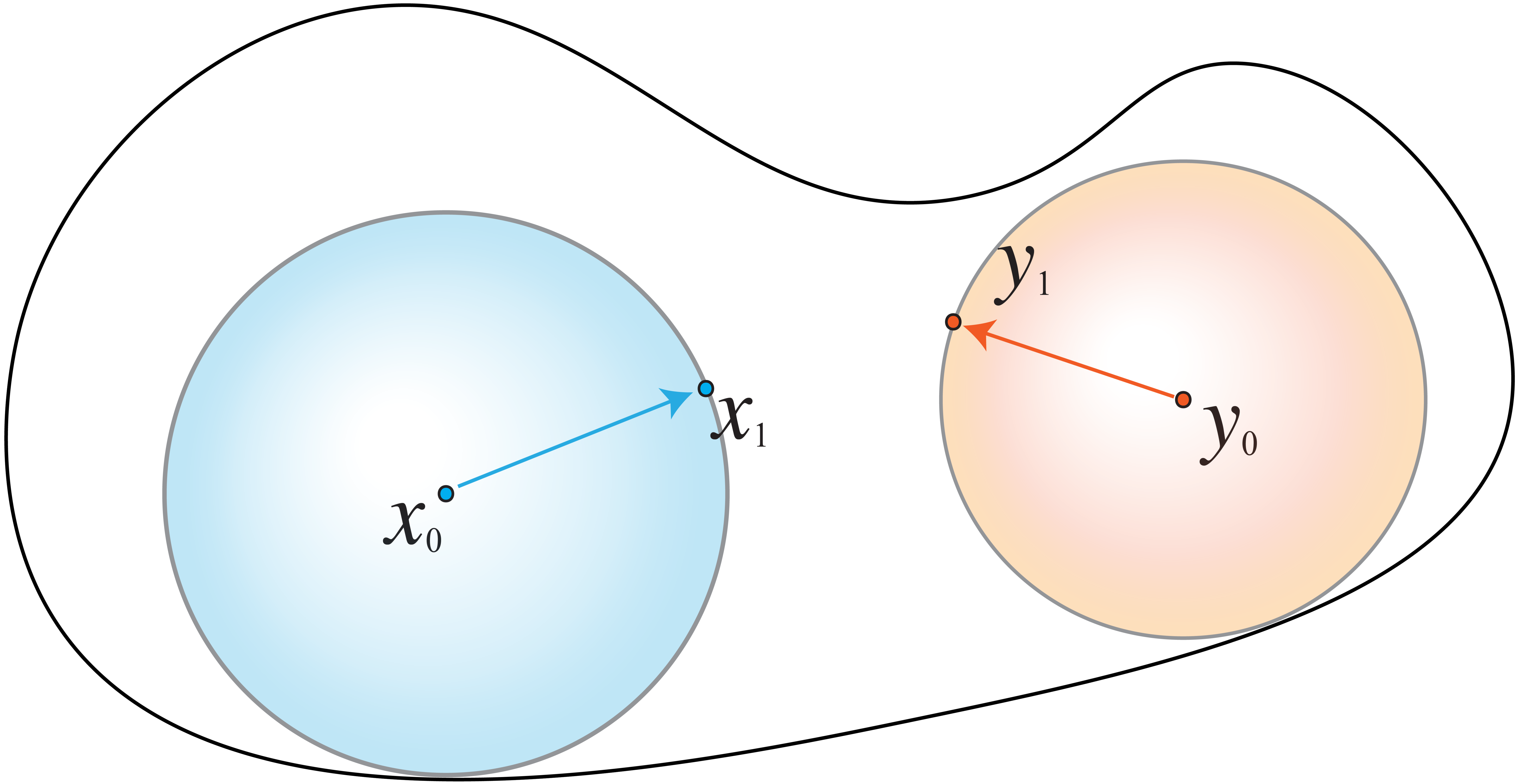
Combine Forward & Reverse

$$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle = 0 + \langle \mathcal{G}(x_1 \leftrightarrow y_0) \rangle$$



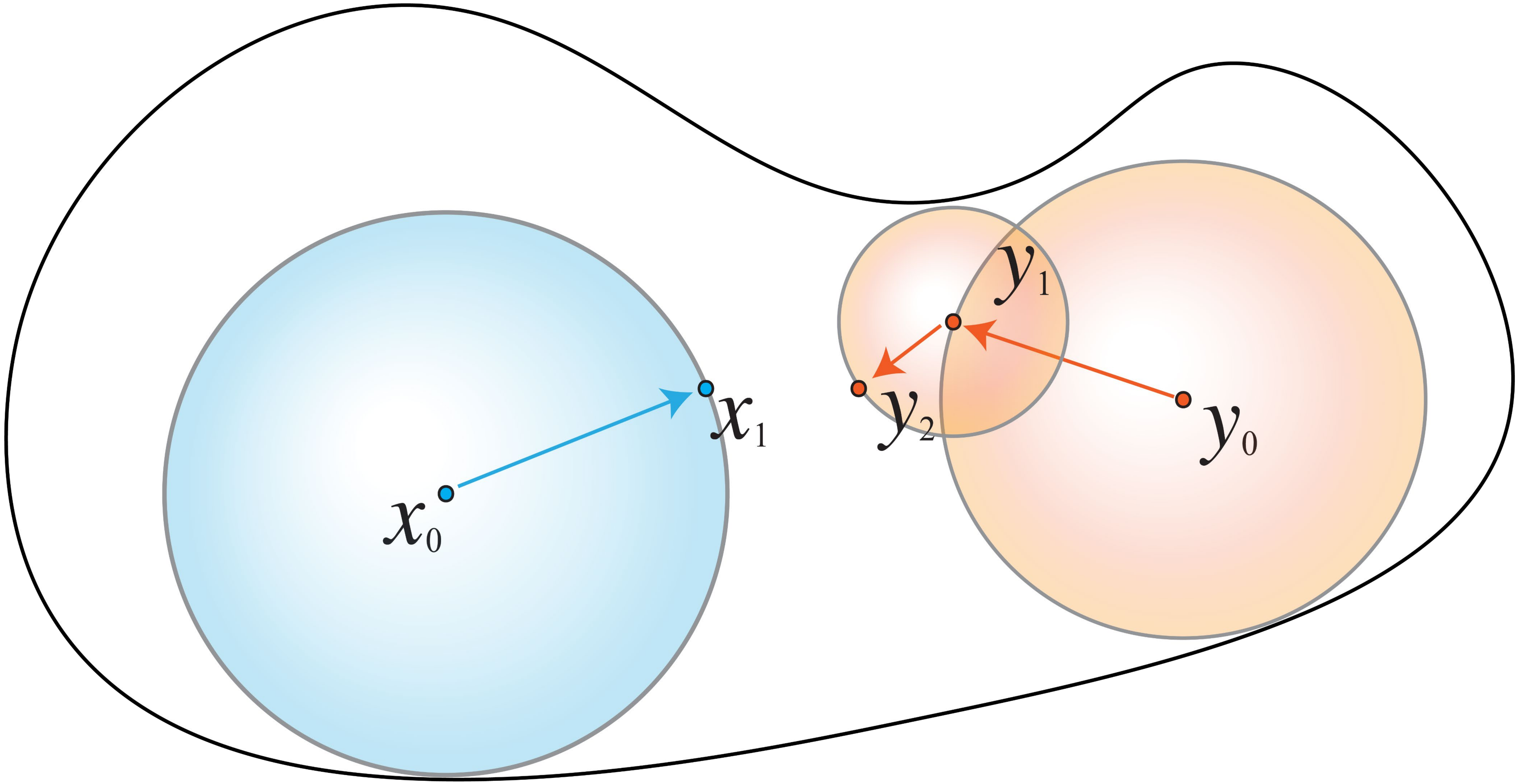
Combine Forward & Reverse

$$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle = 0 + \langle \mathcal{G}(x_1 \leftrightarrow y_1) \rangle$$



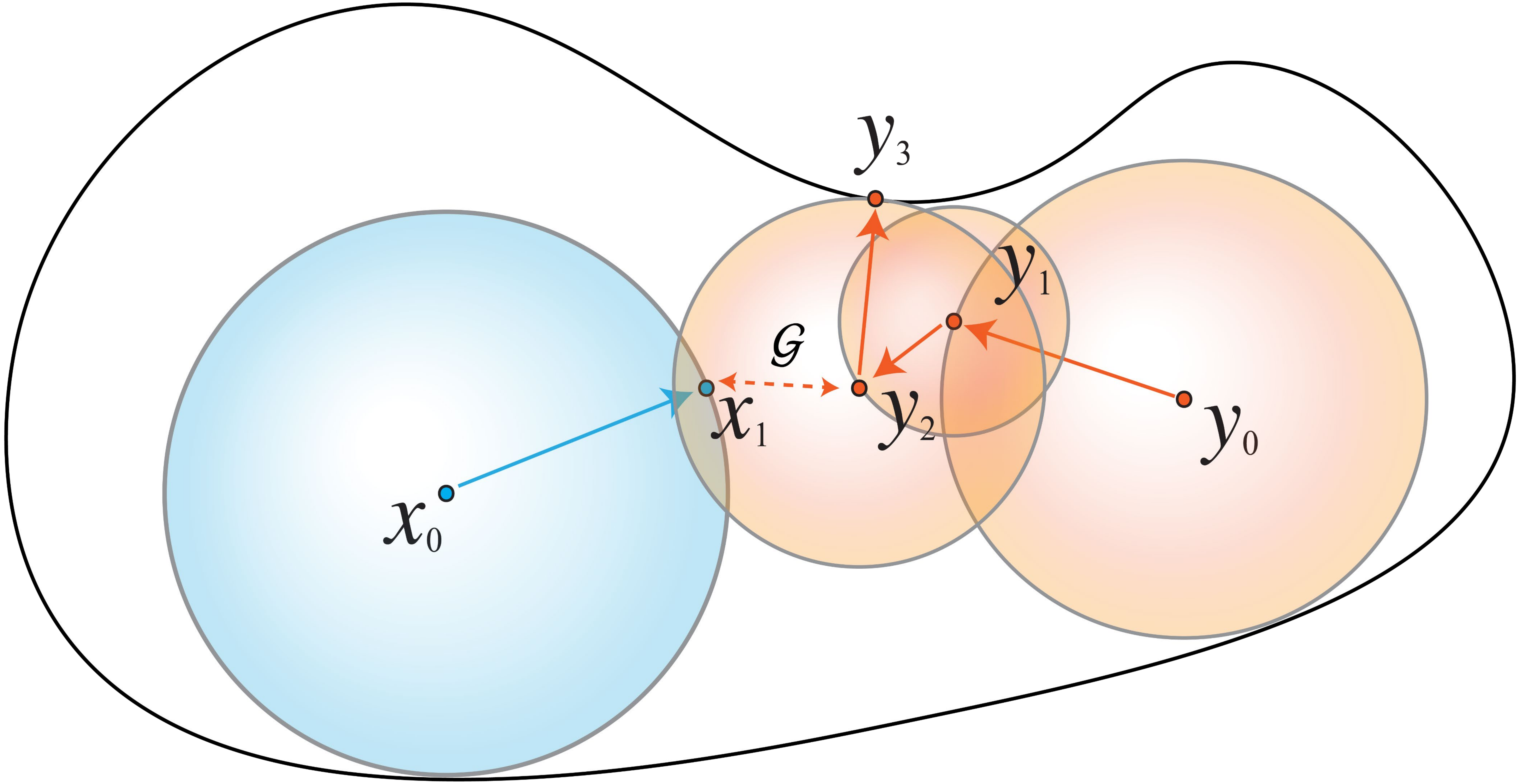
Combine Forward & Reverse

$$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle = 0 + \langle \mathcal{G}(x_1 \leftrightarrow y_2) \rangle$$



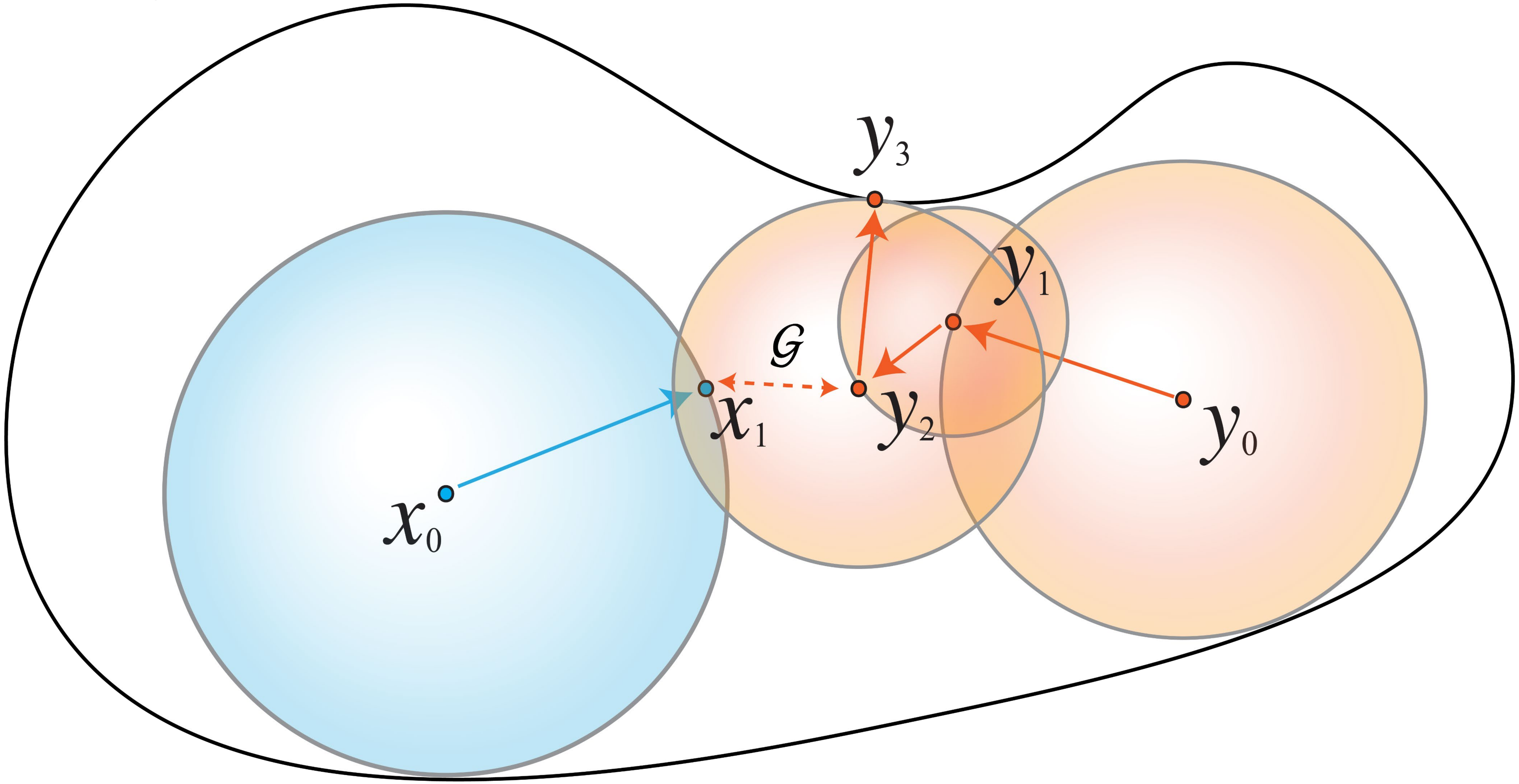
Combine Forward & Reverse

$$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle = \mathcal{G}^{B_{y_2}}(x_1 \leftrightarrow y_2) + \langle \mathcal{G}(x_1 \leftrightarrow y_3) \rangle$$

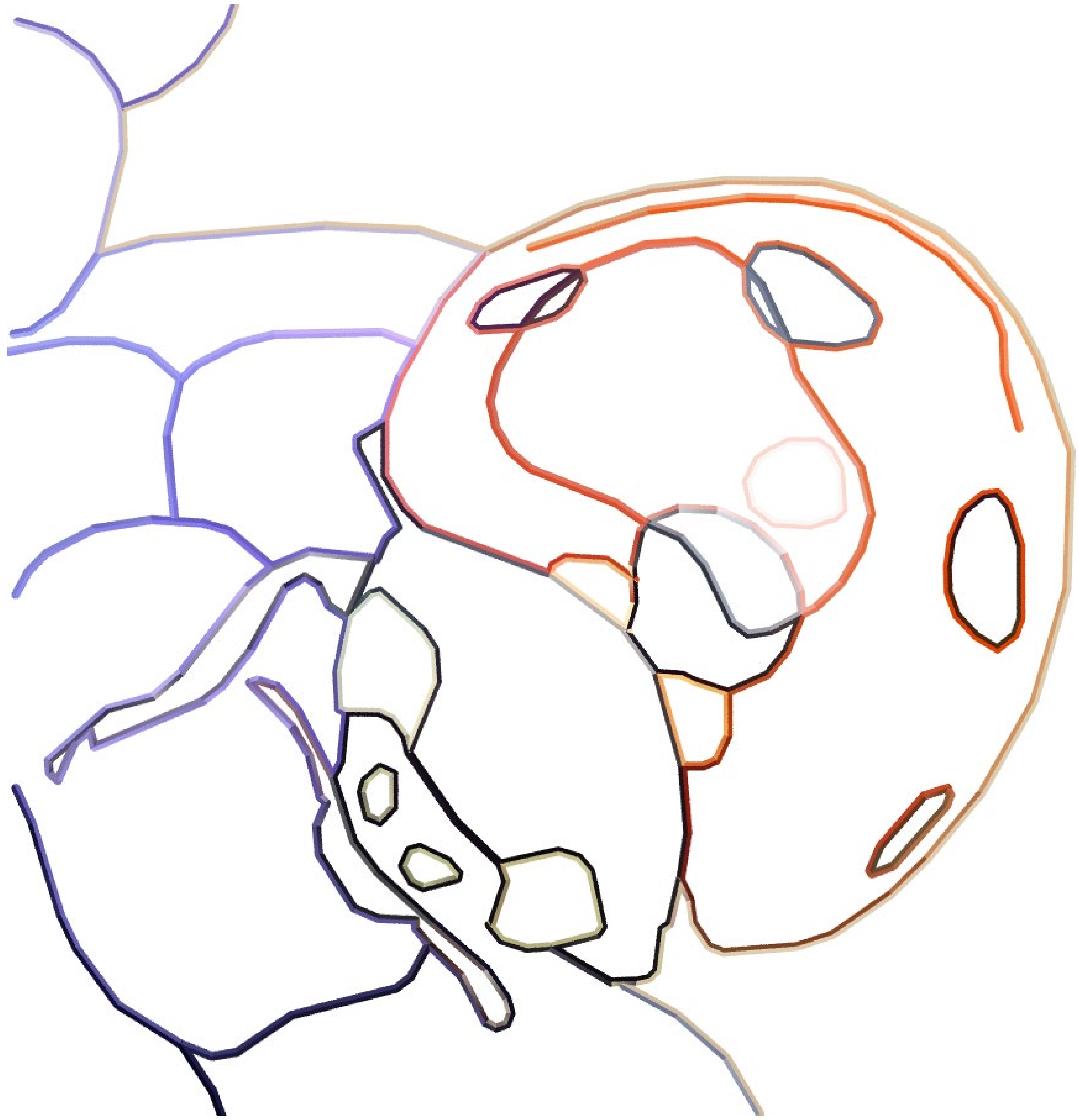


Combine Forward & Reverse

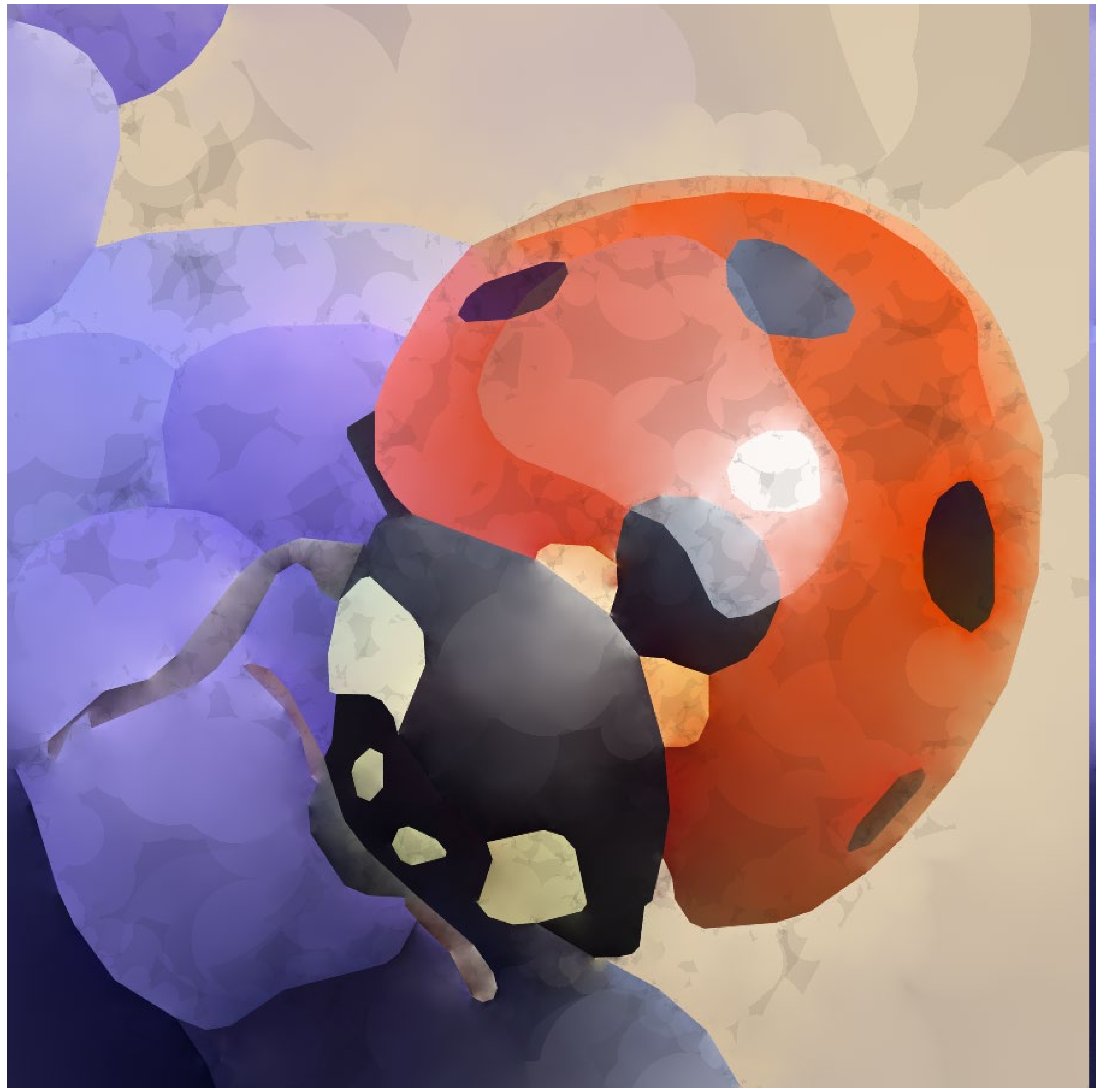
$$\langle \mathcal{G}(x_0 \leftrightarrow y_0) \rangle = \mathcal{G}^{B_{y_2}}(x_1 \leftrightarrow y_2)$$



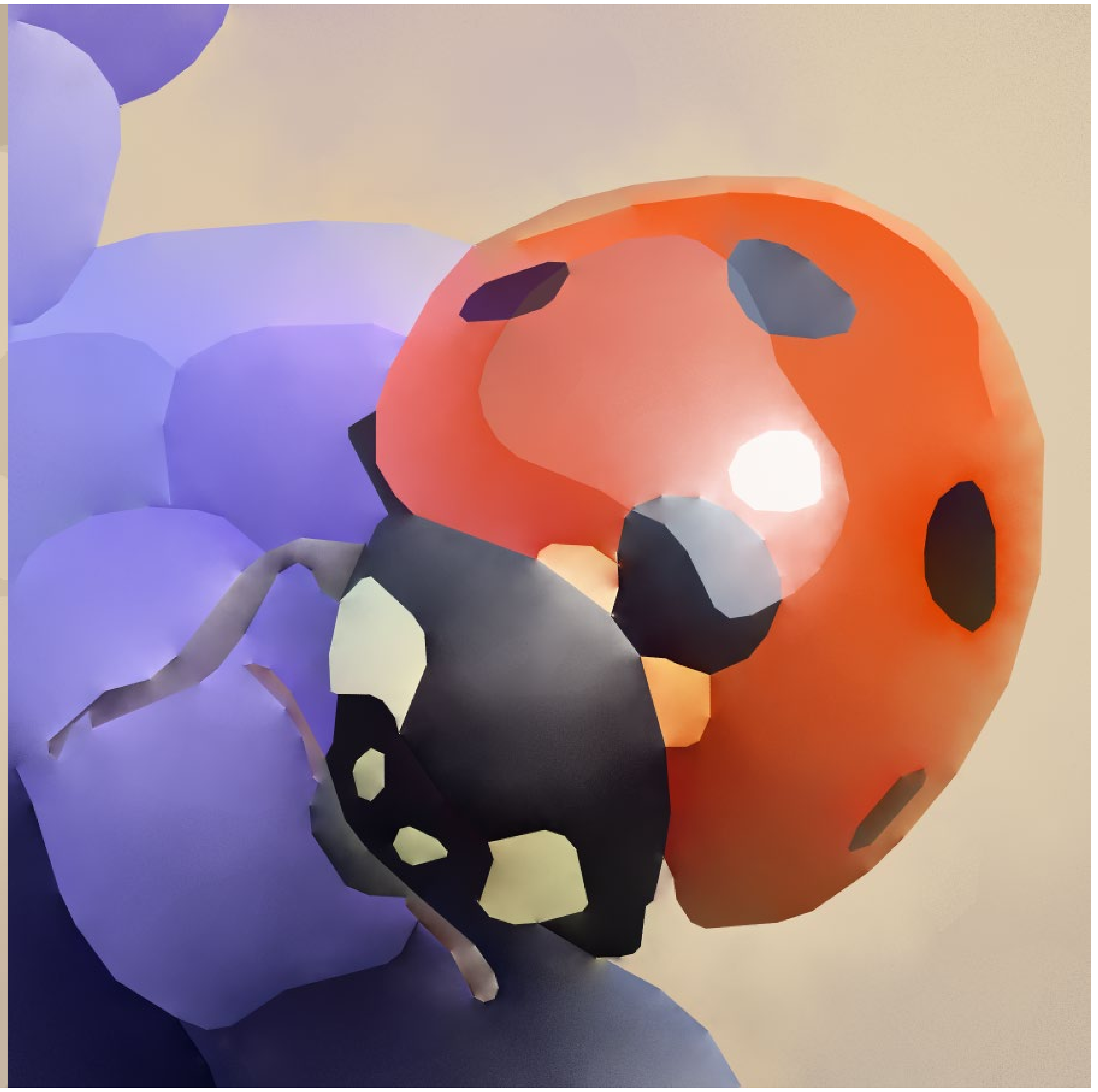
Final Gather



Boundary value



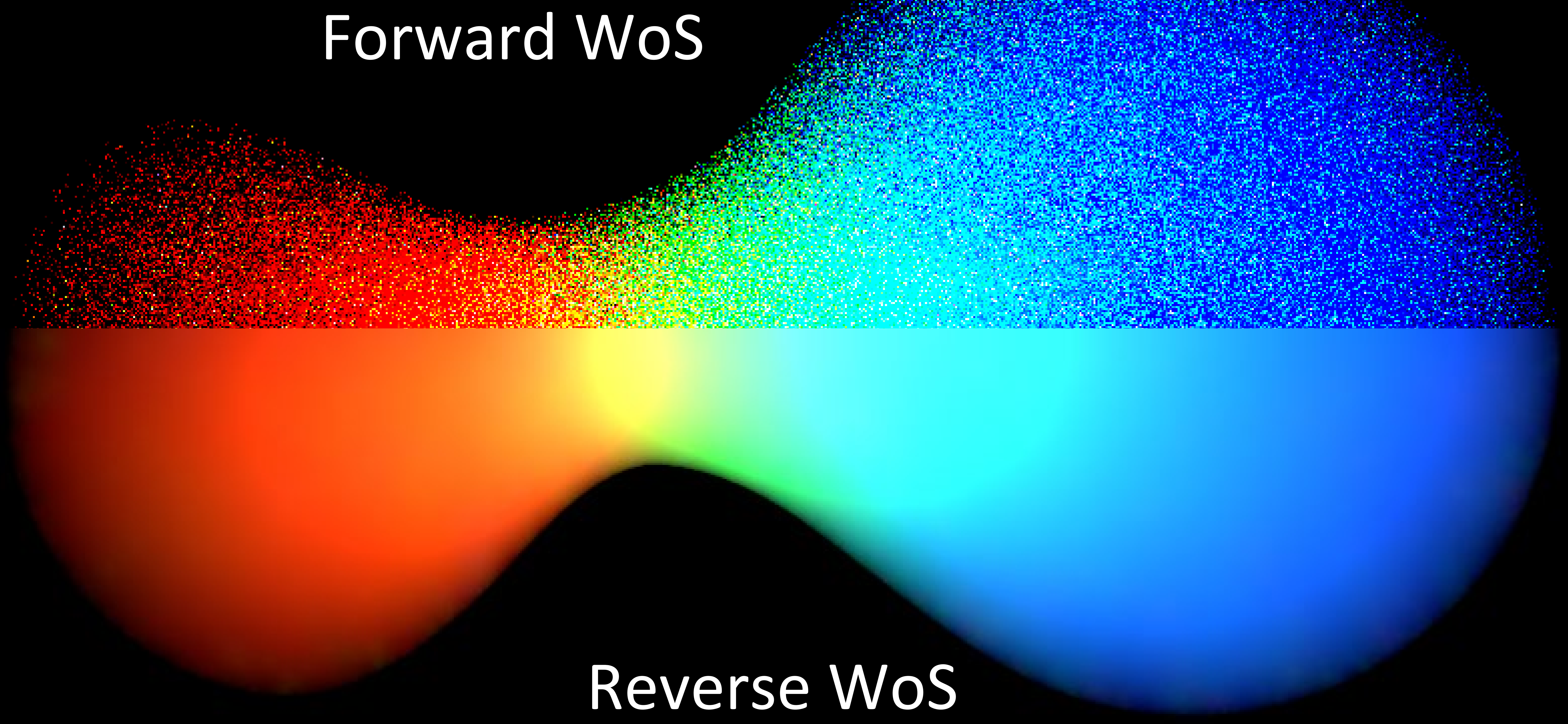
Reverse WoS (Insufficient samples)



Final gather through forward walks

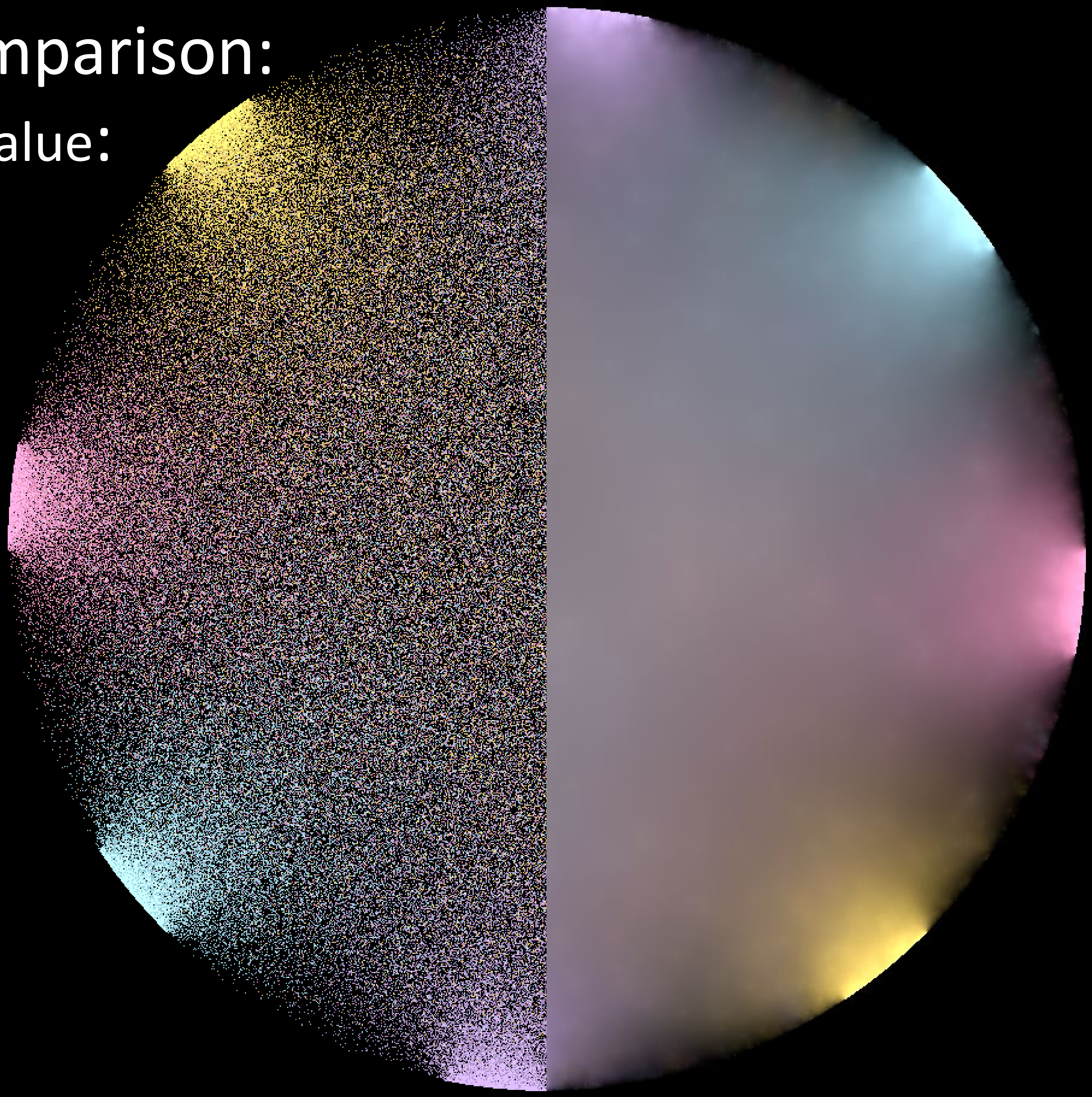
Equal Time Comparison:

Sparse Source Term:



Equal Time Comparison:

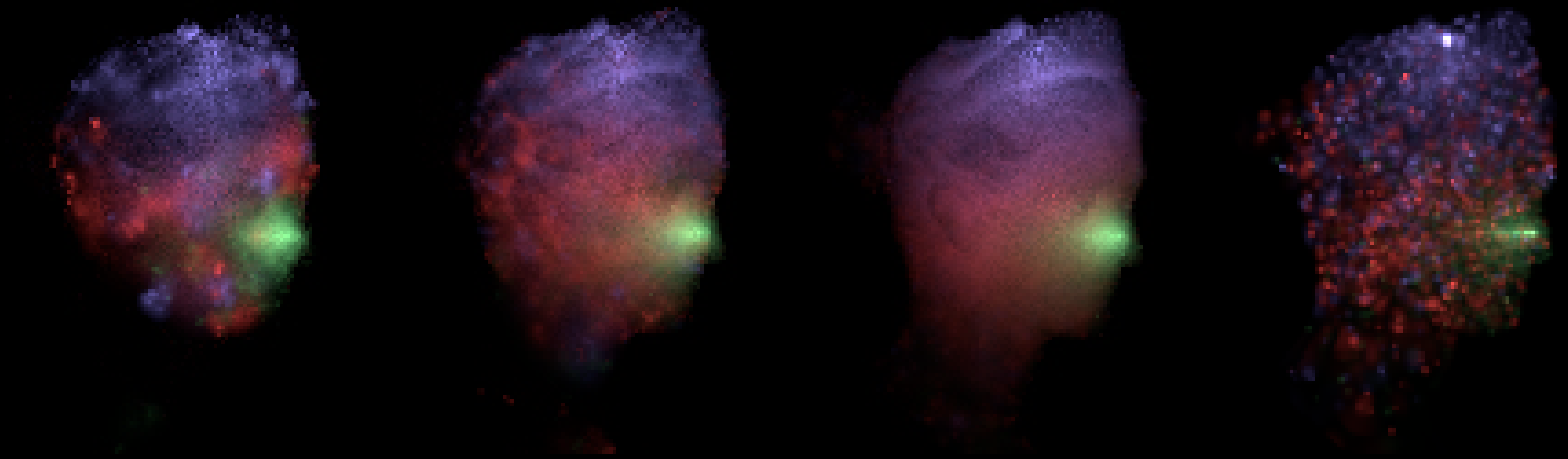
Sparse Boundary Value:



Forward WoS

Reverse WoS

3D Diffusion:



Ours(1000) Ours(10000) Ours(100,000) VPM(100,000)

Other PDEs:

- Rendering equation
- Volume rendering equation

Shell Tracing:

- LEE, R. T. and O'SULLIVAN, C.
“Accelerated light propagation through participating media”.

- MULLER, T., PAPAS, M., GROSS, M., JAROSZ, W., and NOVAK, J.
“Efficient rendering of heterogeneous polydisperse granular media”.

- LEONARD, L., HOHLEIN, K., and WESTERMANN, R.
“Learning multiple-scattering solutions for sphere-tracing”.

https://cs.dartmouth.edu/~wjarosz/publications/muller16efficient.html

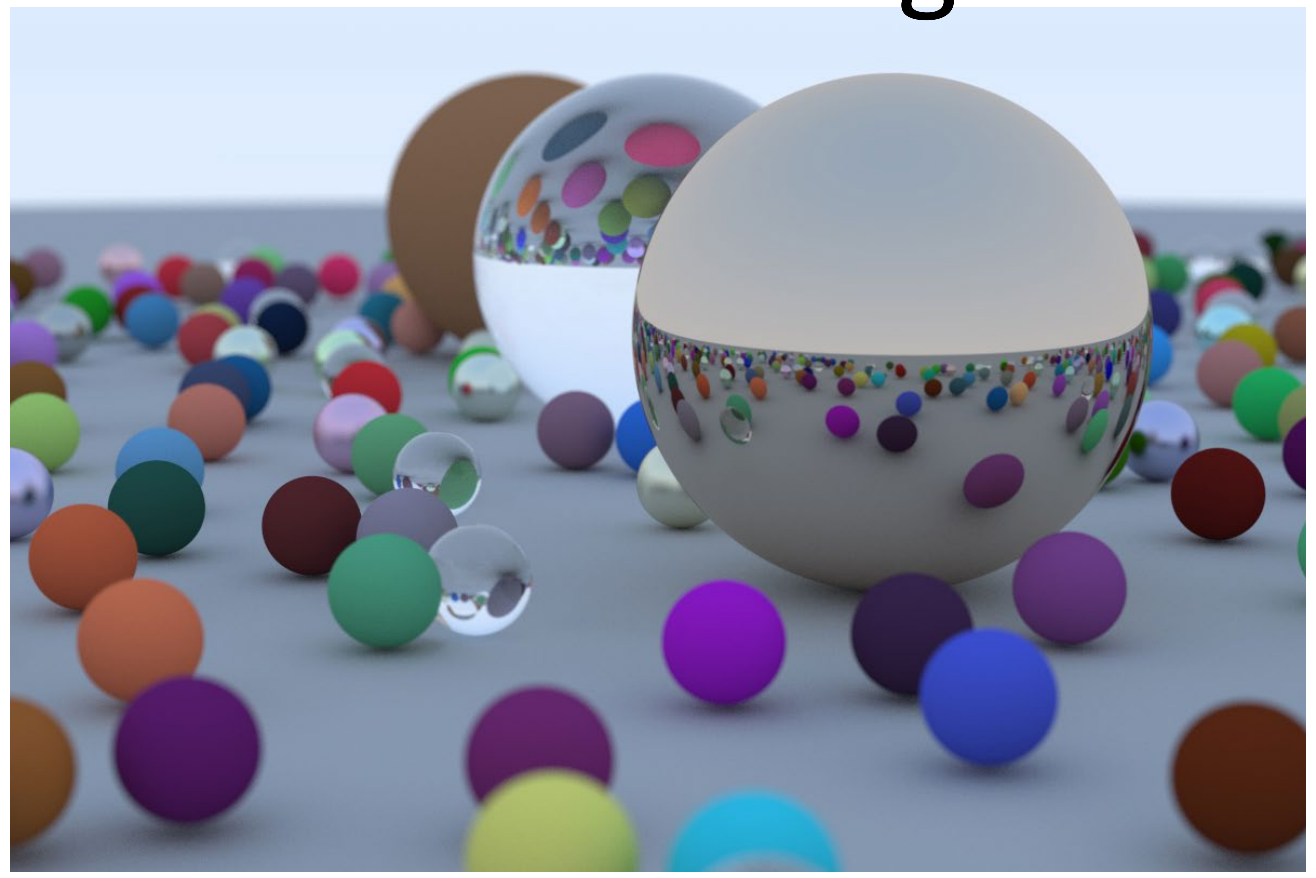
Other PDEs:

- Rendering equation
- Volume rendering equation

- Heat equation (time dependent)

Bidirectional Method

Forward
Path Tracing



raytracing.github.io/books/RayTracingInOneWeekend.html

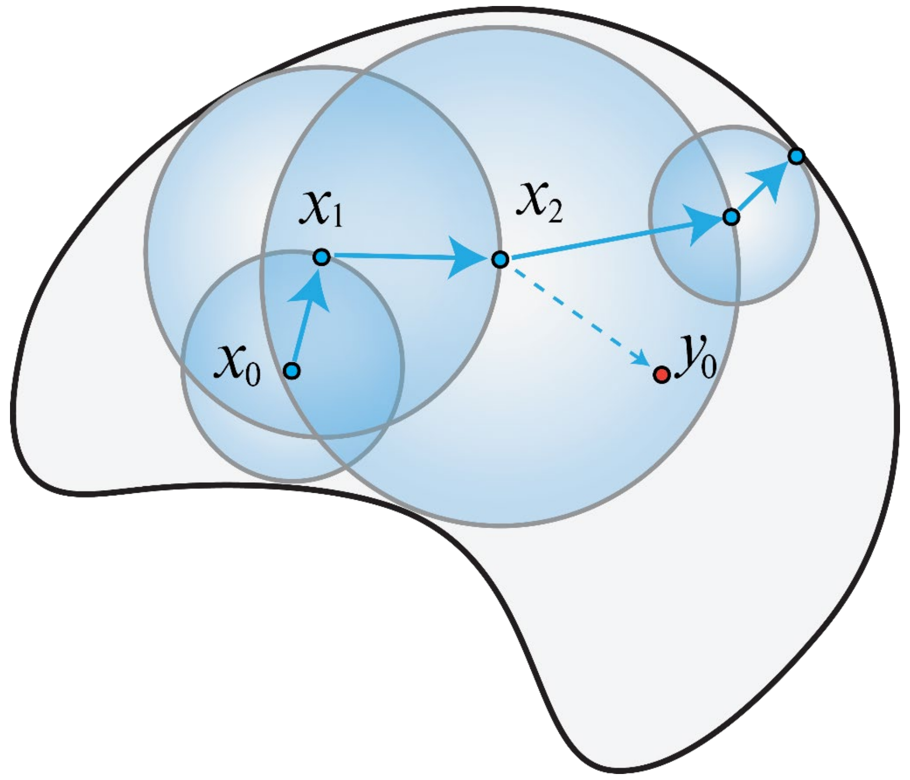
Bidirectional
Bidirectional Path Tracing

graphics.stanford.edu/papers/veach_thesis/thesis.pdf

Reverse
Photon Mapping / VPLs

graphics.ucsd.edu/~henrik/papers/photon_map/

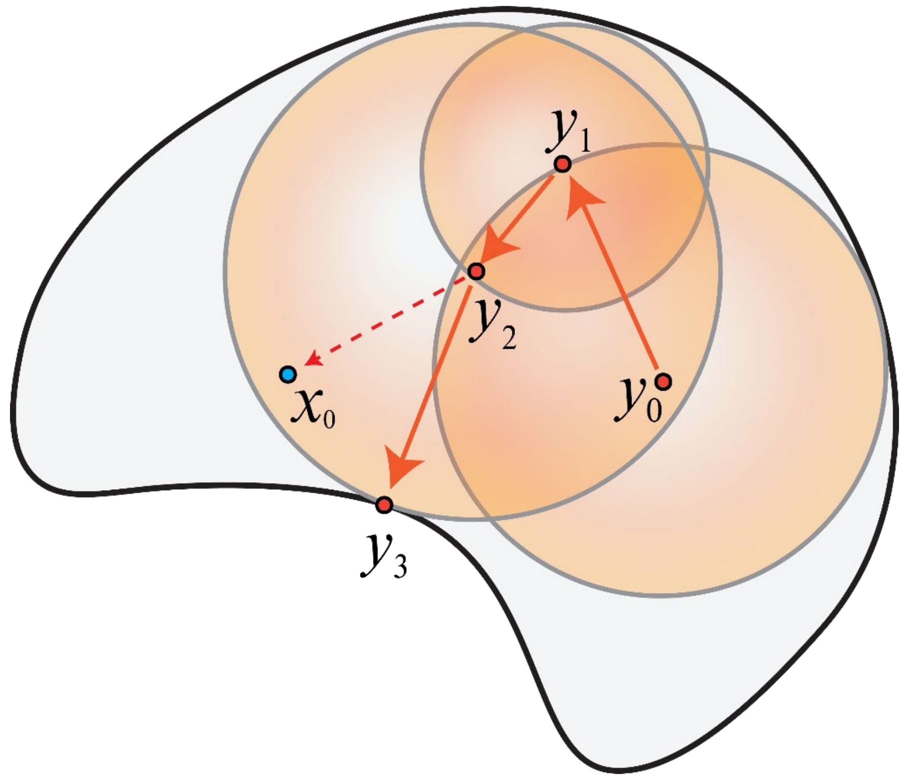
Forward WoS



Bidirectional WoS?

?

Reverse WoS



MIS

Different choices of forward & reverse steps lead to different path spaces

How to perform MIS between all different path spaces?

Thank you!