Error Analysis of Common Sampling Stratgies

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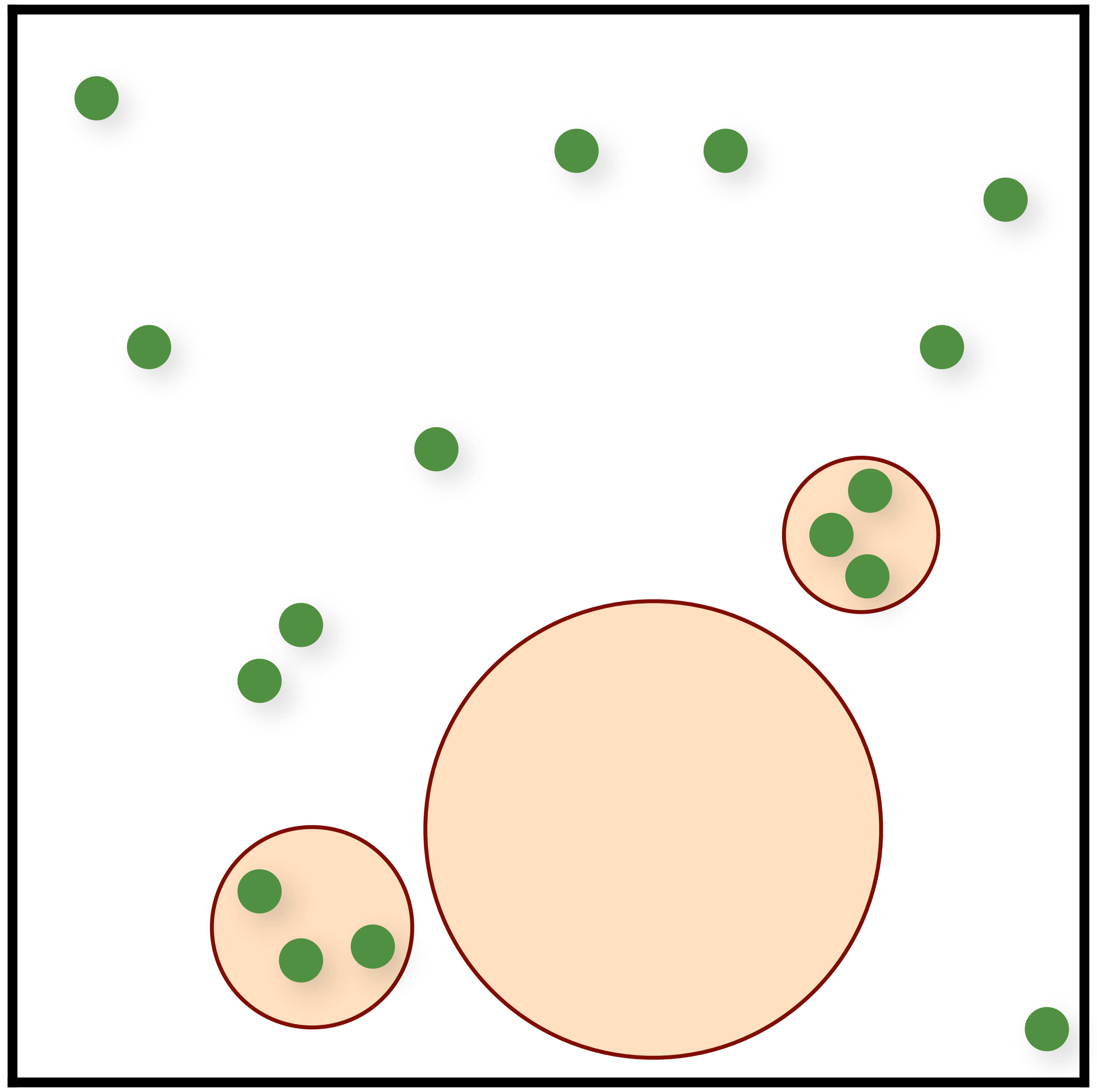
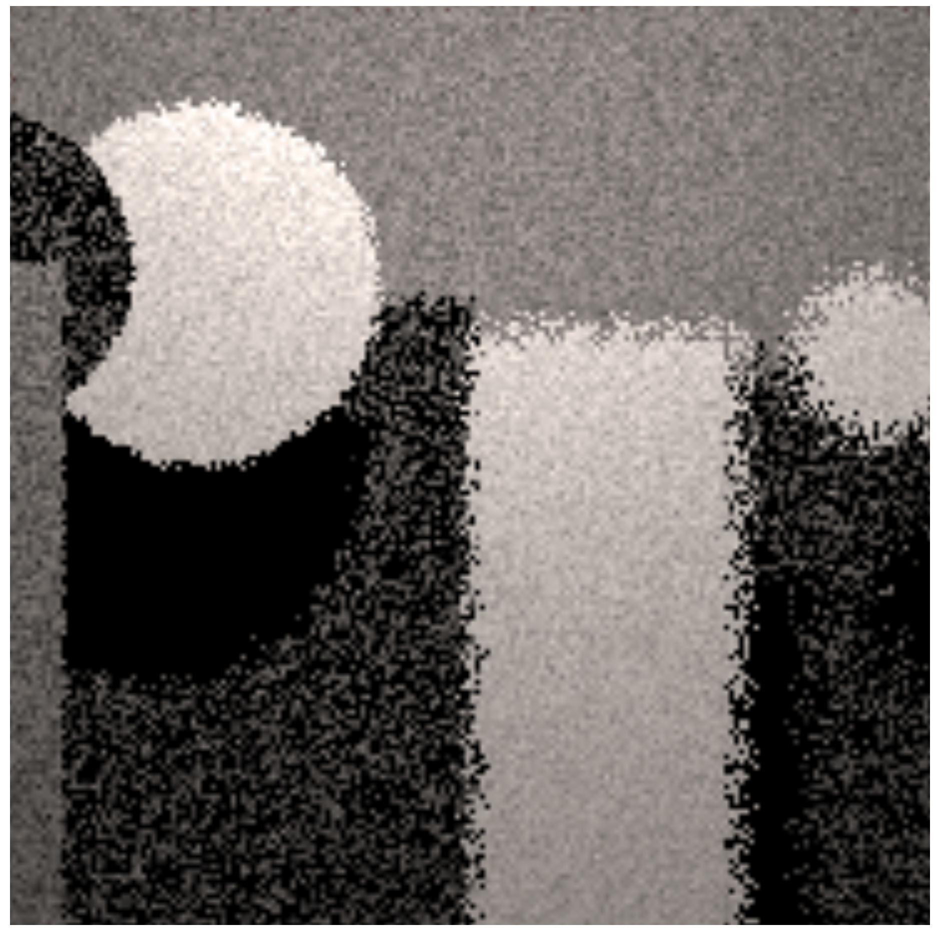
informatik

Random





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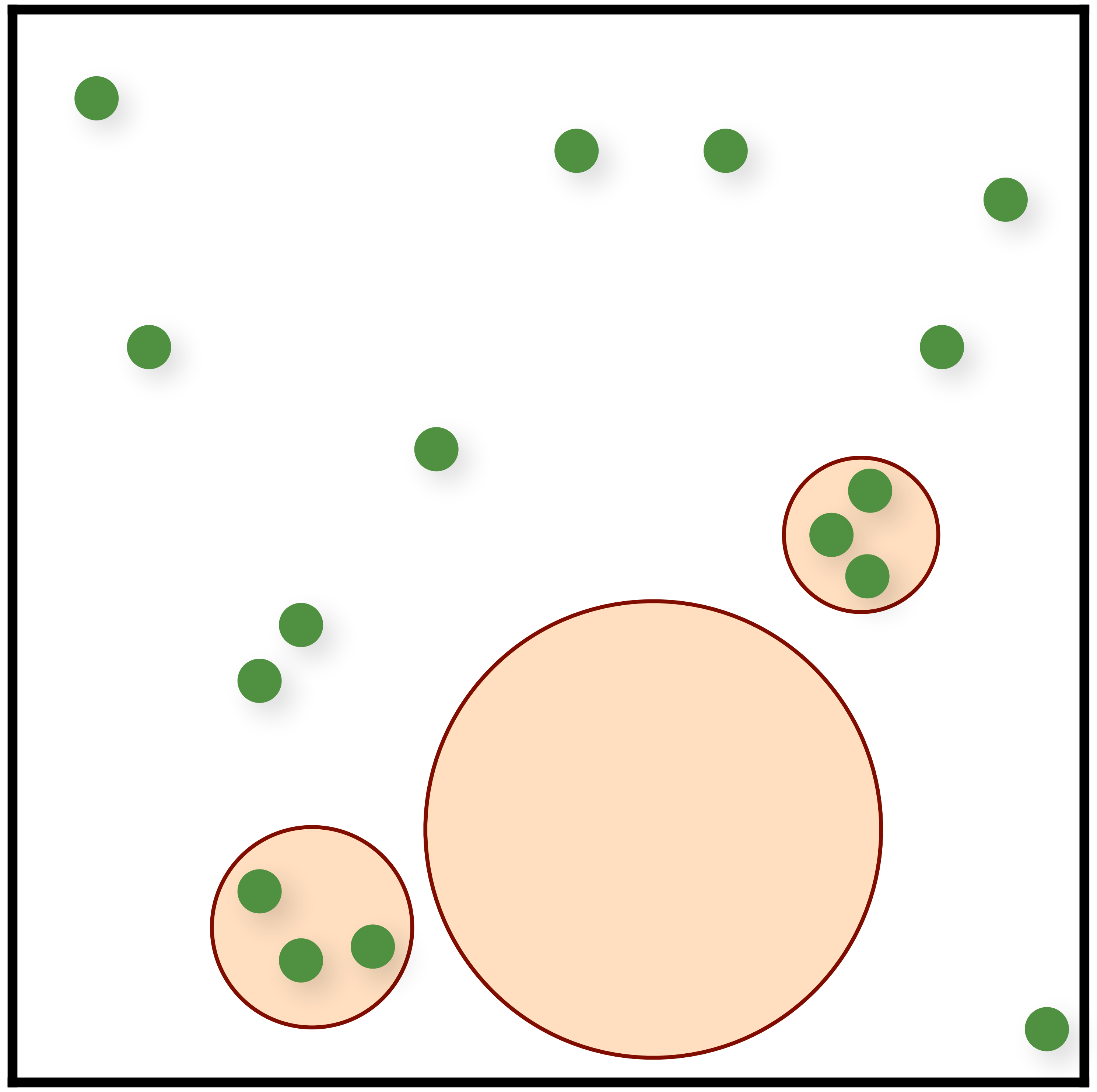

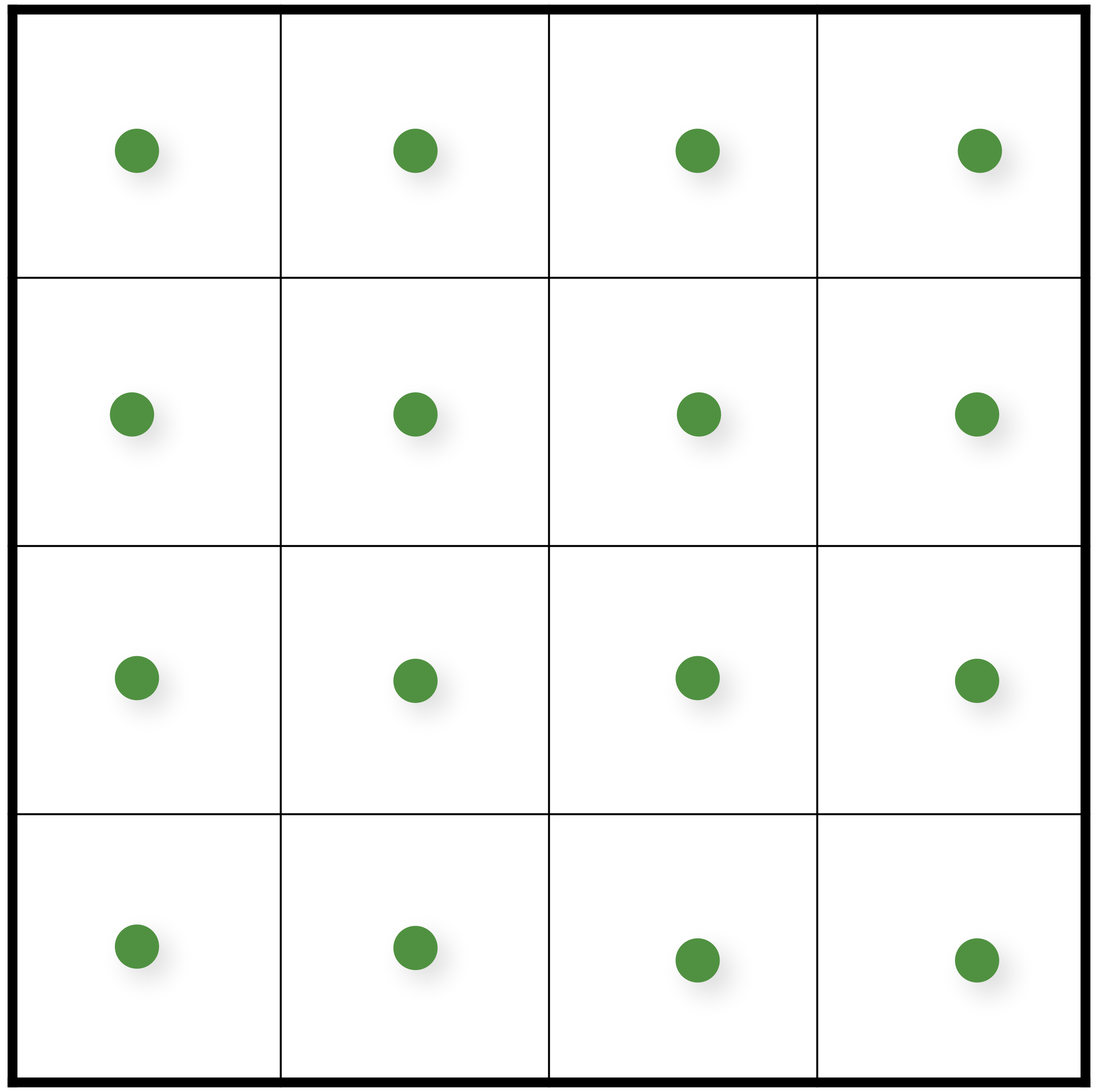

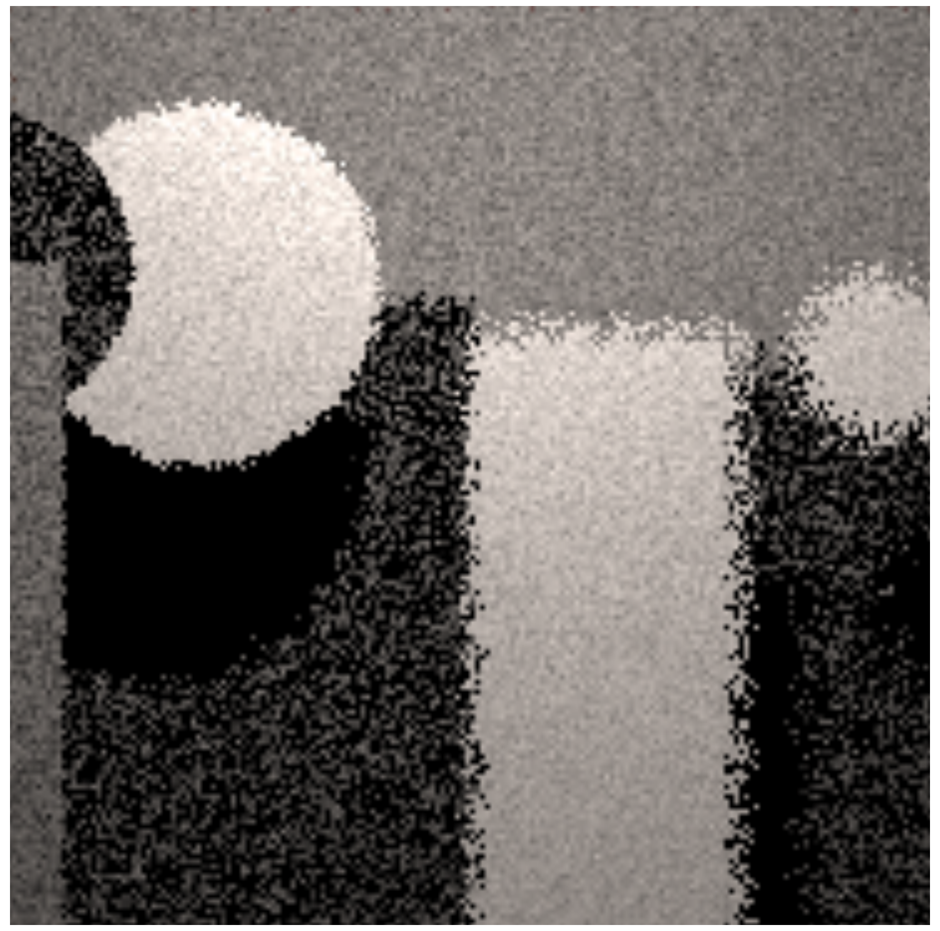

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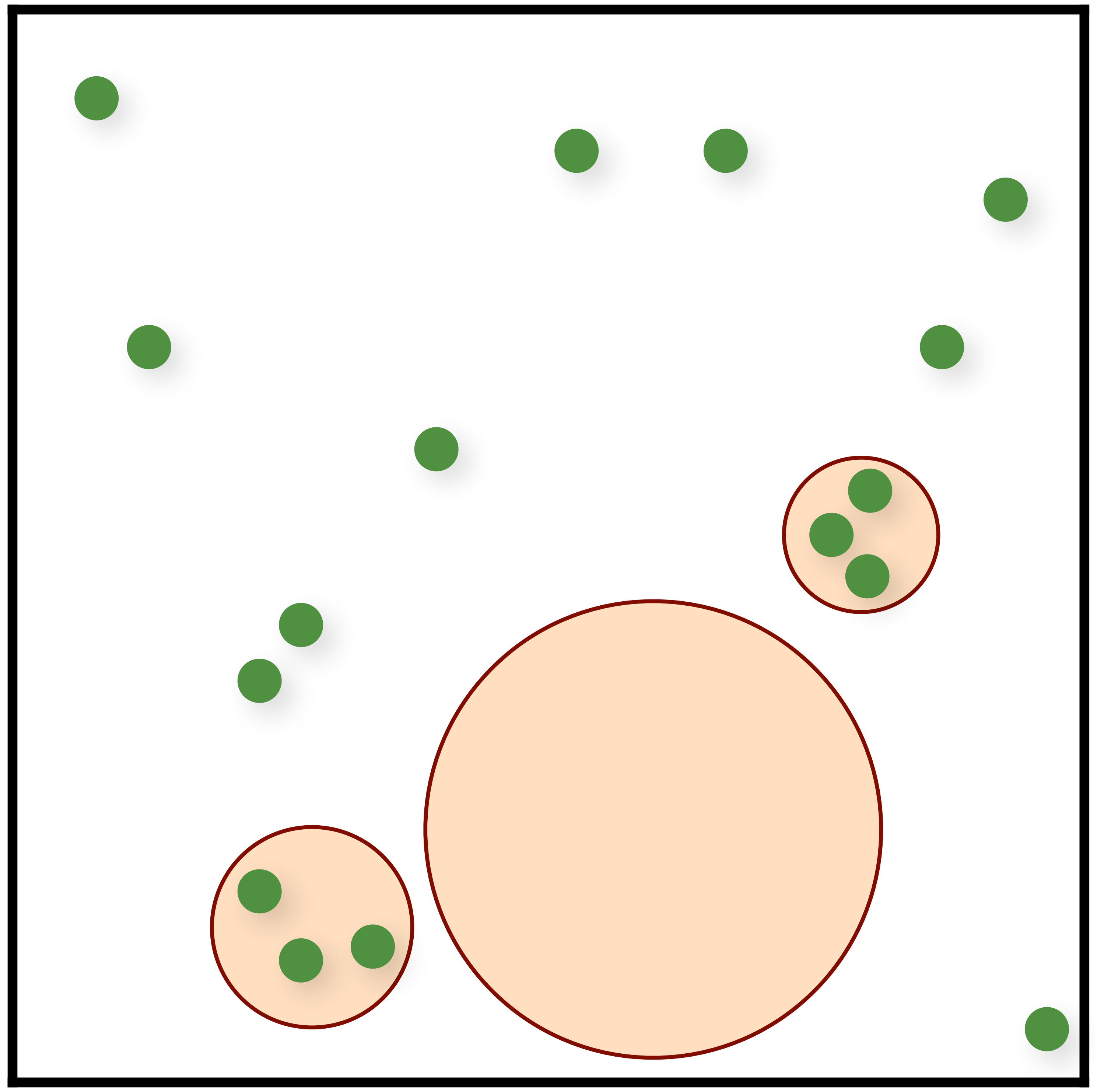
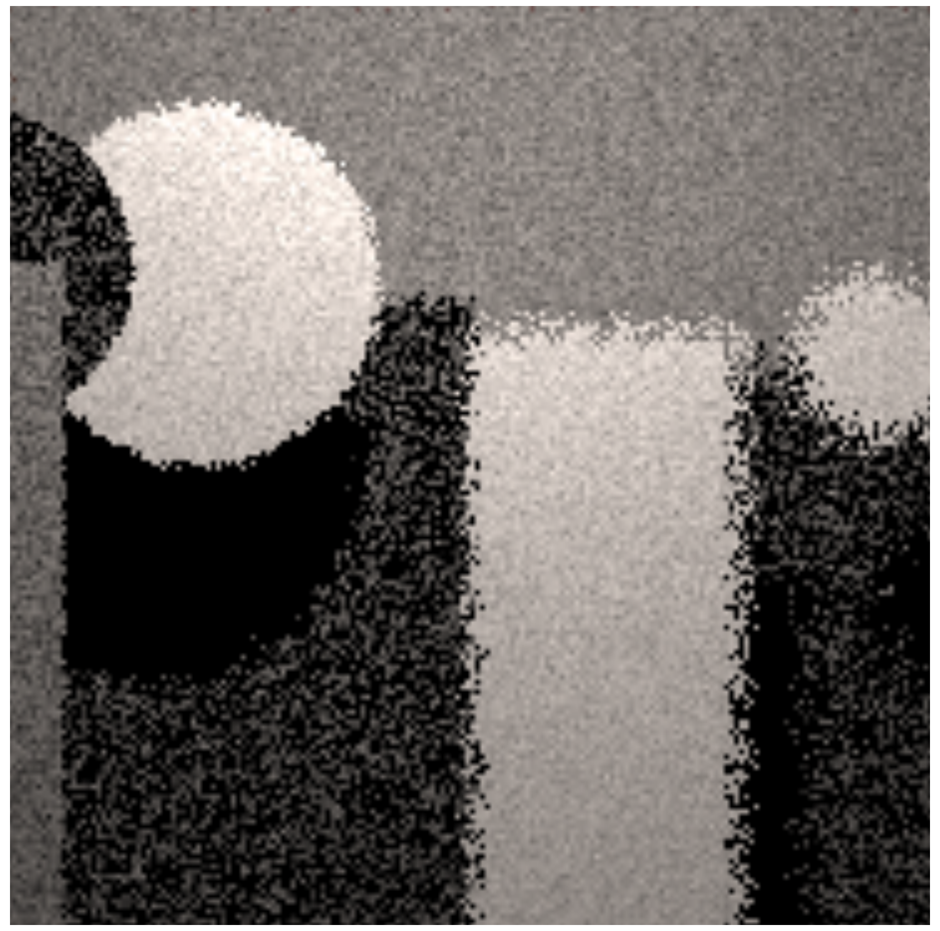
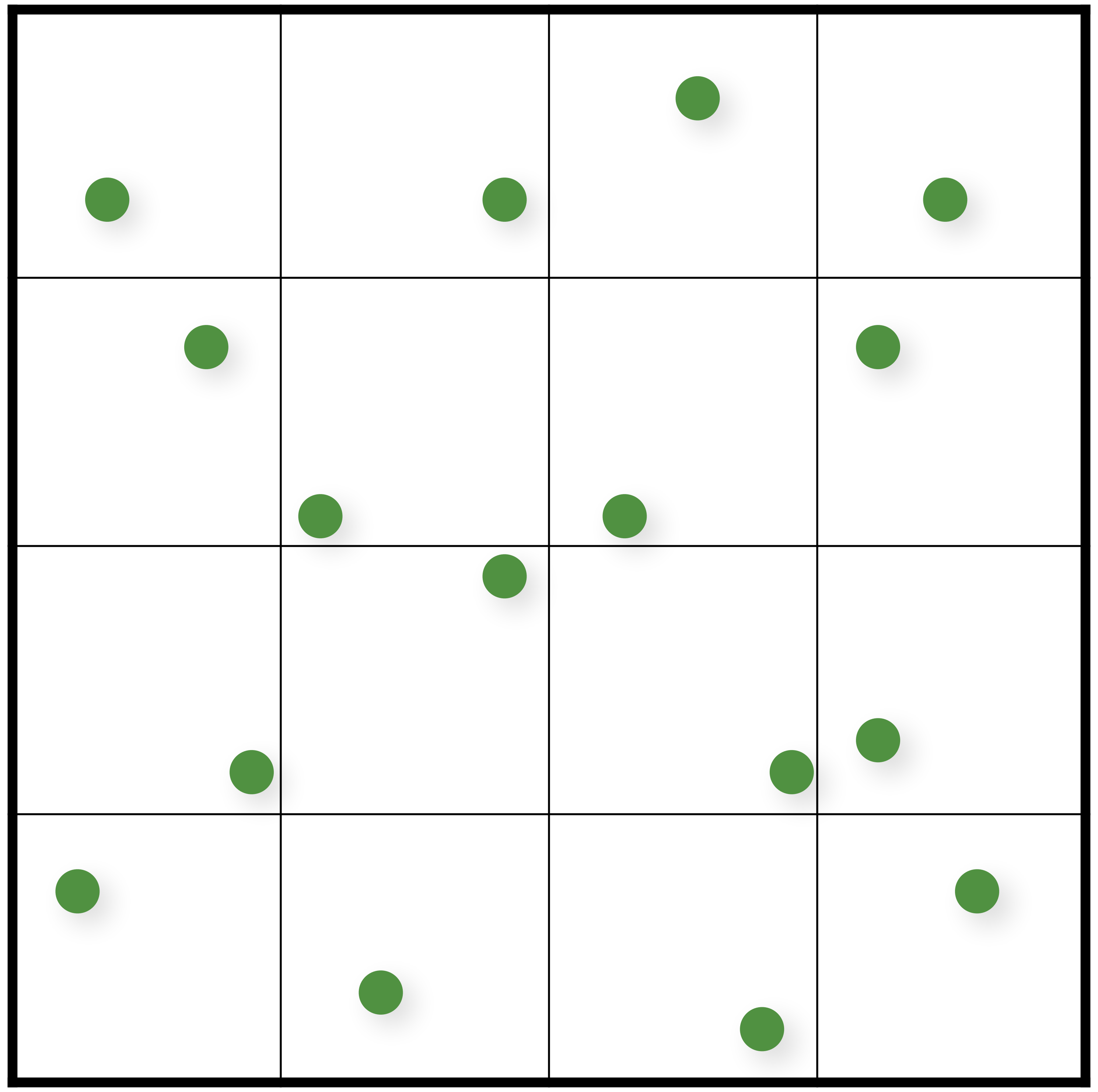
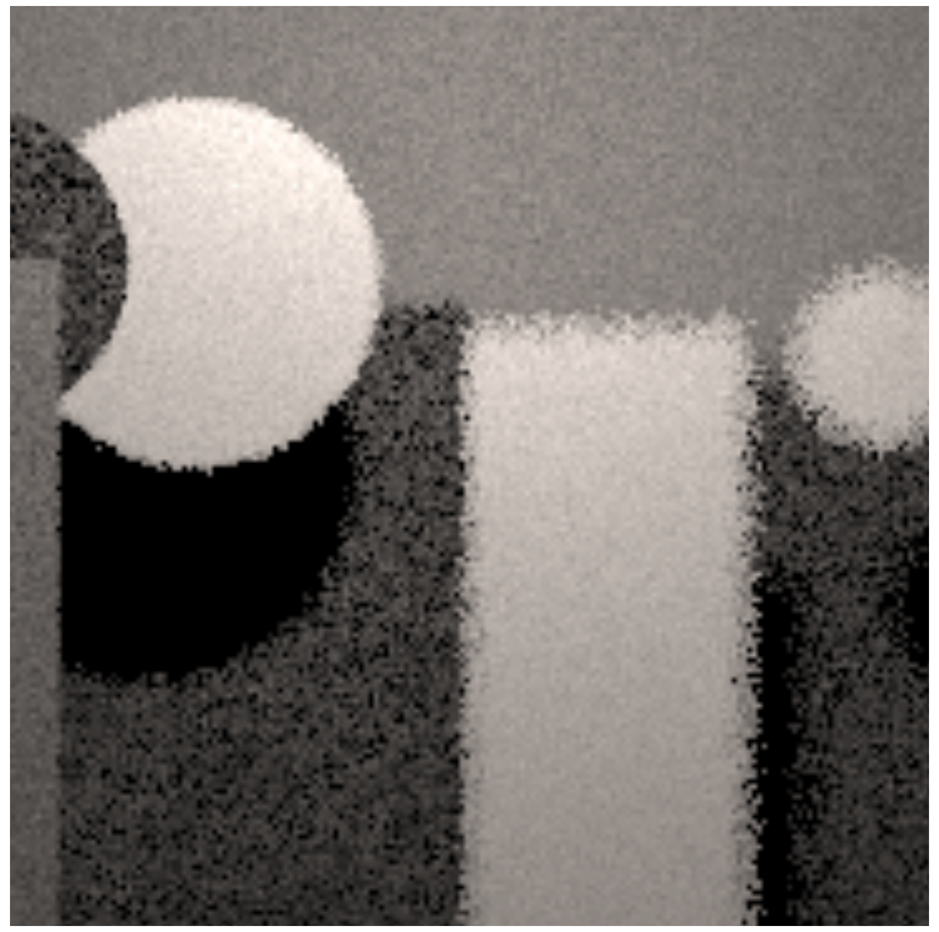




Jitter





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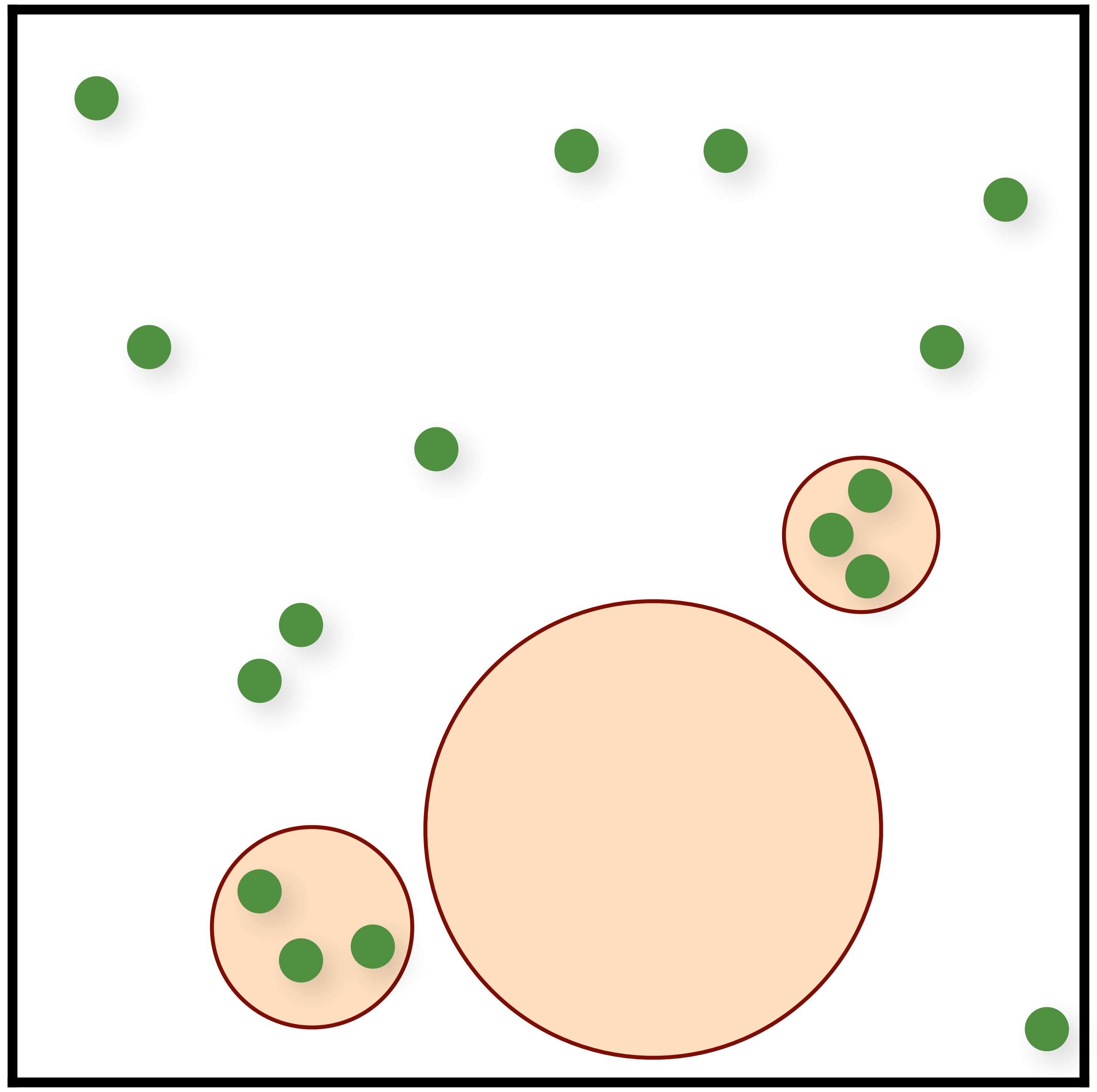
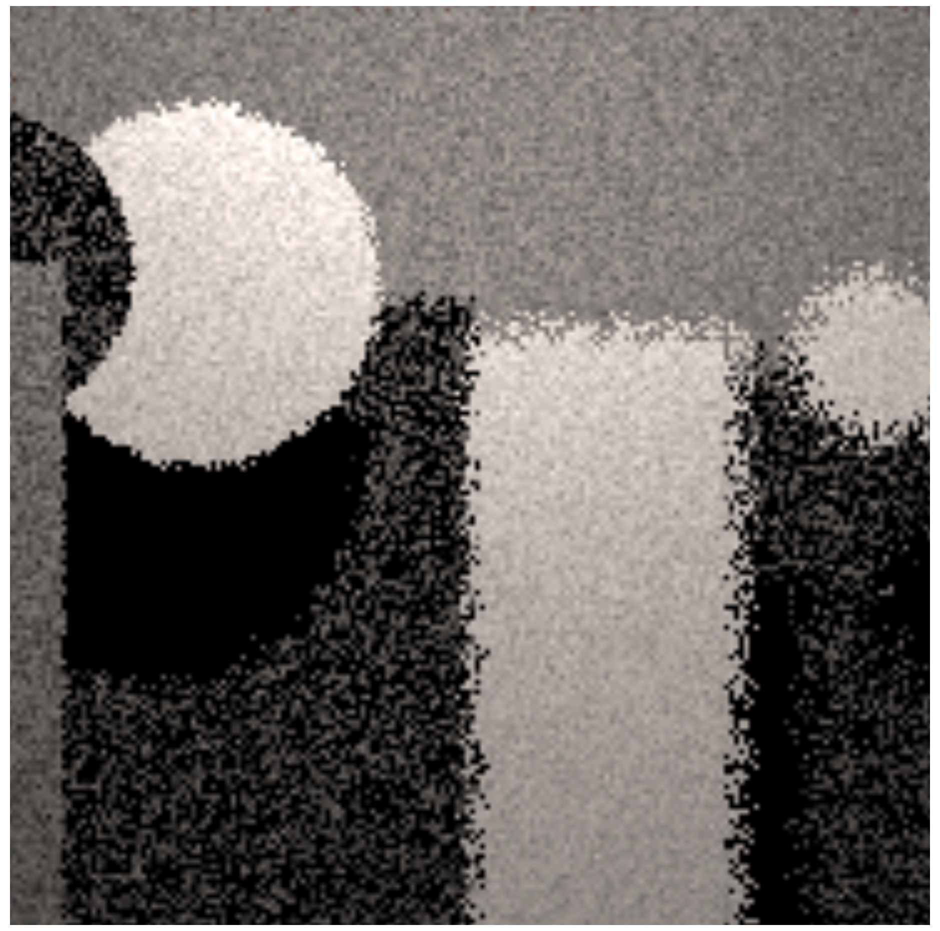
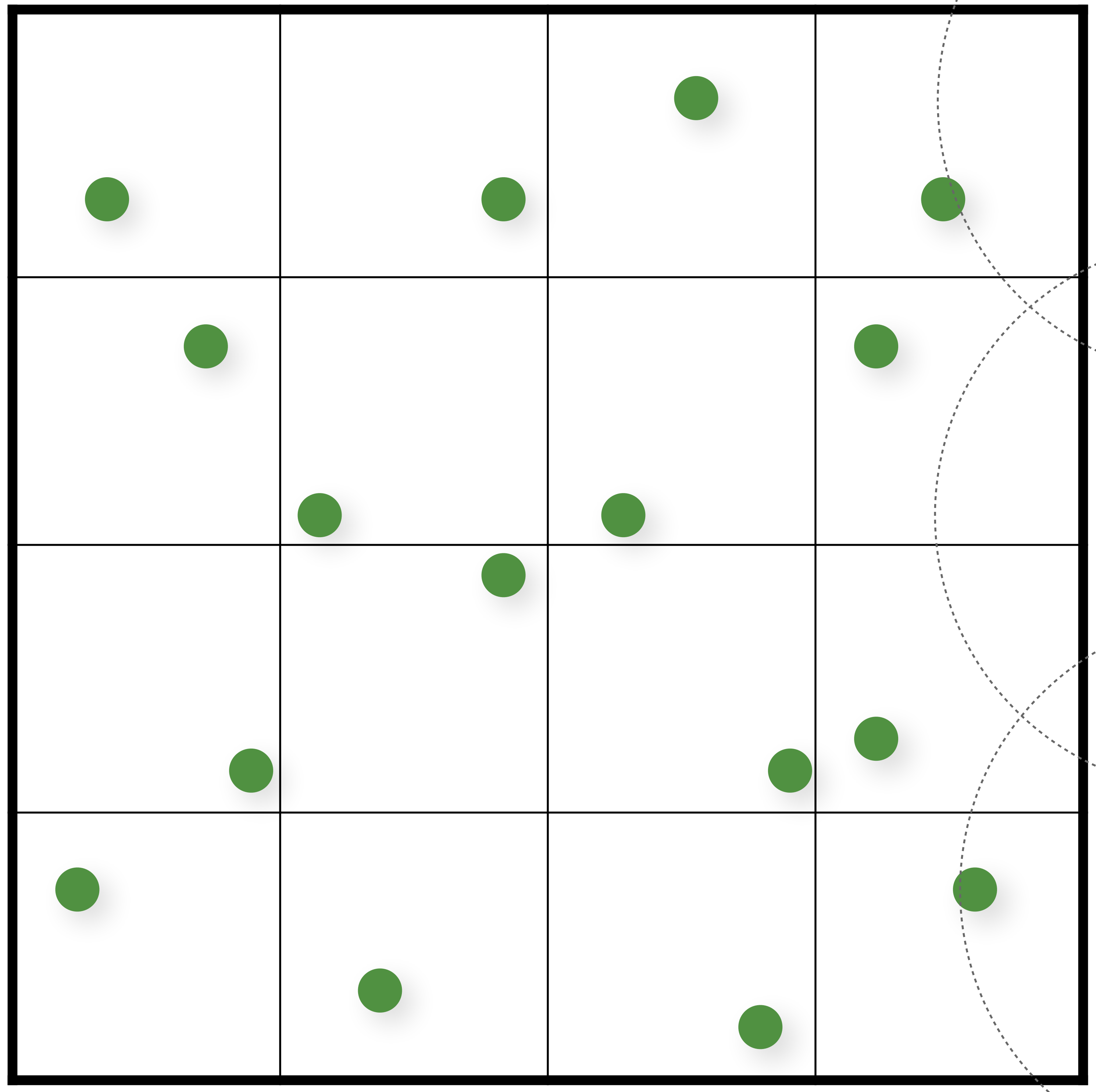
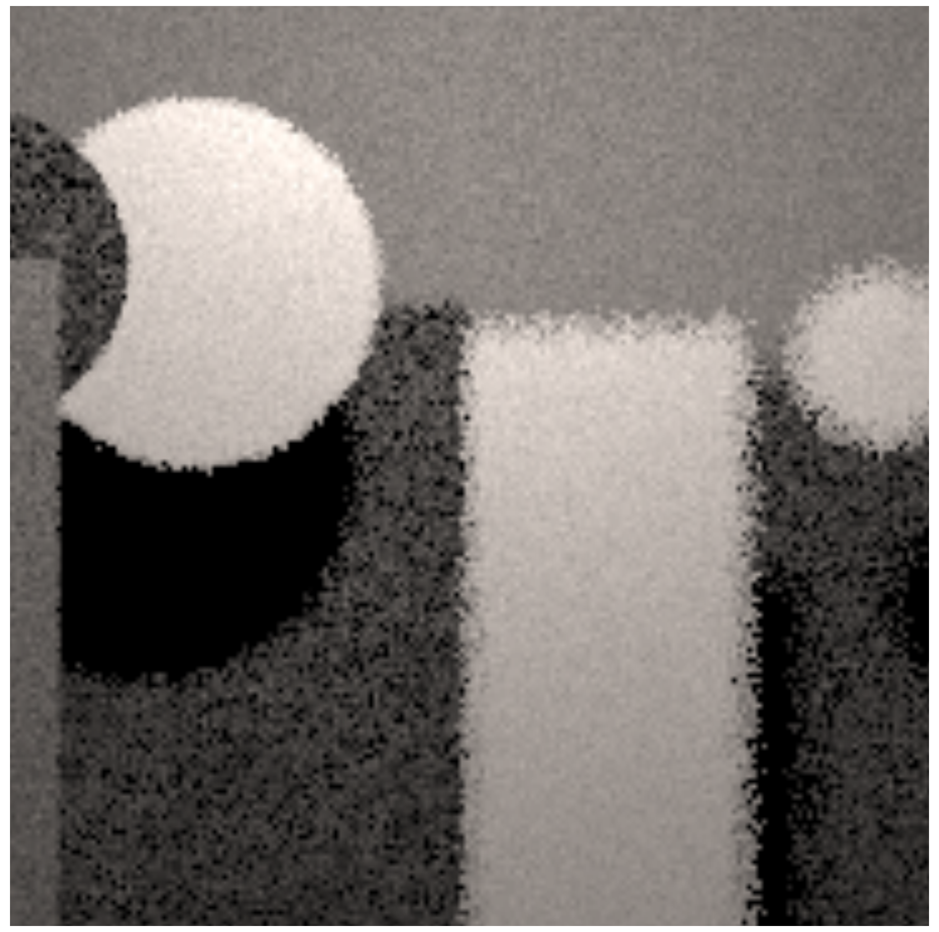
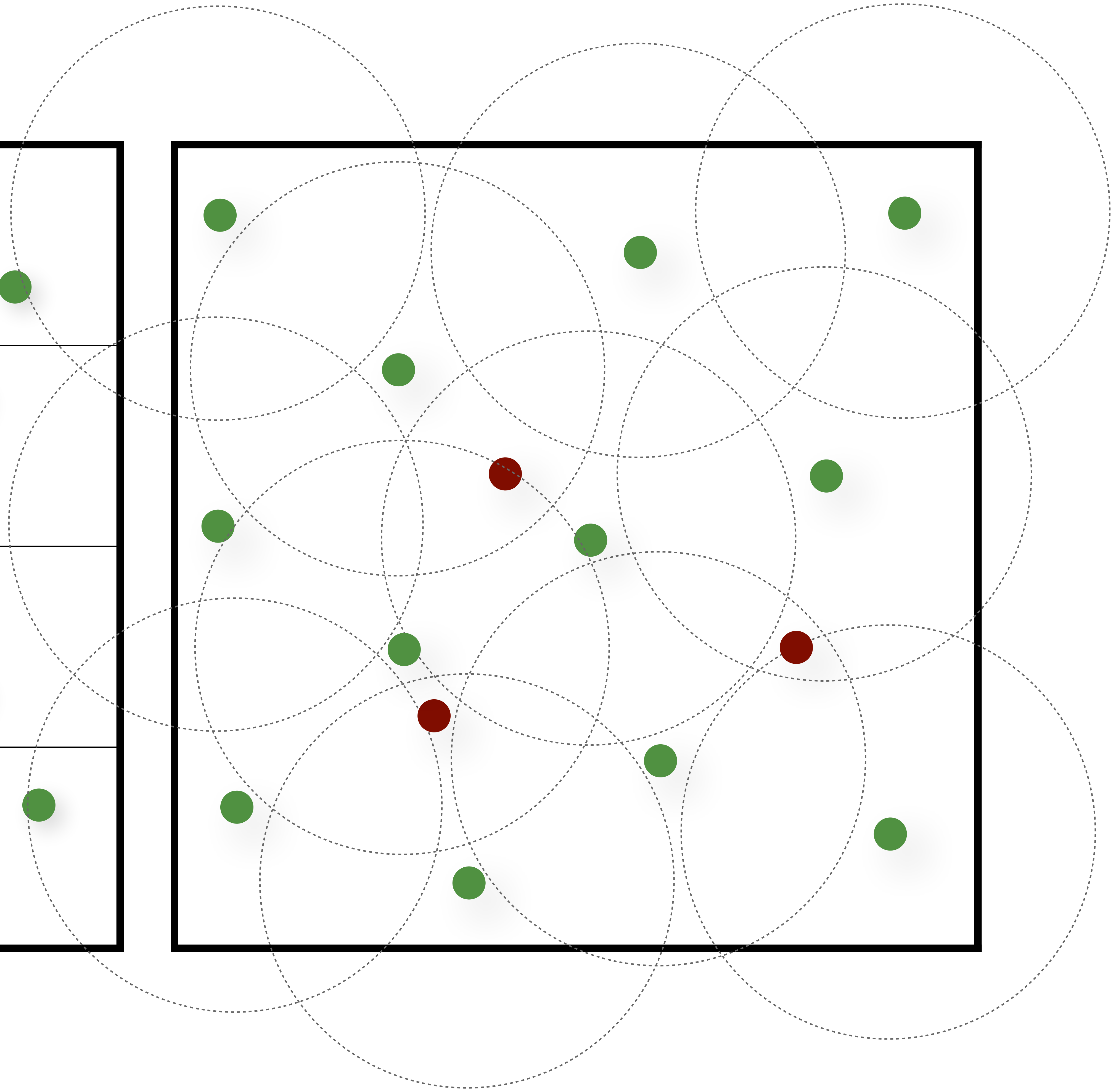




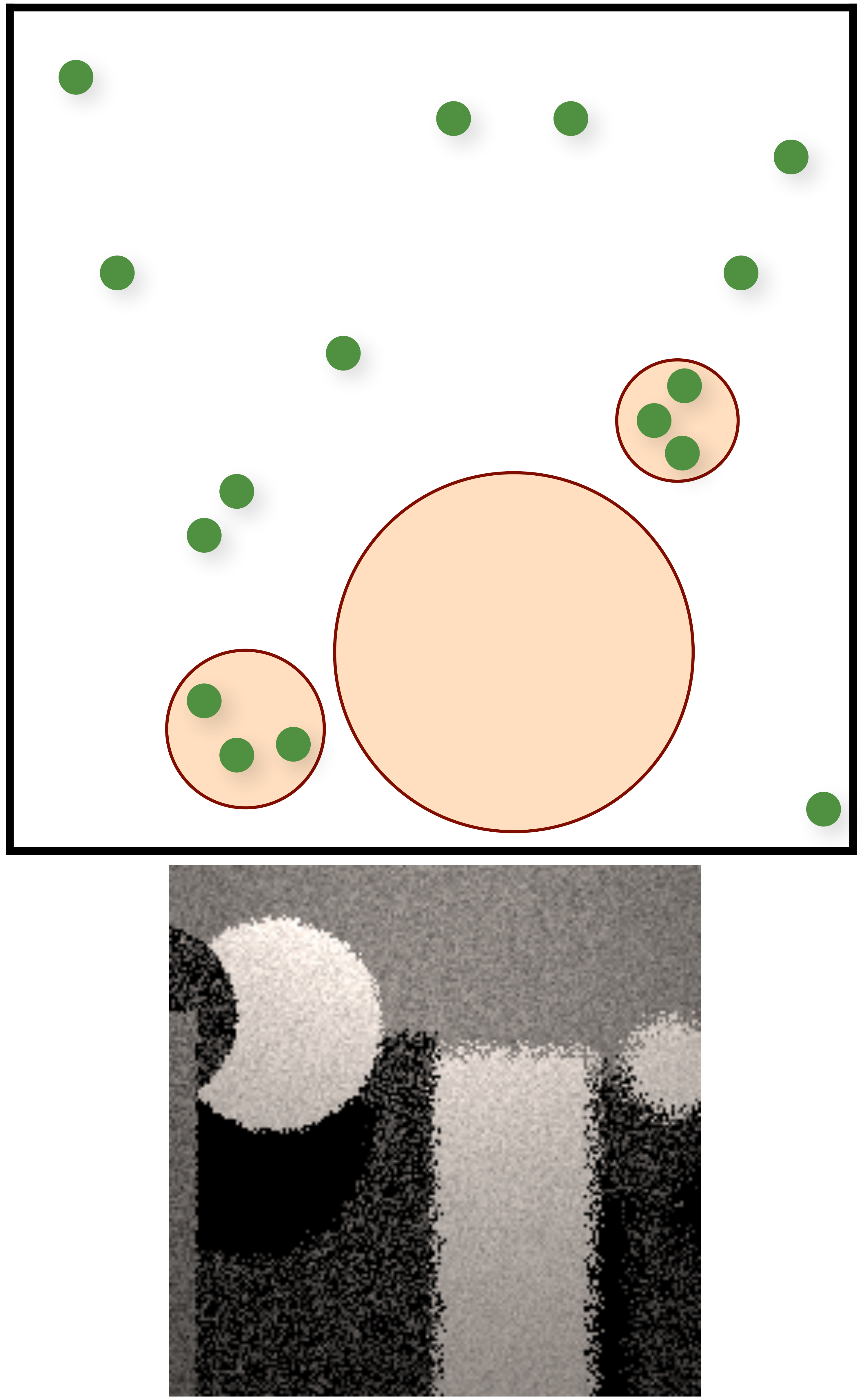
Jitter



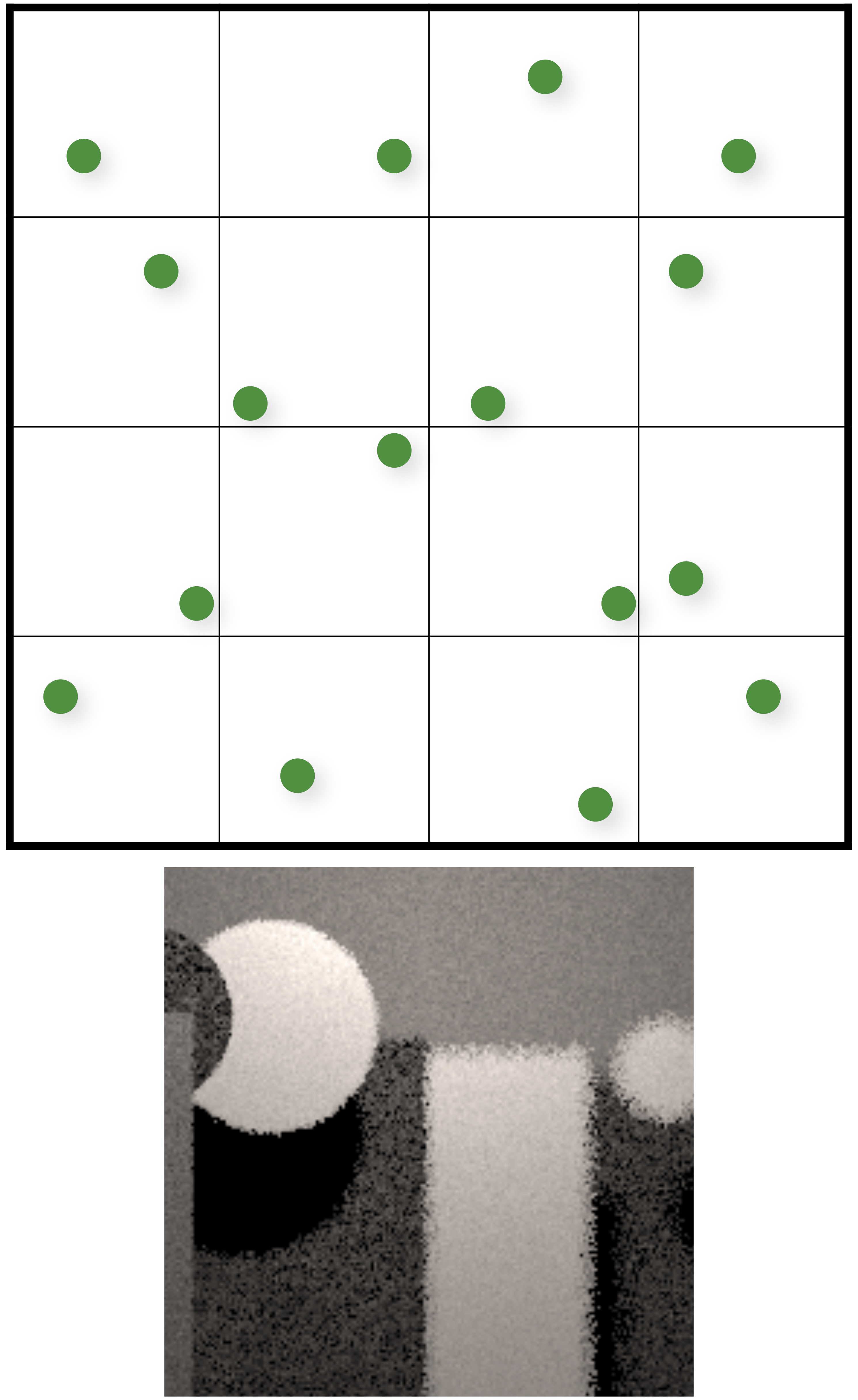




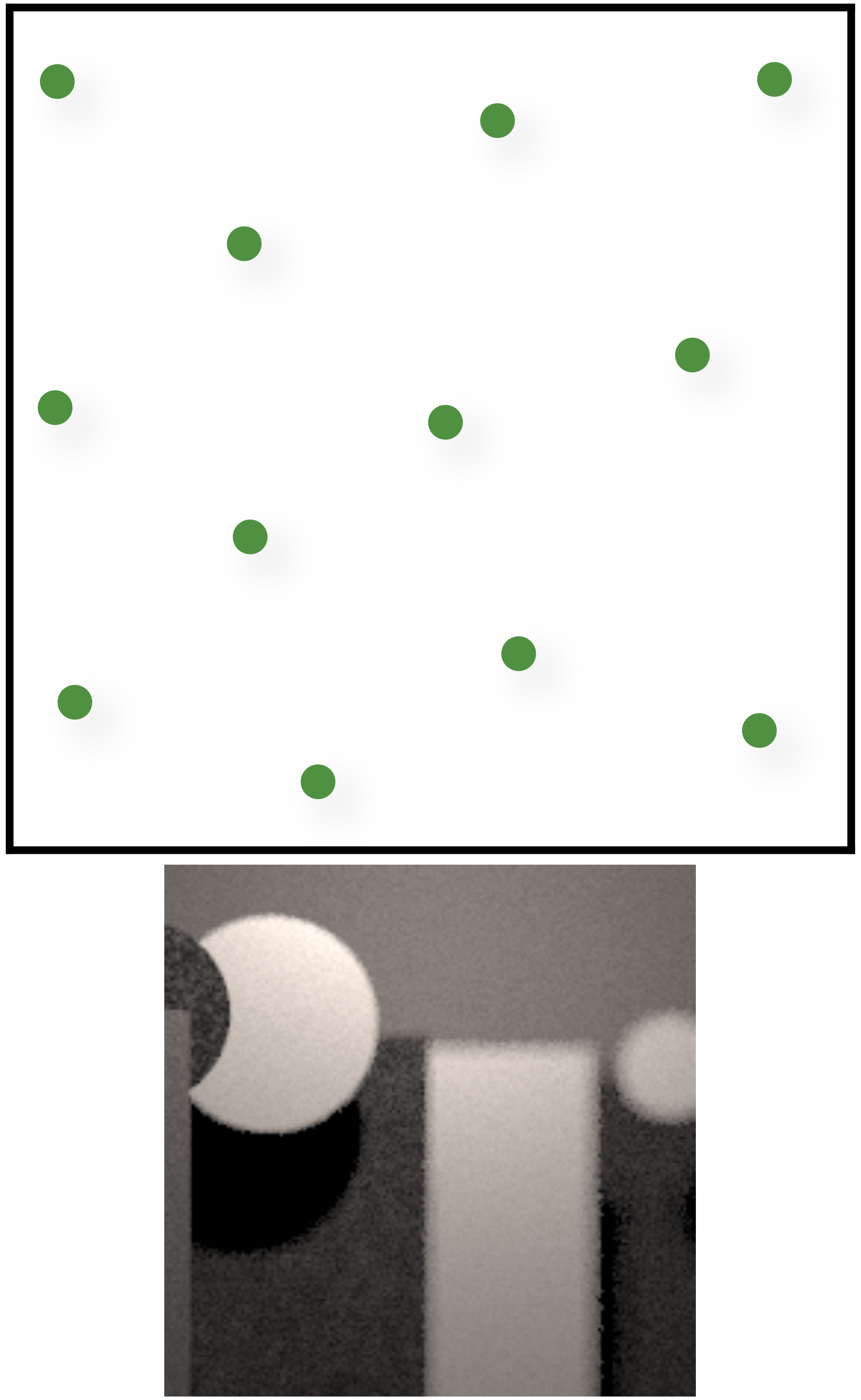
Random



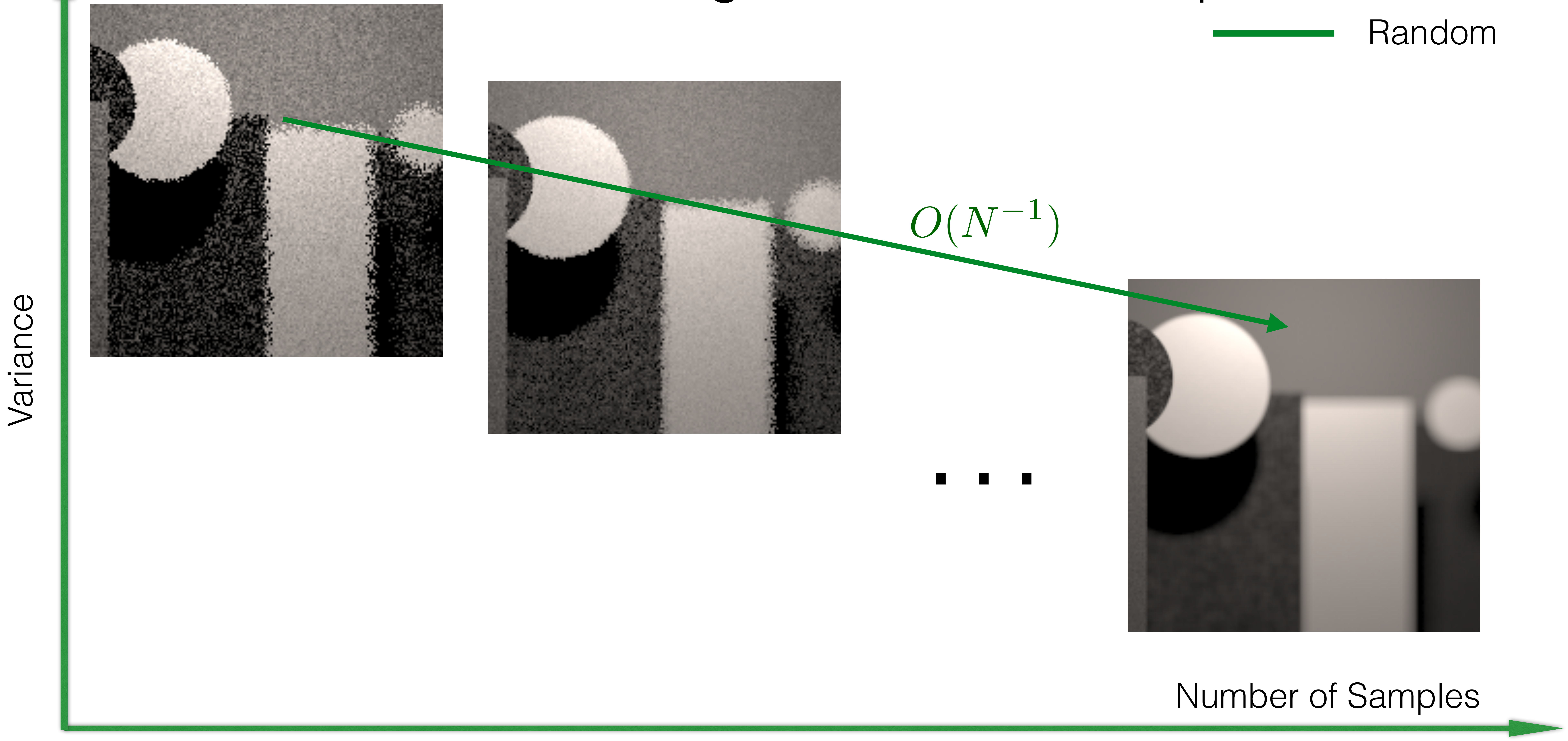
Jitter



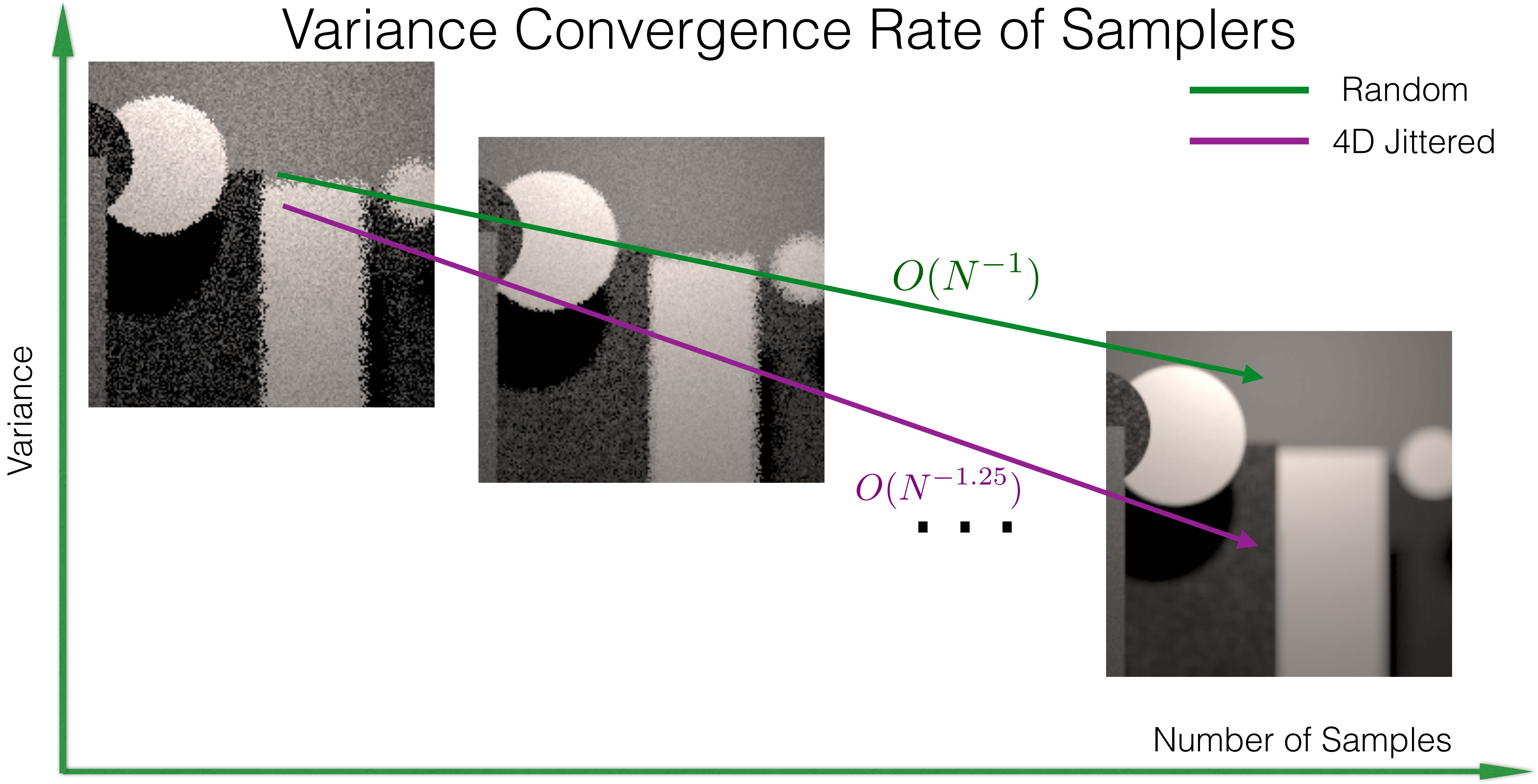
Poisson Disk



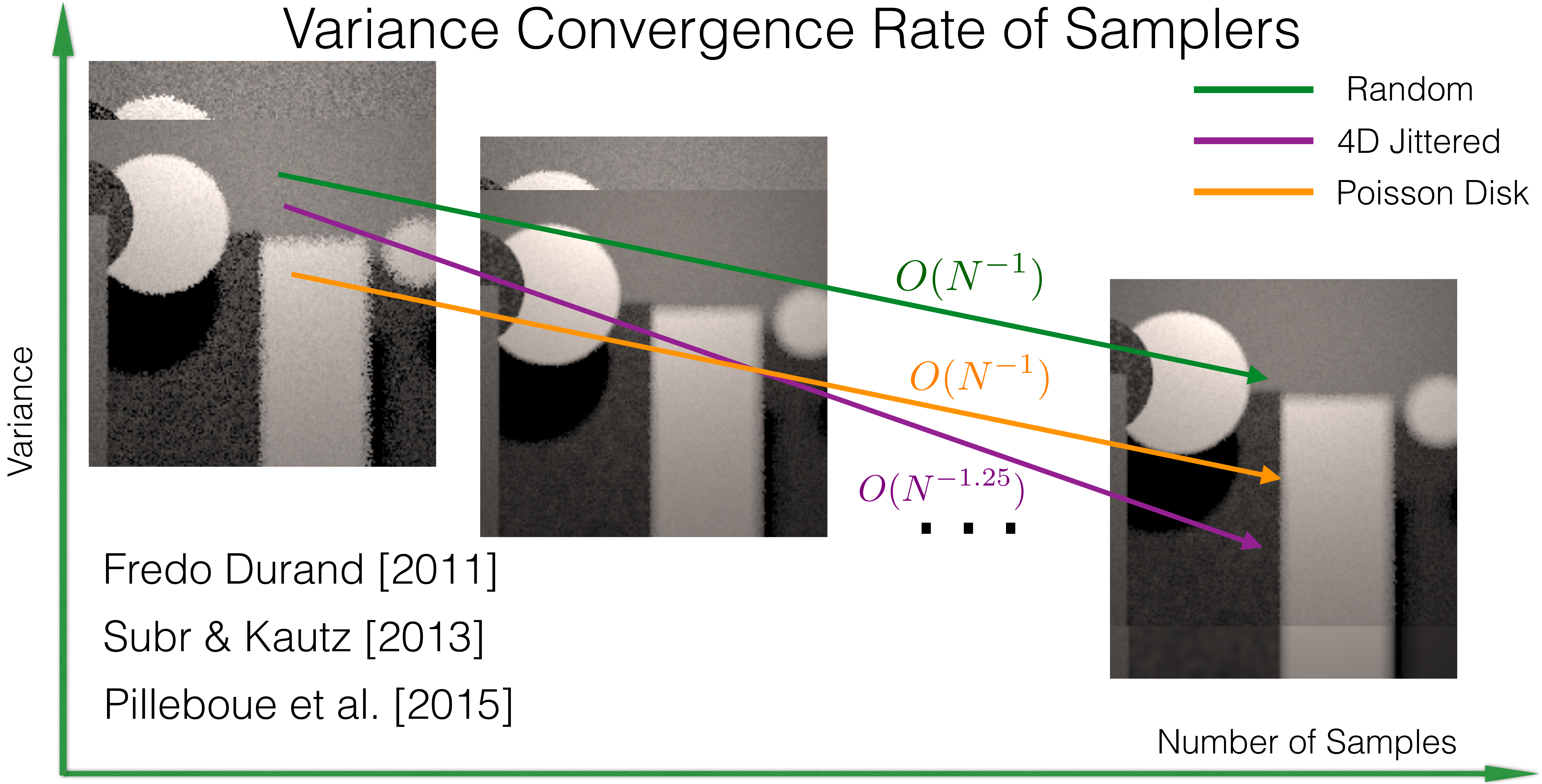
Variance Convergence Rate of Samplers



Variance Convergence Rate of Samplers



Variance Convergence Rate of Samplers



Fredo Durand [2011]

Subr & Kautz [2013]

Pilleboue et al. [2015]

Stratification strategies

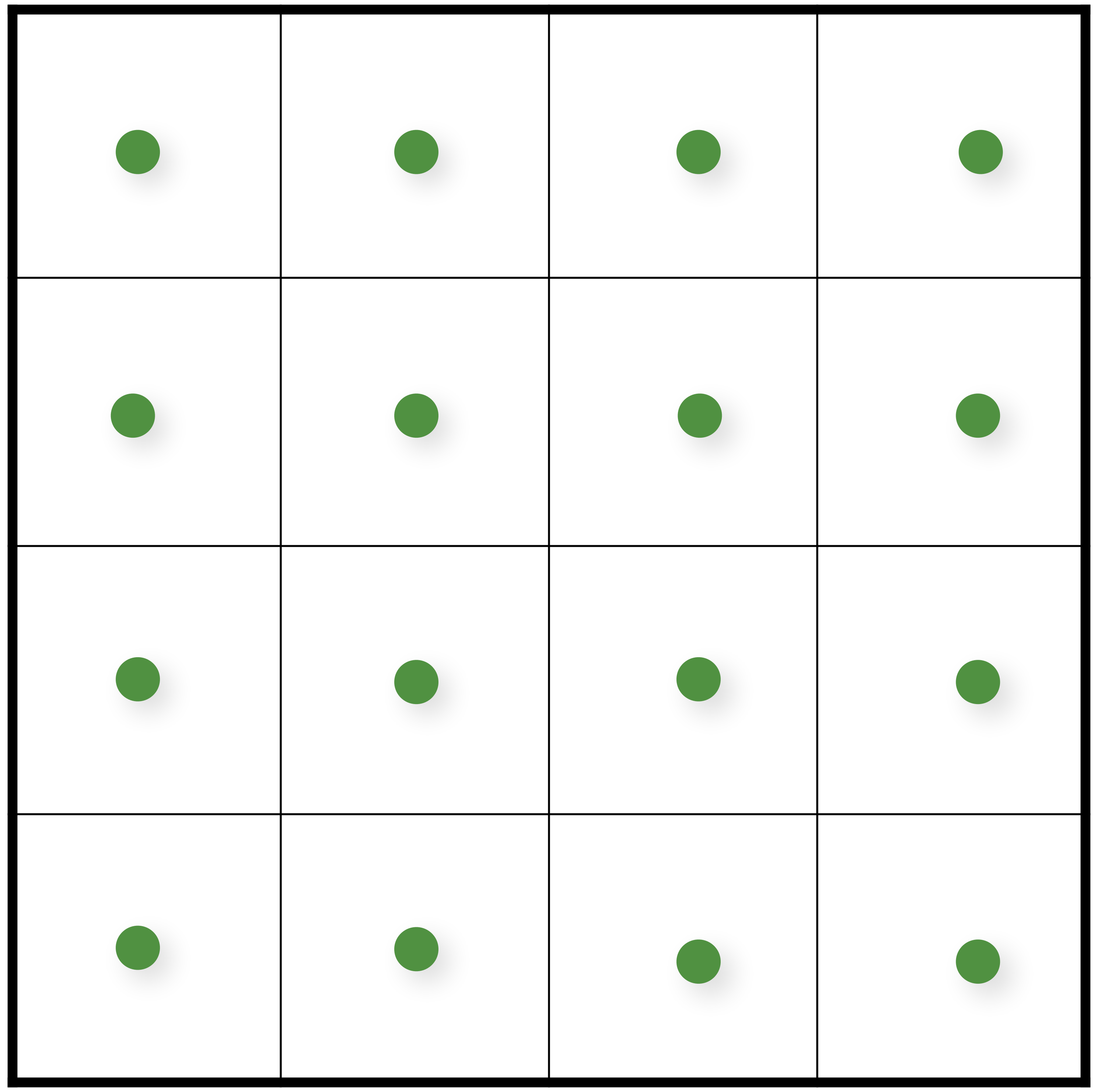
Blue noise sampling and beyond

Importance sampling

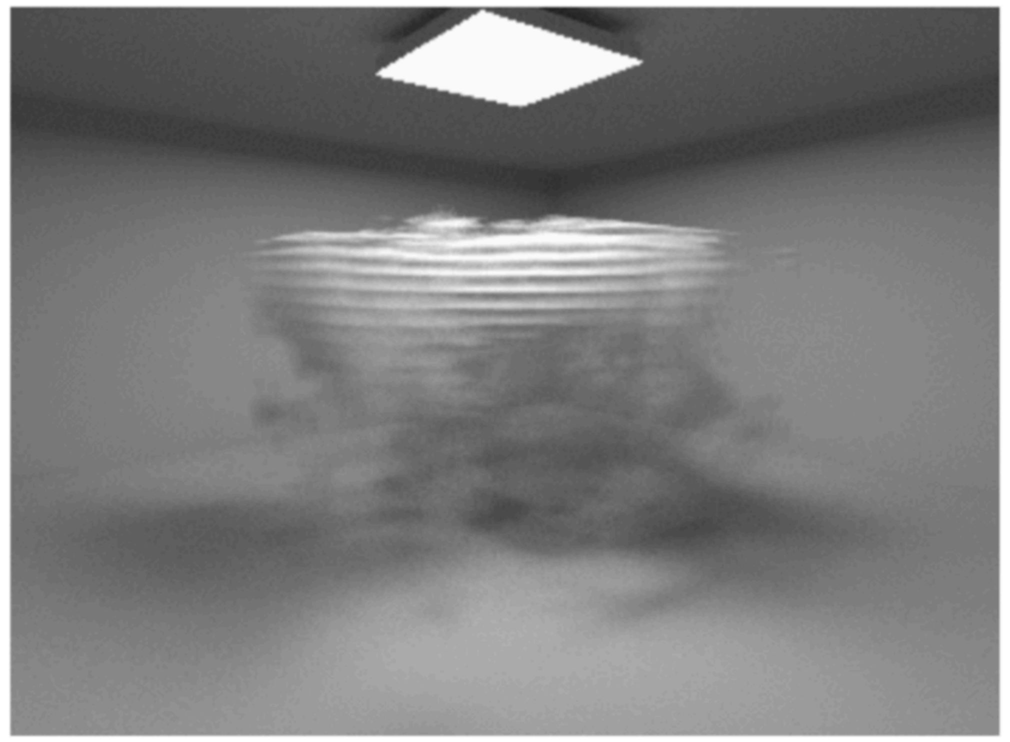
with correlated samples

Regular grid samples

Pauly et al. [2000]



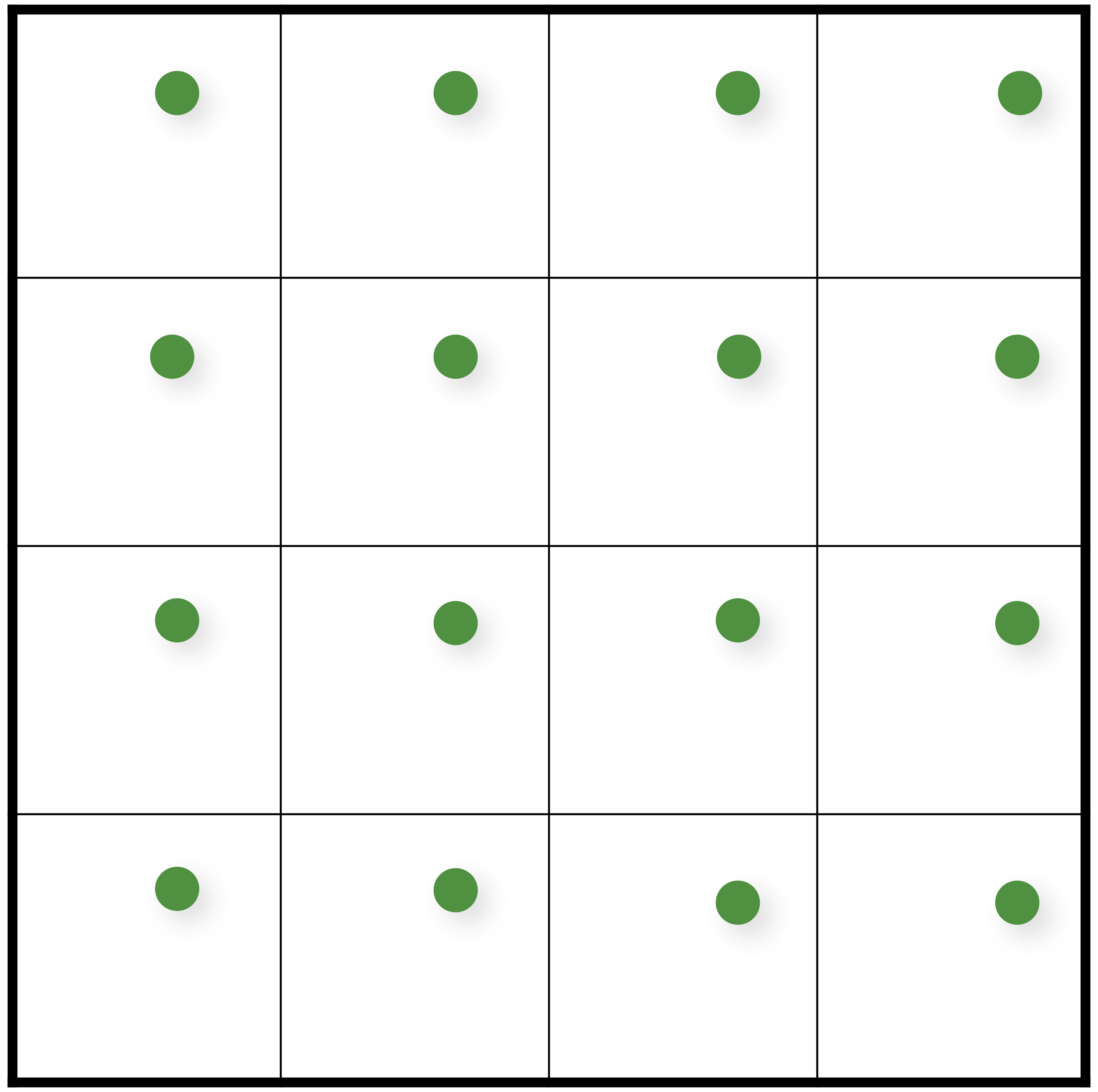
Regular grid



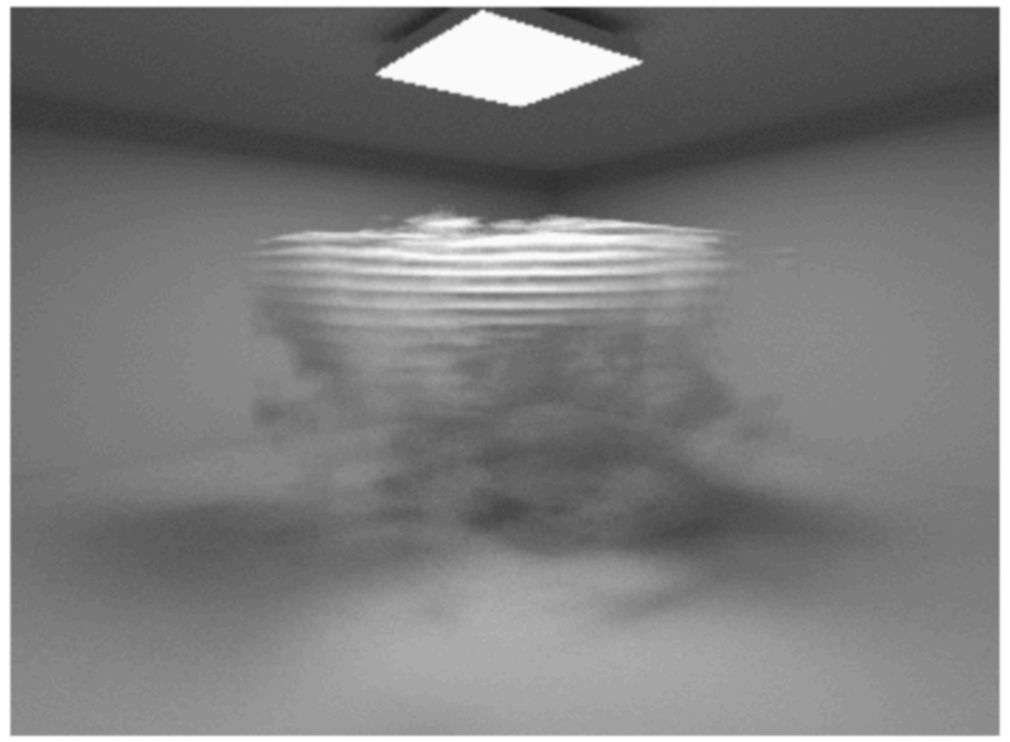
Regular

Uniformly jittered regular grid

Pauly et al. [2000]



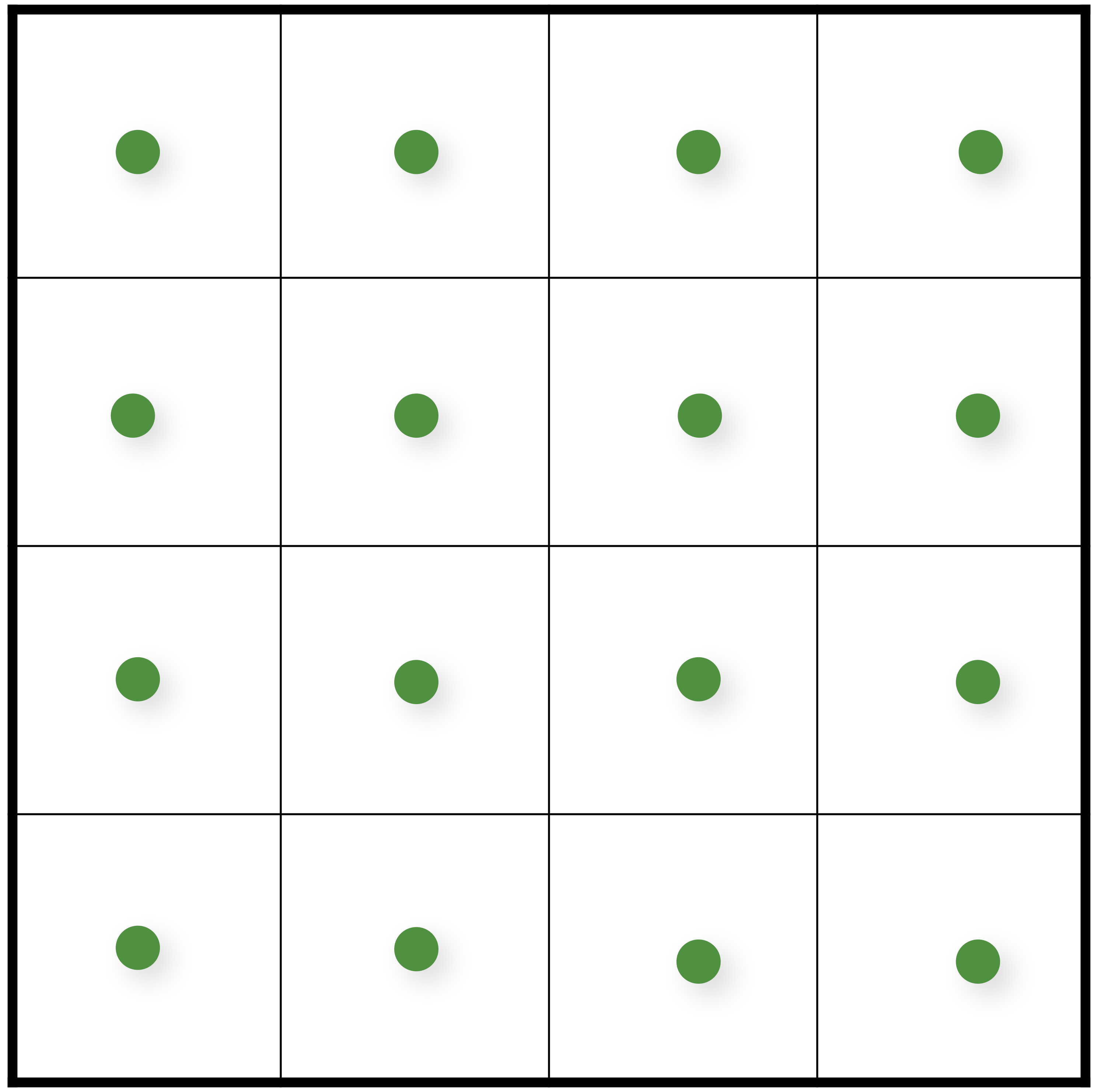
Uniform jitter



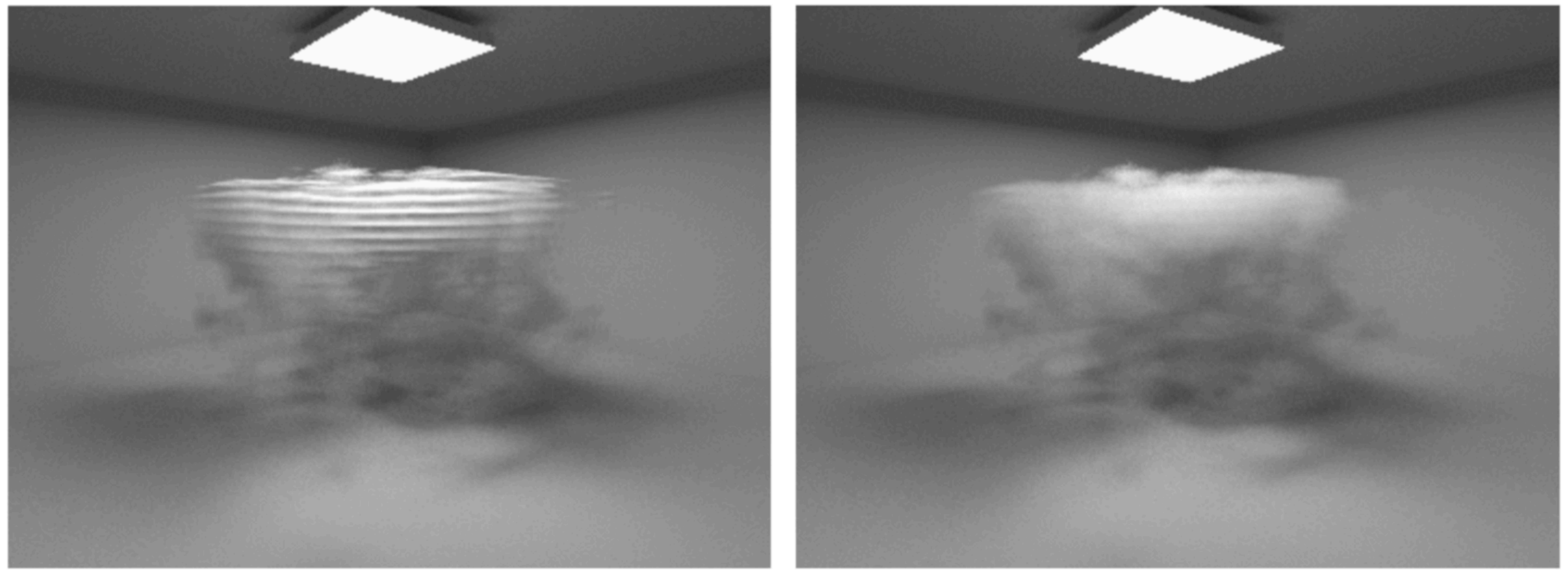
Regular

Randomly jittered samples

Pauly et al. [2000]

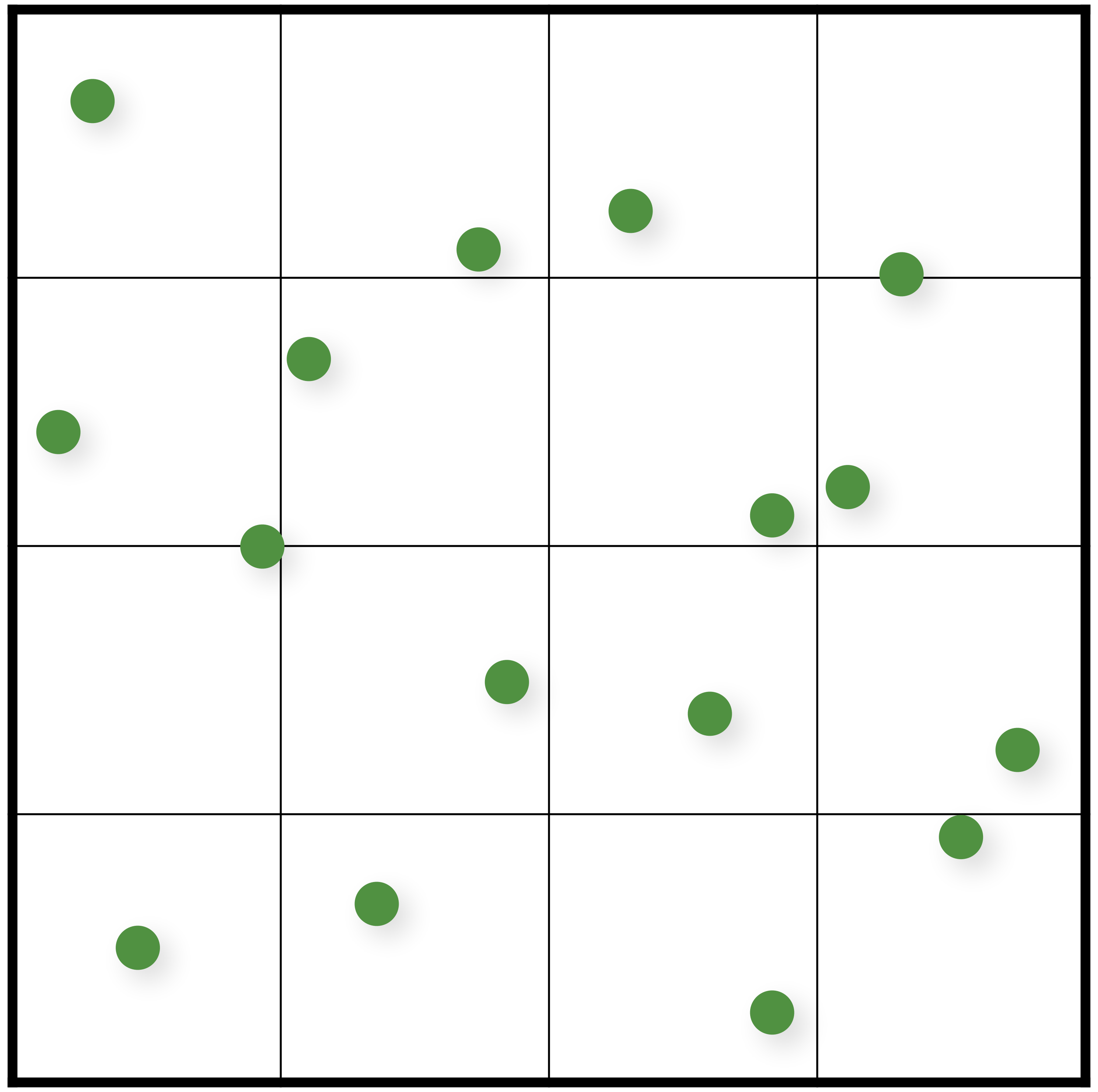


Random jitter



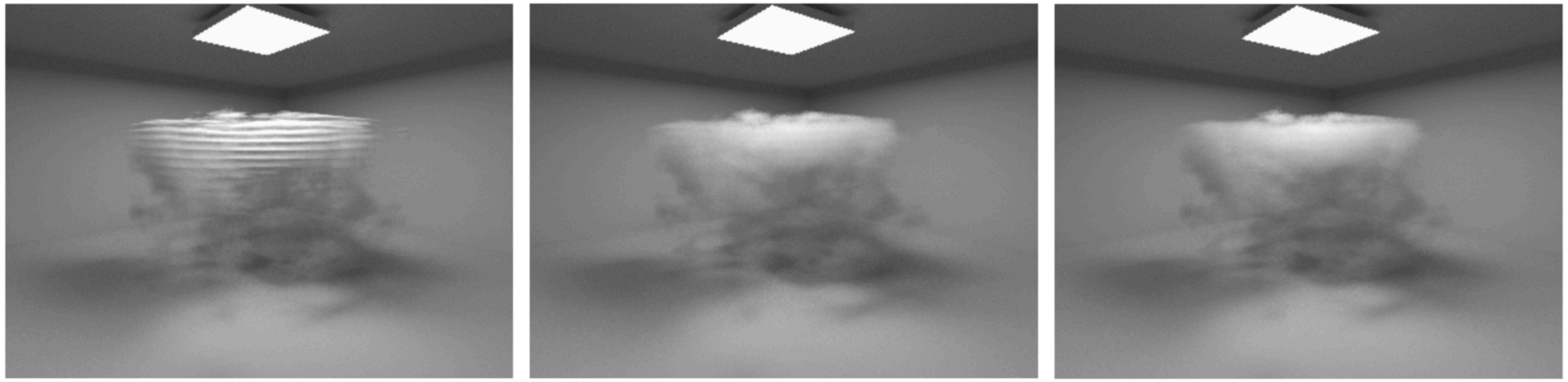
Regular Uniform jitter

Randomly jittered samples



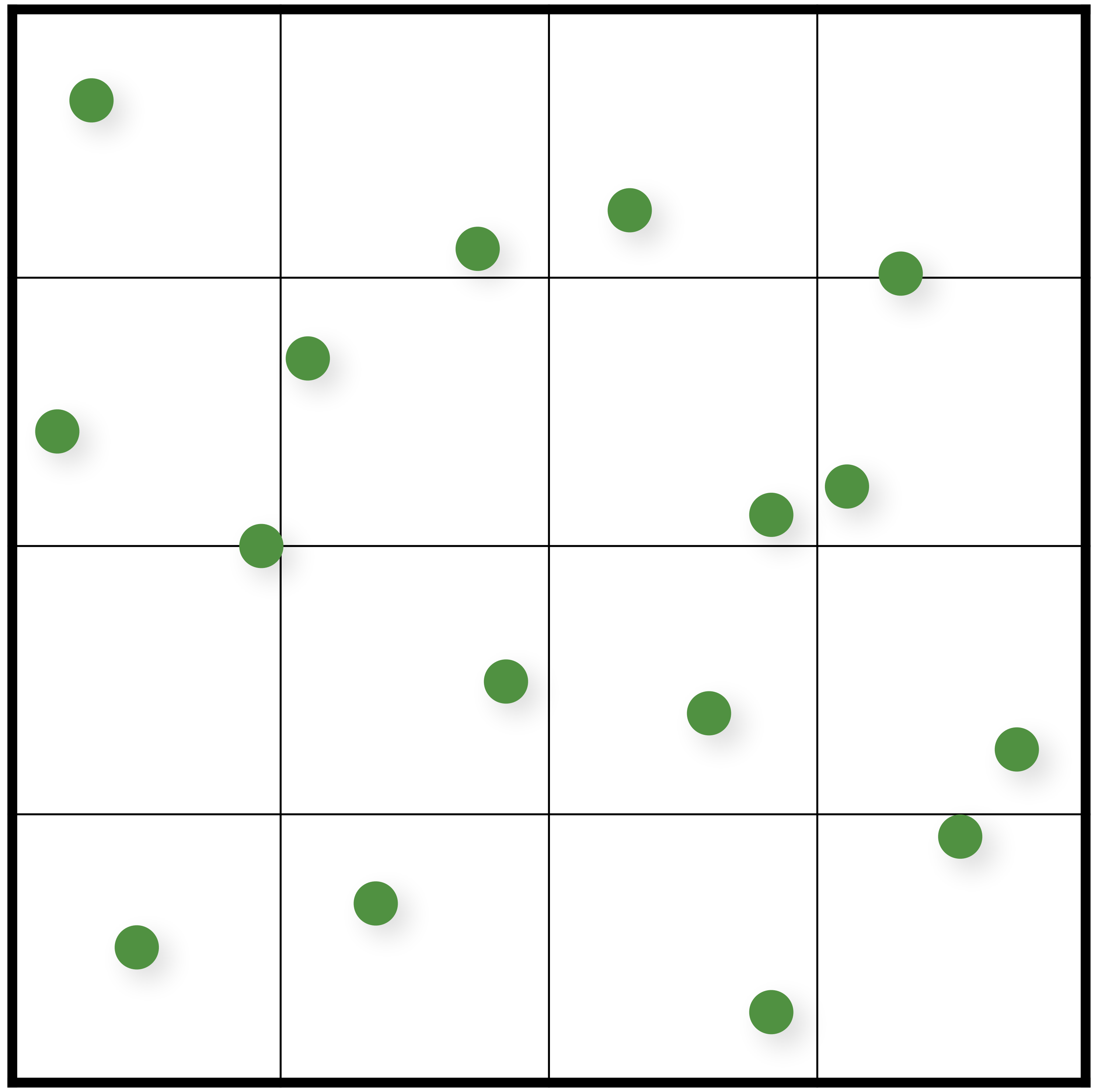
Random jitter

Pauly et al. [2000]

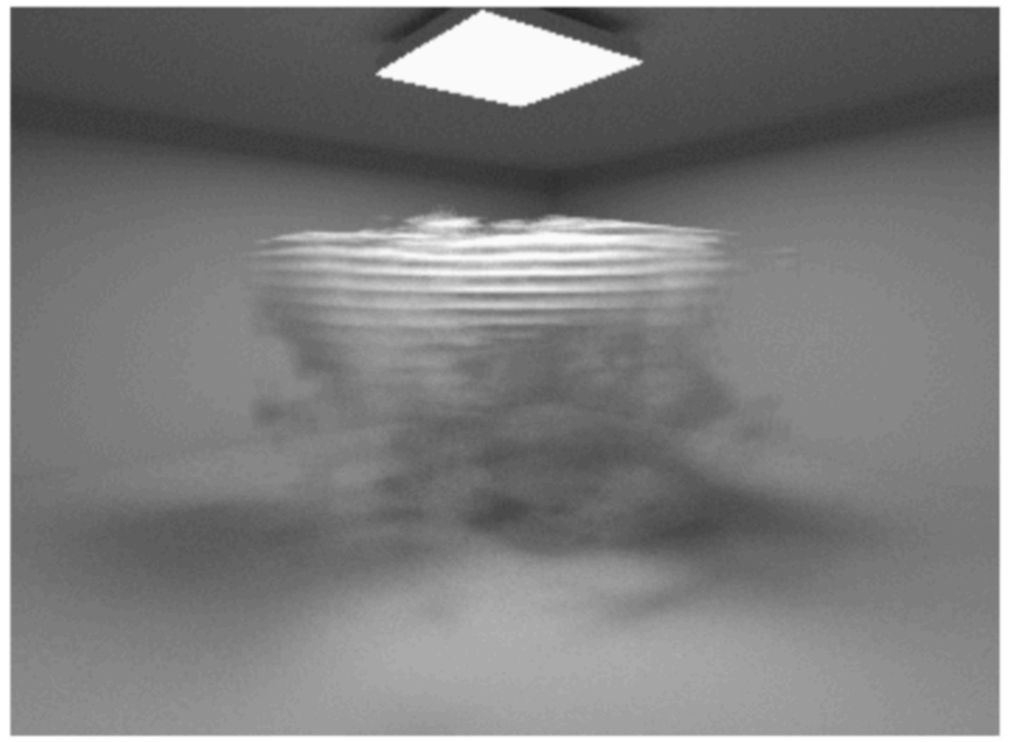


Regular Uniform jitter Random jitter

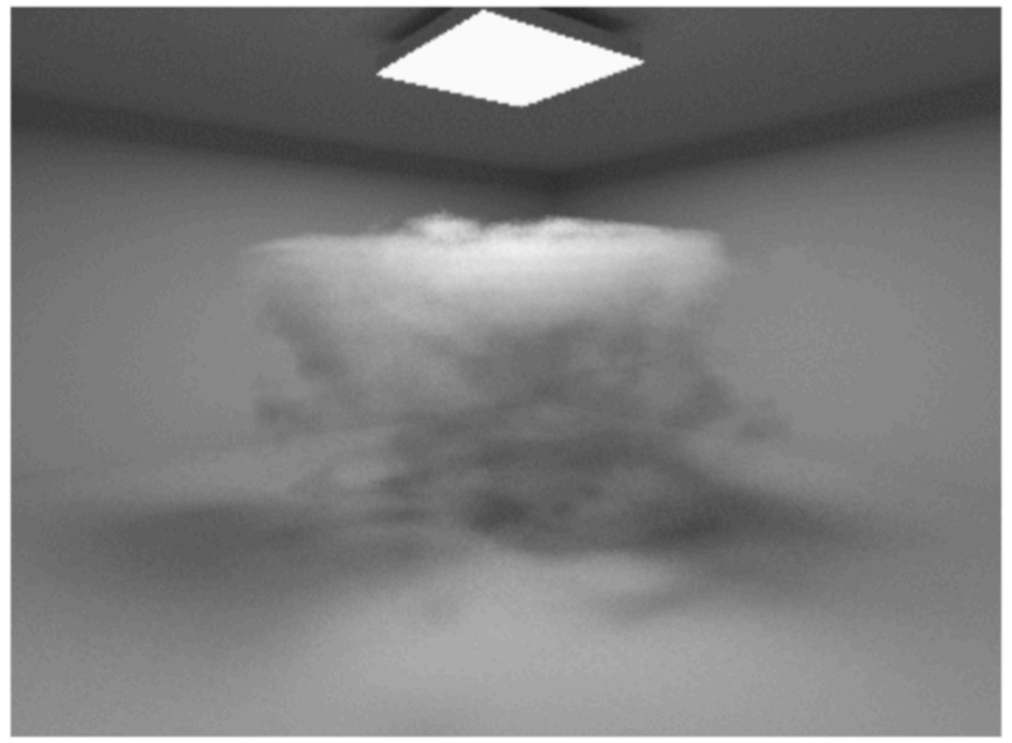
Randomly jittered samples



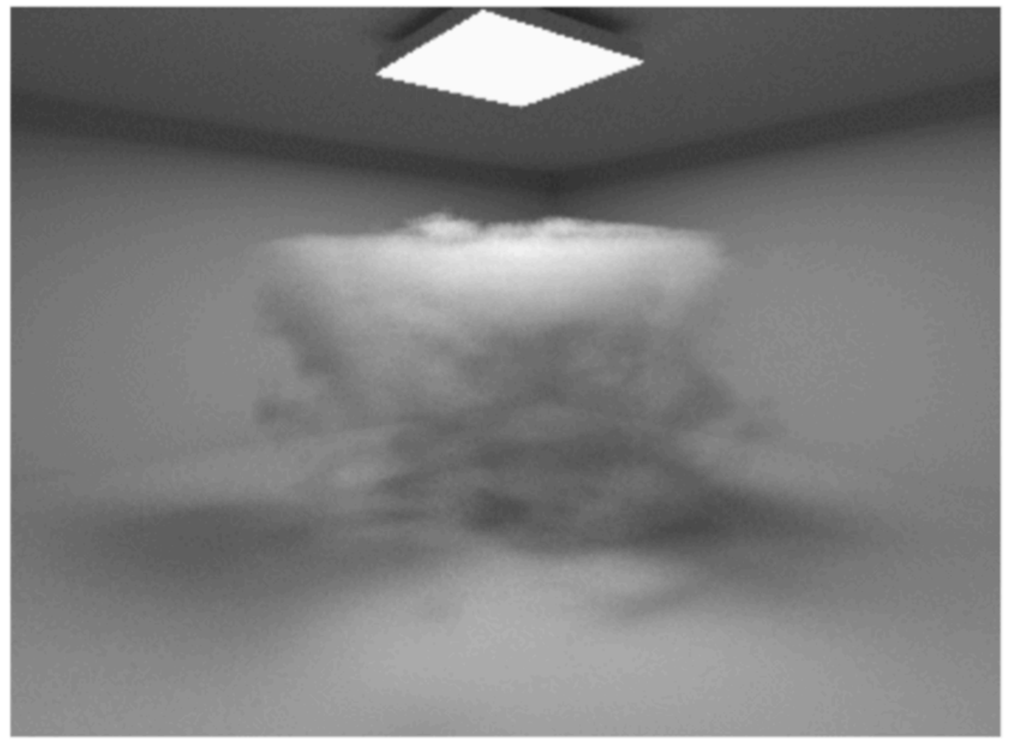
Pauly et al. [2000]



Regular

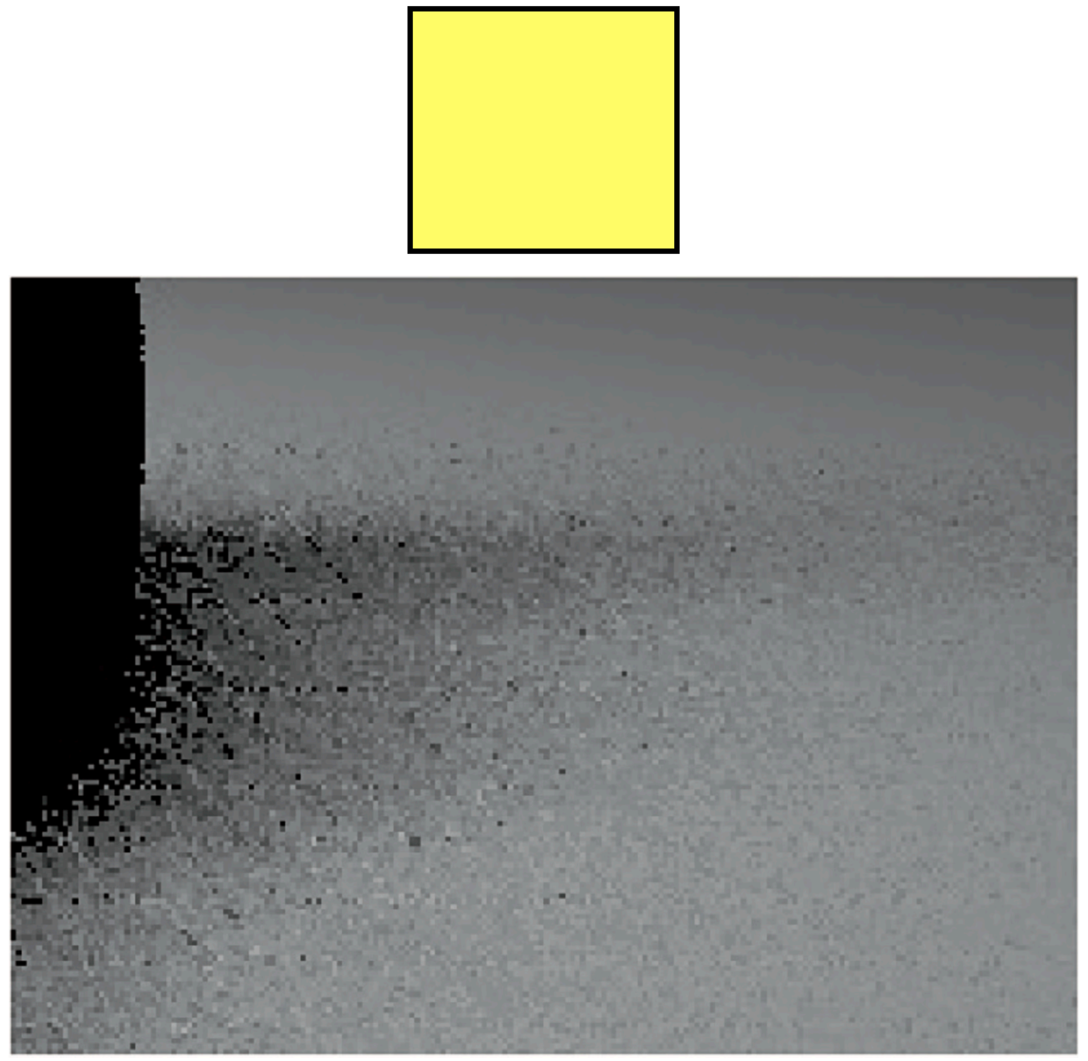


Uniform jitter

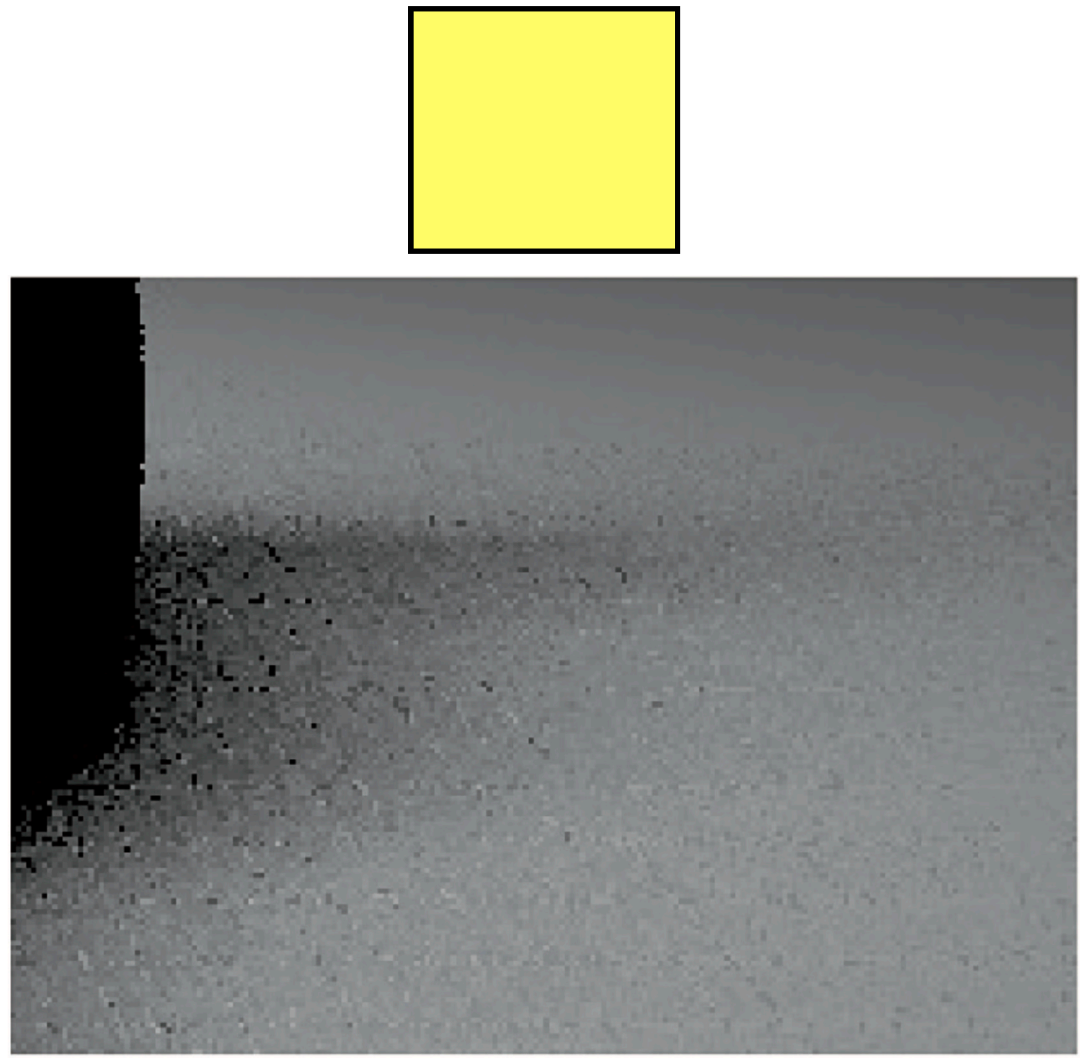


Random jitter

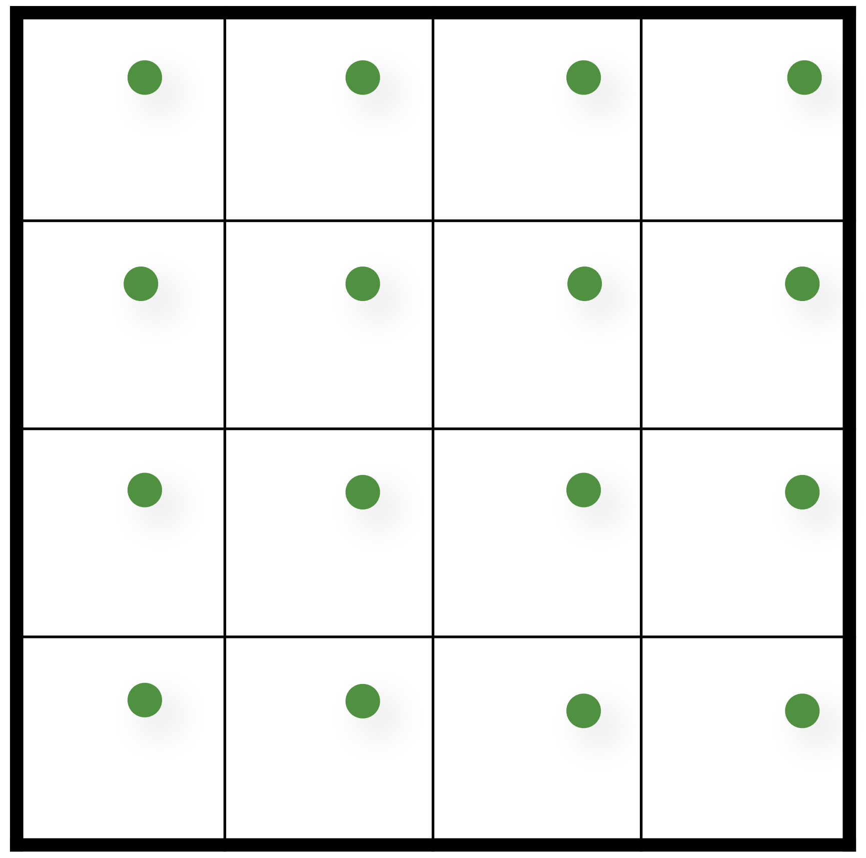
Randomly jittered samples



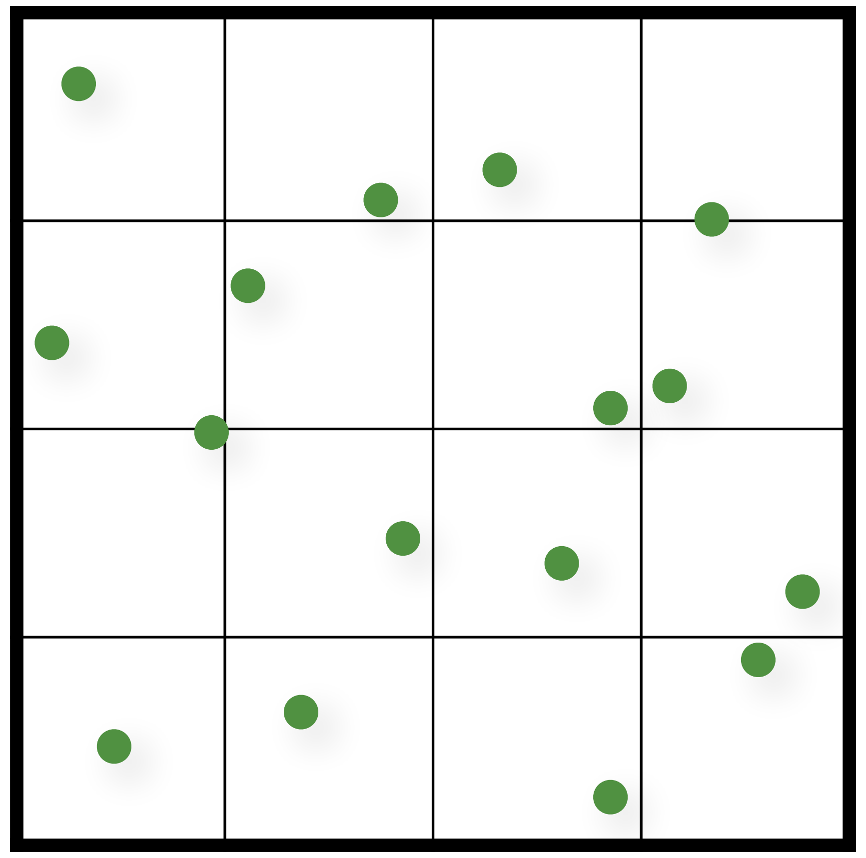
Uniform jitter
(RMS 13.4%)



Random jitter
(RMS 10.4%)



Uniform jitter



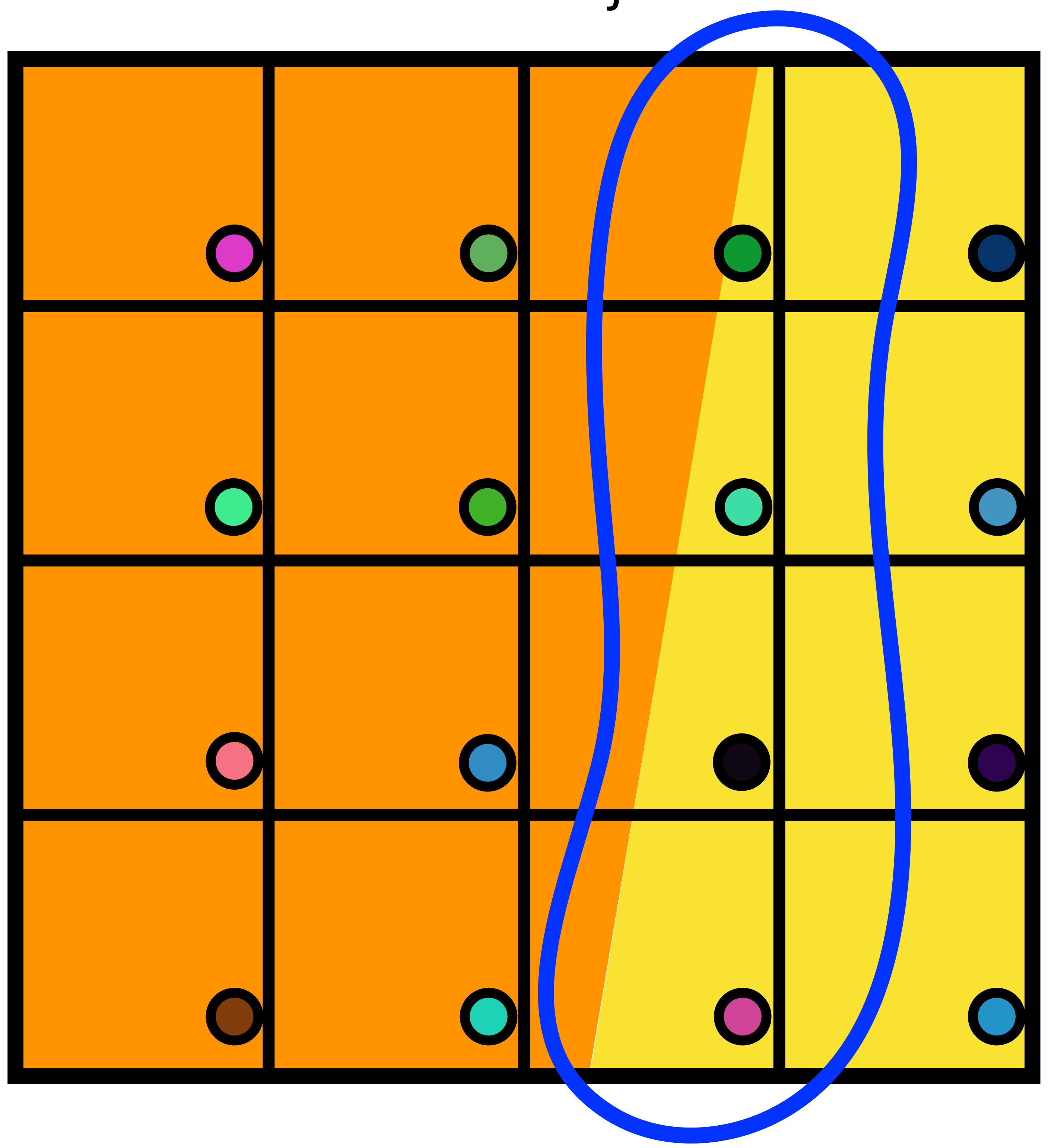
Random jitter

Ramamoorthi et al. [2012]

Square area light source

Uniform jitter

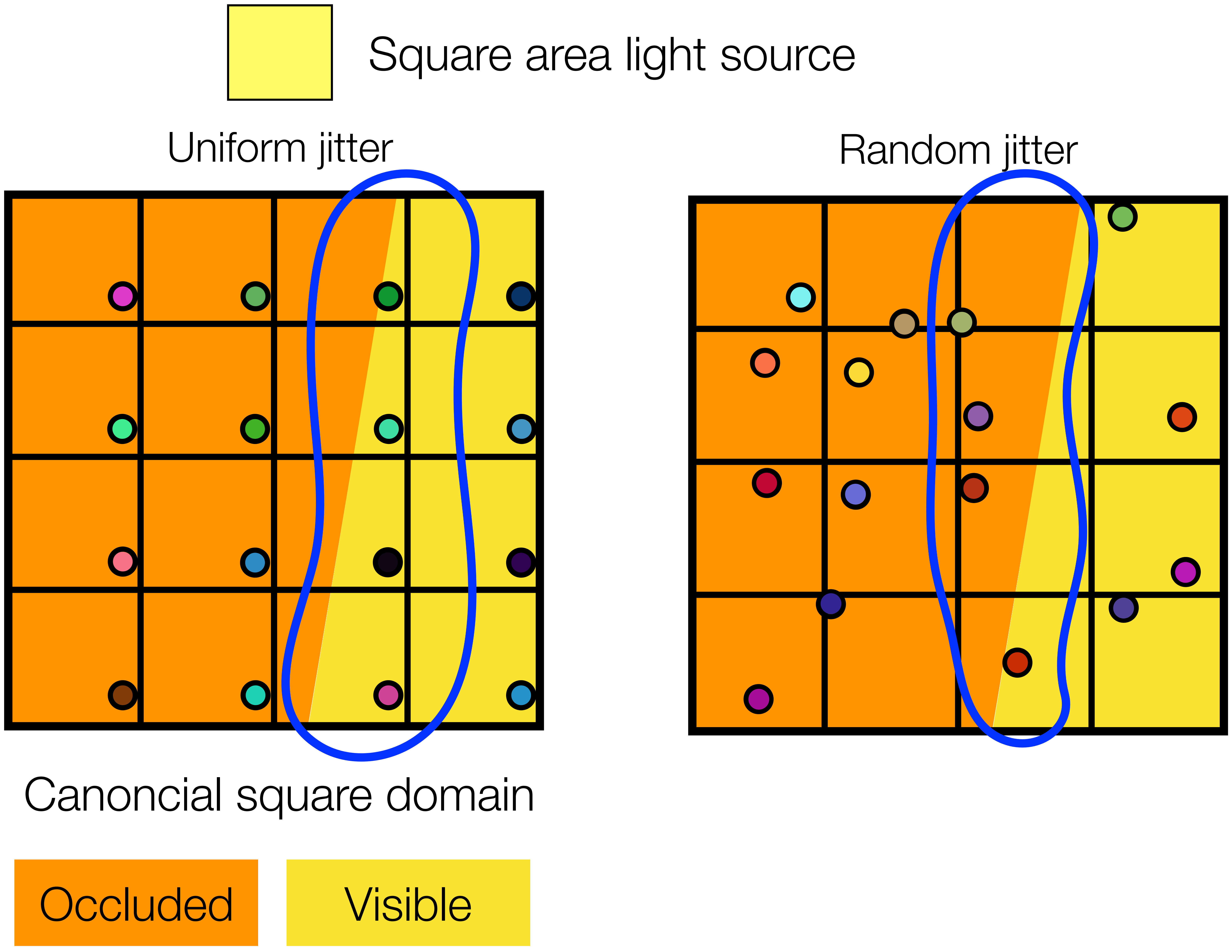
Random jitter



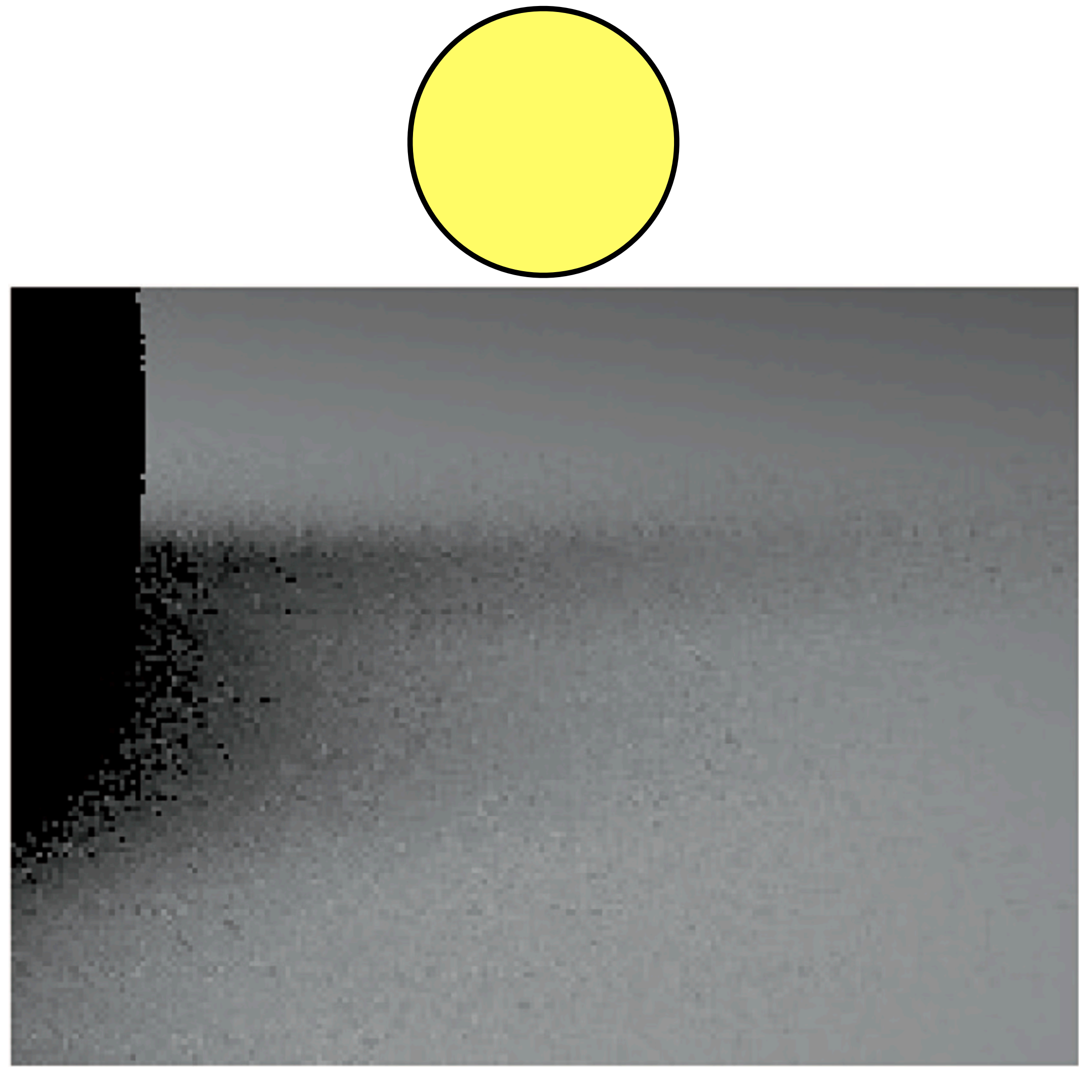
Canoncial square domain

Occluded

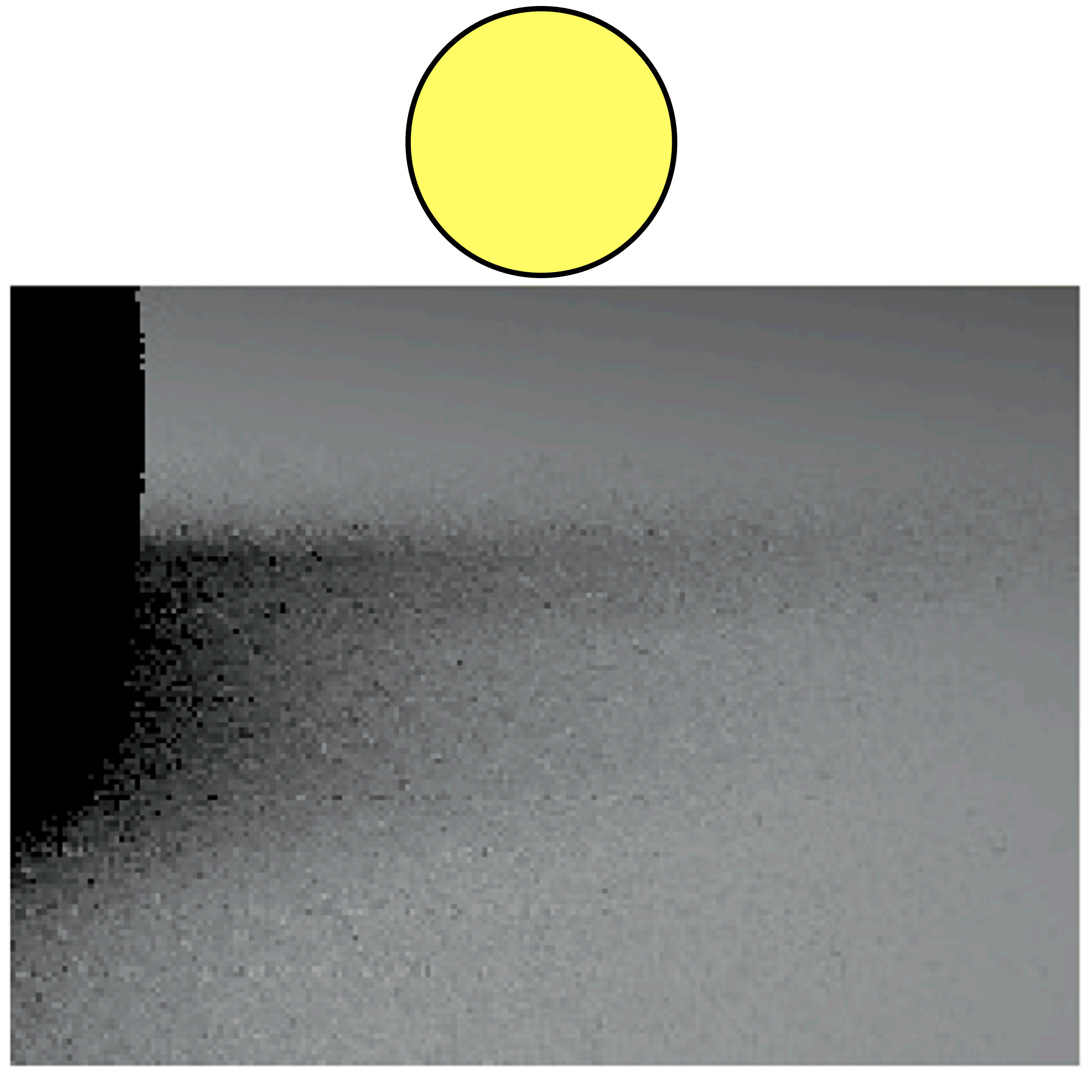
Visible


Square area light source
Uniform jitter
Random jitter
Canoncial square domain
Occluded
Visible

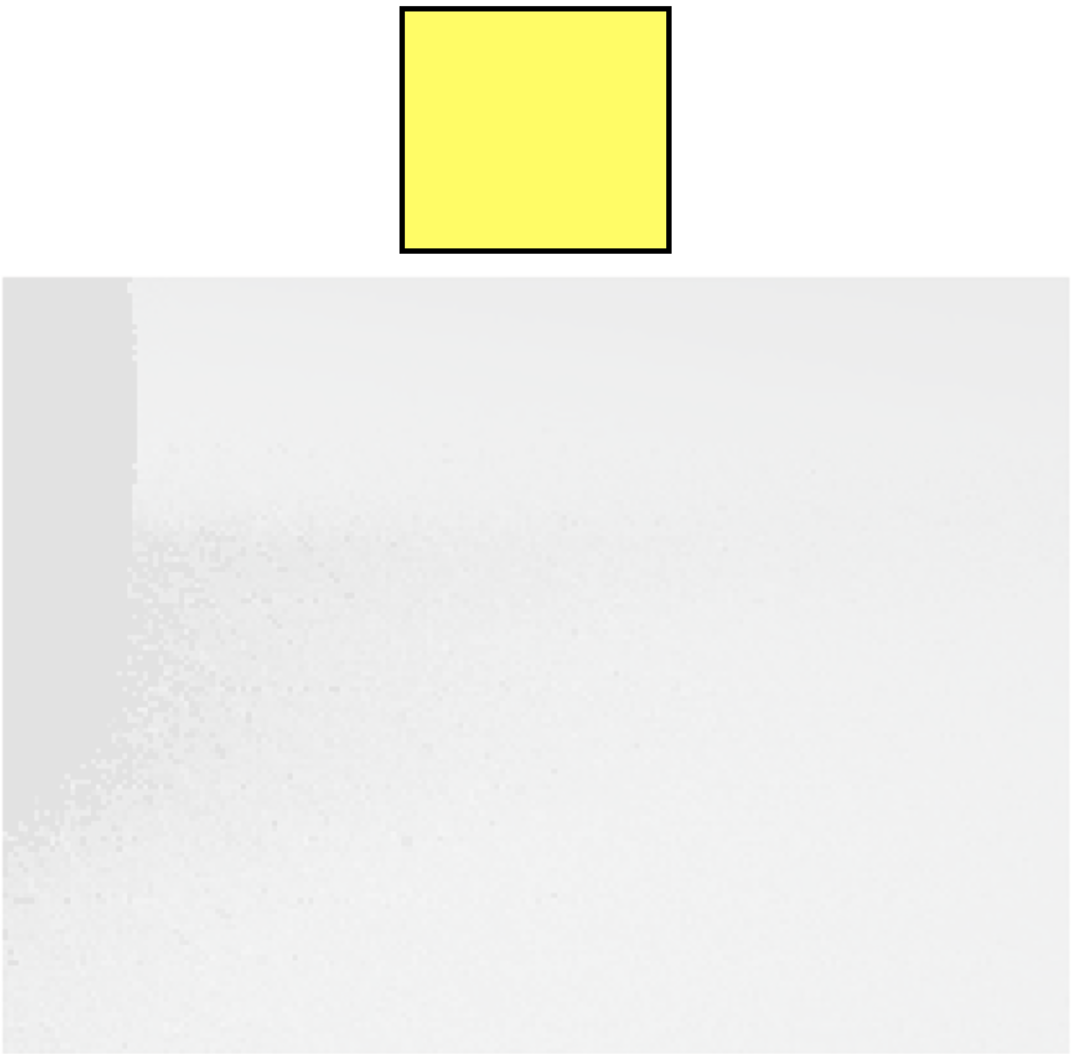
Randomly jittered samples



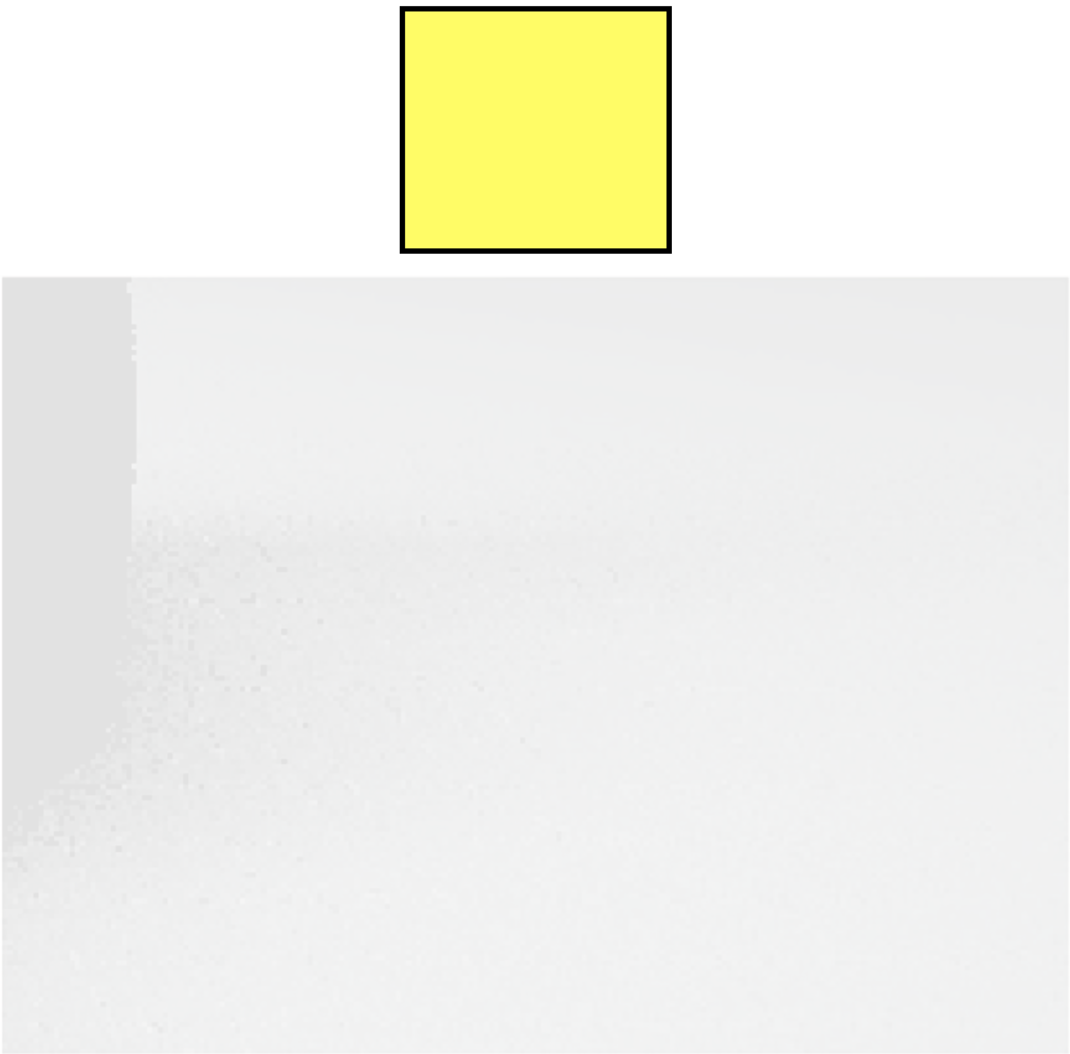
Uniform jitter
(RMS 6.59%)



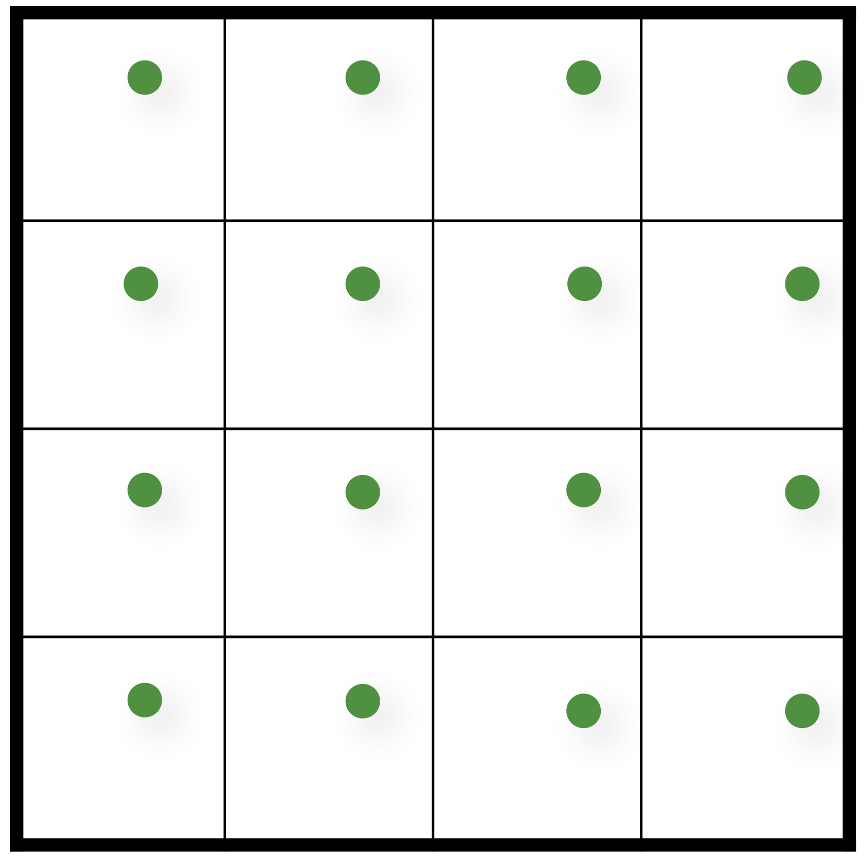
Random jitter
(RMS 8,32%)



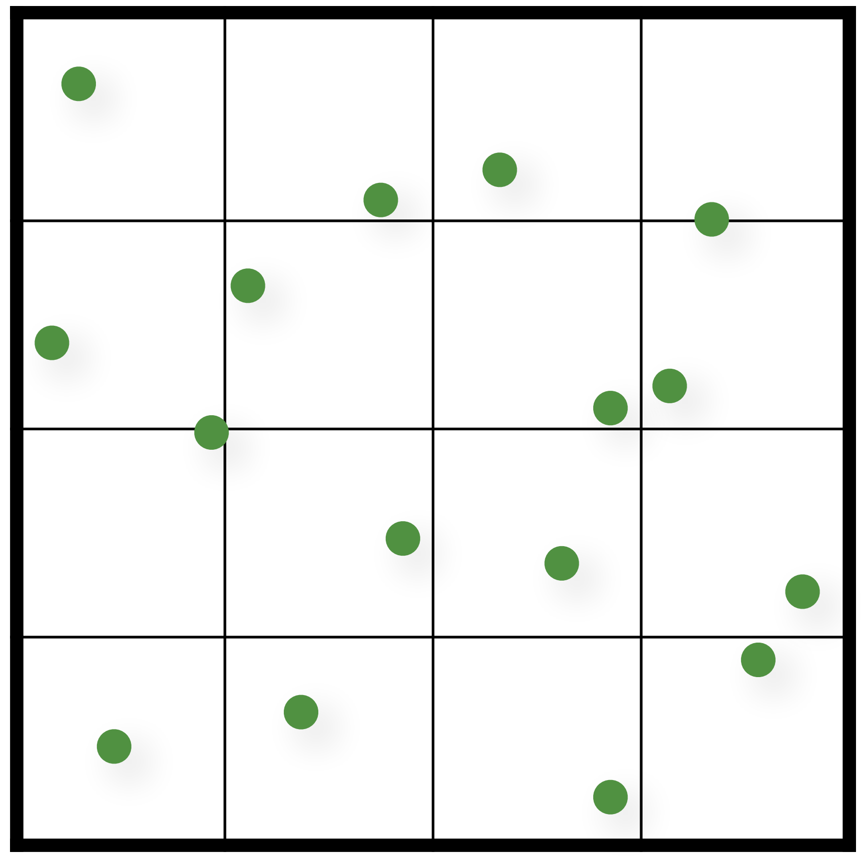
Uniform jitter
(RMS 13.4%)



Random jitter
(RMS 10.4%)



Uniform jitter

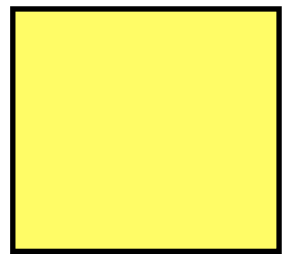


Random jitter

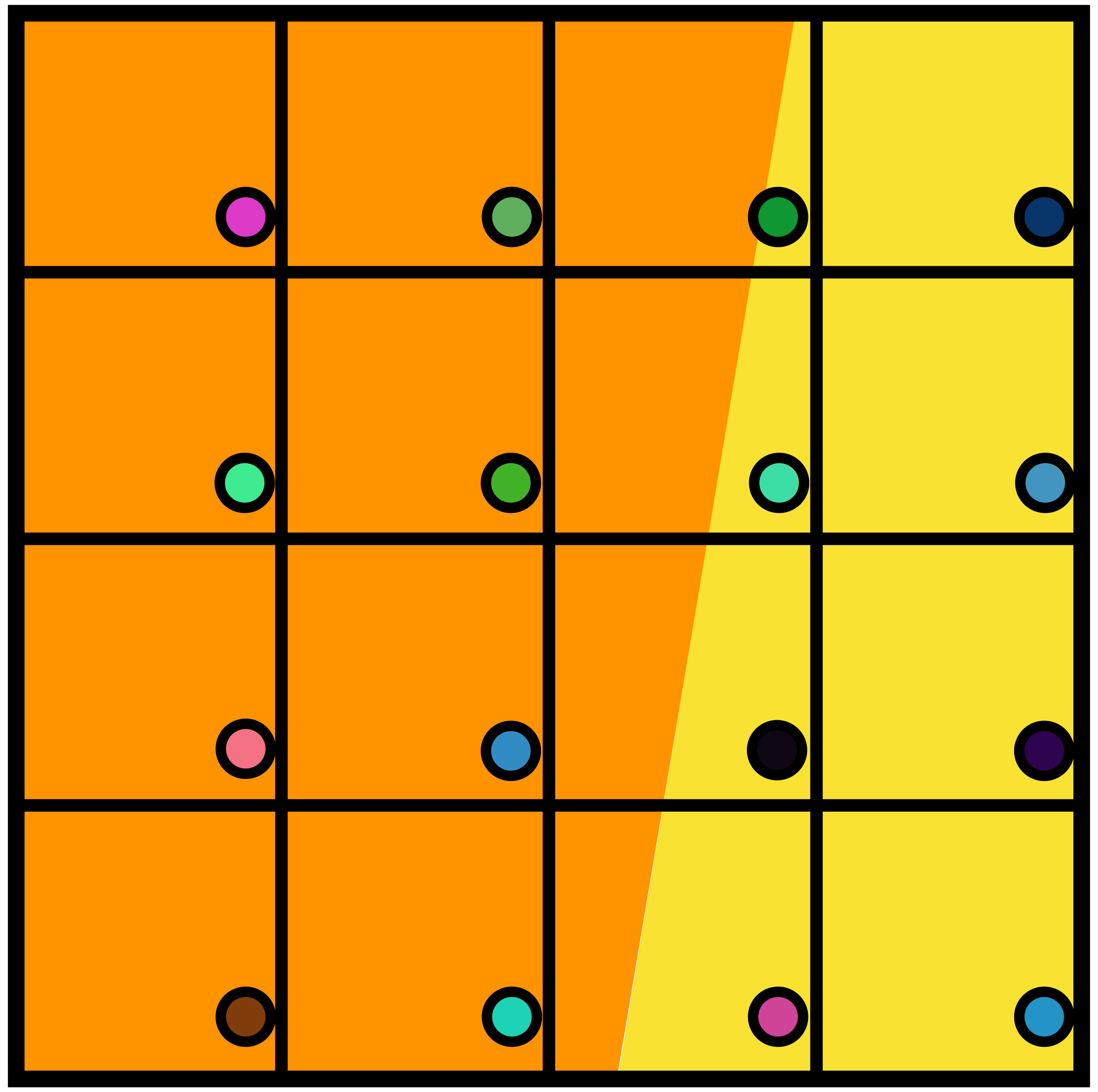
Ramamoorthi et al. [2012]

Polar mapping performs **better** for some samplers compared to **concentric mapping**

observed by **Andrew Kensler [2013]**



Square area light source



Canoncial square domain

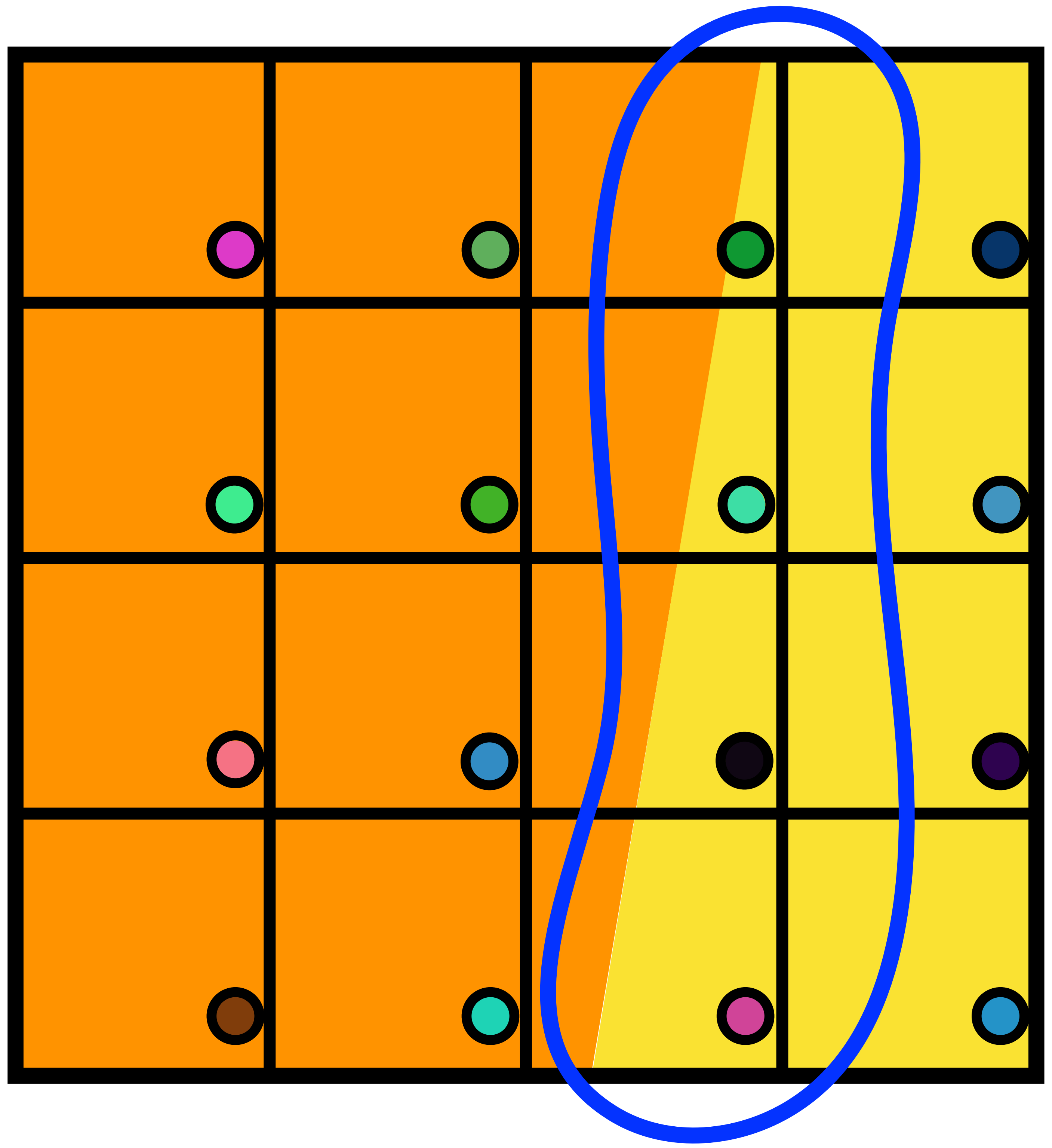
Occluded

Visible

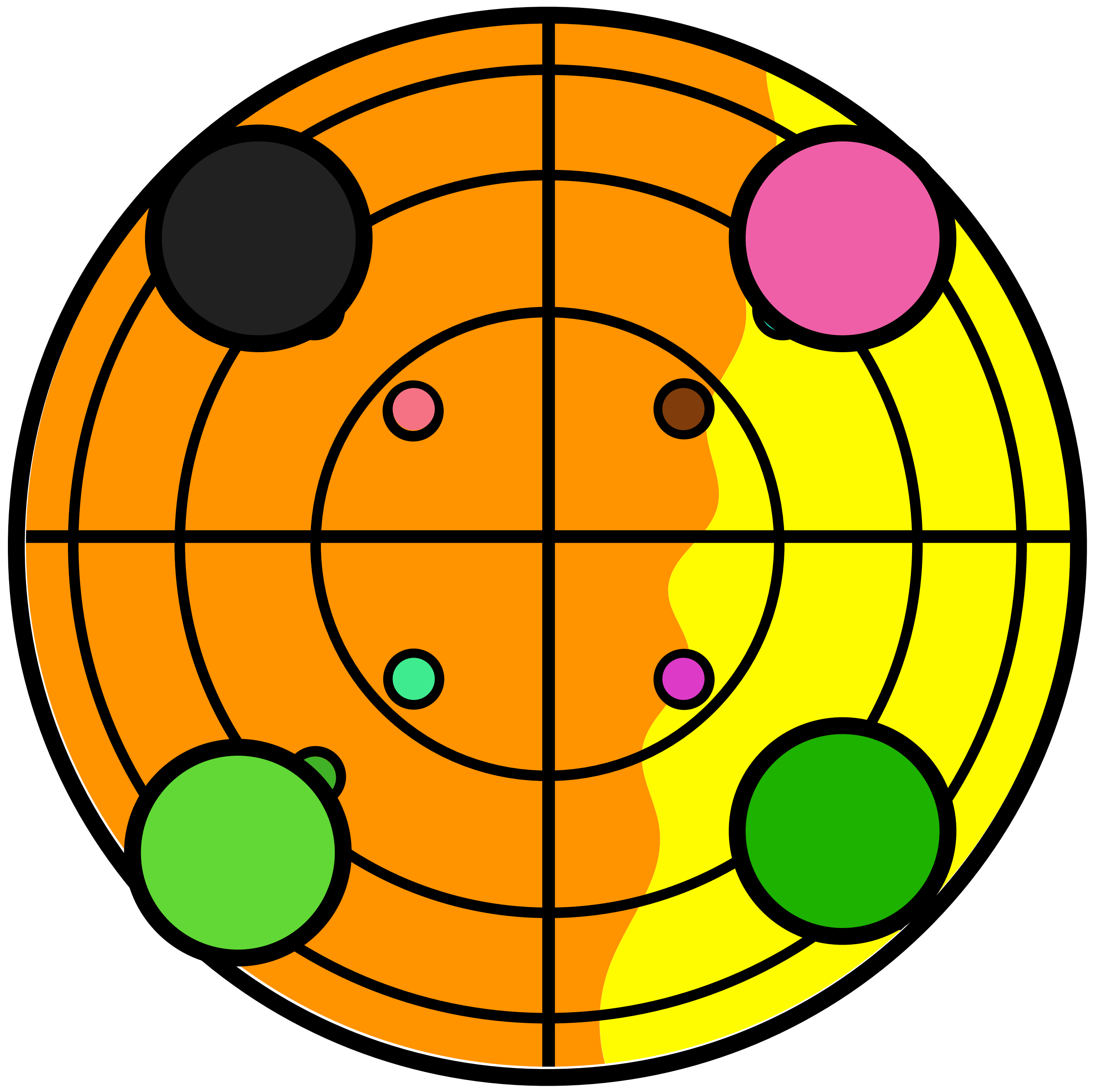
Per Christensen [2018]

Square area light source

Disk area light source



Canoncial square domain



Polar mapping

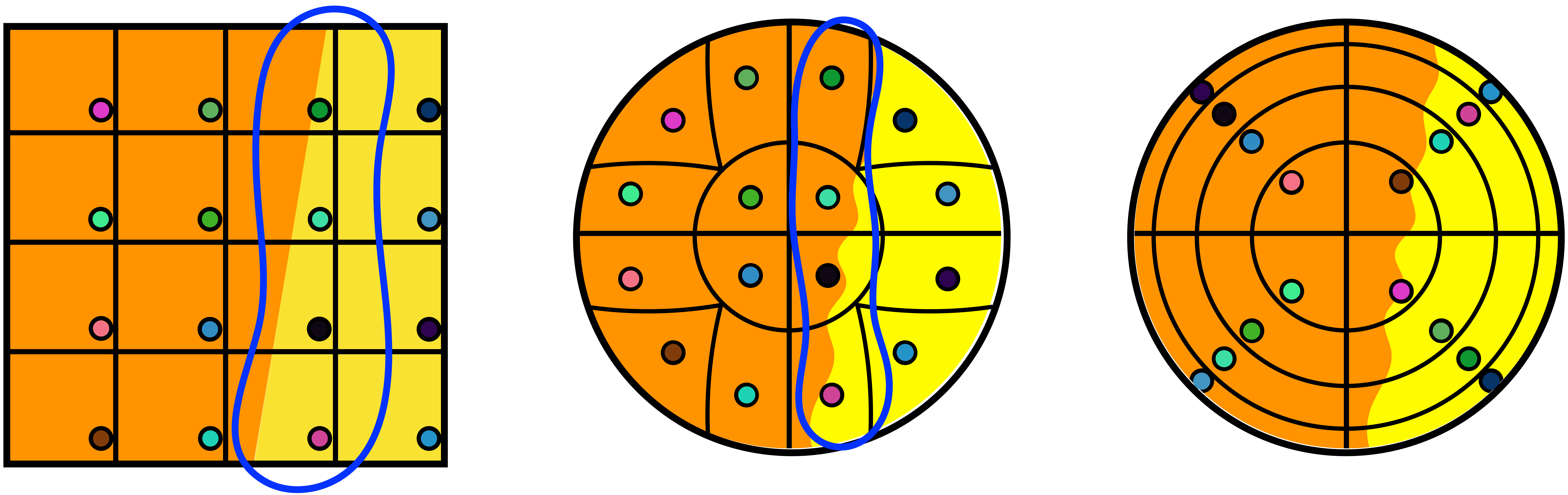
Occluded

Visible

Per Christensen [2018]

Square area light source

Disk area light source



Canoncial square domain

Concentric mapping

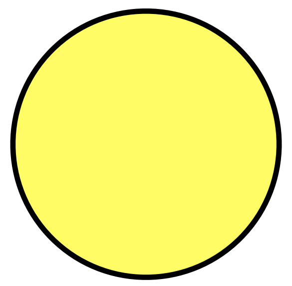
Polar mapping

Occluded

Visible

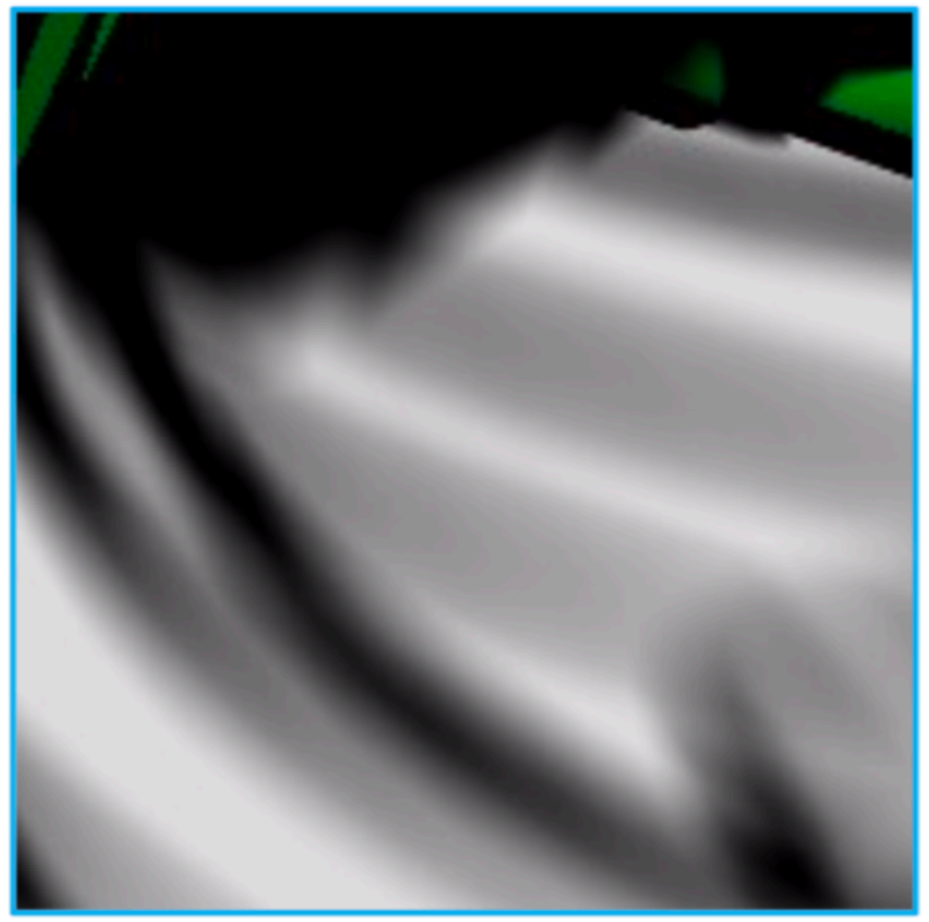
Shirley and Chiu [1997]

Per Christensen [2018]

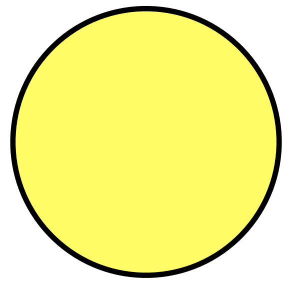


Disk area light source

Reference

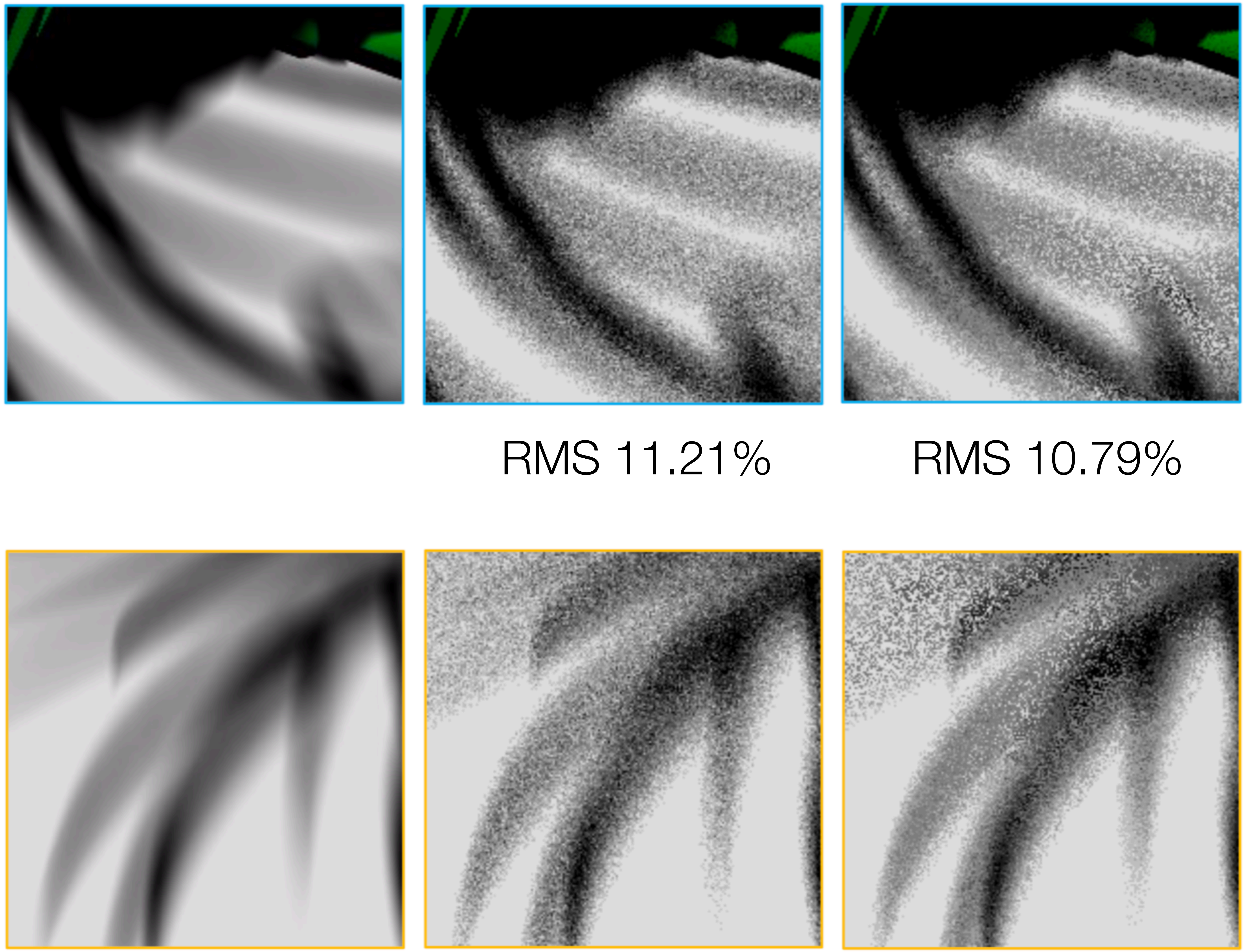


Cengiz Oztireli [2016]



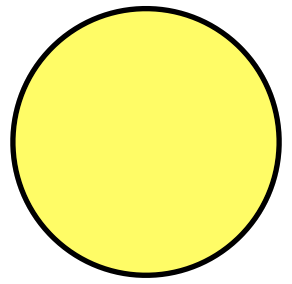
Disk area
light source

Reference | Random jitter | Uniform jitter



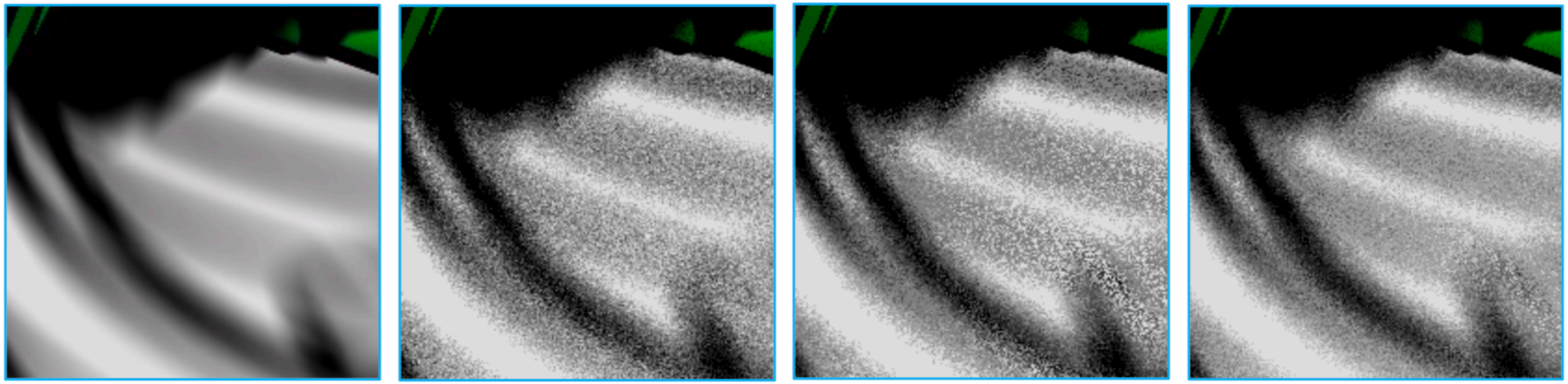
RMS 10.92% | RMS 11.77%

Cengiz Oztireli [2016]

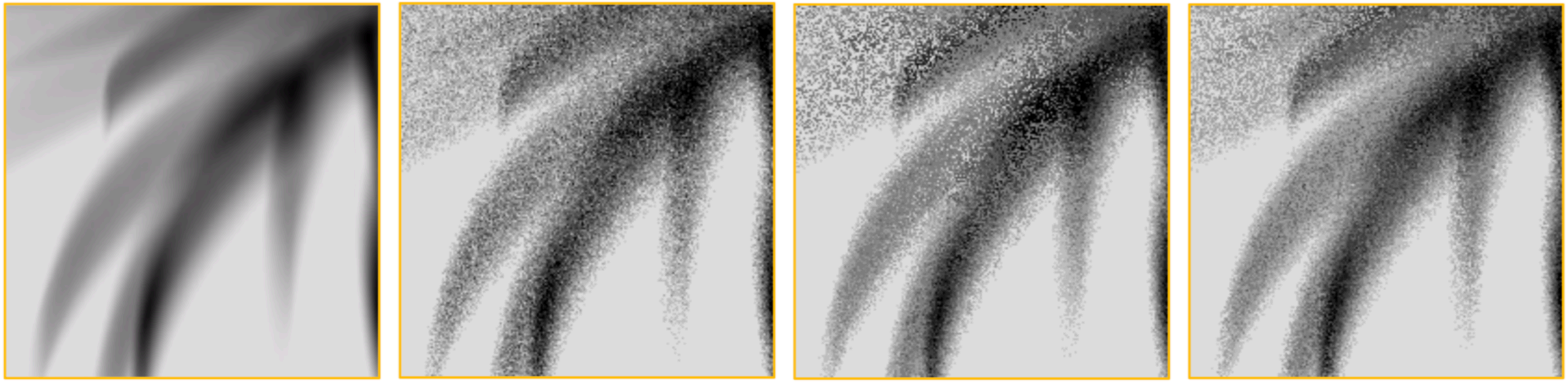


Disk area light source

	Reference	Random jitter	Uniform jitter	Isotropic jitter



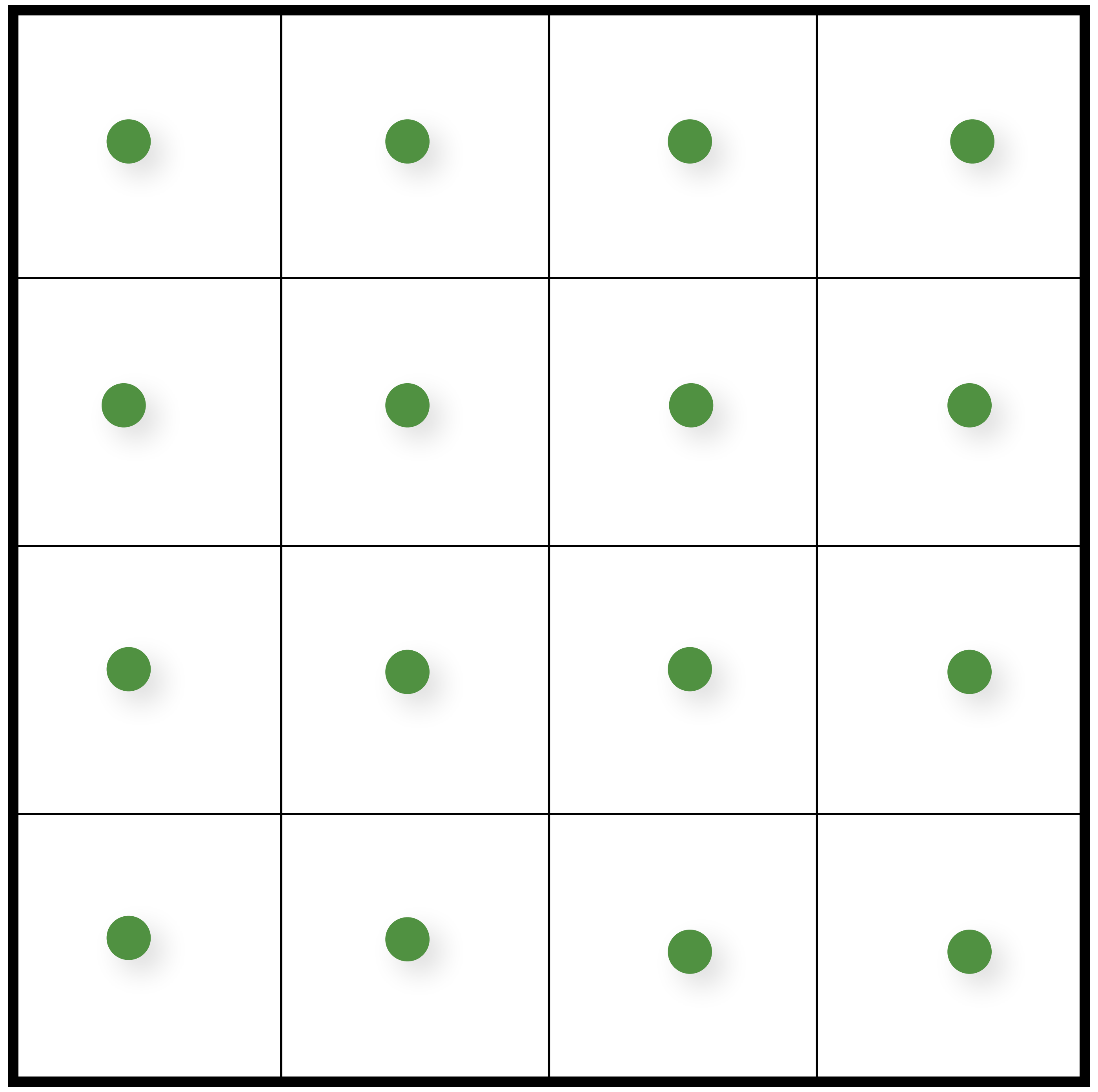
RMS 11.21% | RMS 10.79% | RMS 8.00%



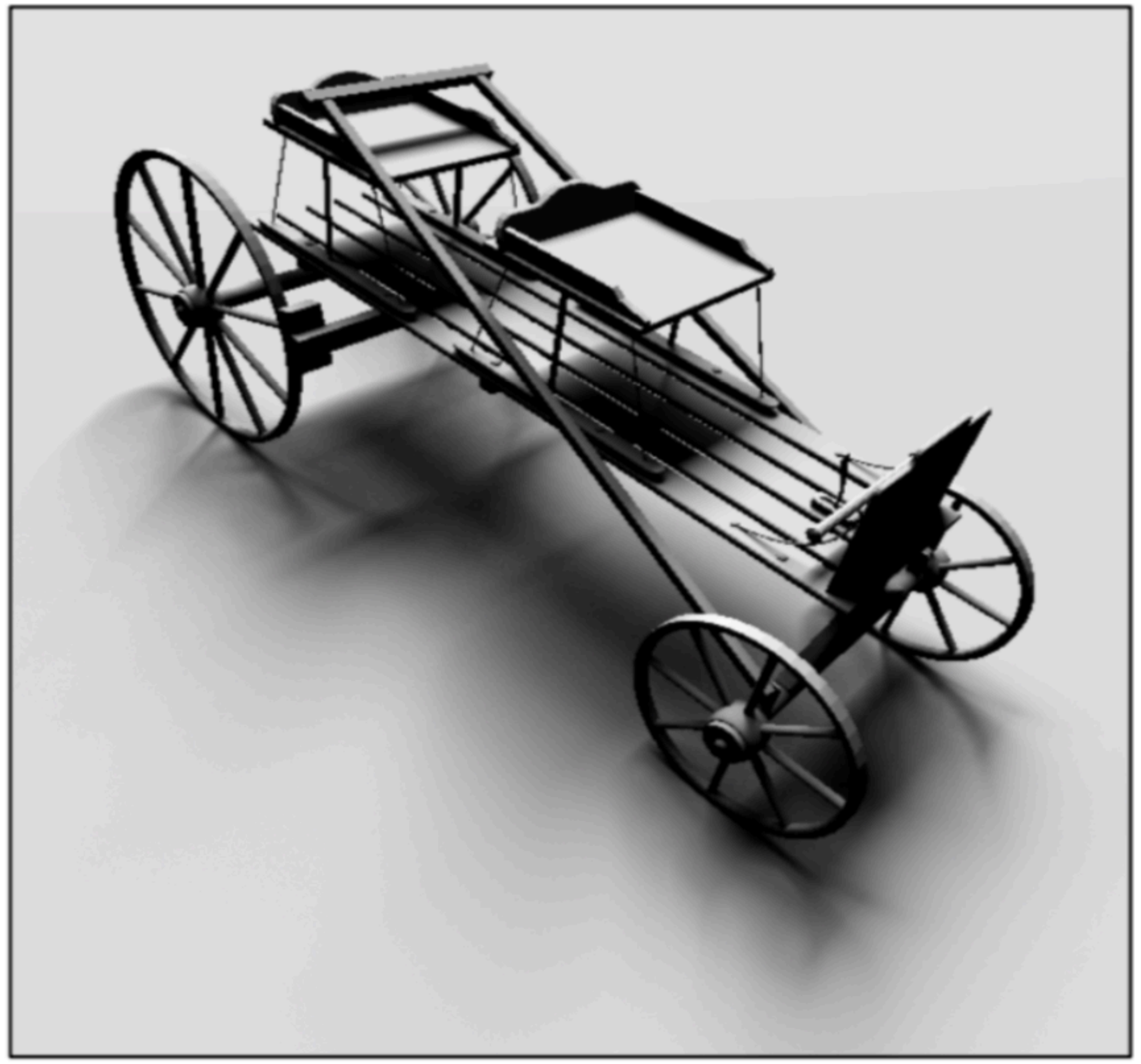
RMS 10.92% | RMS 11.77% | RMS 8,77%

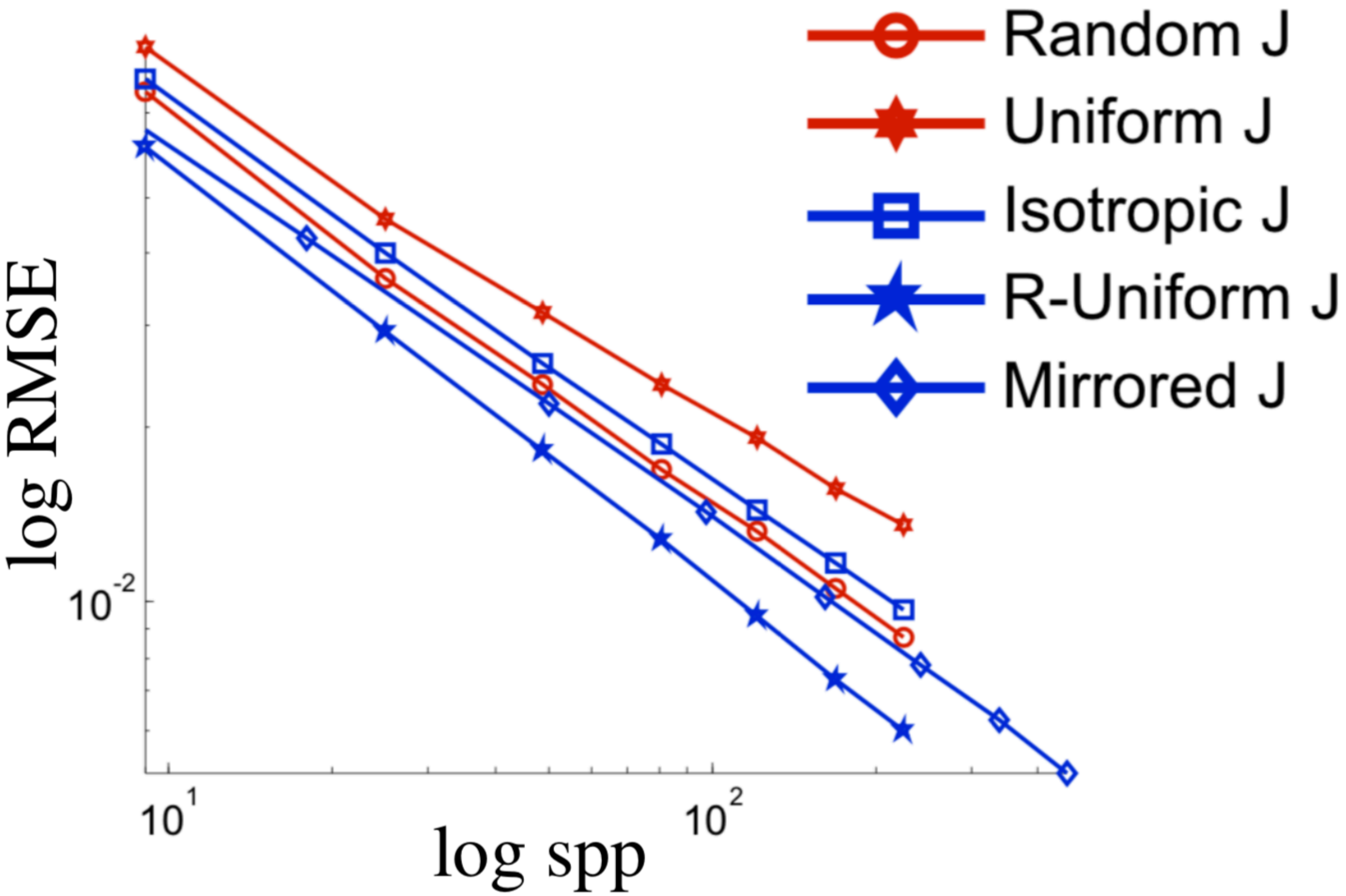
Cengiz Oztireli [2016]

Isotropic jitter = uniform jitter + random rotation



Rotated uniform jitter better for not too complex shadows





Cengiz Oztireli [2016]

Stratification strategies

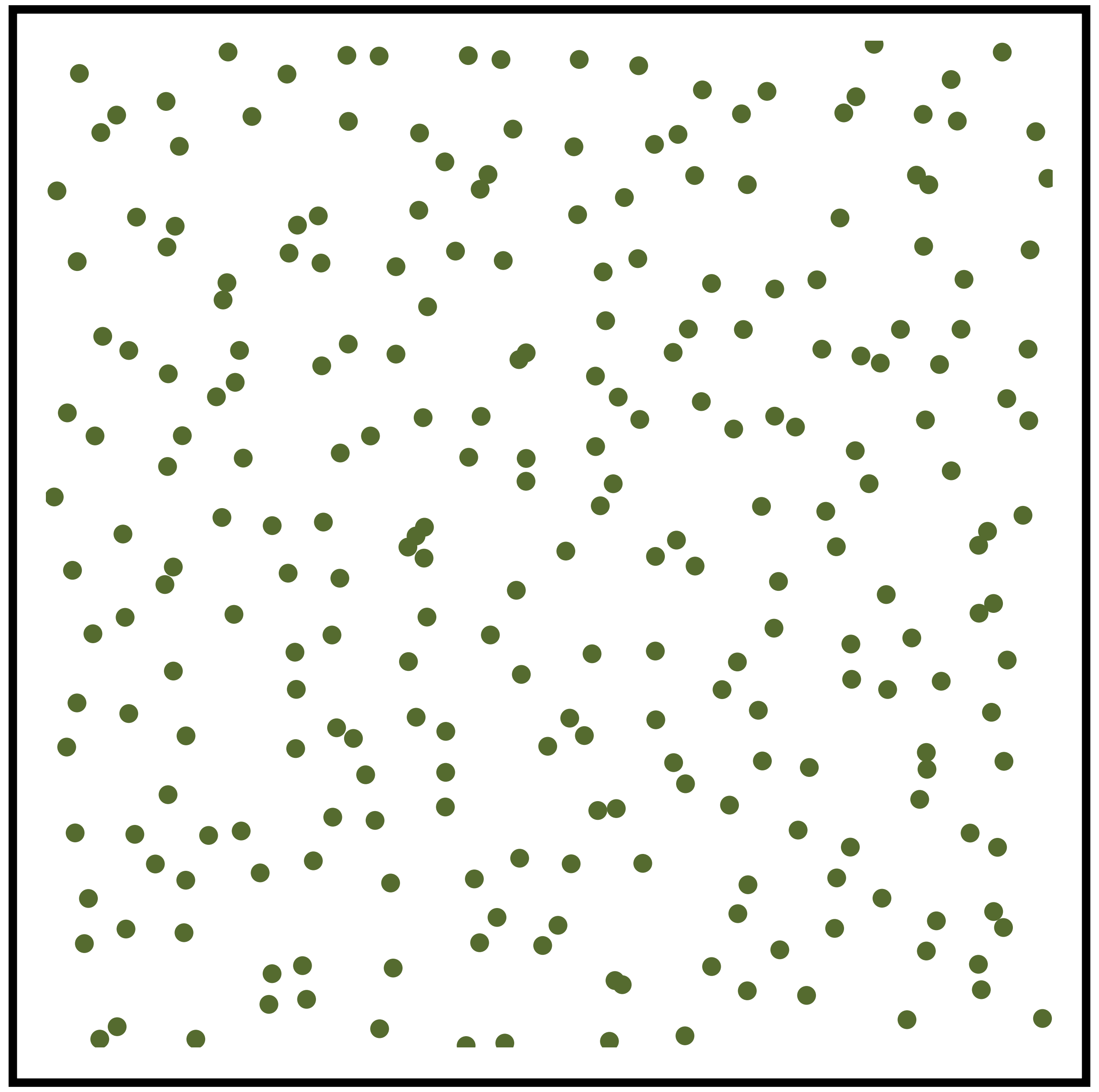
Blue noise sampling and beyond

Fourier analysis of sample correlations

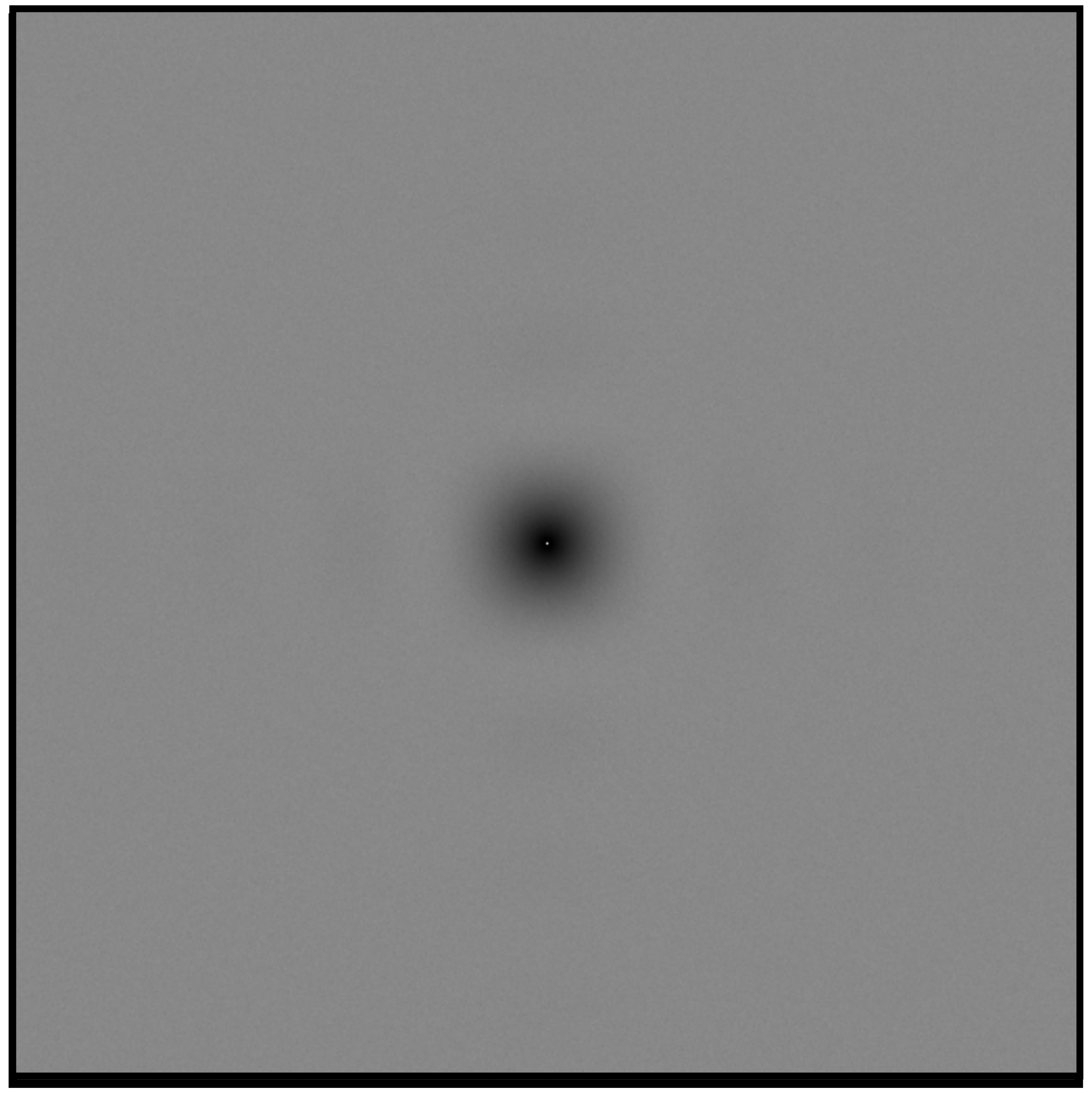
Expected power spectrum for random samples

Expected power spectrum for jittered samples

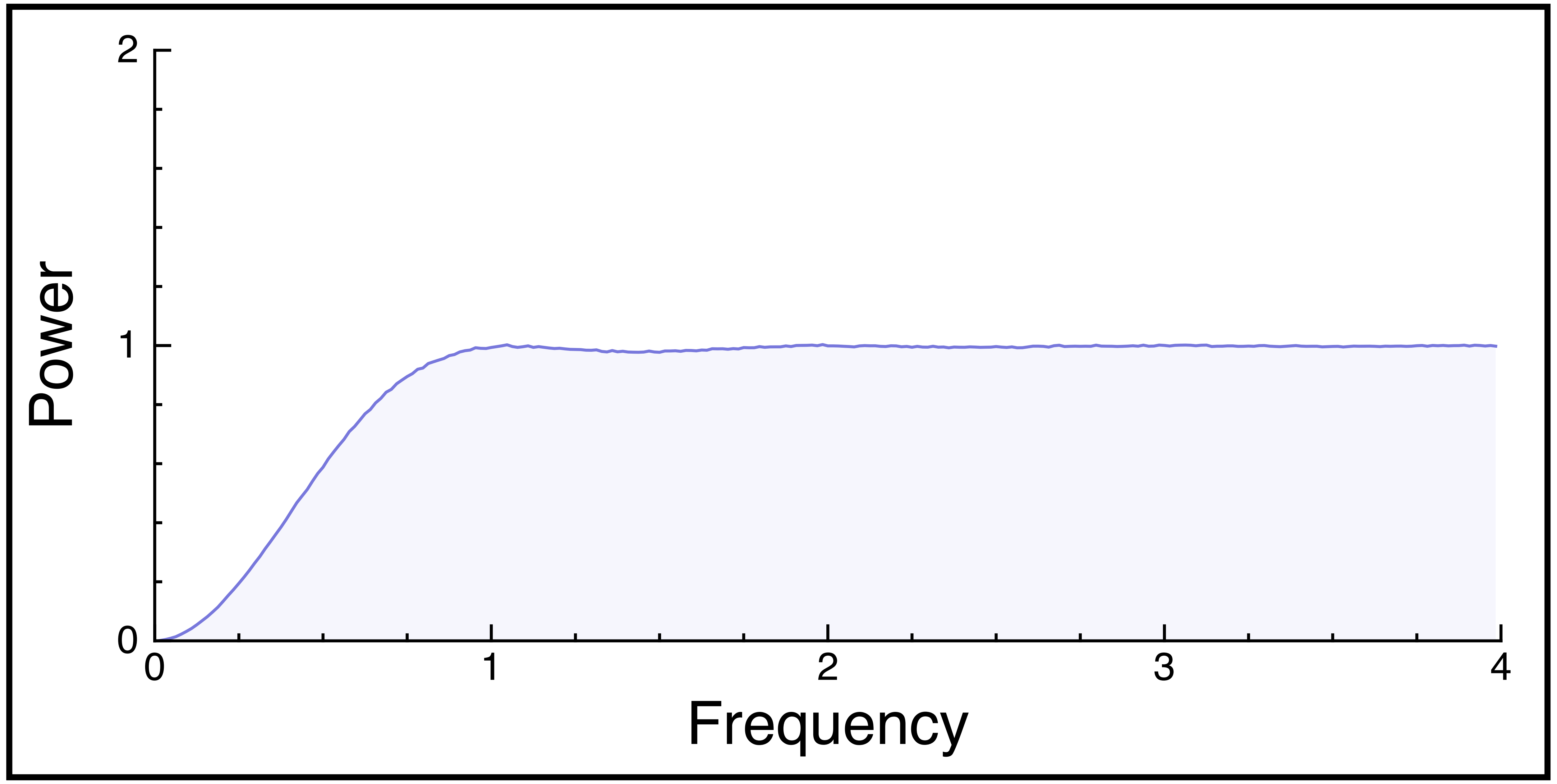
Samples



Expected power spectrum

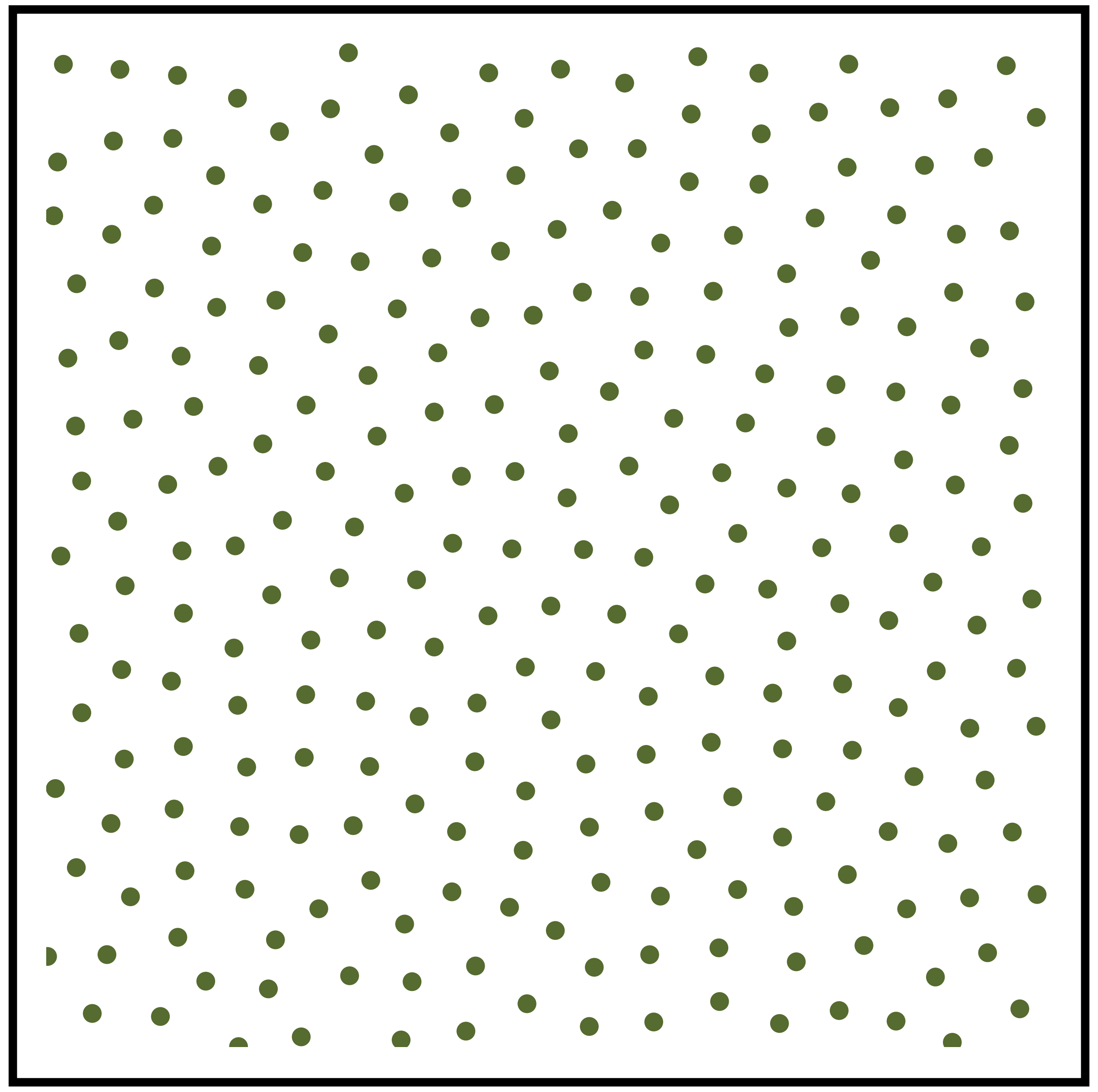


Radial mean

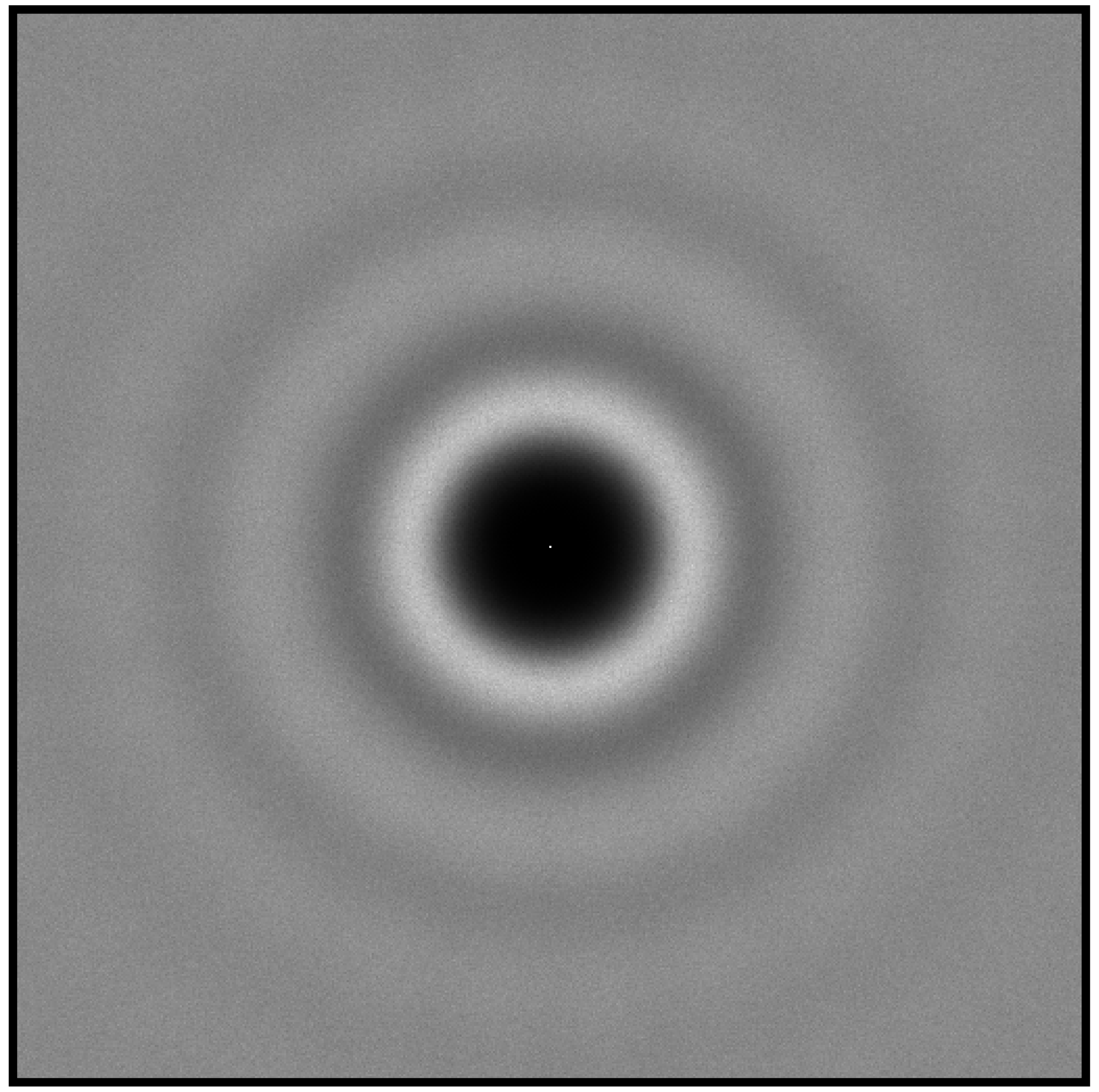


Expected power spectrum for blue noise samples

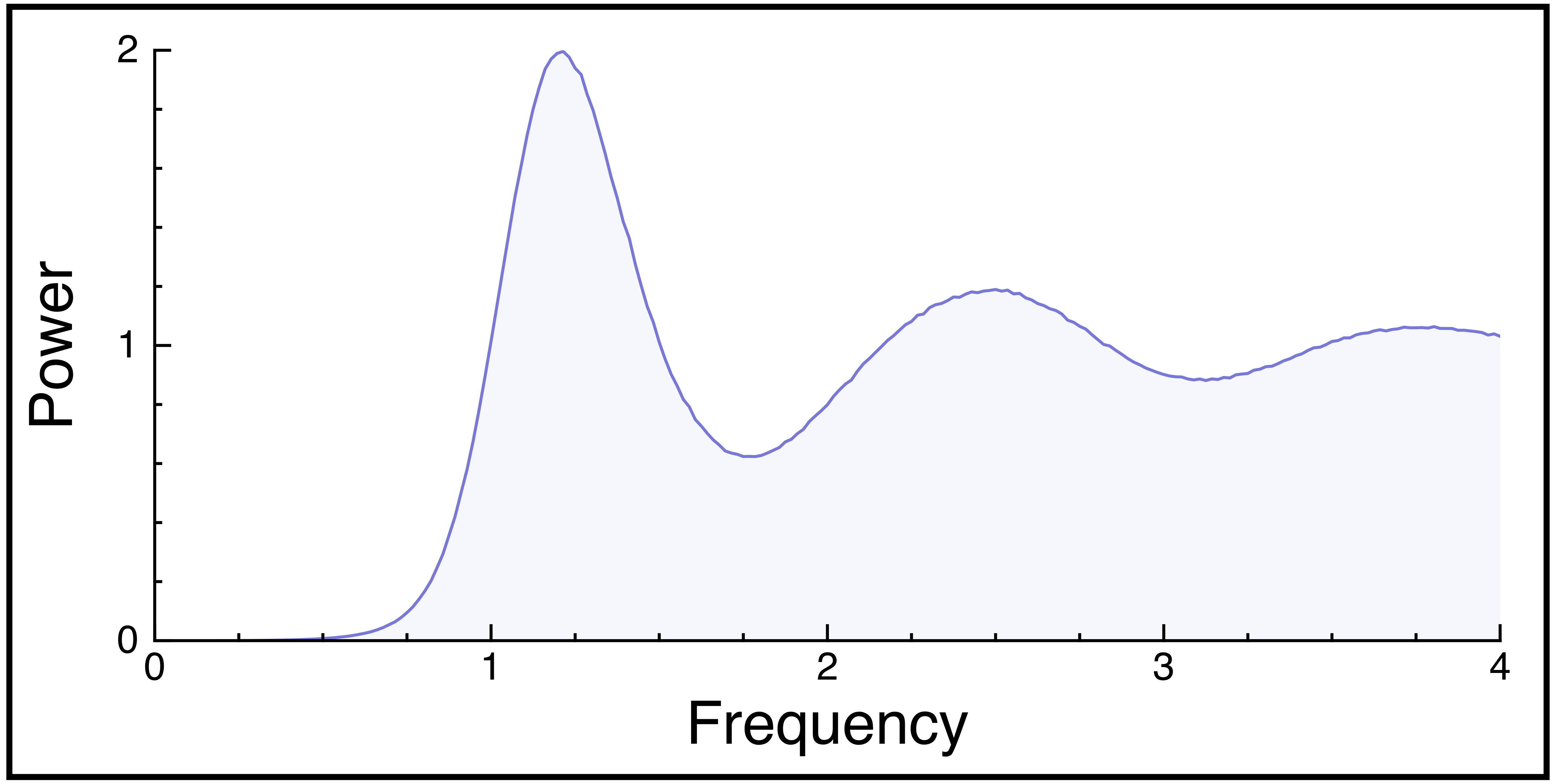
Samples



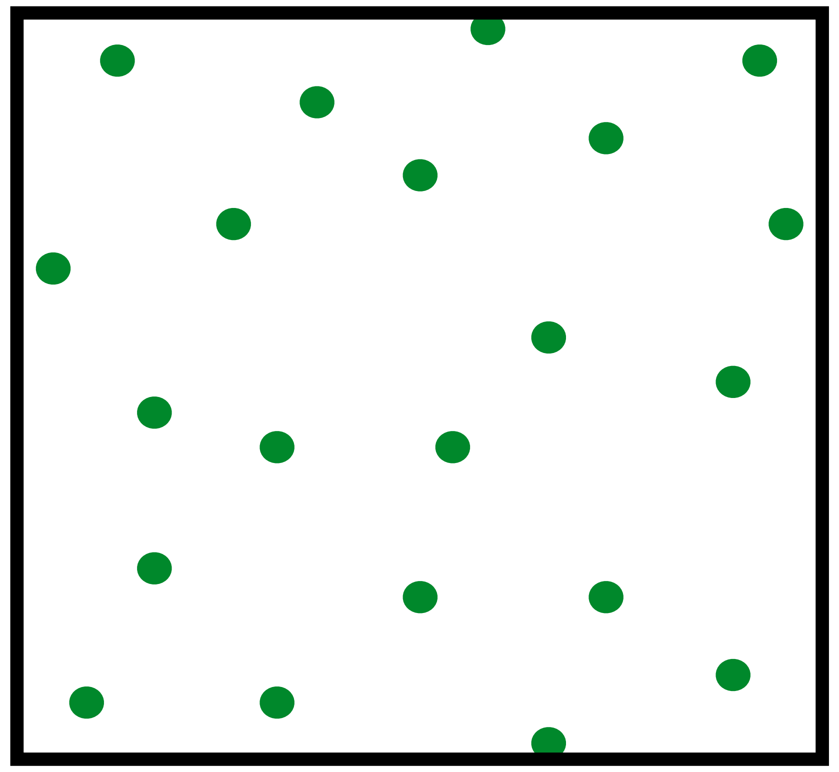
Expected power spectrum



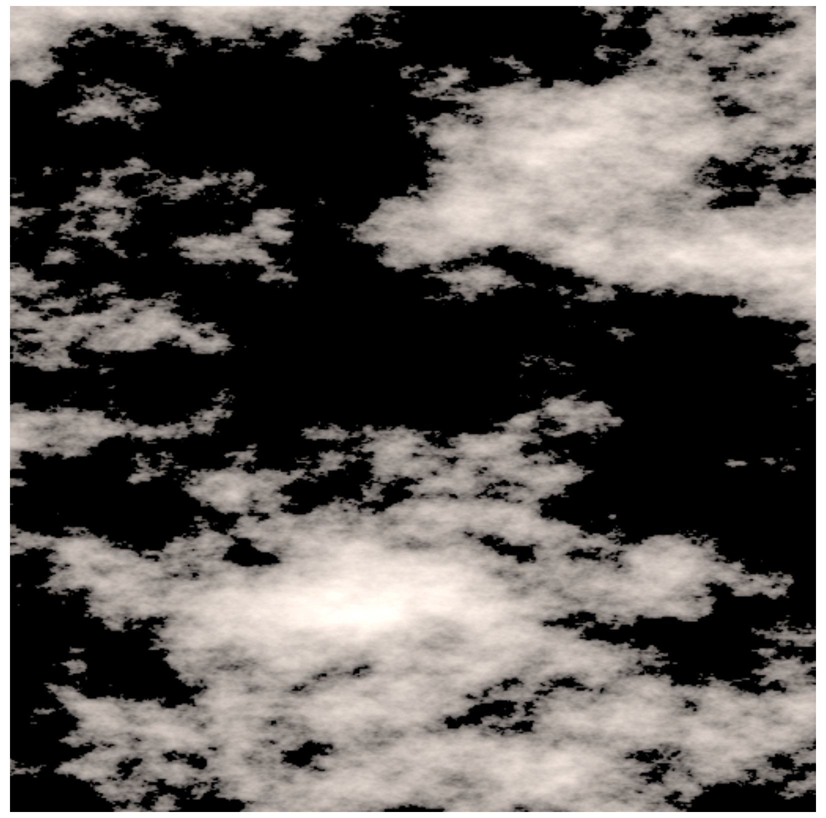
Radial mean



Variance in terms of power spectra



$f(\vec{x})$



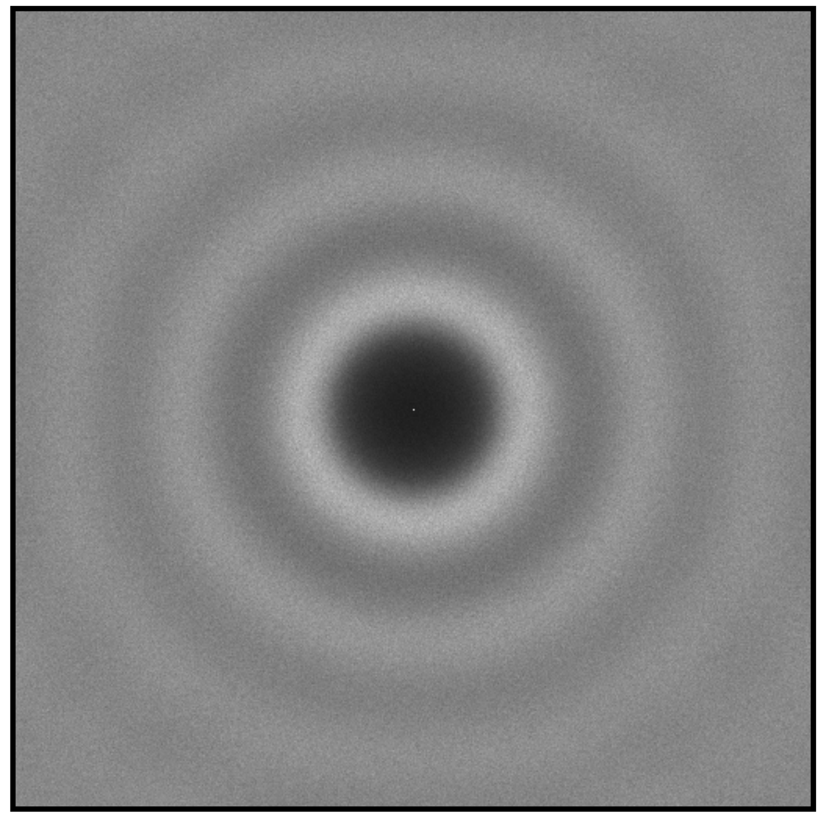
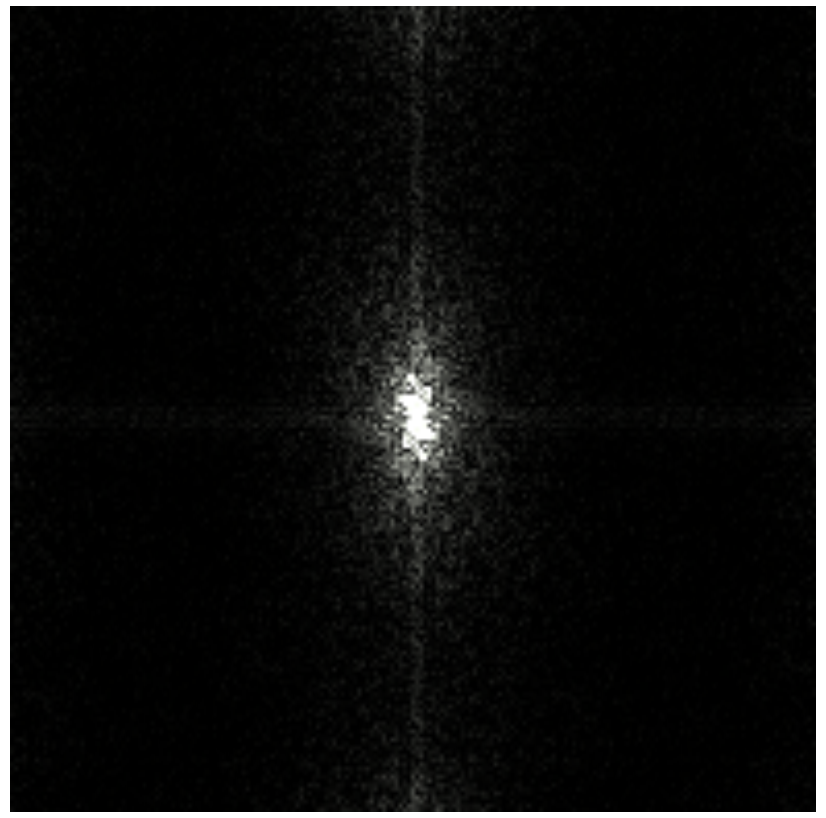
Fredo Durand [2011]

Pillebuoe et al. [2015]

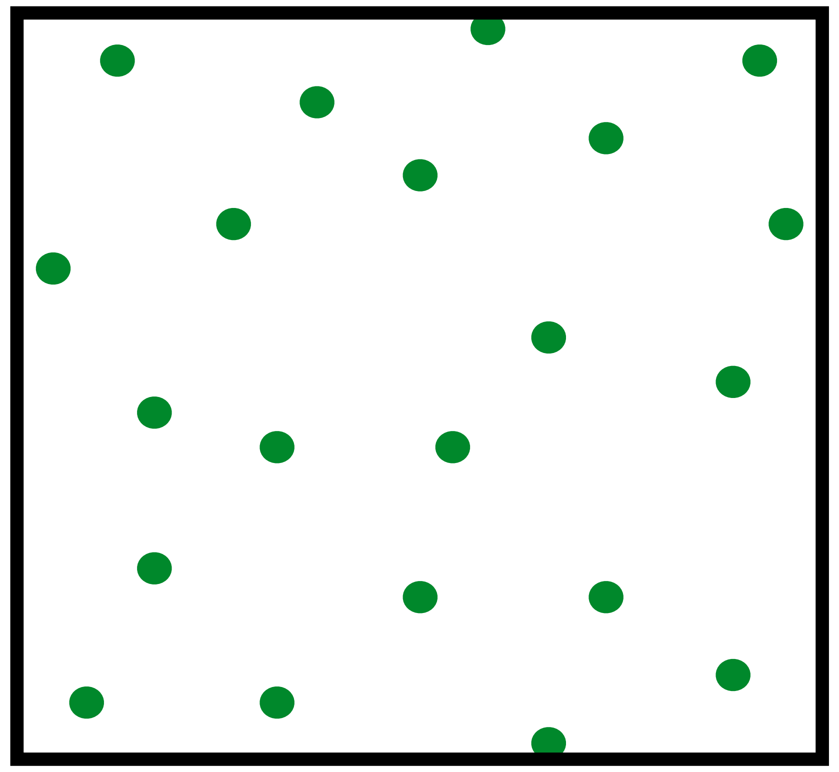
Variance in terms of power spectra

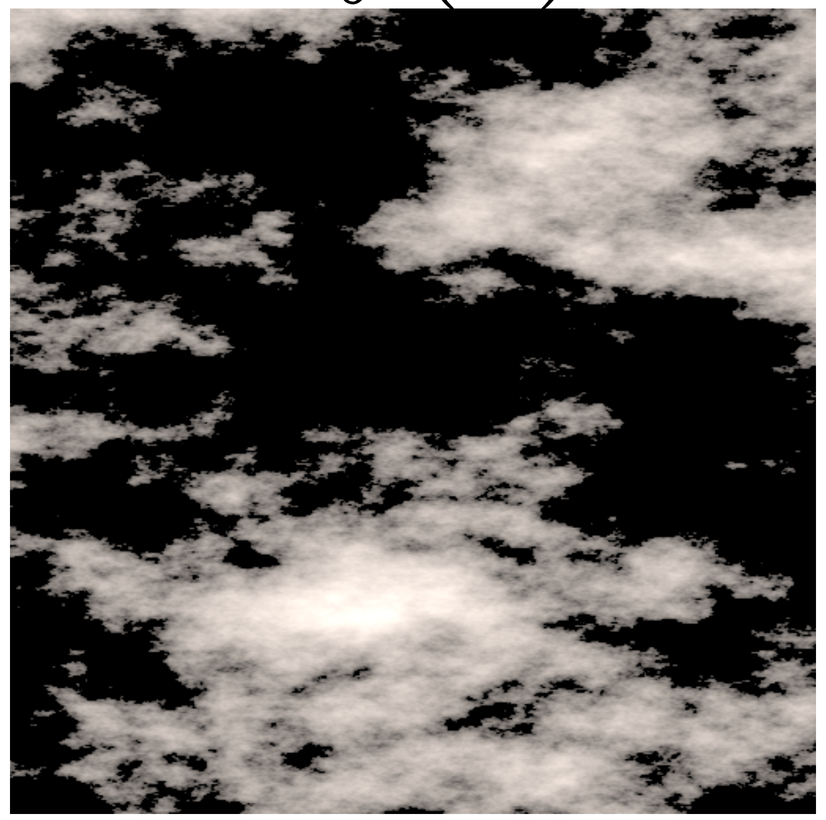
Samples' expected power spectrum

Integrand power spectrum

$\mathrm{Var}(I_N) \propto$  $\times$ 

$f(\vec{x})$





Fredo Durand [2011]

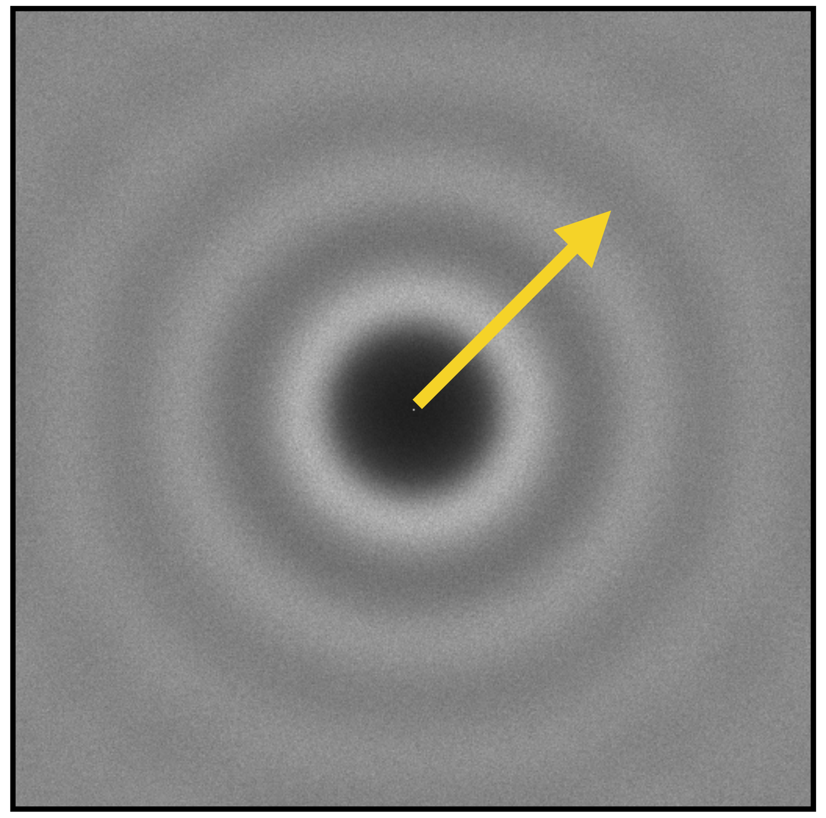
Pillebuoe et al. [2015]

Variance in terms of power spectra

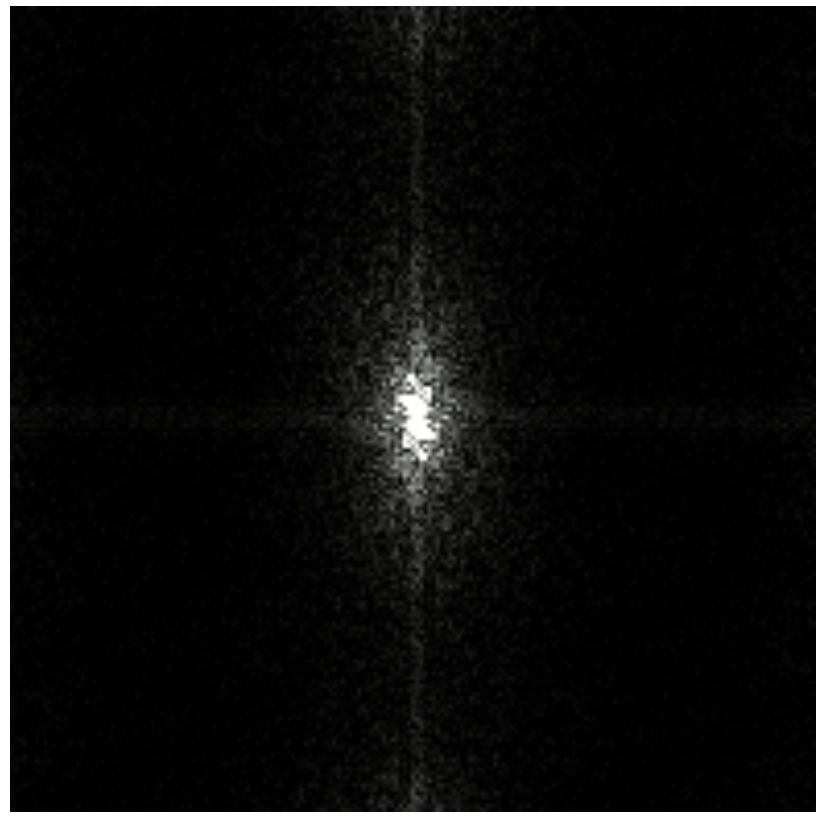
Samples' expected power spectrum

Integrand power spectrum

$$\mathrm{Var}(I_N) \propto$$



$$\times$$



Fredo Durand [2011]

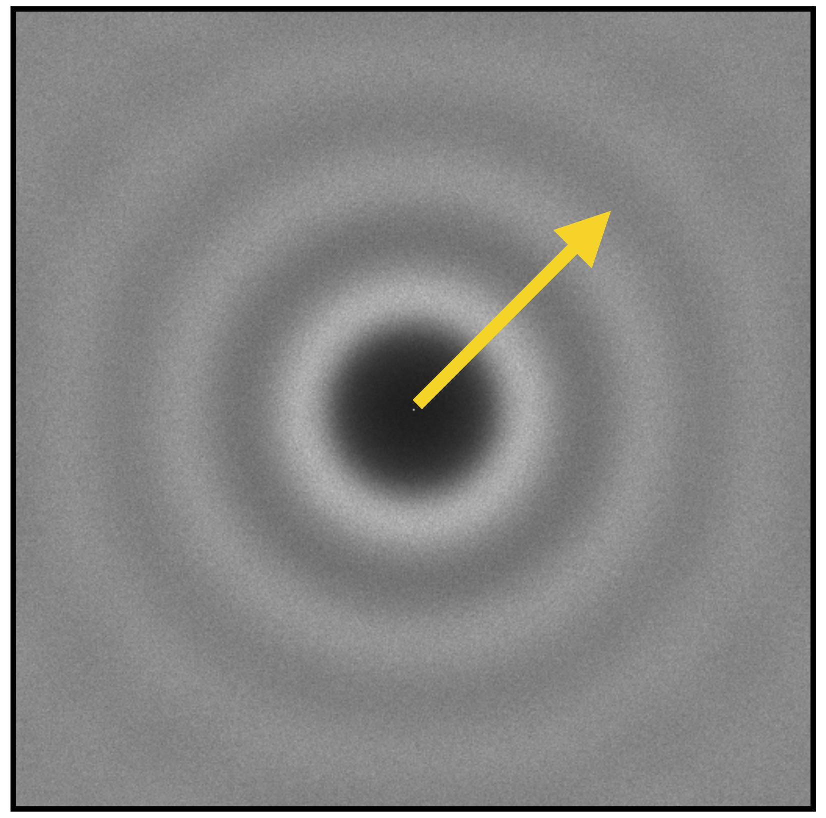
Pillebuoe et al. [2015]

Variance in terms of power spectra

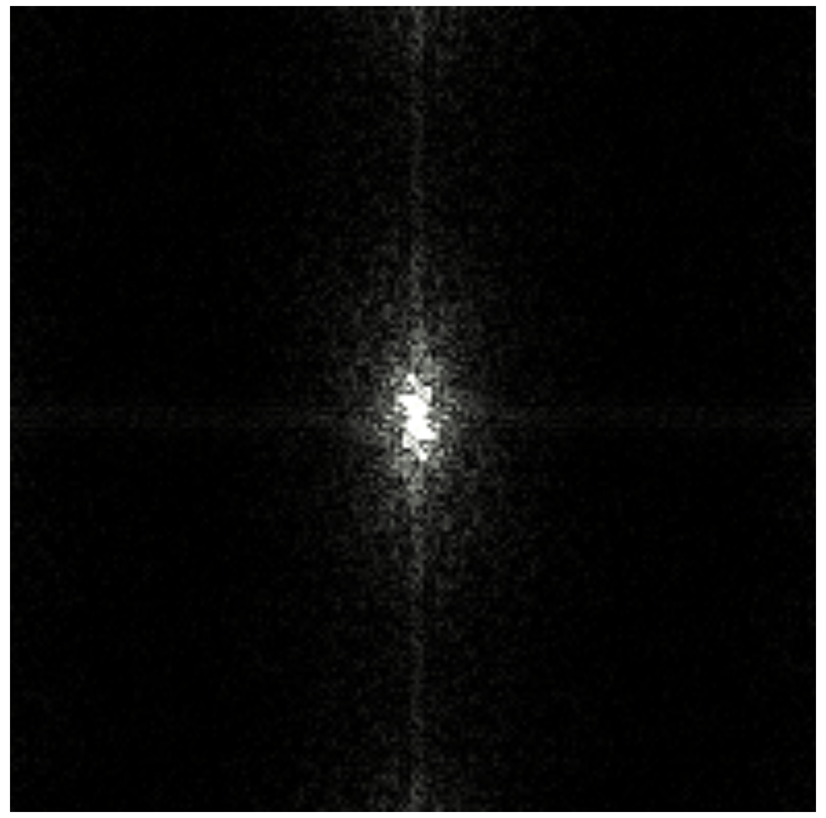
Samples' expected power spectrum

Integrand power spectrum

$$\mathrm{Var}(I_N) \propto$$



$$\times$$



Fredo Durand [2011]

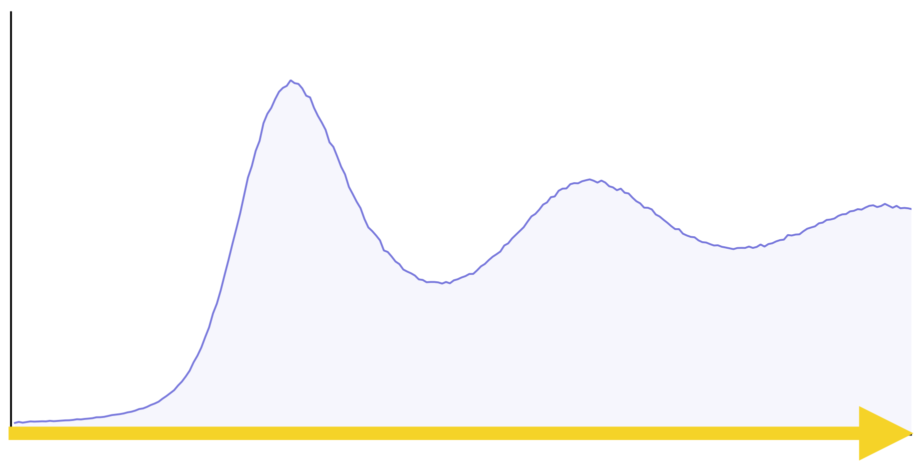
Pillebuoe et al. [2015]

Variance in terms of power spectra

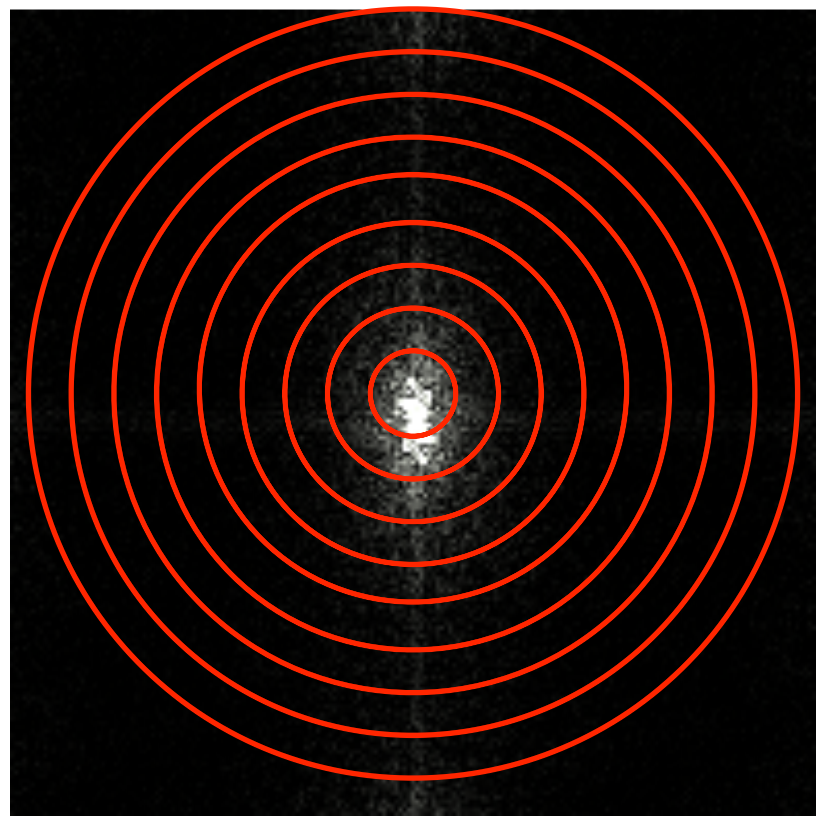
Samples' expected power spectrum

Integrand power spectrum

$\mathrm{Var}(I_N) \propto$



$\times$

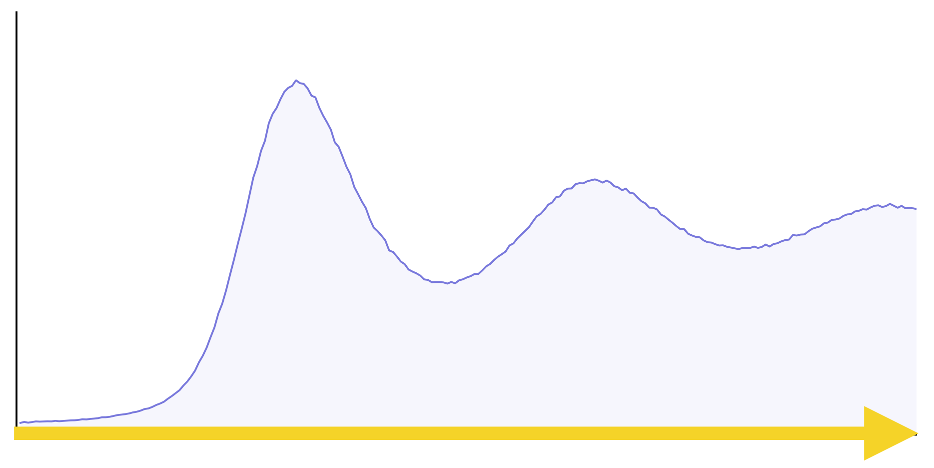
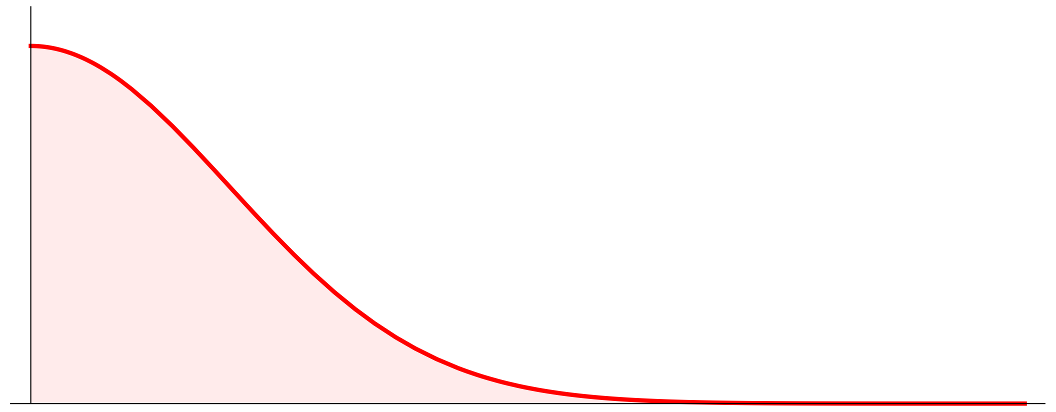


Pillebuoe et al. [2015]

Variance in terms of power spectra

Samples' expected power spectrum

Integrand power spectrum

$\mathrm{Var}(I_N) \propto$  $\times$ 

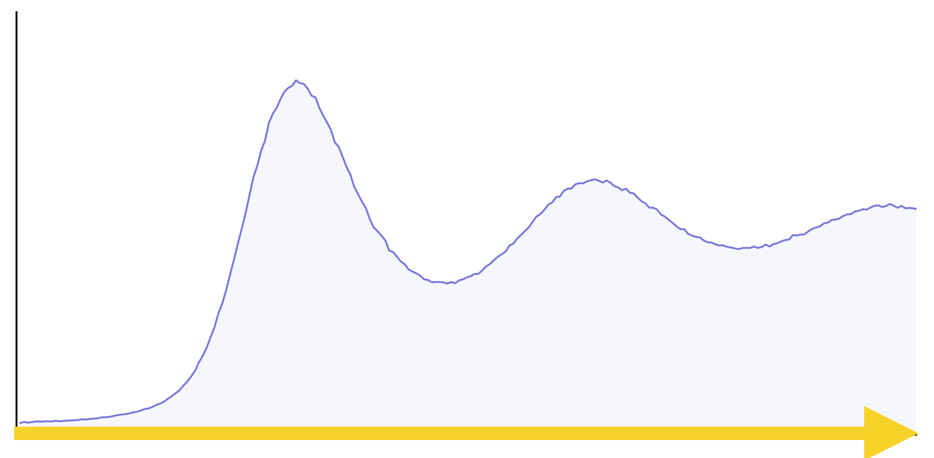
Pilleboue et al. [2015]

Convergence rate depends on the low frequency region

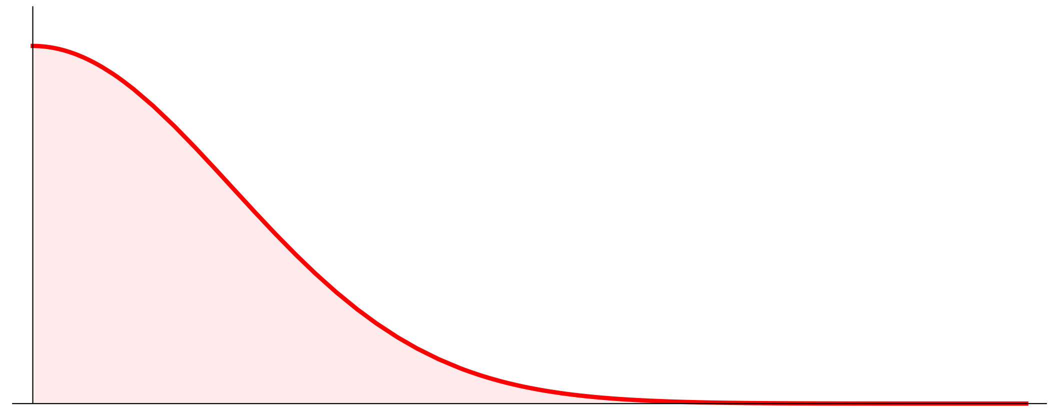
Samples' expected power spectrum

Integrand power spectrum

$\mathrm{Var}(I_N) \propto$



$\times$



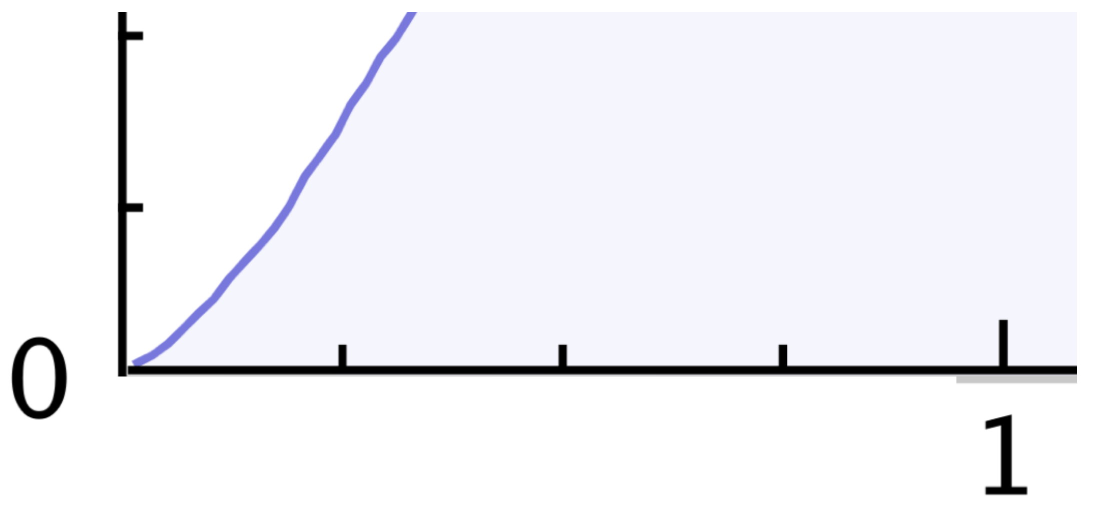
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

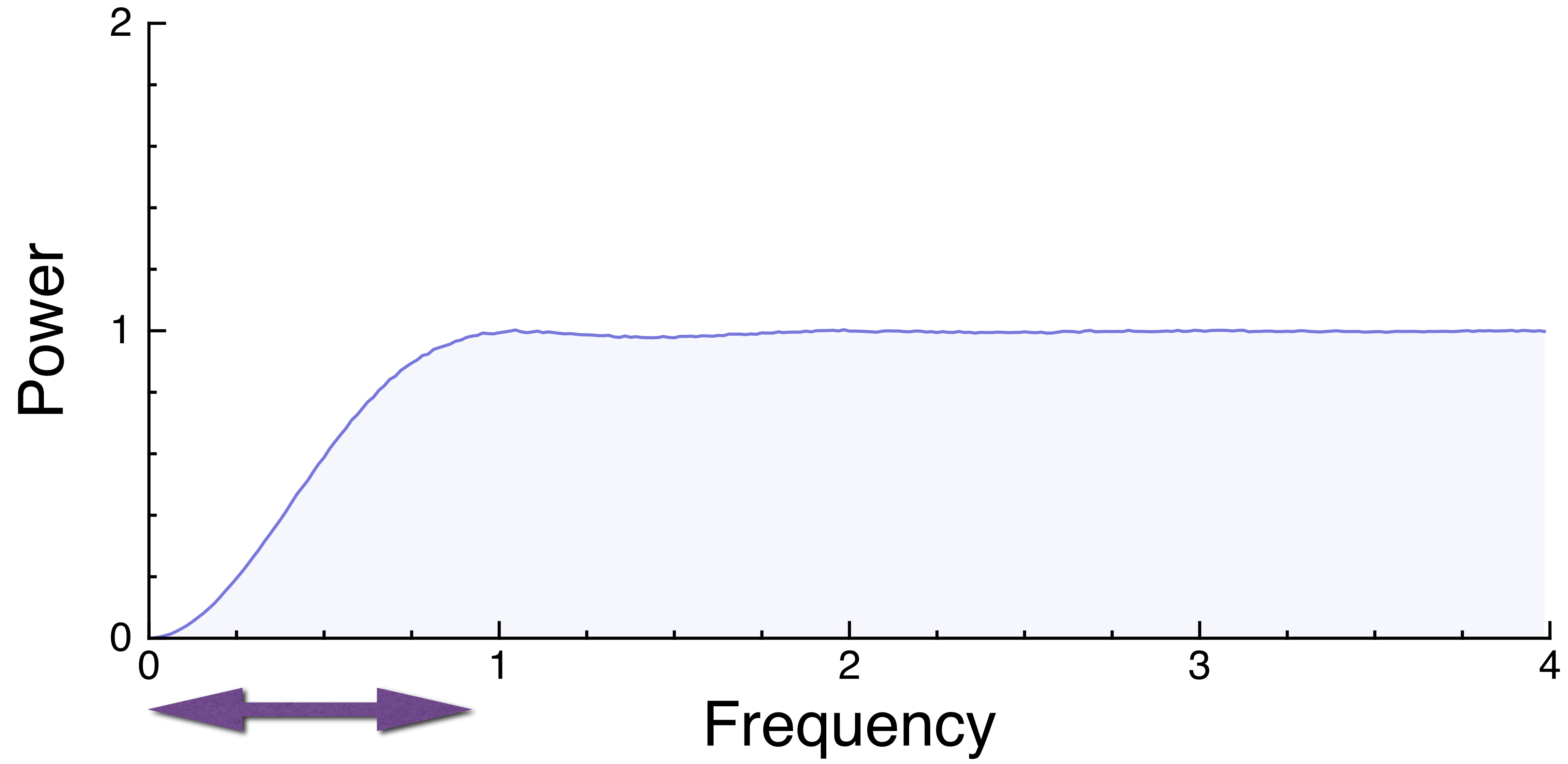
Pilleboue et al. [2015]

Jittered samples converges faster than Poisson Disk

Zoom-in

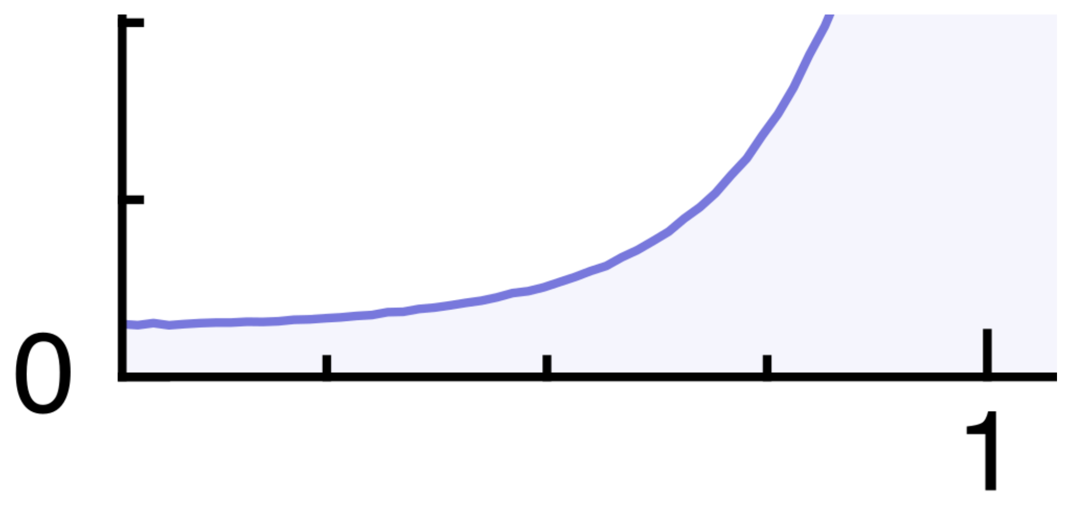
Jitter

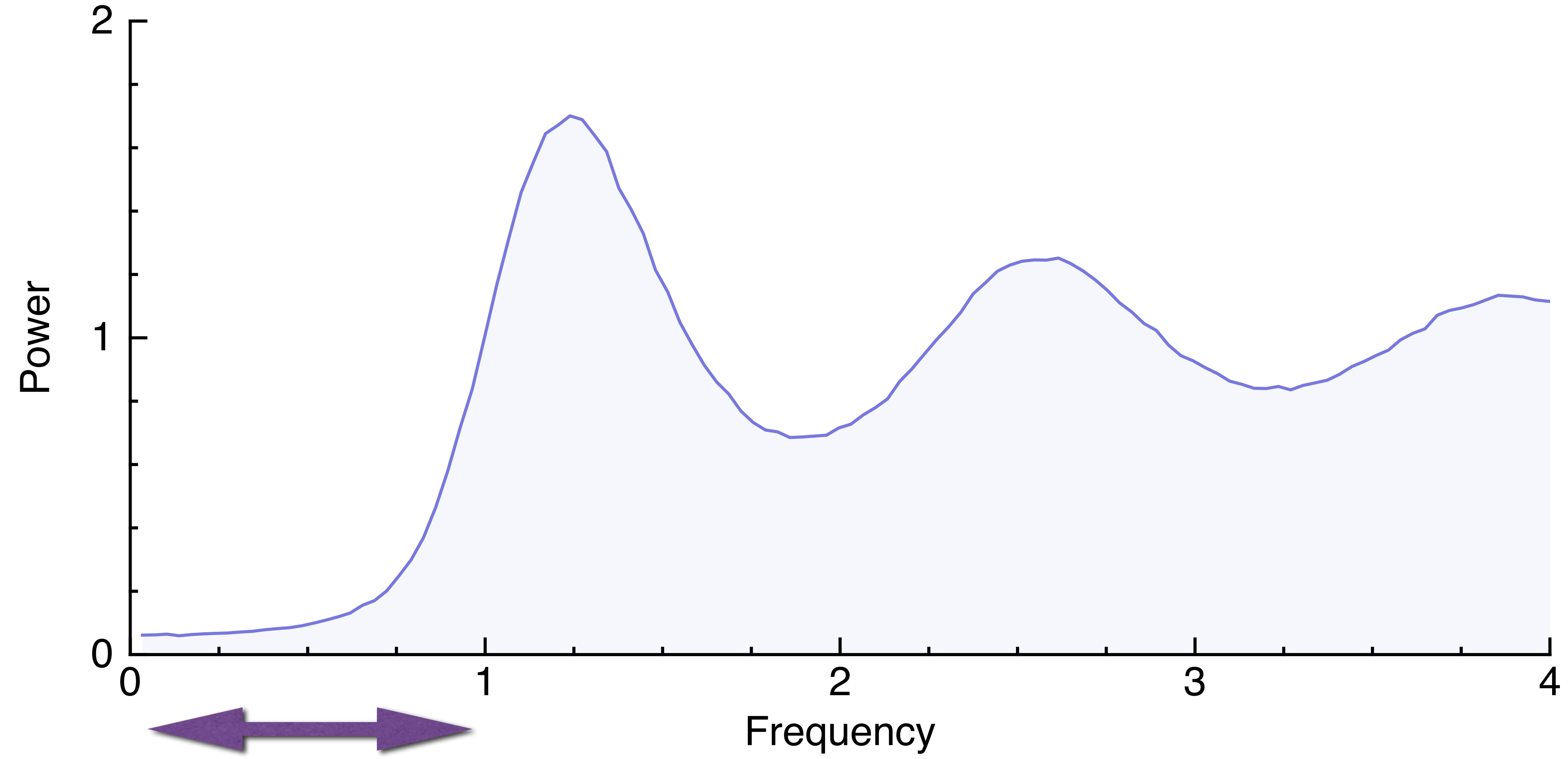




$$O(N^{-1.5})$$

Poisson Disk





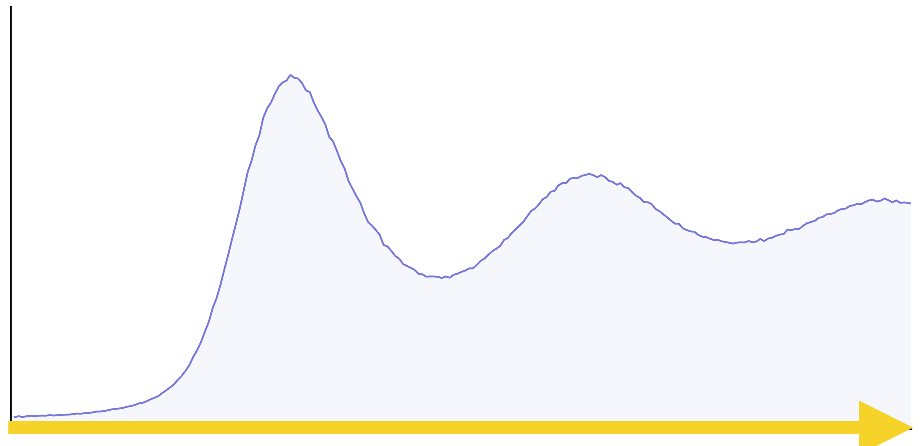
$$O(N^{-1})$$

Convergence rate depends on the low frequency region

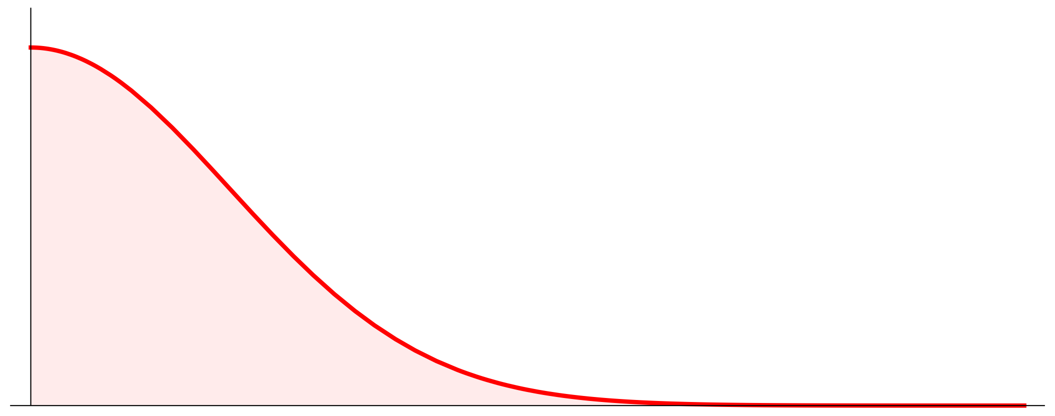
Samples' expected power spectrum

Integrand power spectrum

$$\mathrm{Var}(I_N) \propto$$

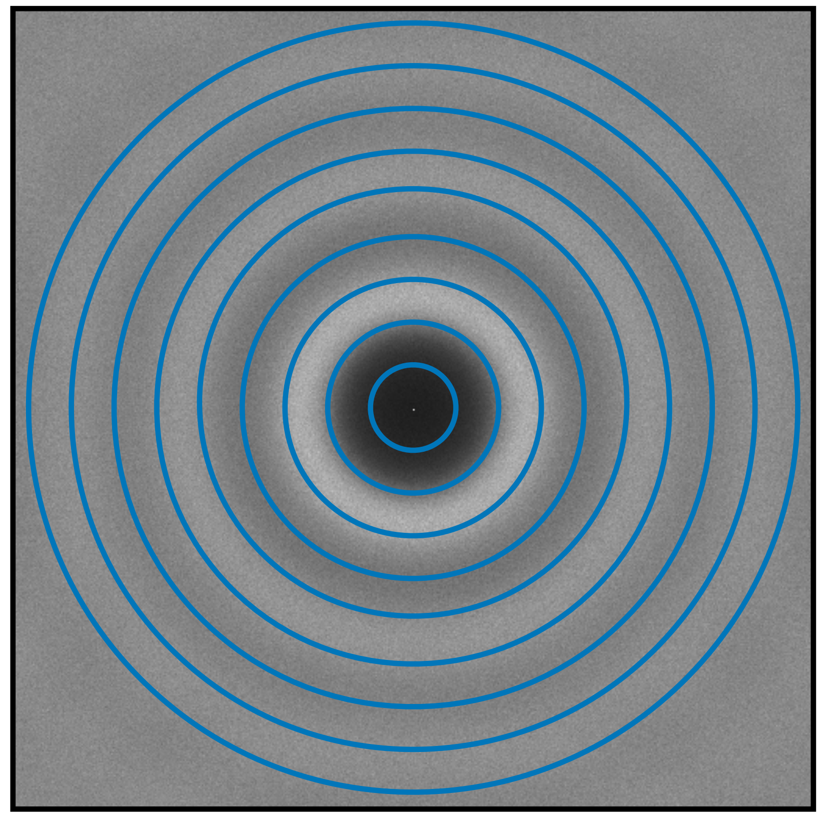


$\times$



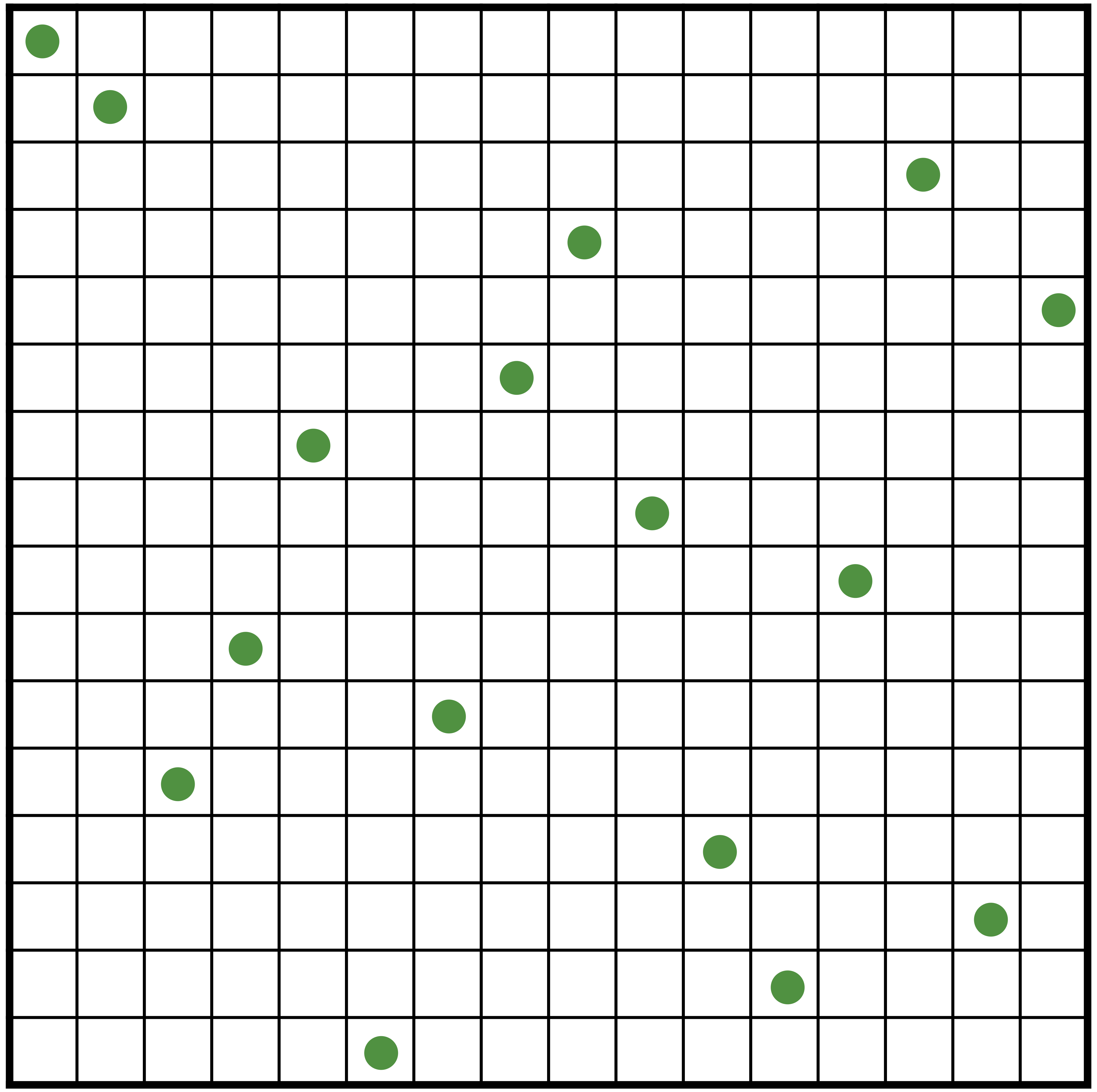
Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Isotropic Spectrum
Poisson Disk



Pilleboue et al. [2015]

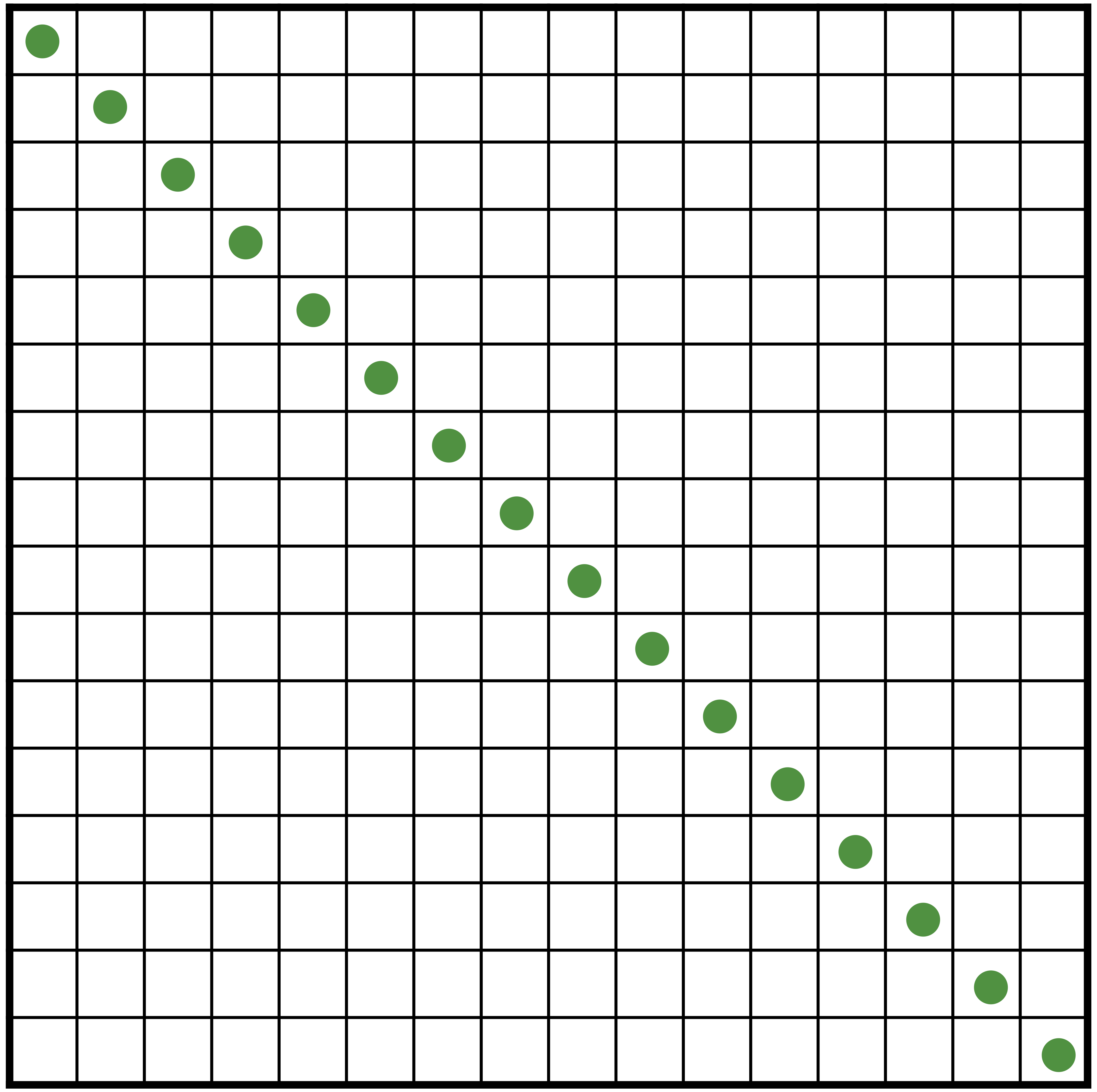
Latin Hypercube Sampler (N-rooks)



Slide after Wojciech Jarosz

Latin Hypercube Sampler (N-rooks)

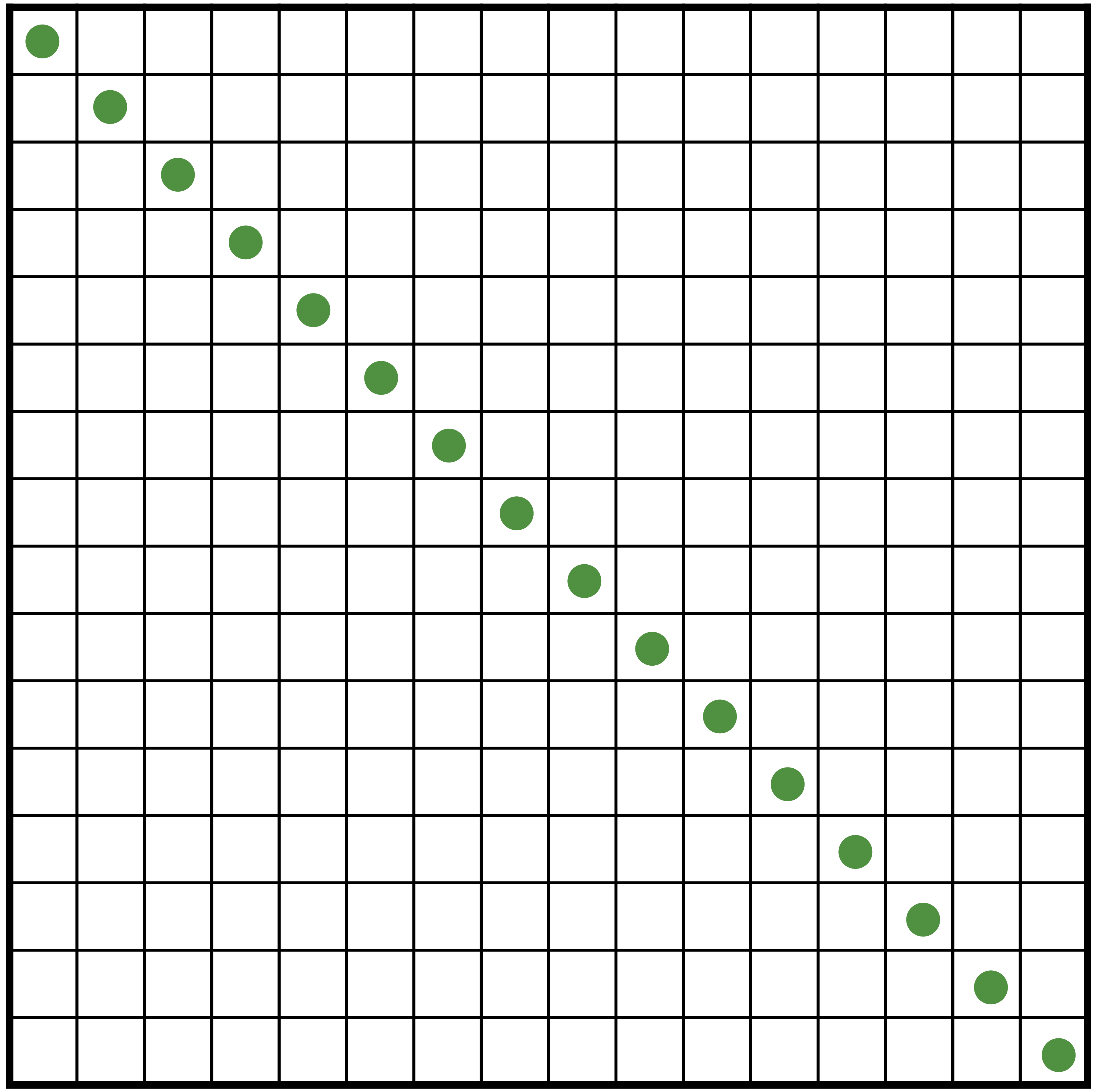
Initialize



Slide after Wojciech Jarosz

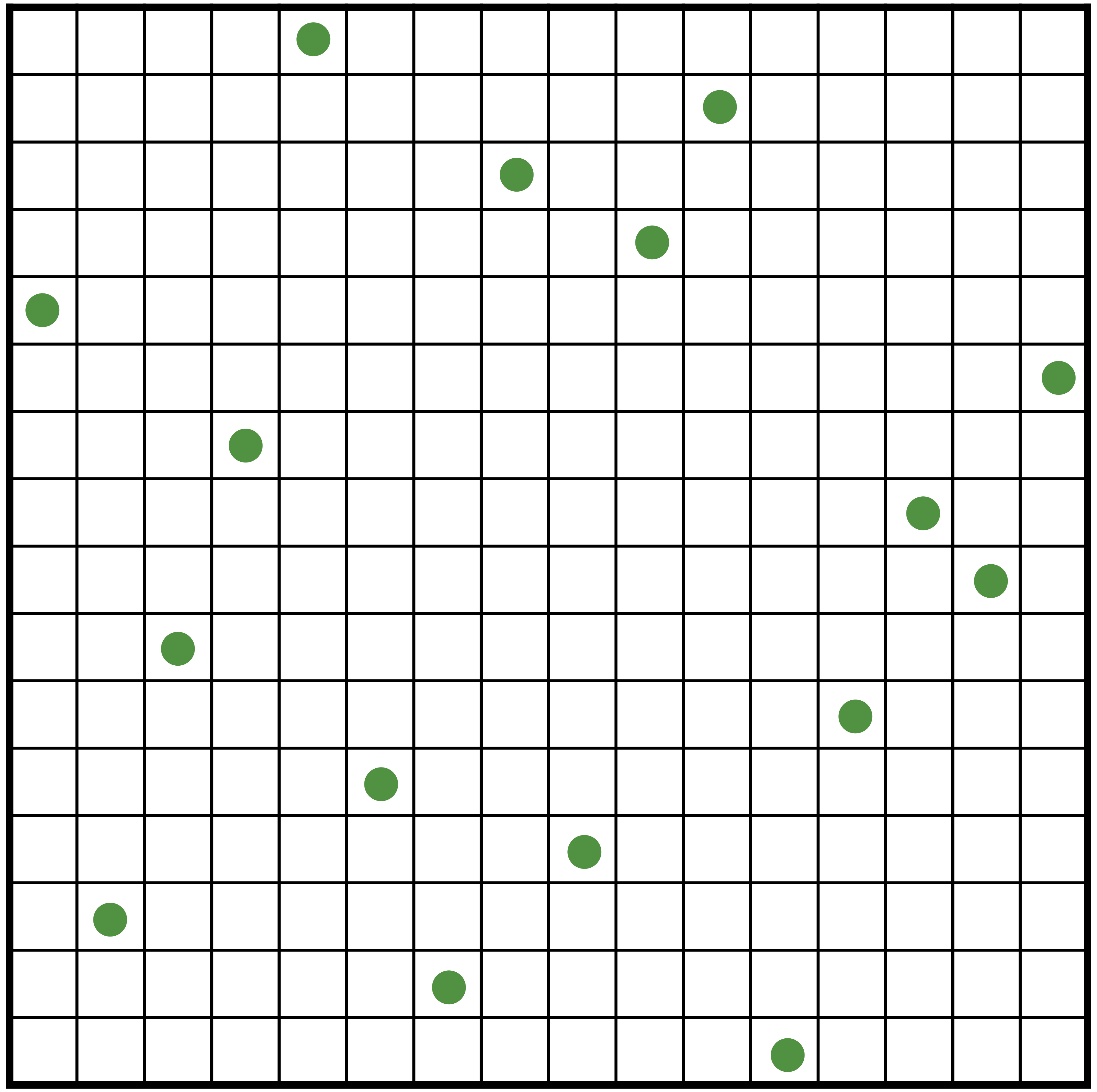
Latin Hypercube Sampler (N-rooks)

Shuffle rows



Slide after Wojciech Jarosz

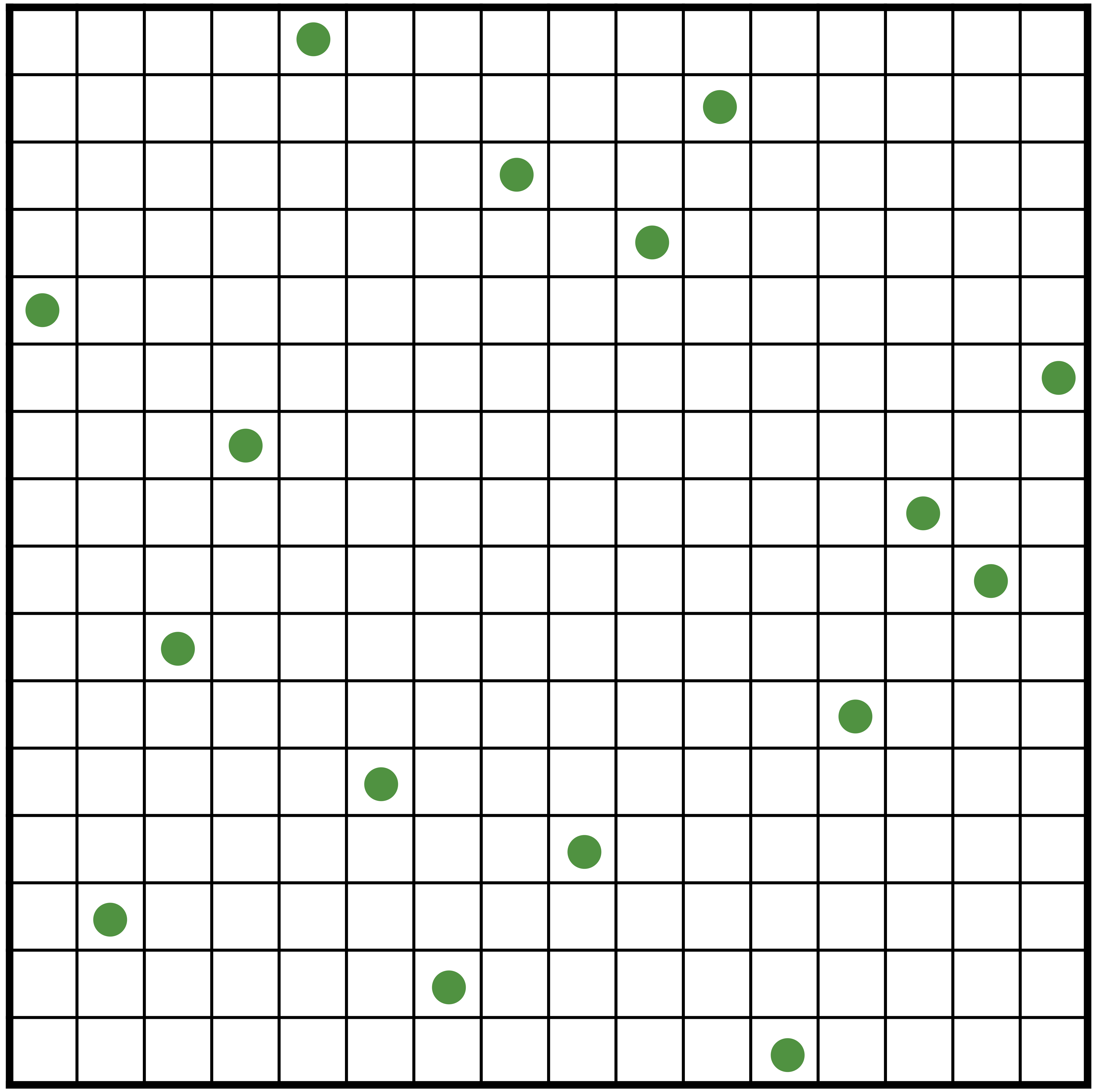
Latin Hypercube Sampler (N-rooks)



Slide after Wojciech Jarosz

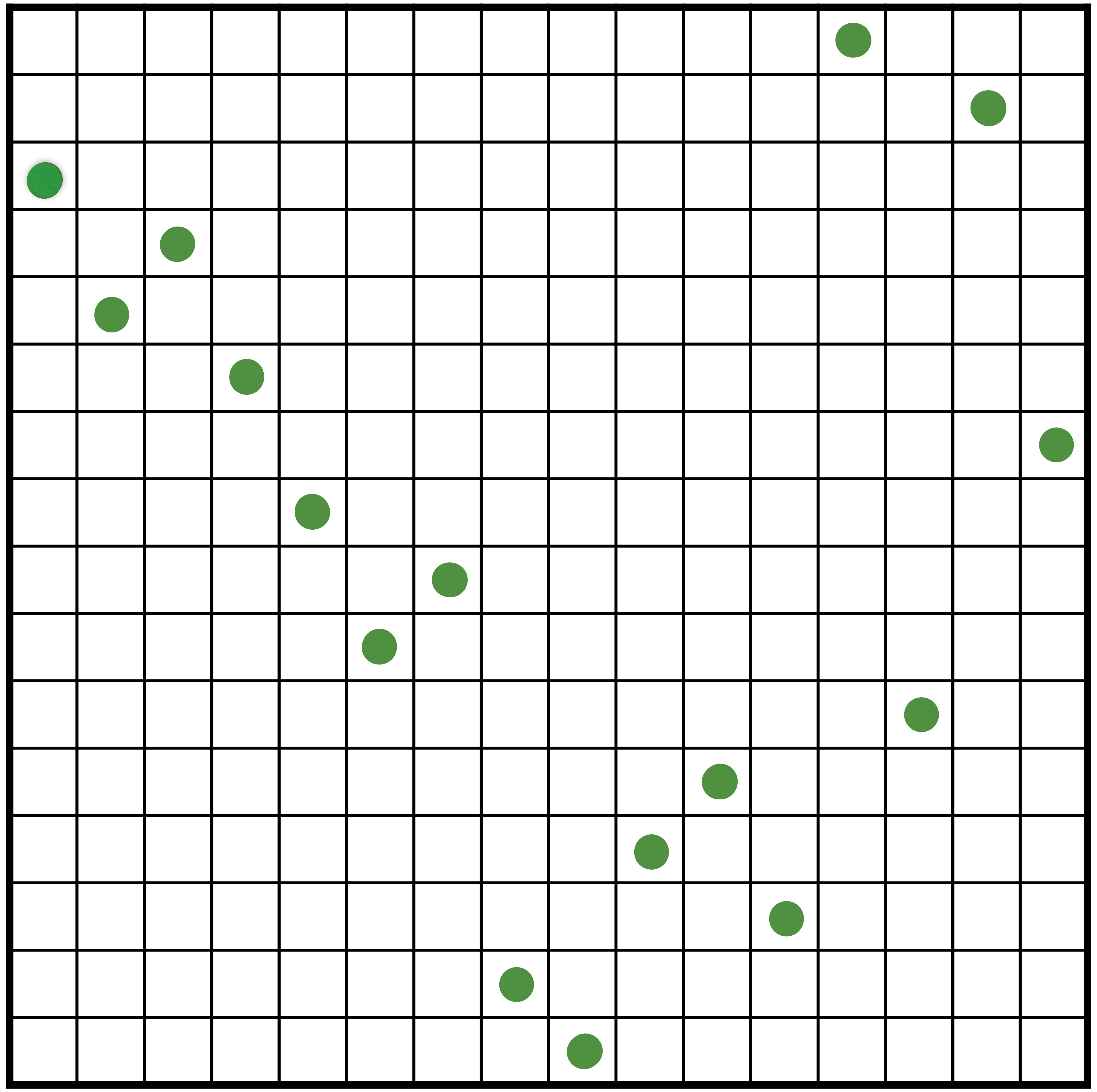
Latin Hypercube Sampler (N-rooks)

Shuffle columns



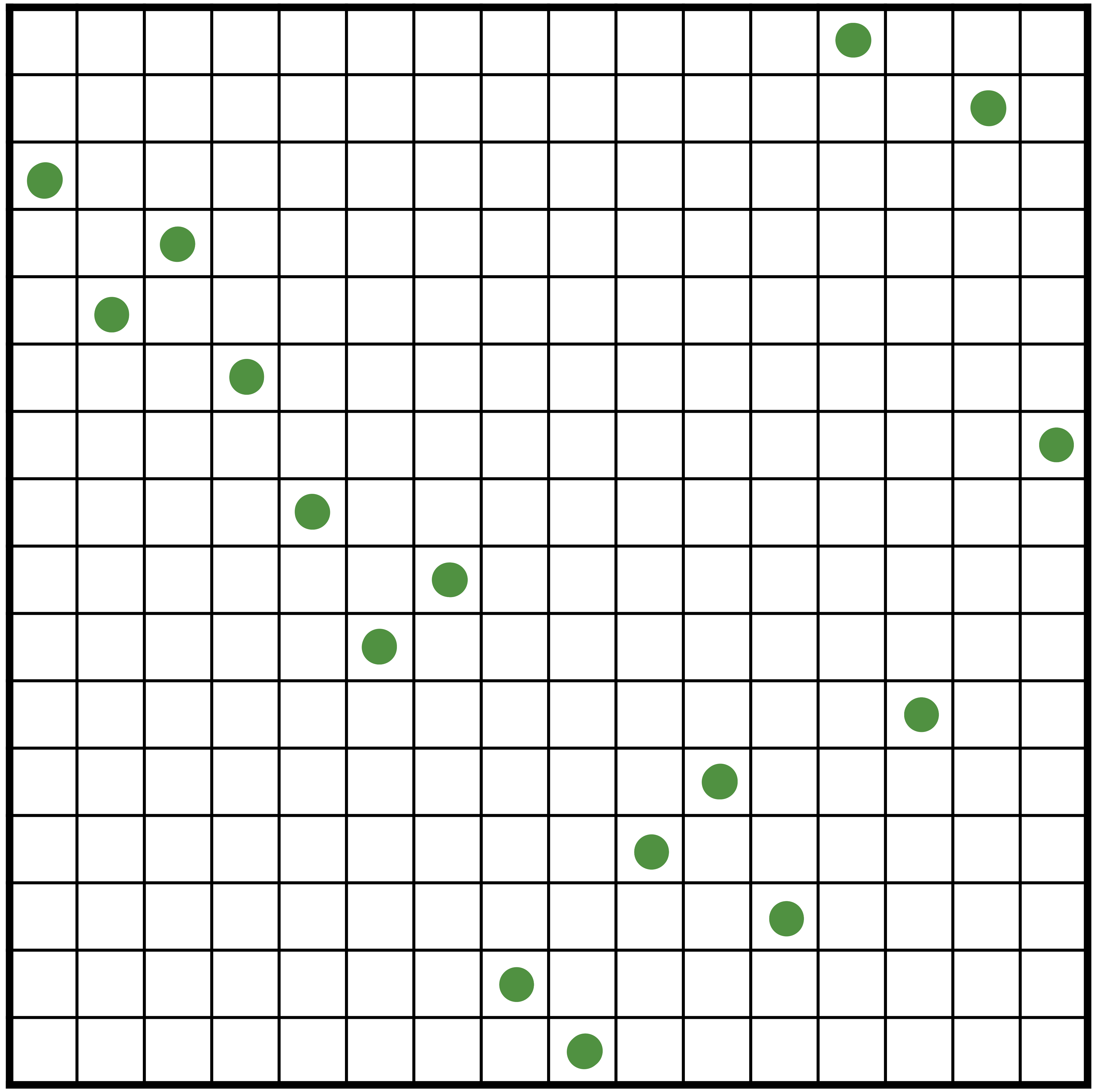
Slide after Wojciech Jarosz

Latin Hypercube Sampler (N-rooks)



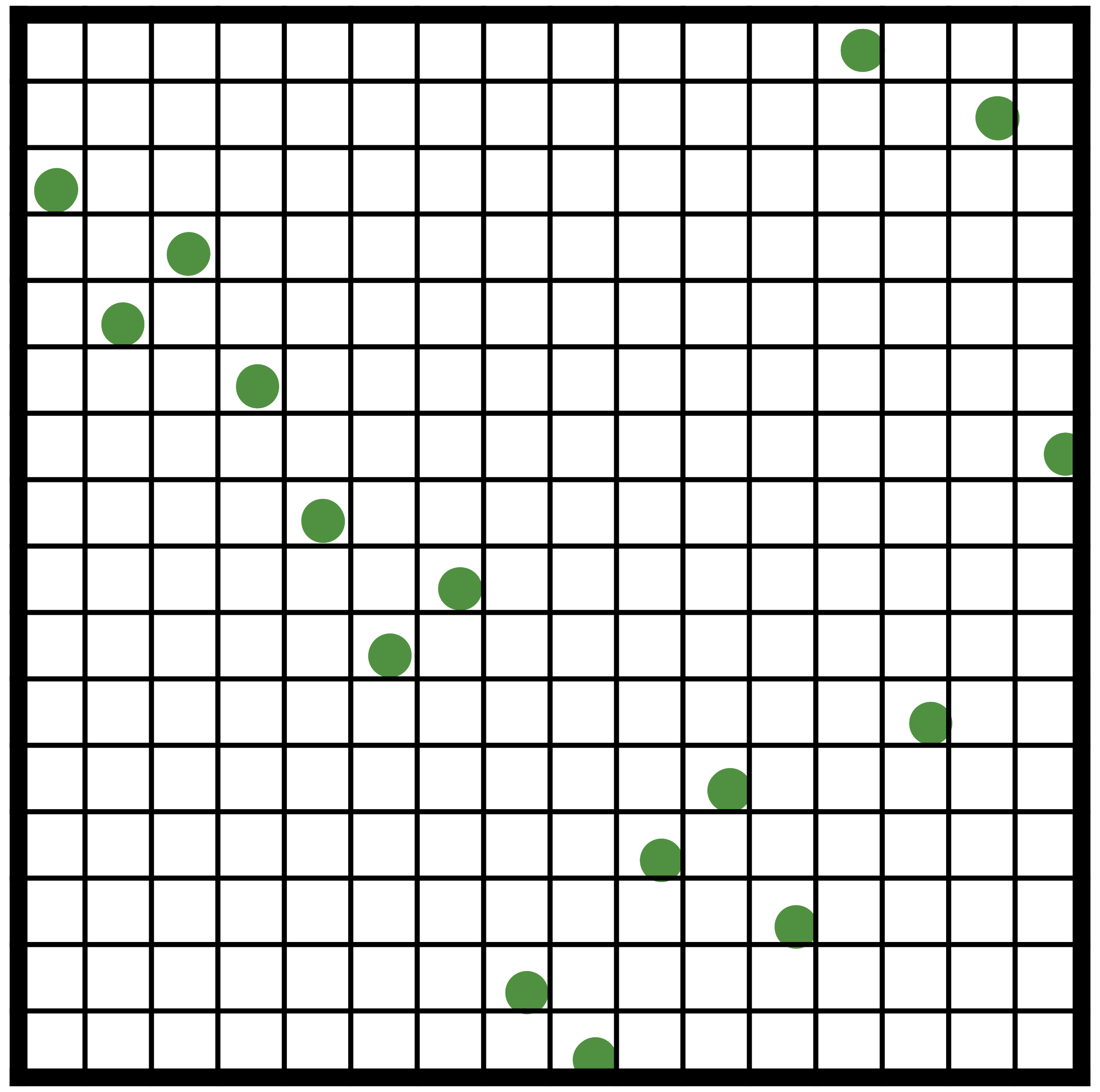
Slide after Wojciech Jarosz

Latin Hypercube Sampler (N-rooks)

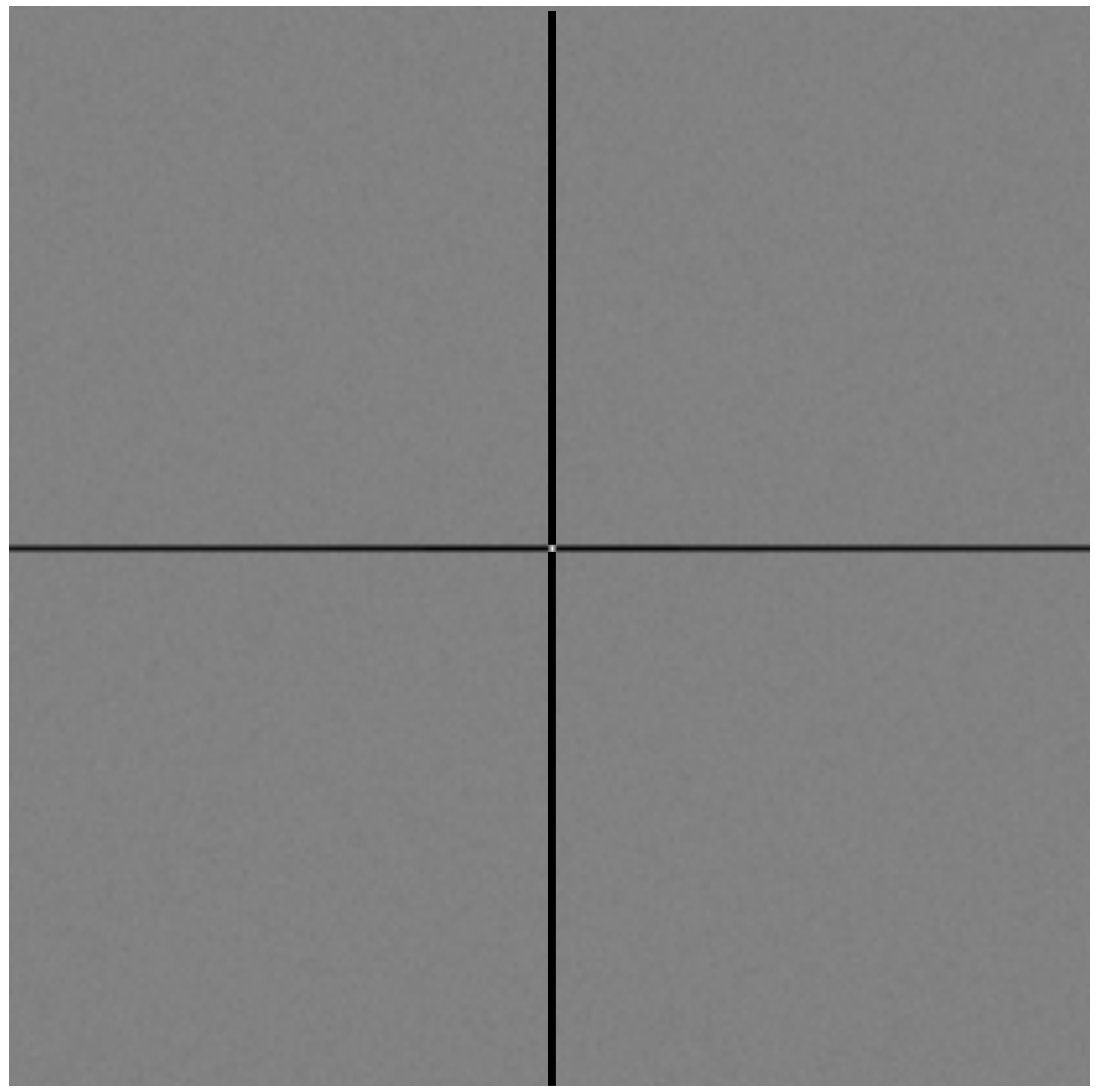


Slide after Wojciech Jarosz

Anisotropic Sampling Power Spectra

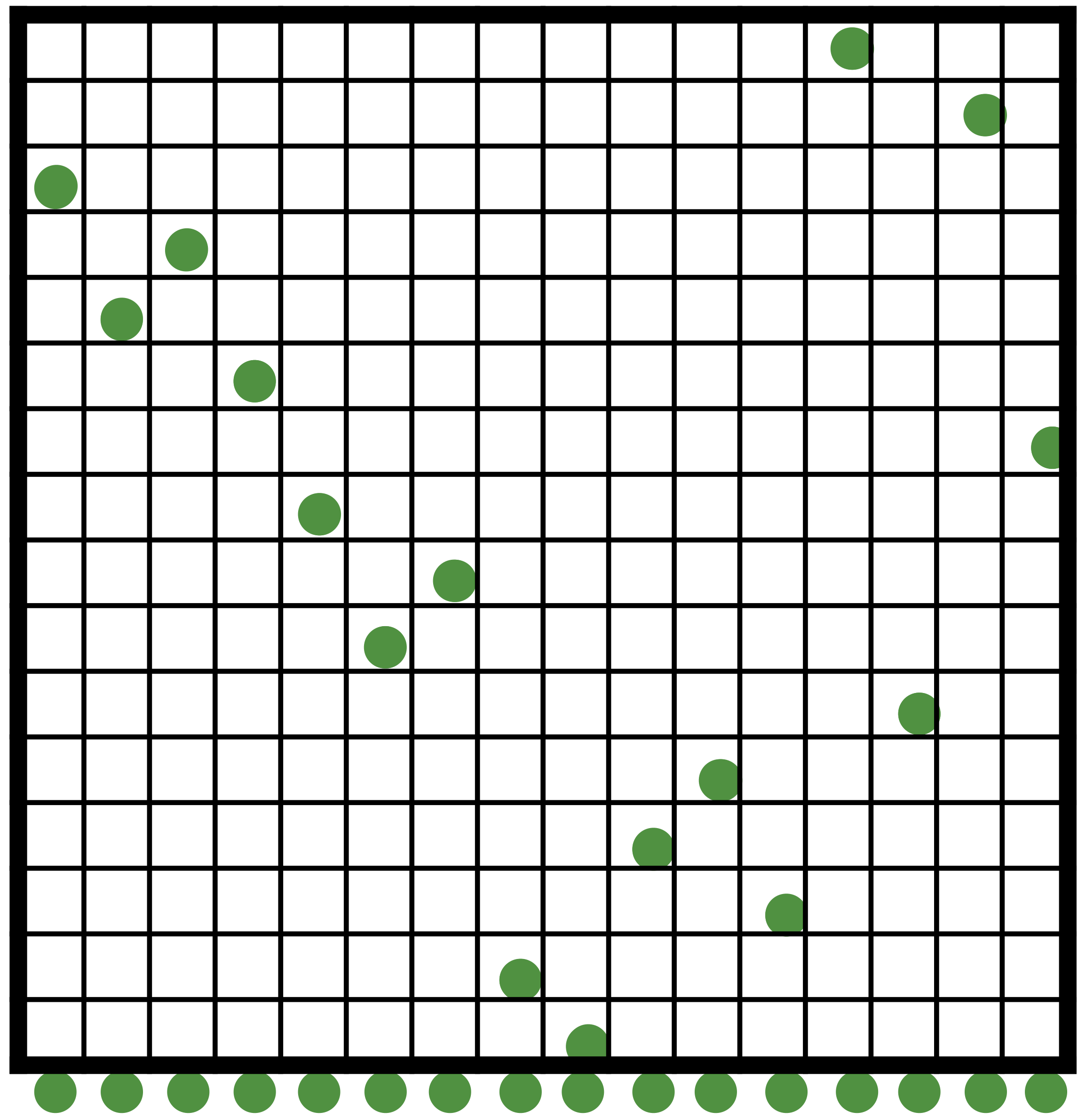


N-rooks /
Latin Hypercube

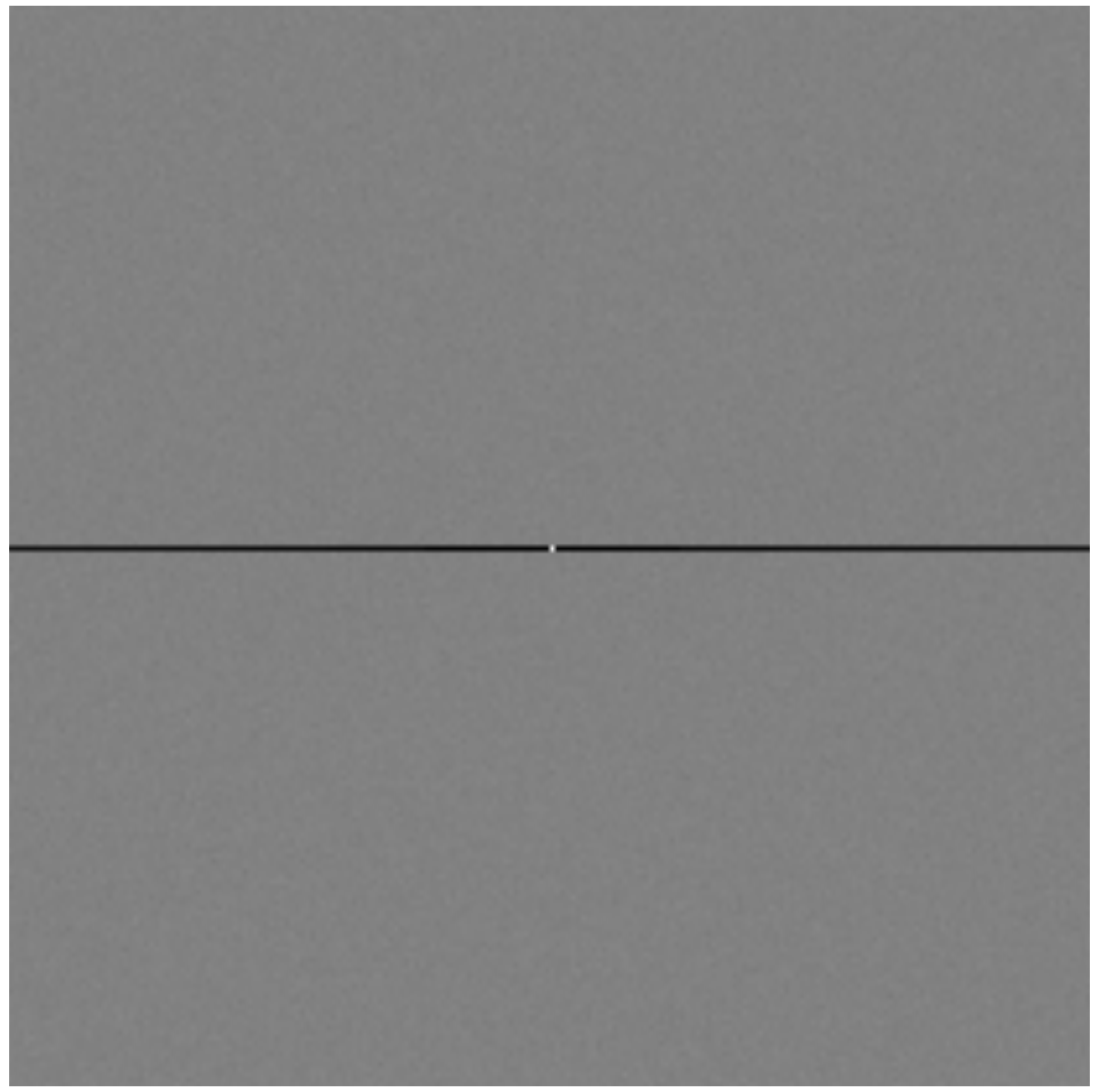


N-rooks
Spectrum

Anisotropic Sampling Power Spectra

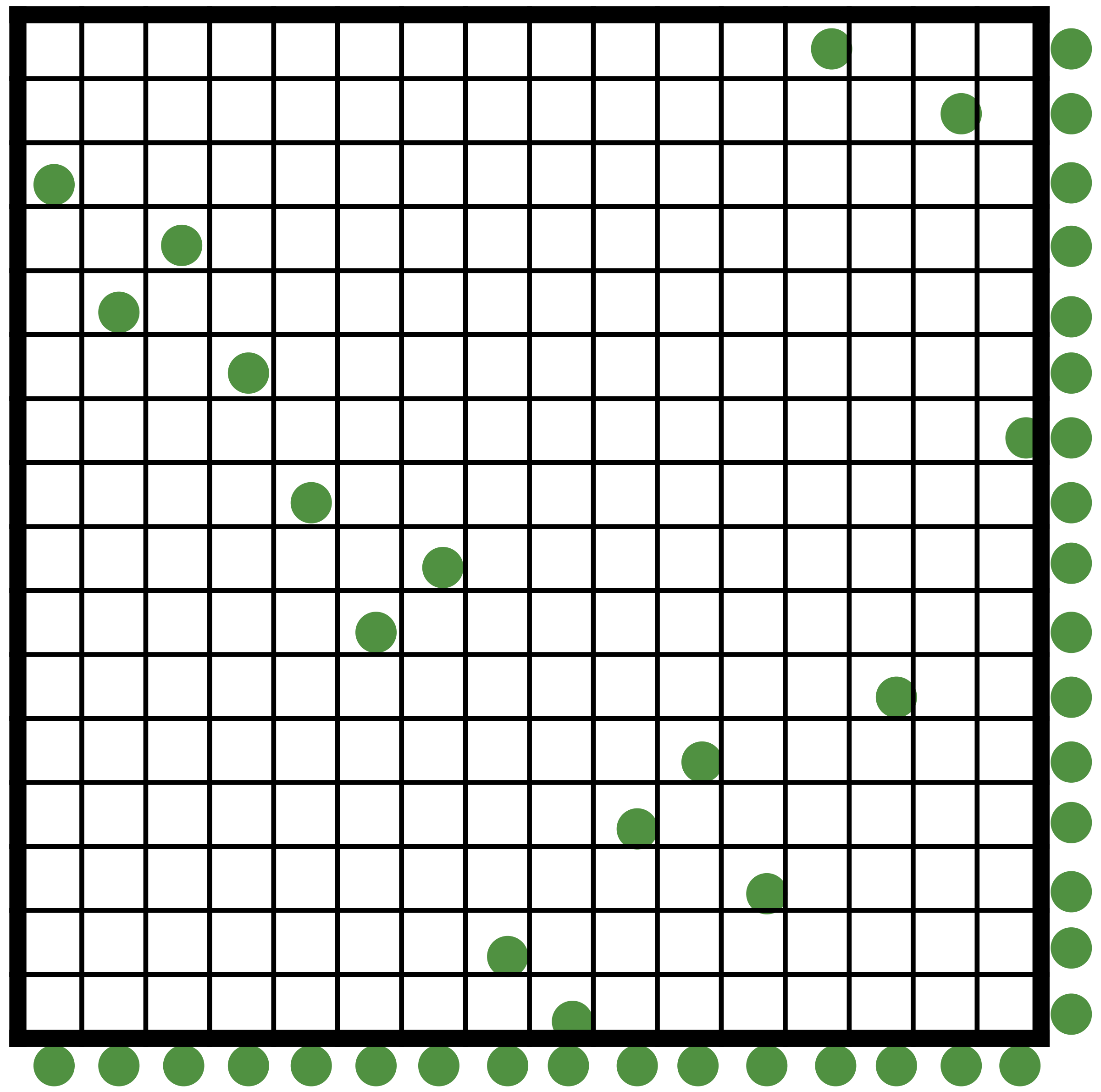


N-rooks /
Latin Hypercube

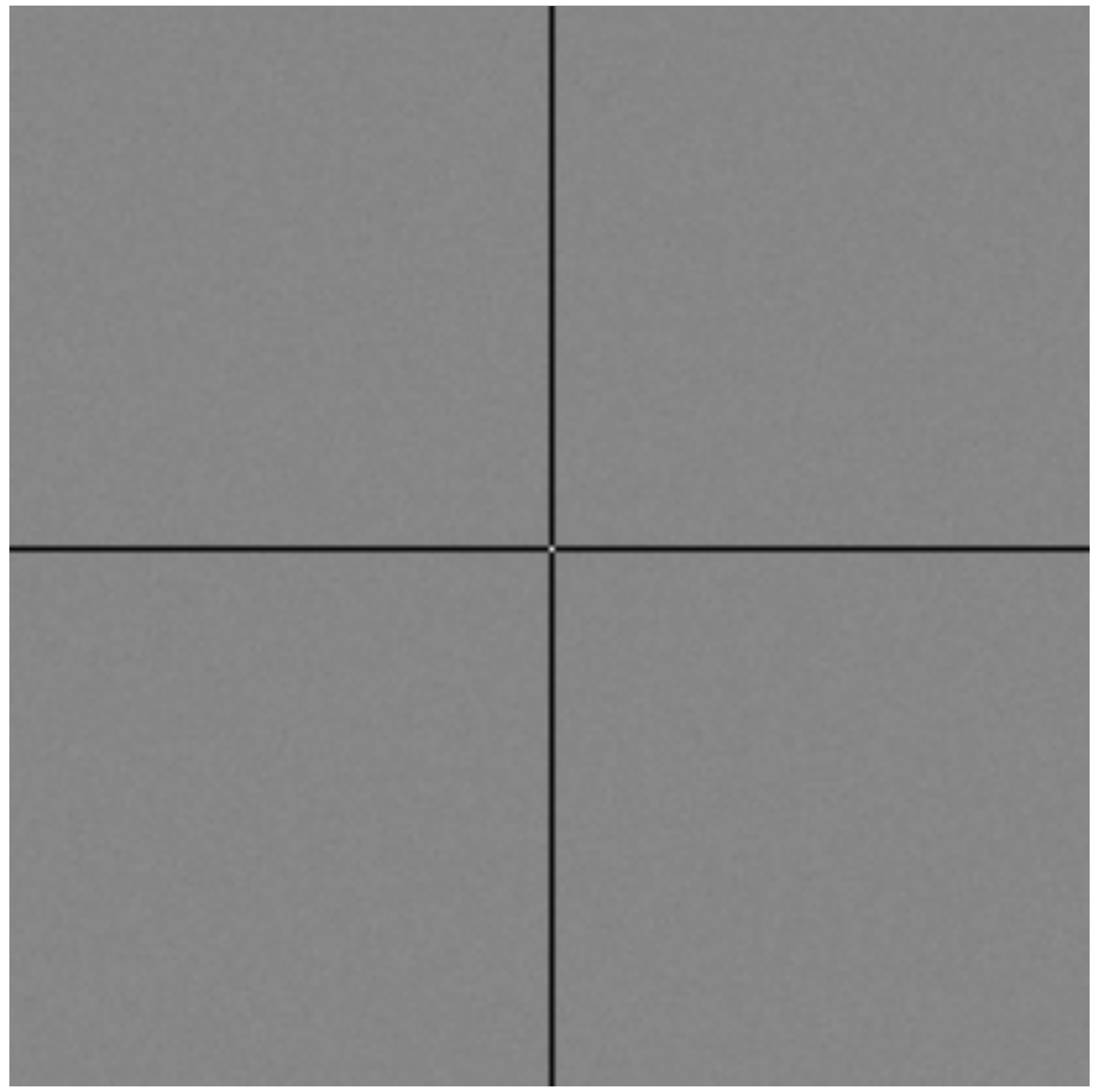


Spectrum

Anisotropic Sampling Power Spectra

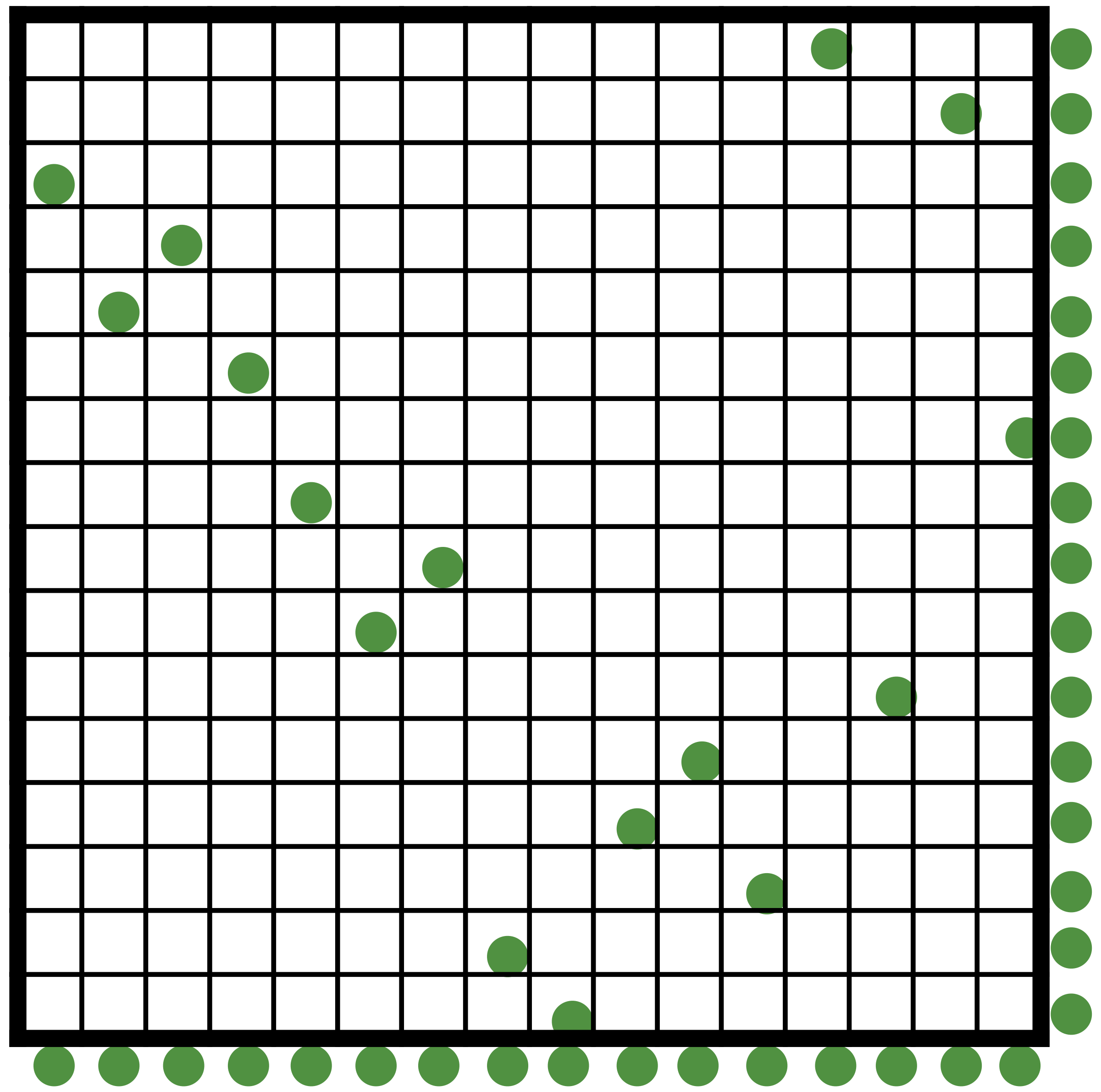


N-rooks /
Latin Hypercube

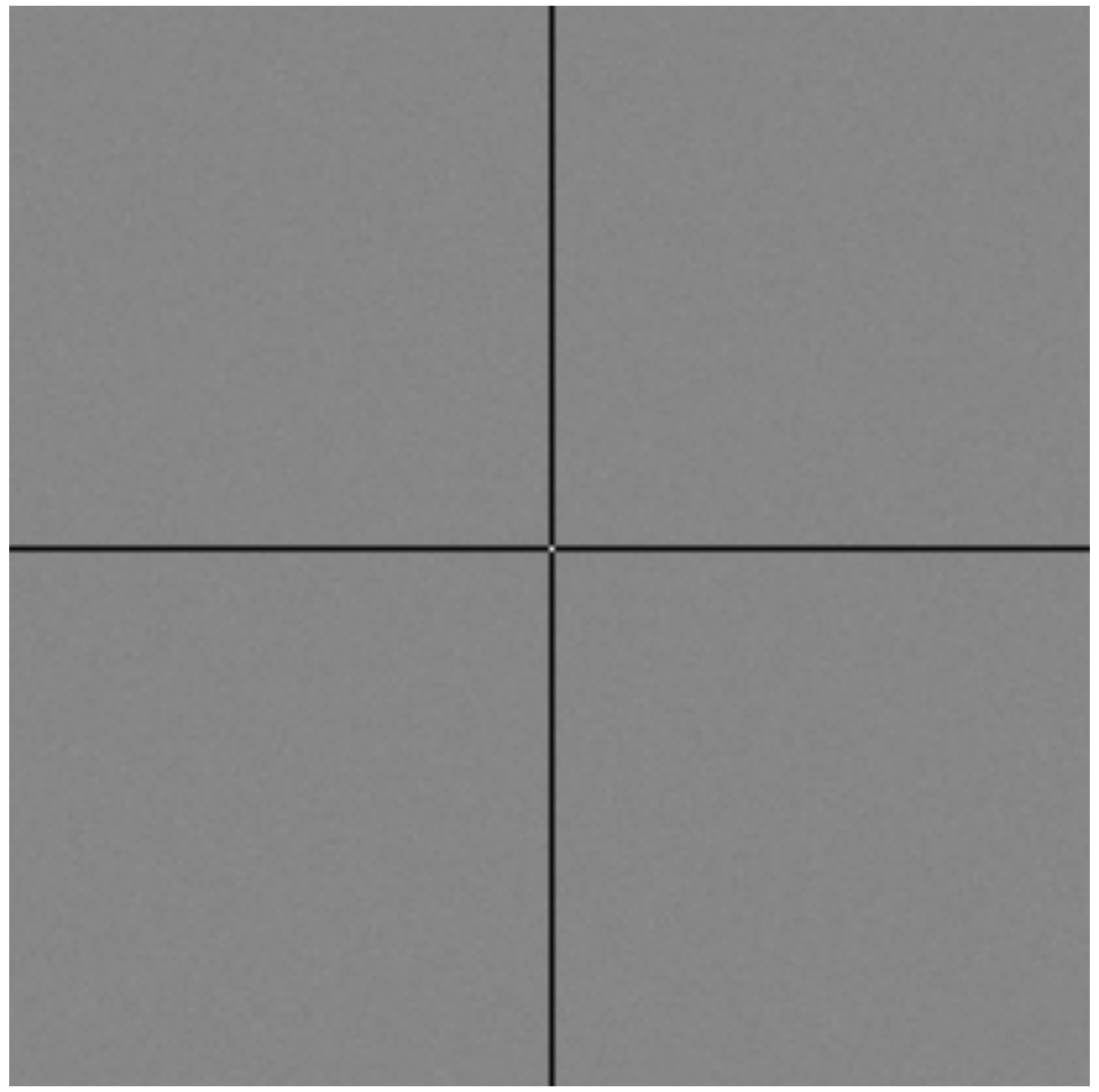


N-rooks
Spectrum

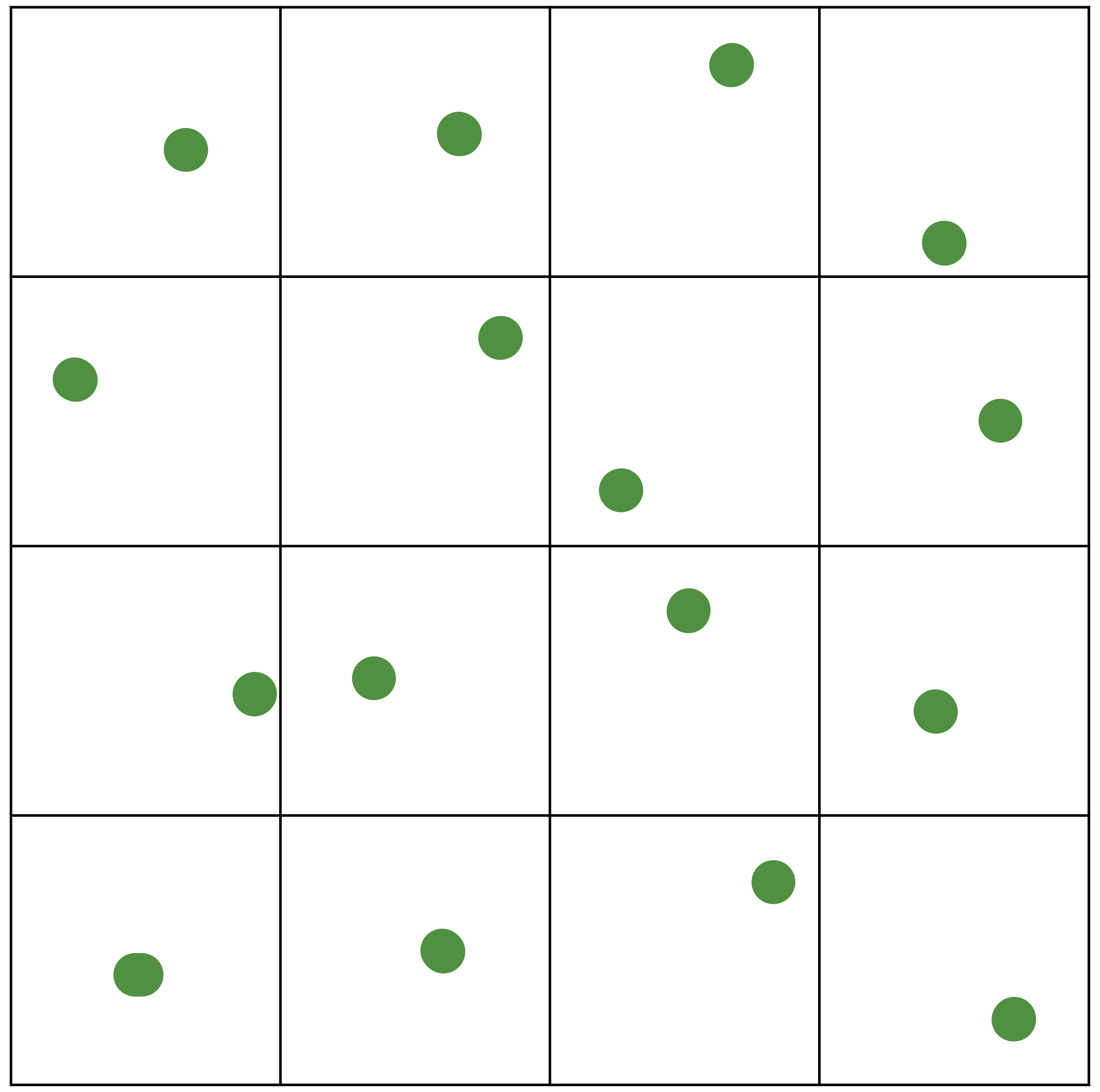
Anisotropic Sampling Power Spectra



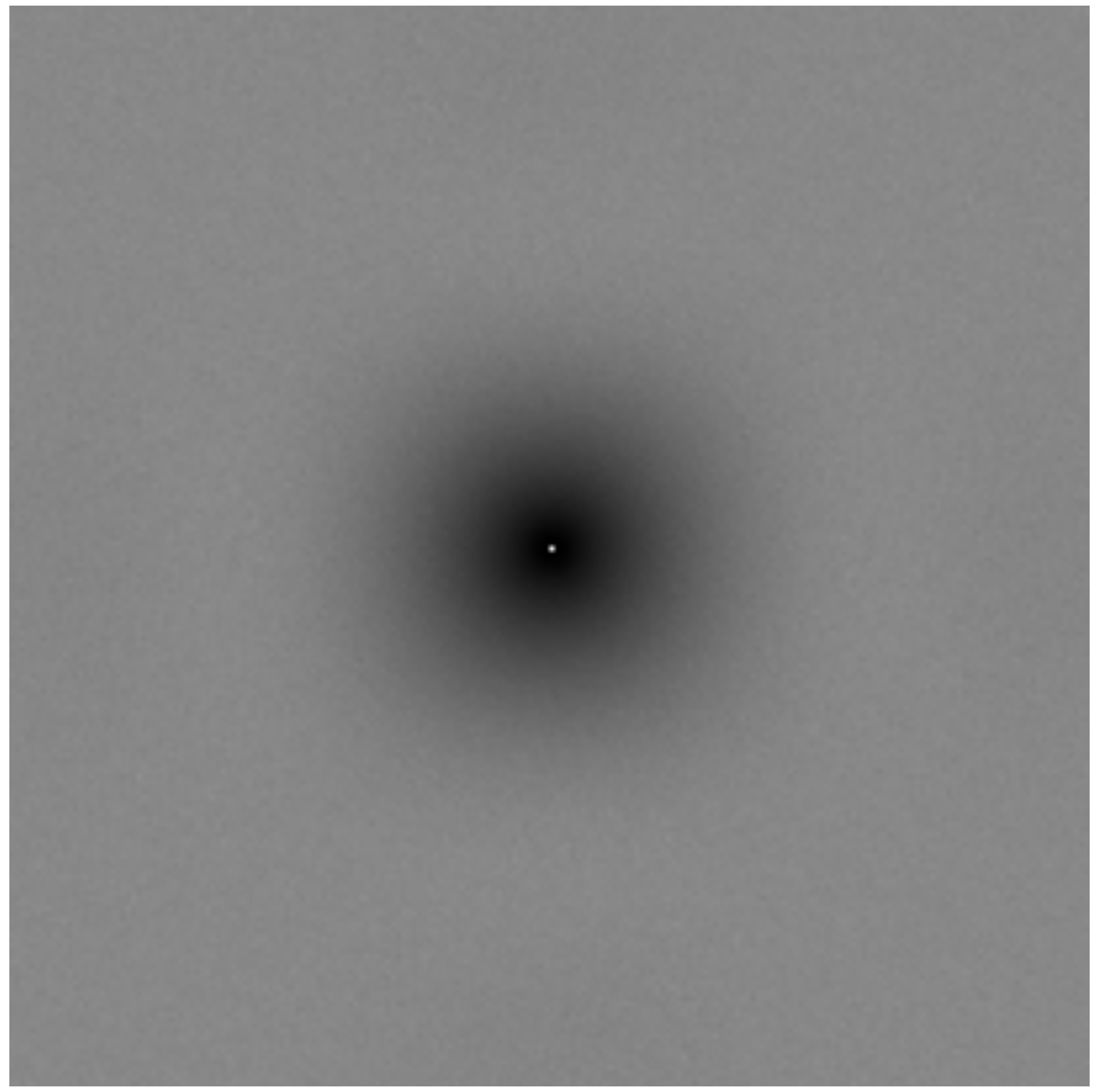
N-rooks / Latin Hypercube



N-rooks Spectrum

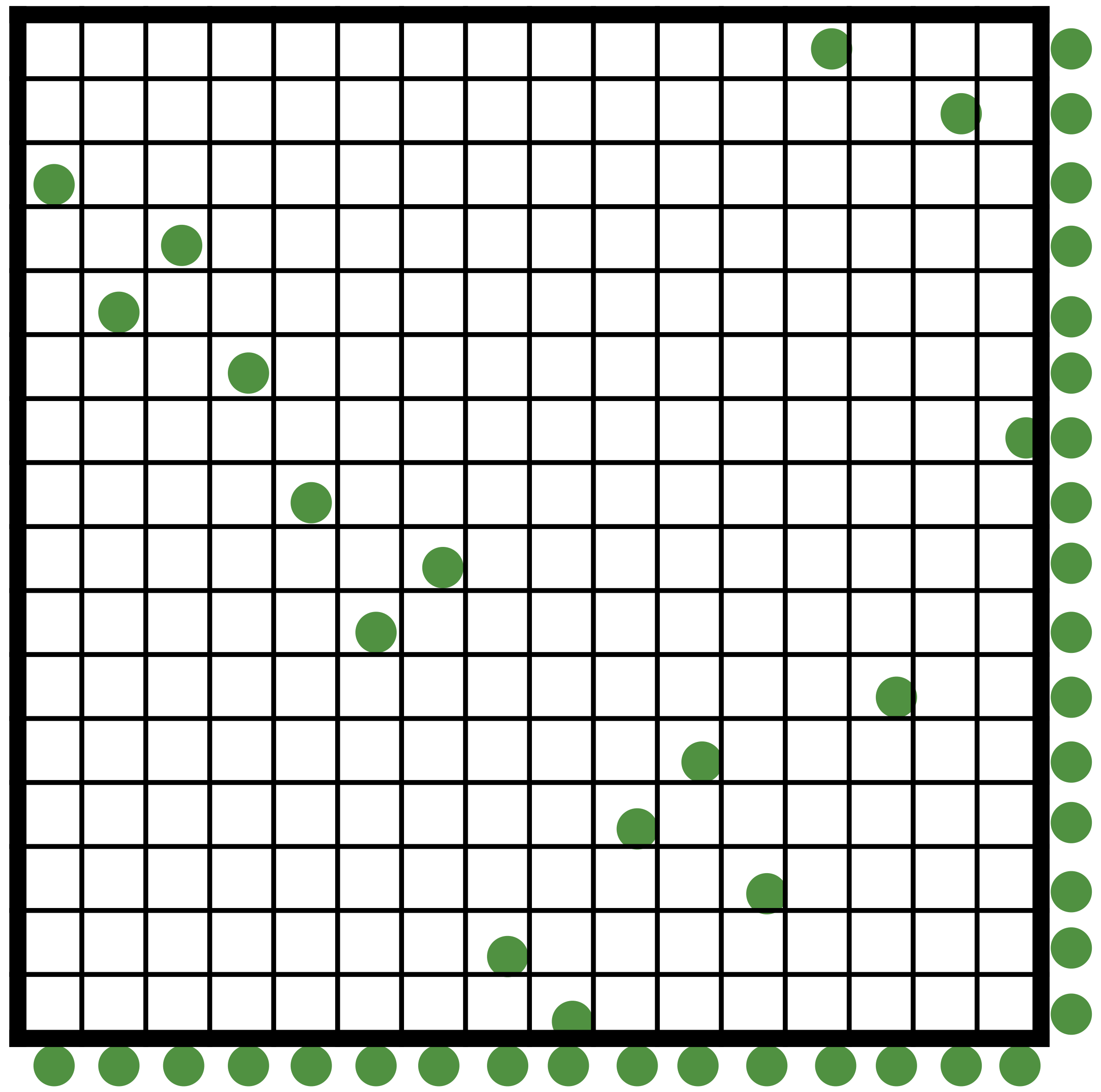


Jitter

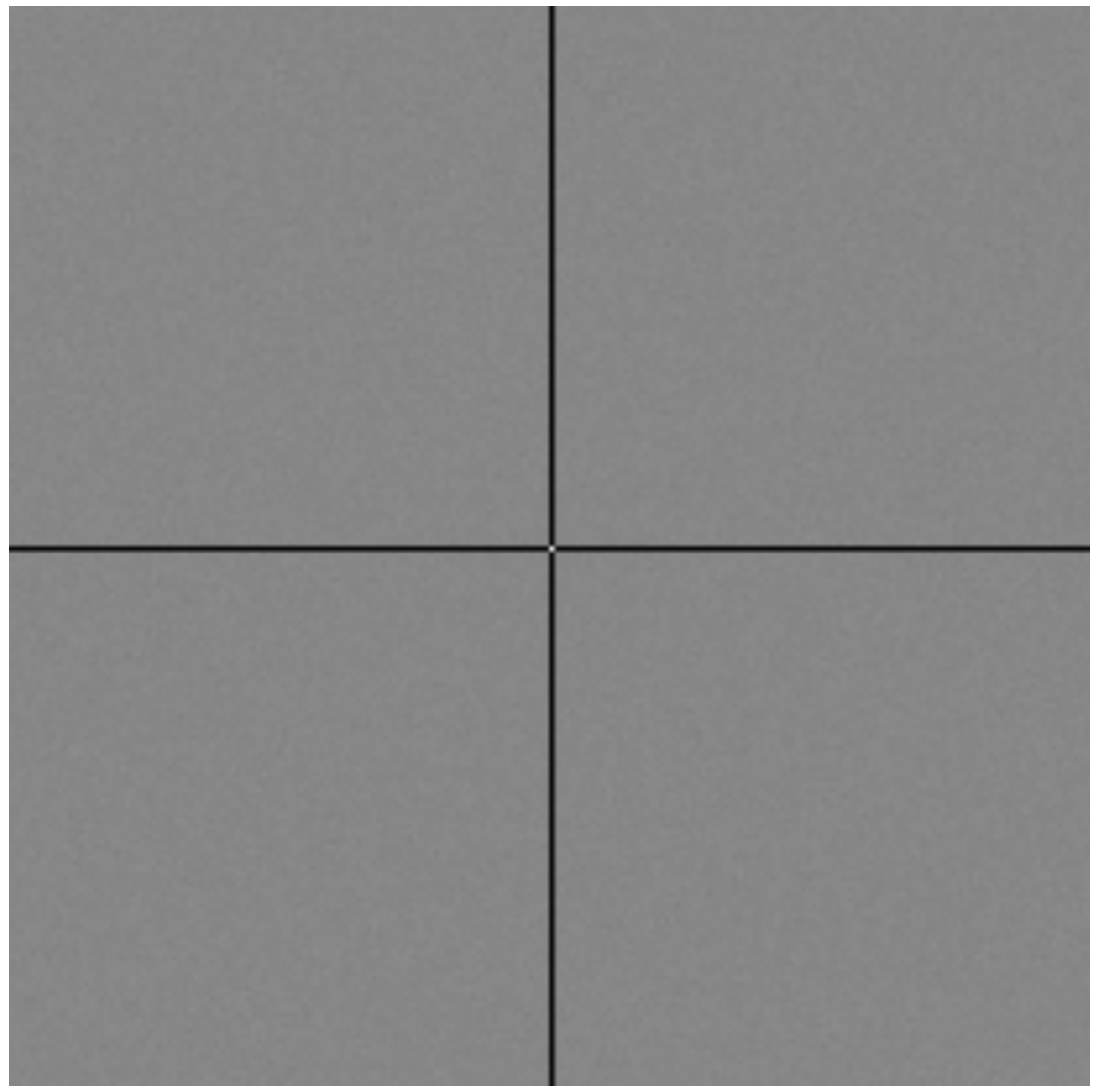


Jitter Spectrum

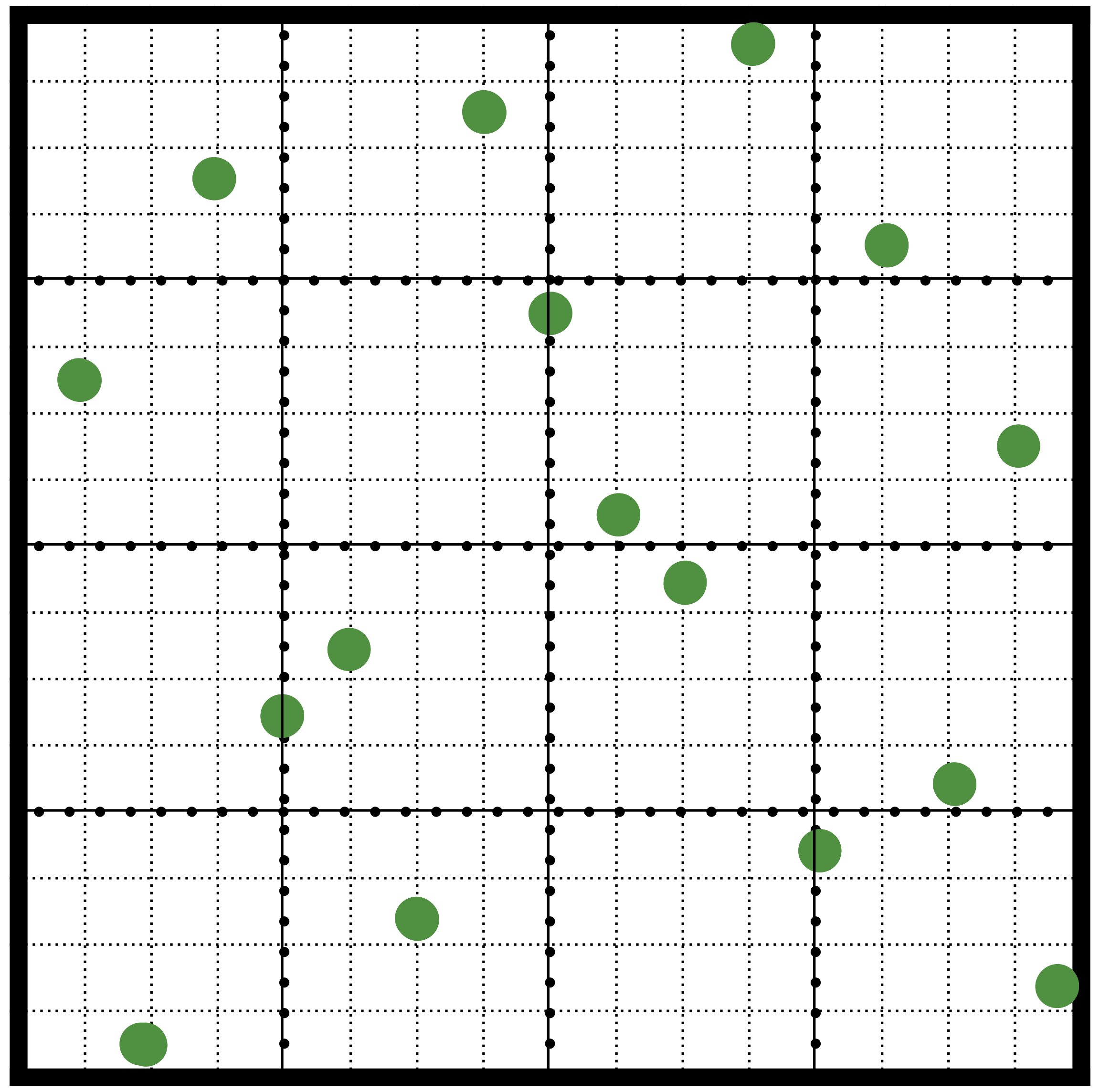
Anisotropic Sampling Power Spectra



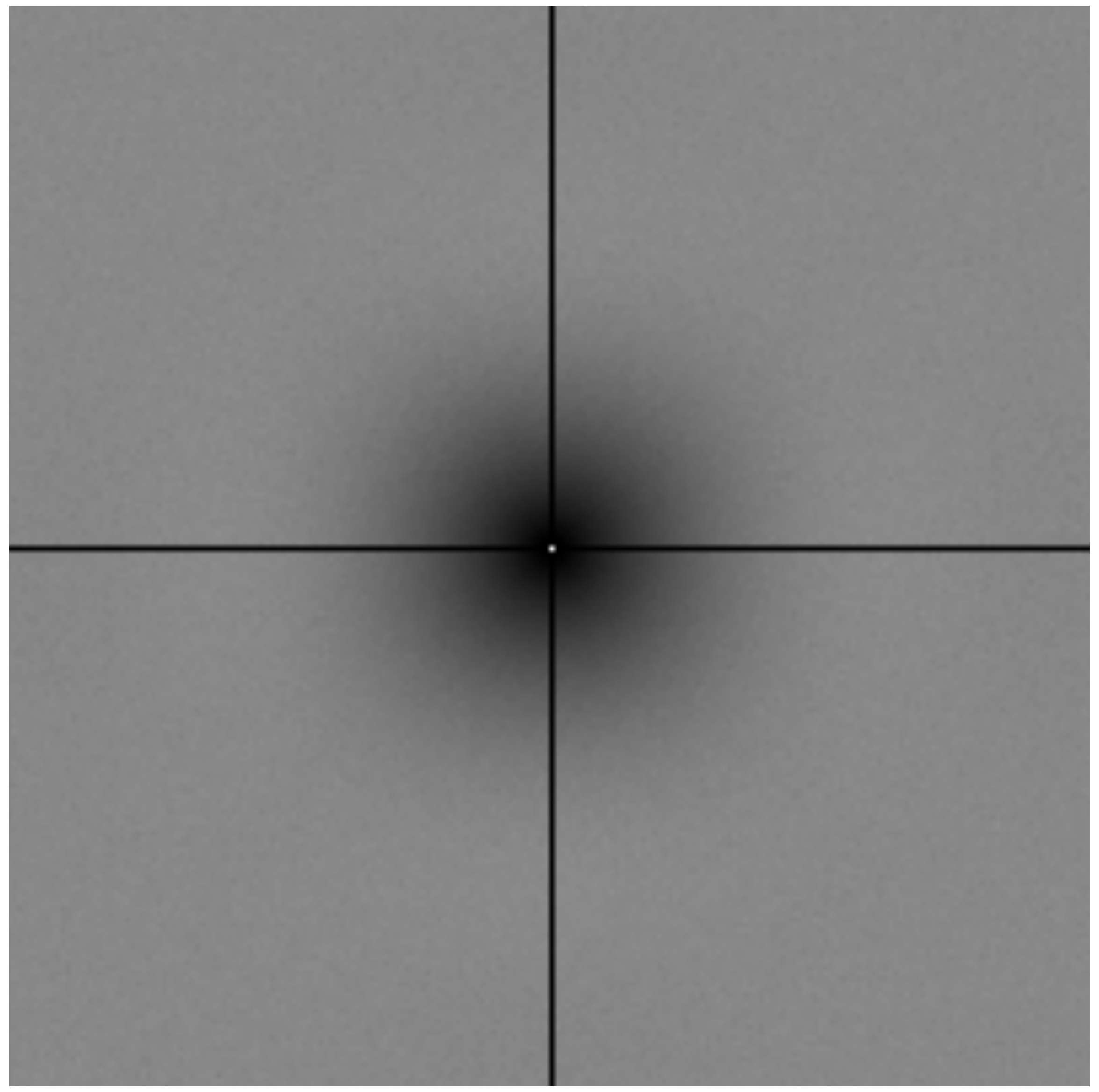
N-rooks /
Latin Hypercube



N-rooks
Spectrum



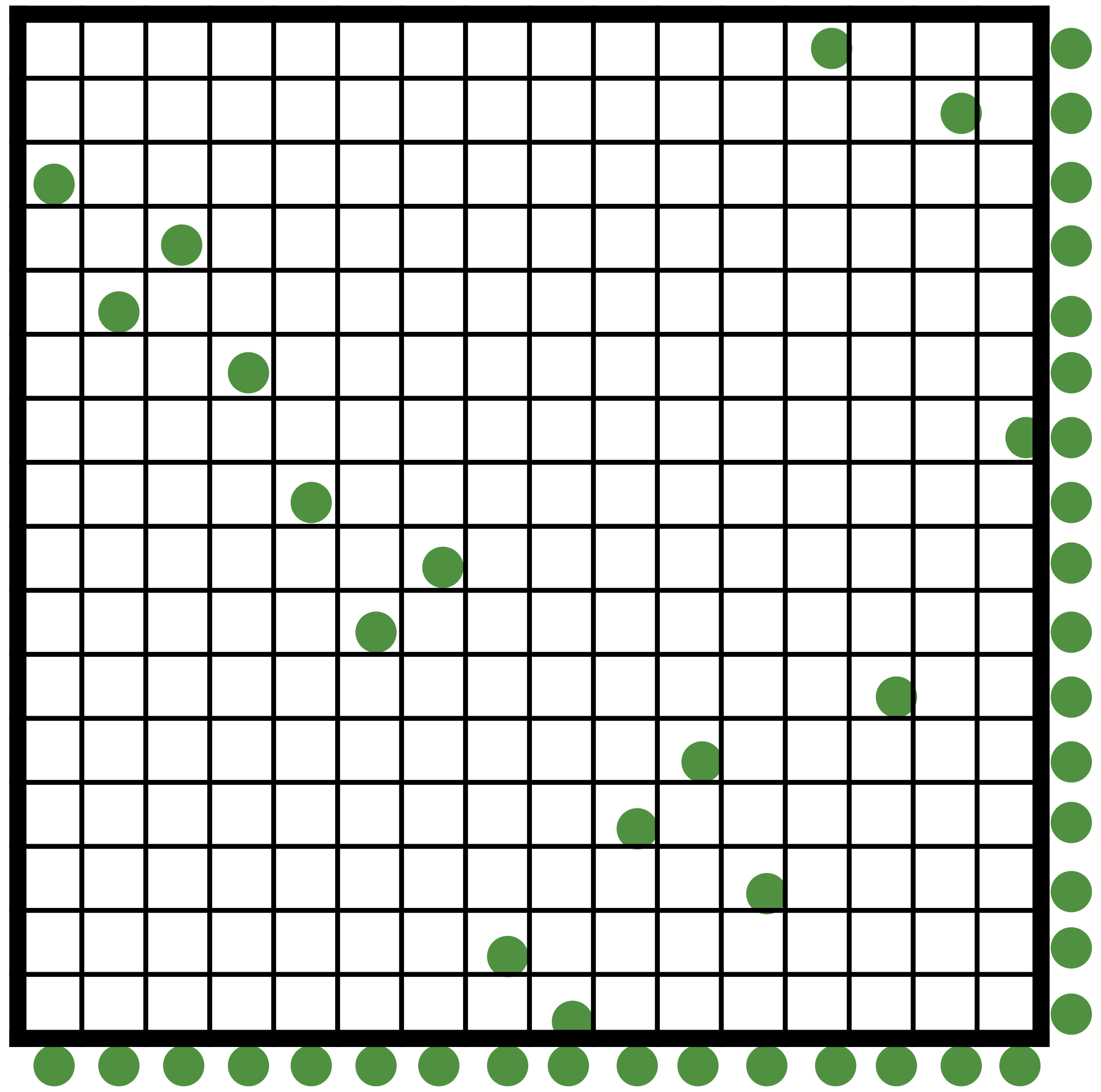
Multi-Jitter



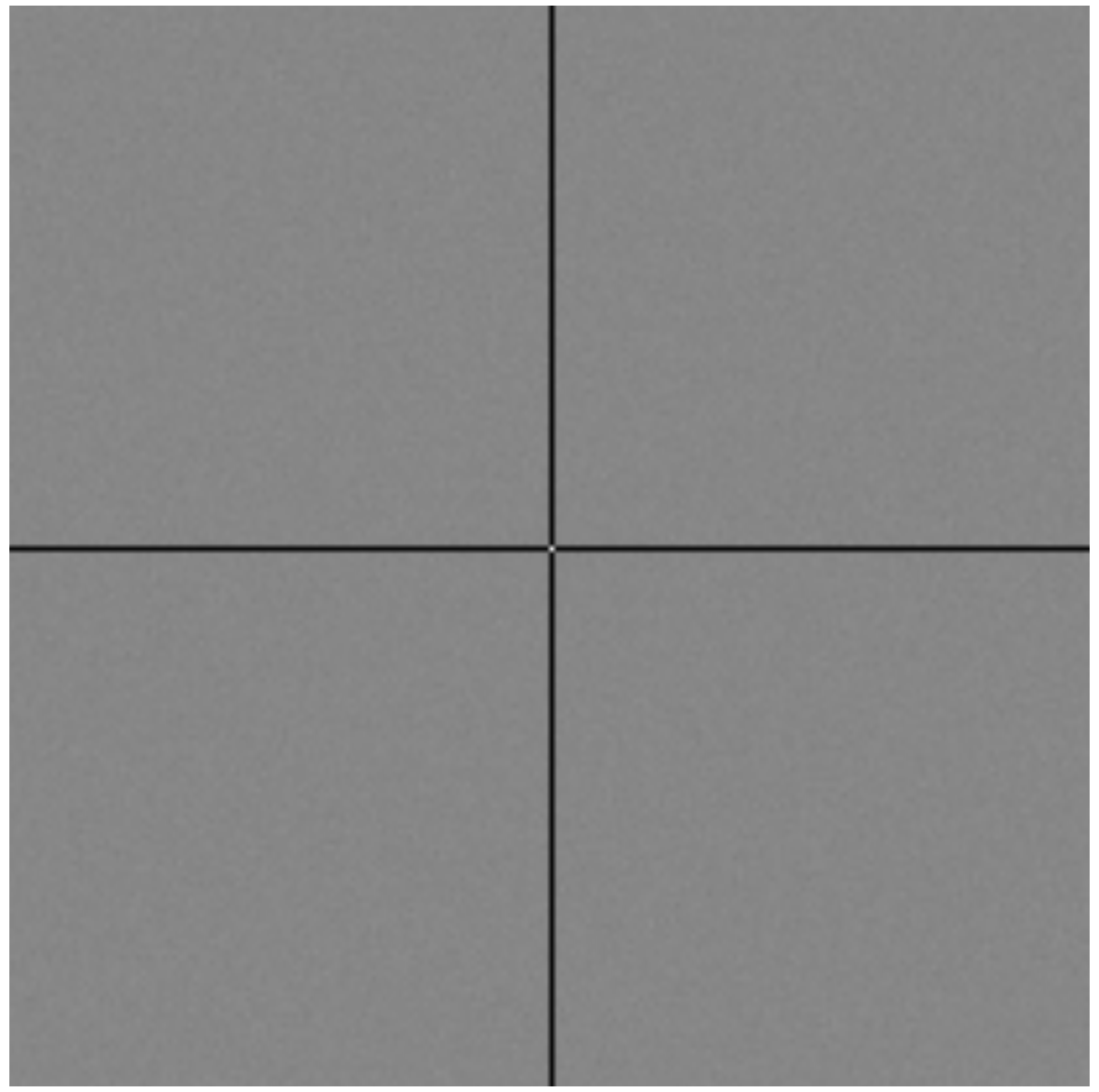
Multi-Jitter
Spectrum

Chiu et al. [1993]

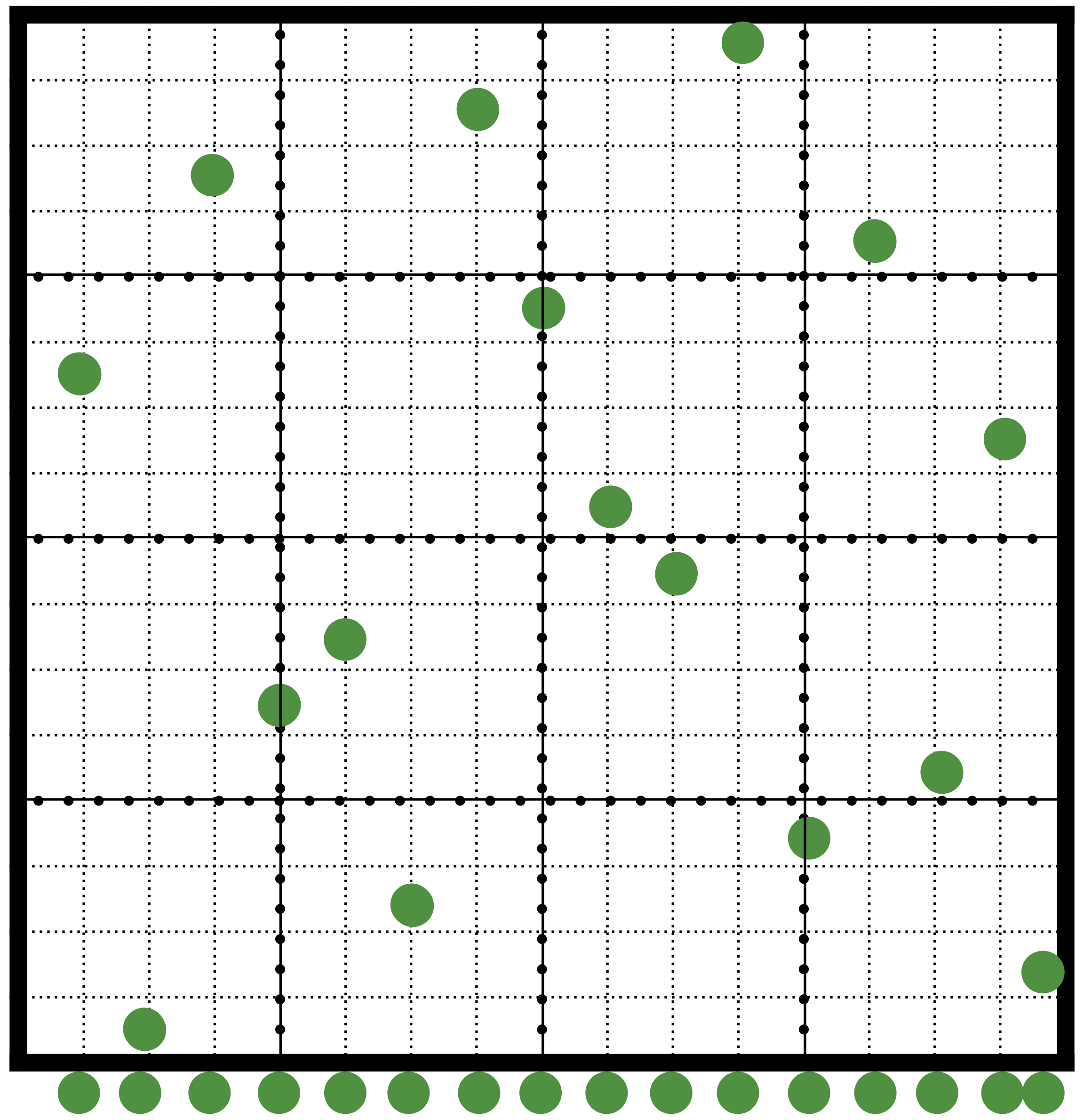
Anisotropic Sampling Power Spectra



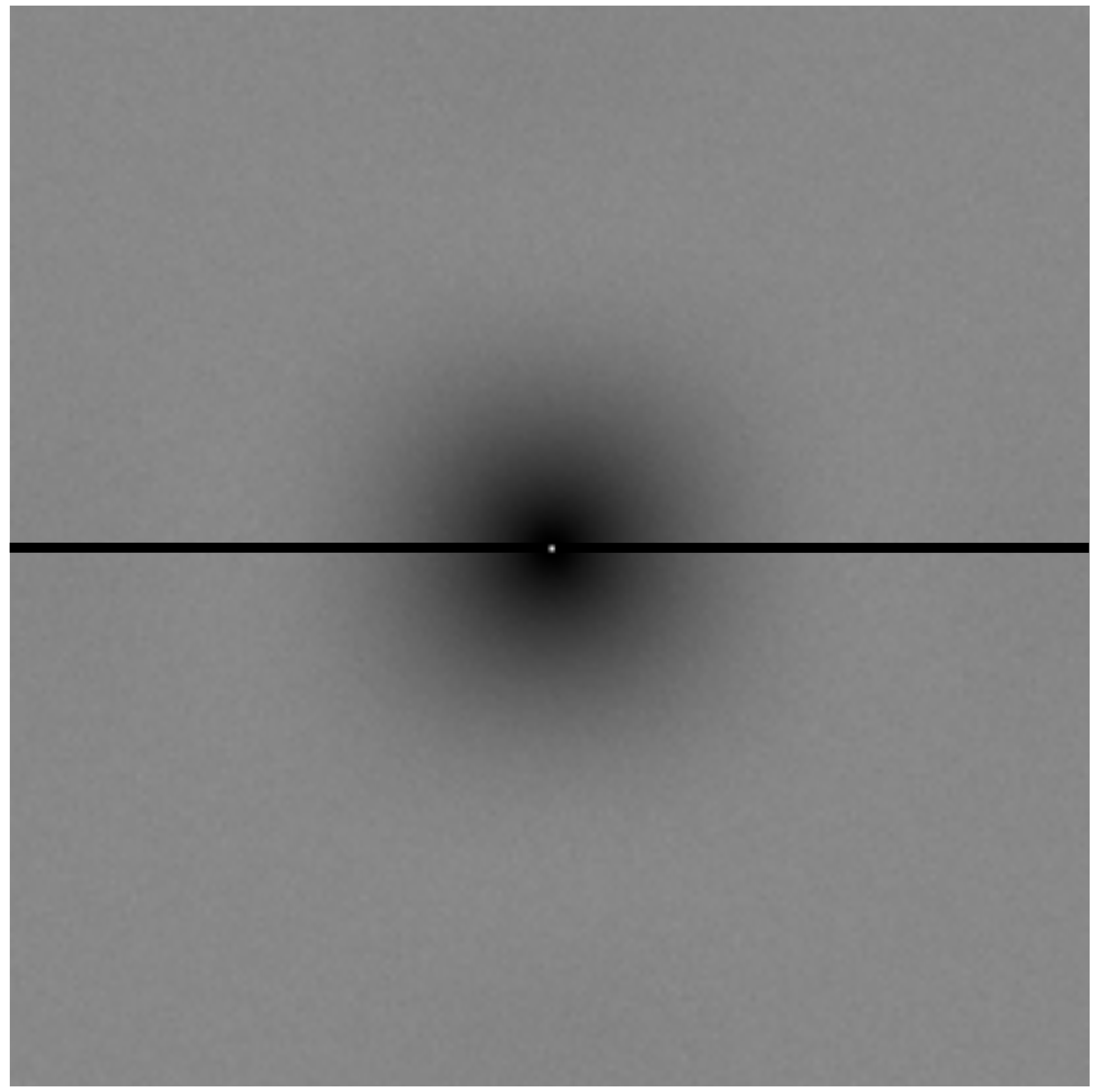
N-rooks /
Latin Hypercube



N-rooks
Spectrum



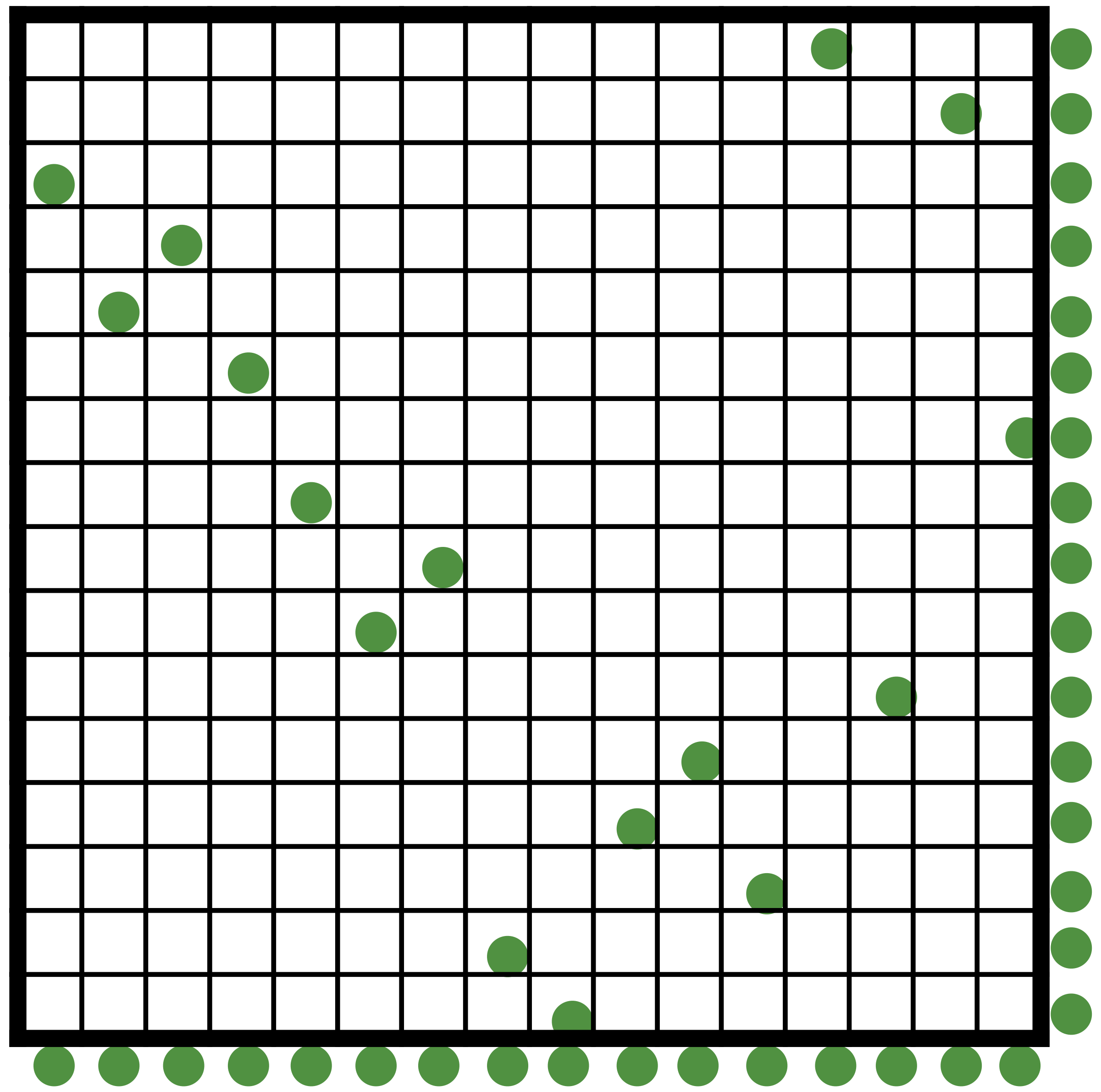
Multi-jitter



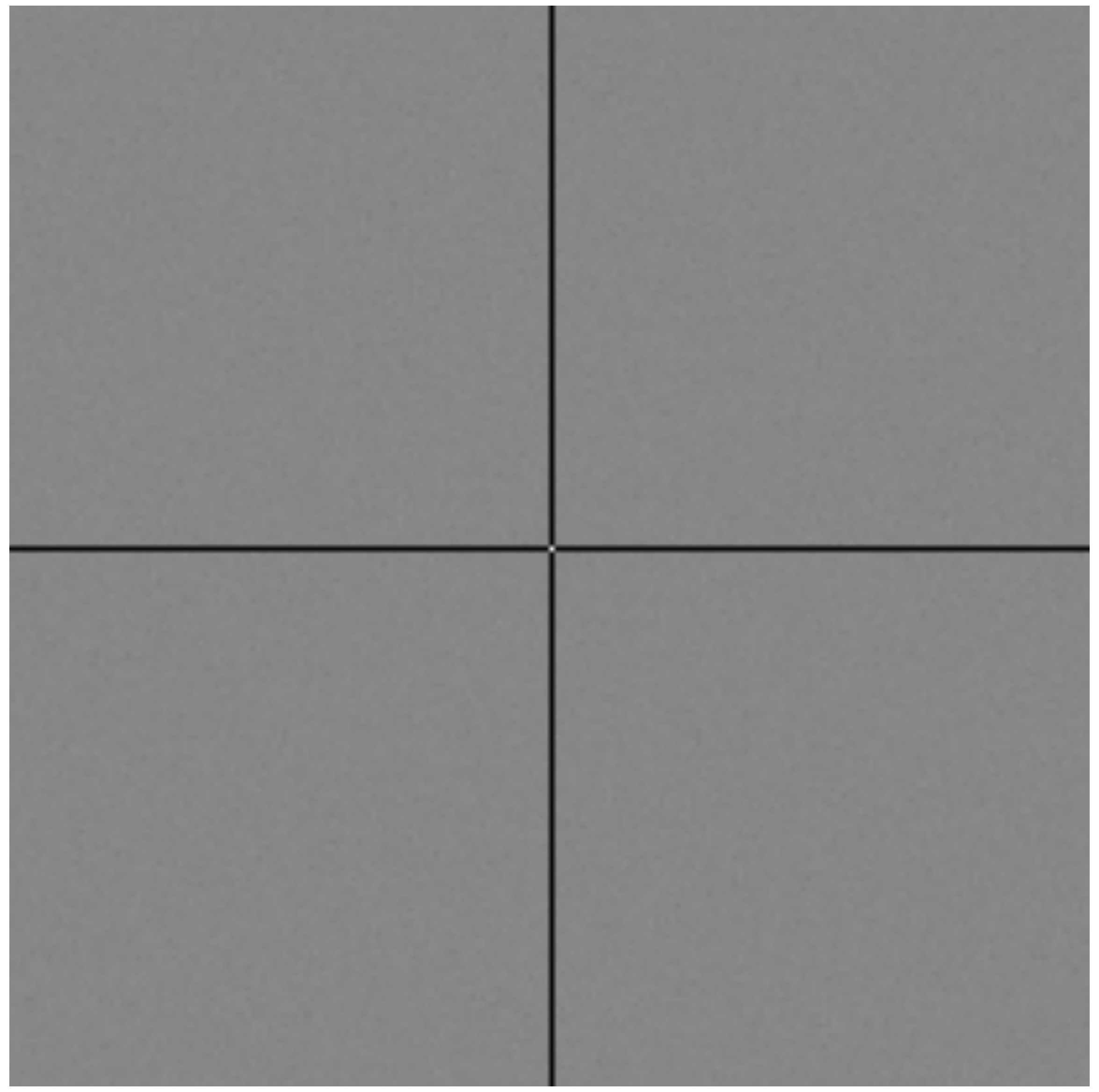
Multi-Jitter
Spectrum

Chiu et al. [1993]

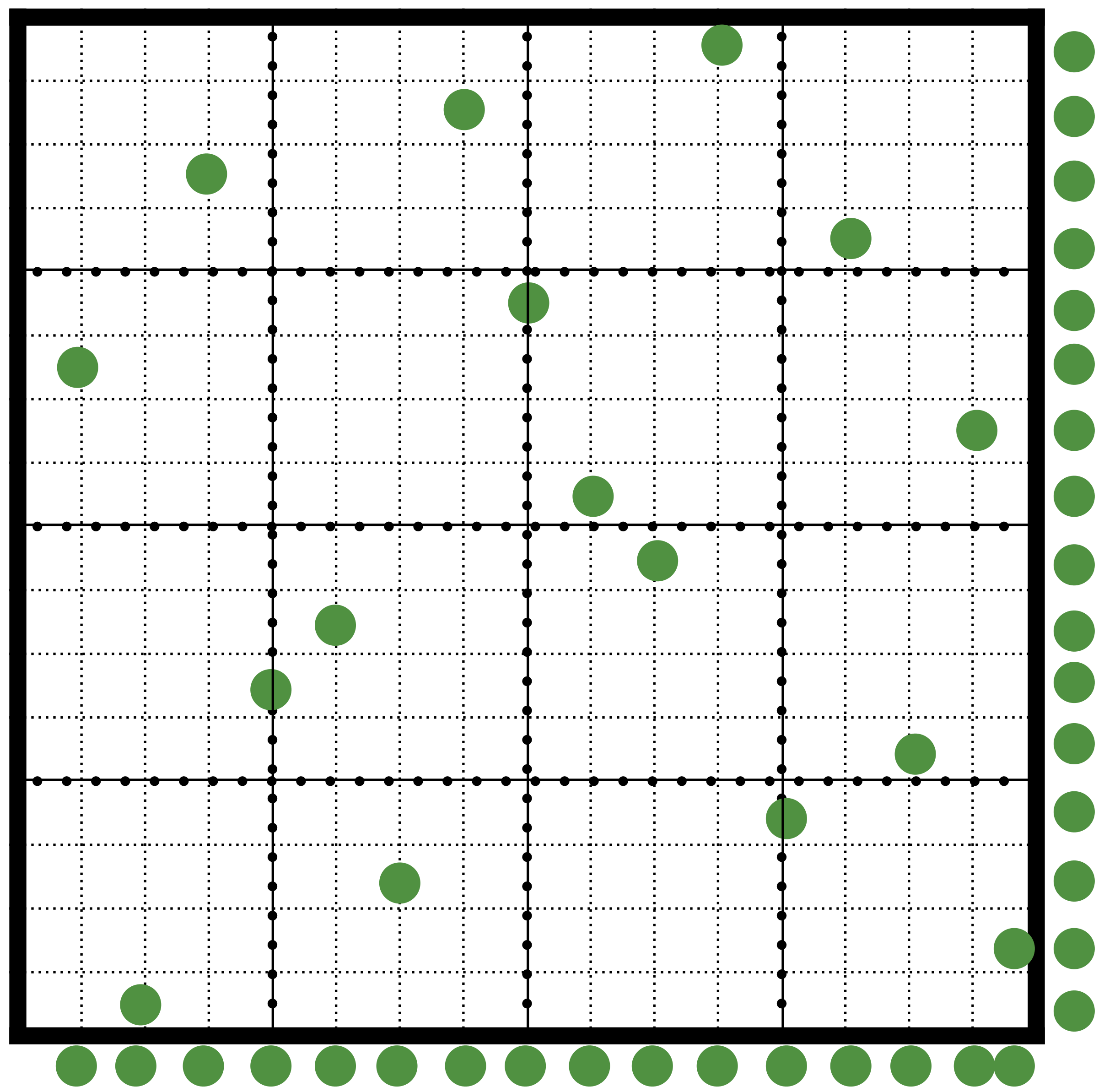
Anisotropic Sampling Power Spectra



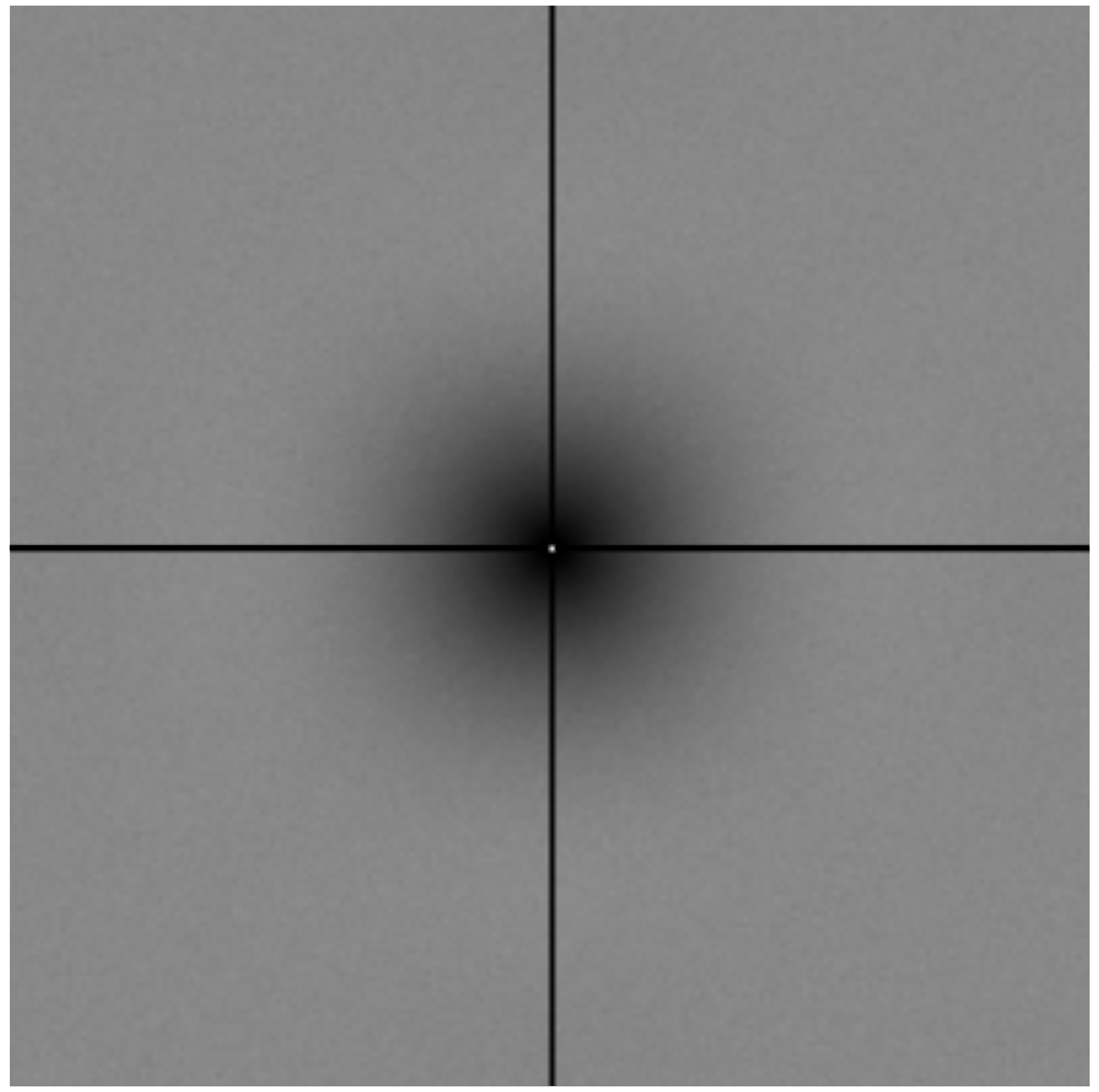
N-rooks /
Latin Hypercube



N-rooks
Spectrum



Multi-jitter

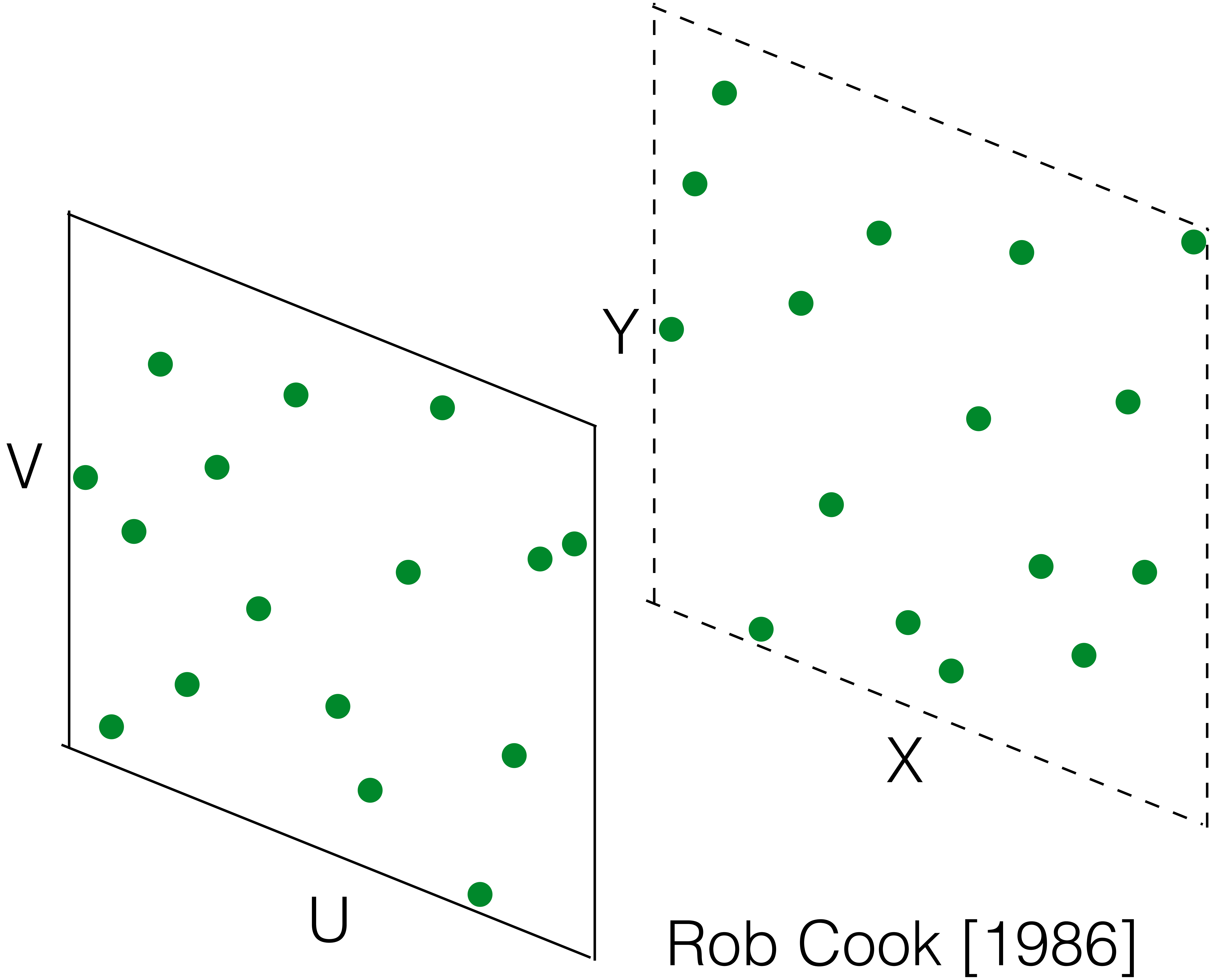


Multi-Jitter
Spectrum

Chiu et al. [1993]

Sampling in Higher Dimensions

4D Sampling



Rob Cook [1986]

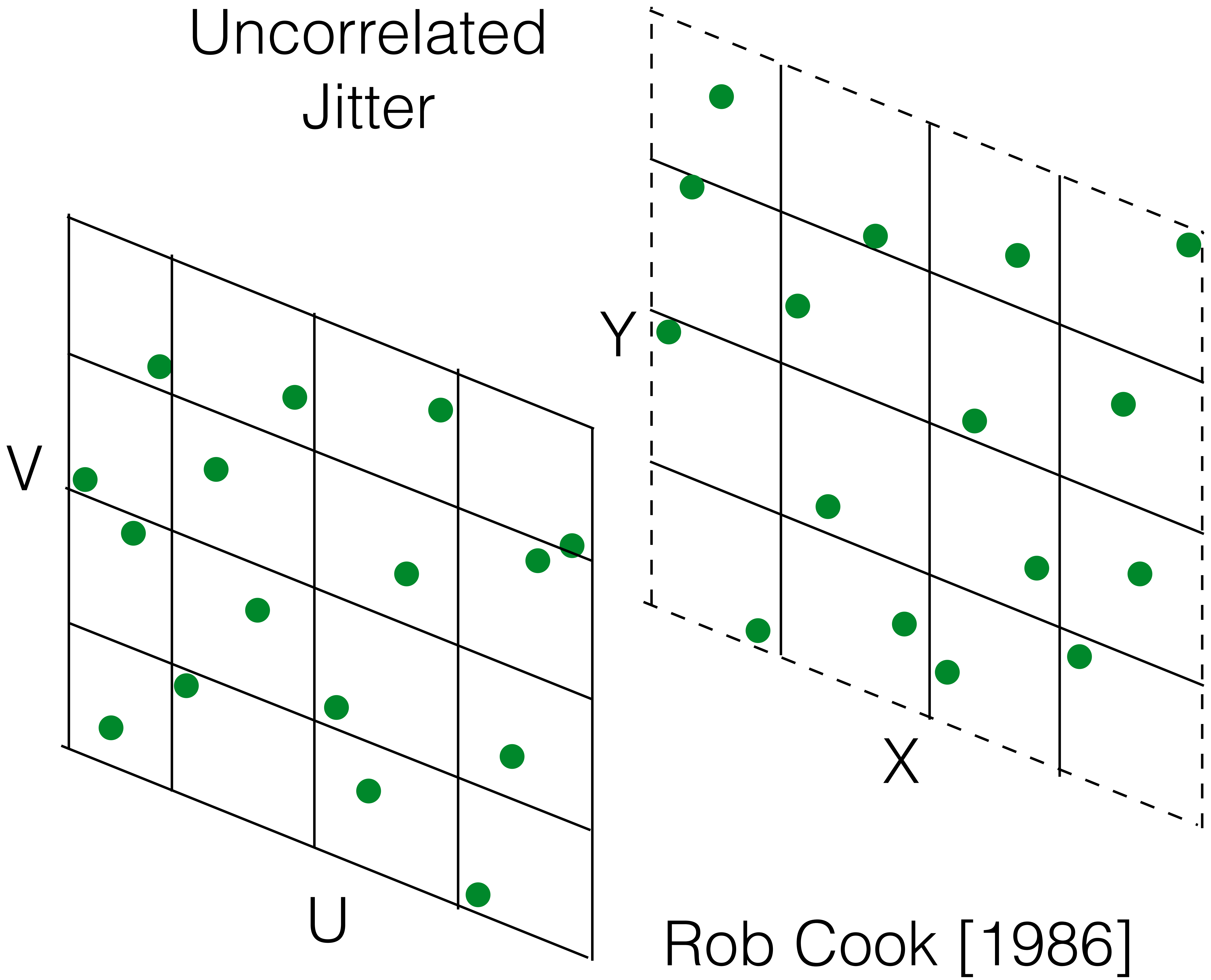
2D	2D
(x_1, y_1)	(u_1, v_1)
(x_2, y_2)	(u_2, v_2)
(x_3, y_3)	(u_3, v_3)
(x_4, y_4)	(u_4, v_4)
$\vdots$	$\vdots$

4D

$$(x_1, y_1, u_3, v_3)$$
$$(x_2, y_2, u_1, v_1)$$
$$(x_3, y_3, u_4, v_4)$$
$$(x_4, y_4, u_2, v_2)$$
$$\vdots$$

4D Sampling

Uncorrelated Jitter



Rob Cook [1986]

2D

$$(x_1, y_1)$$
$$(x_2, y_2)$$
$$(x_3, y_3)$$
$$(x_4, y_4)$$
$$\vdots$$

2D

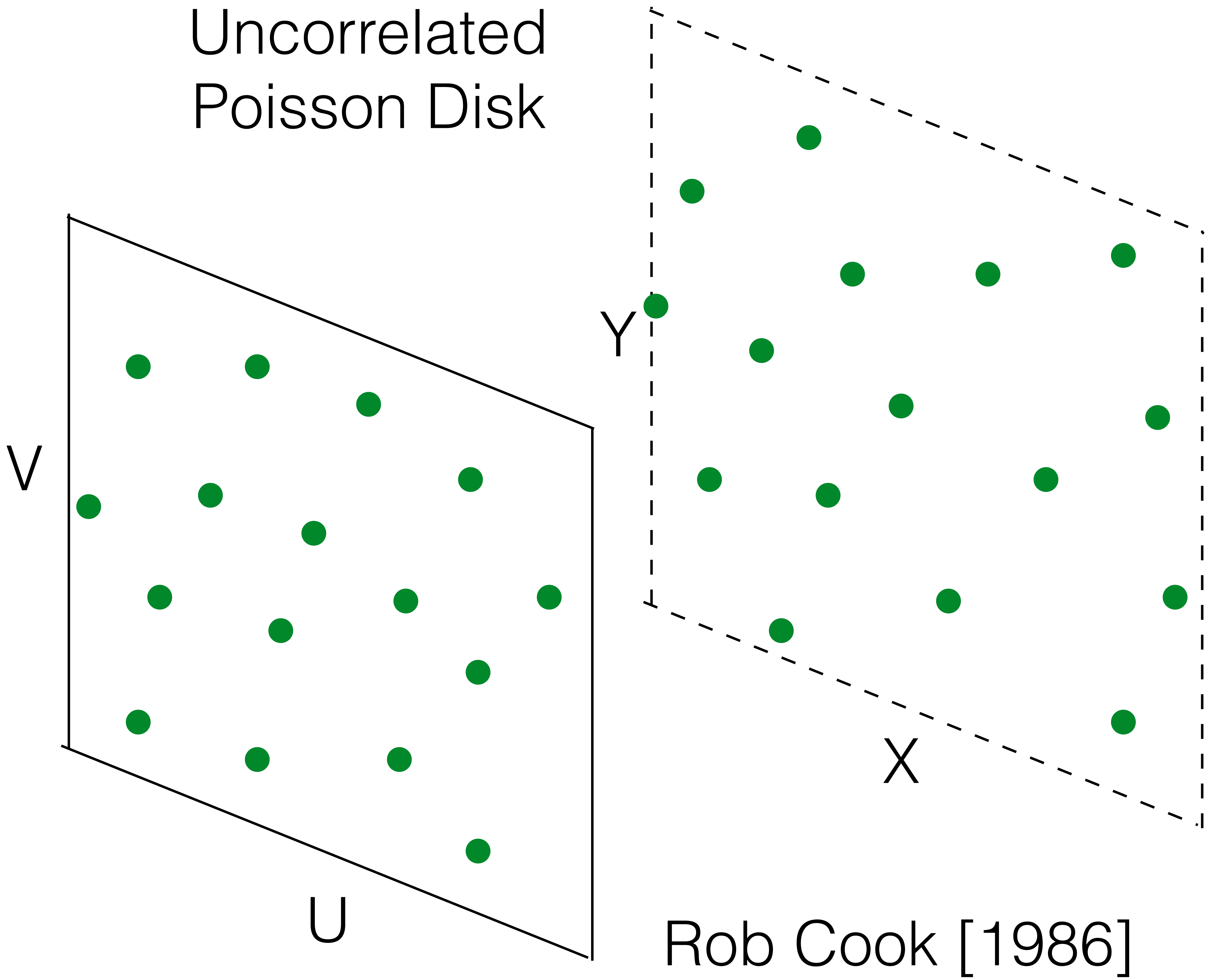
$$(u_1, v_1)$$
$$(u_2, v_2)$$
$$(u_3, v_3)$$
$$(u_4, v_4)$$
$$\vdots$$

4D

$$(x_1, y_1, u_3, v_3)$$
$$(x_2, y_2, u_1, v_1)$$
$$(x_3, y_3, u_4, v_4)$$
$$(x_4, y_4, u_2, v_2)$$
$$\vdots$$

4D Sampling

Uncorrelated
Poisson Disk



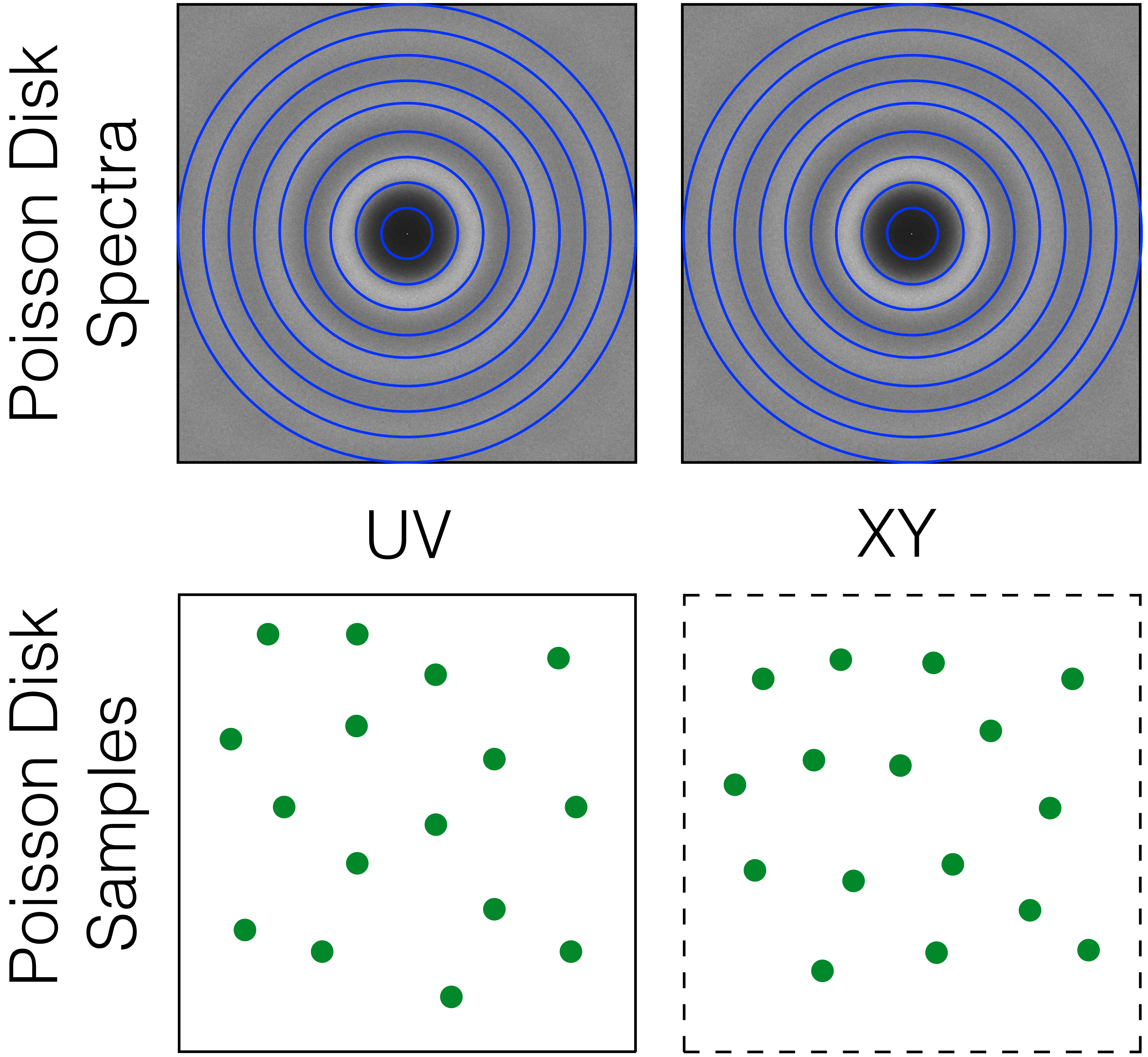
Rob Cook [1986]

2D	2D
(x_1, y_1)	(u_1, v_1)
(x_2, y_2)	(u_2, v_2)
(x_3, y_3)	(u_3, v_3)
(x_4, y_4)	(u_4, v_4)
$\vdots$	$\vdots$

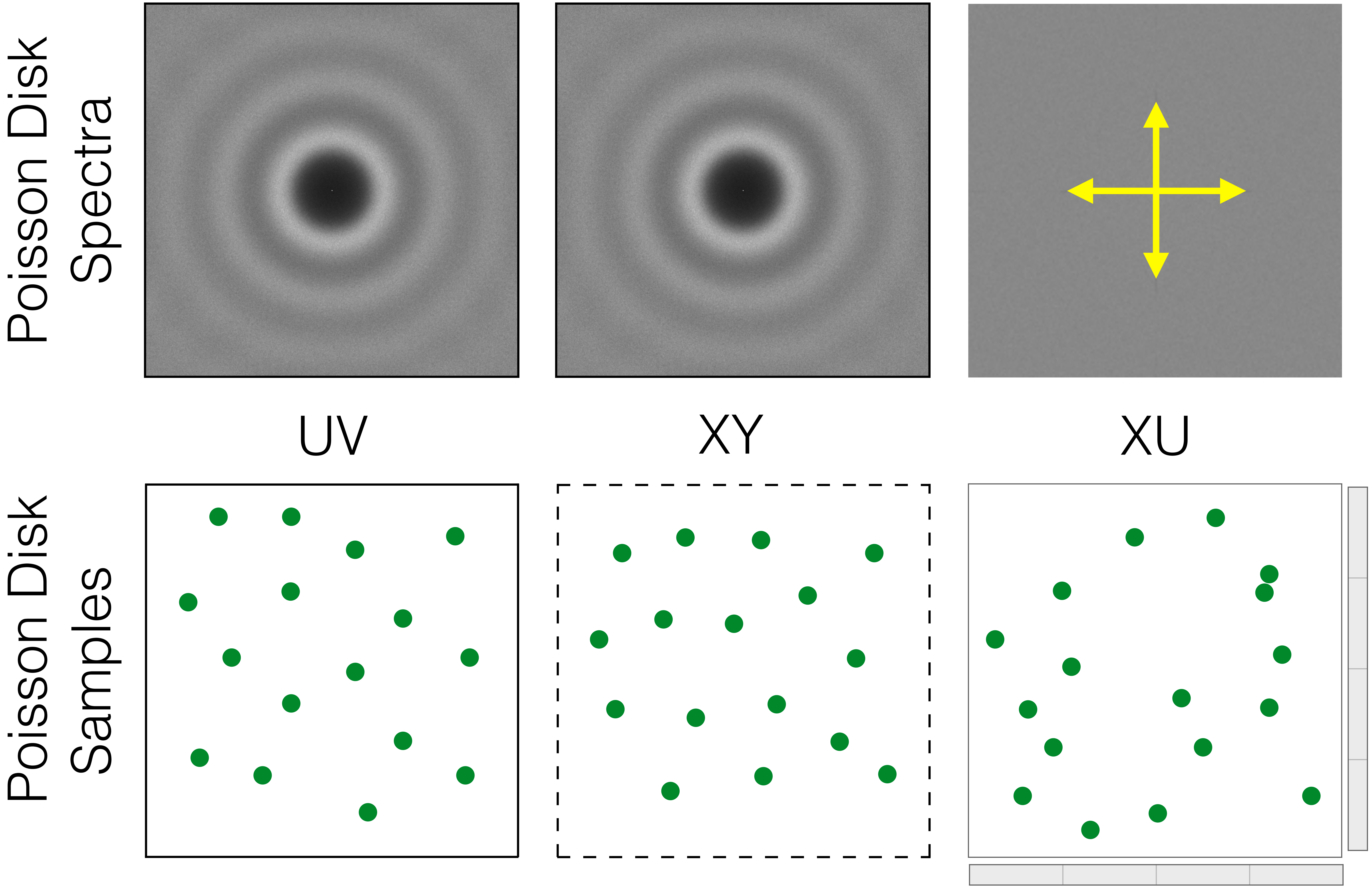
4D

$$(x_1, y_1, u_3, v_3)$$
$$(x_2, y_2, u_1, v_1)$$
$$(x_3, y_3, u_4, v_4)$$
$$(x_4, y_4, u_2, v_2)$$
$$\vdots$$

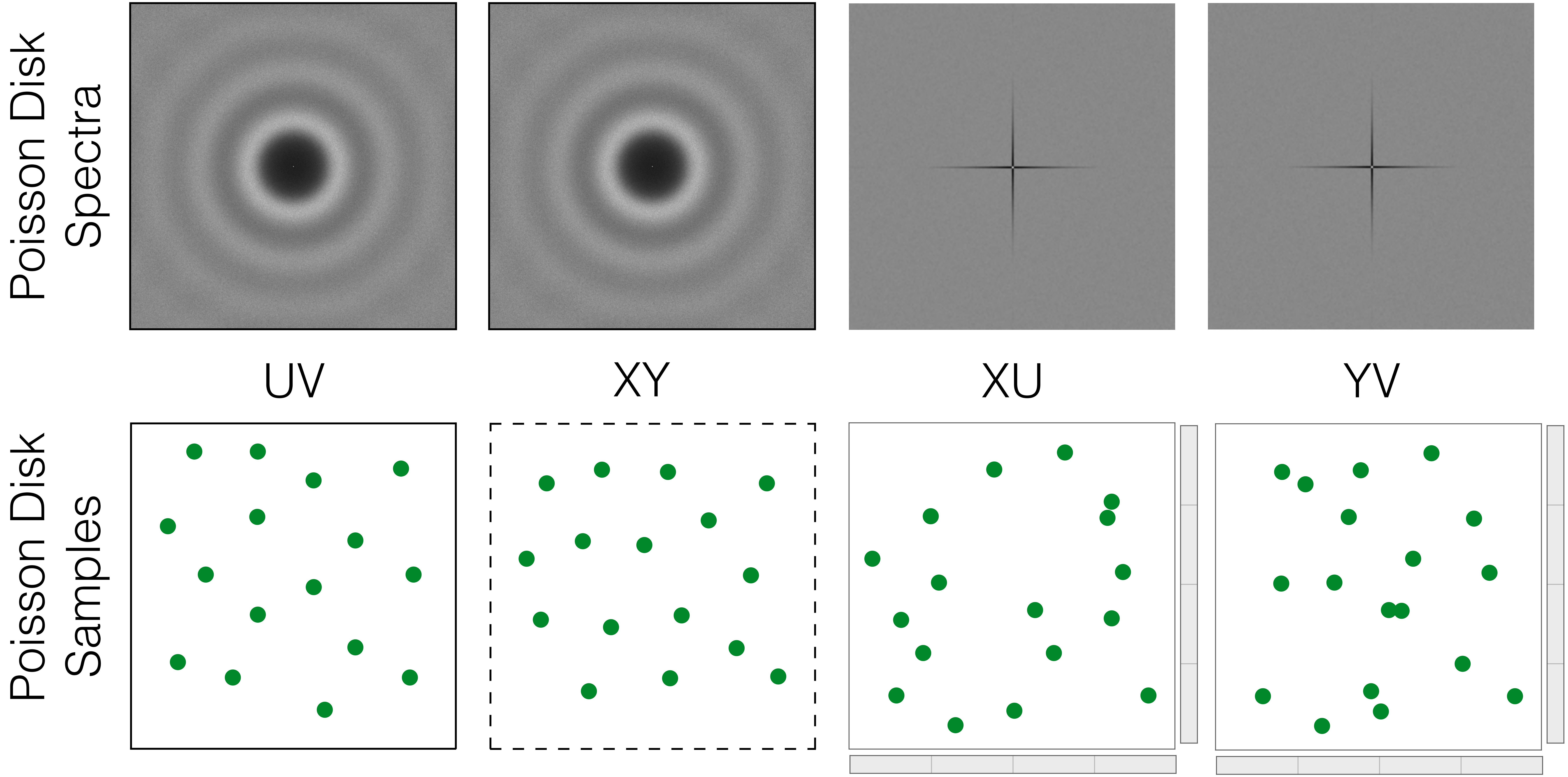
4D Sampling Spectra along Projections



4D Sampling Spectra along Projections

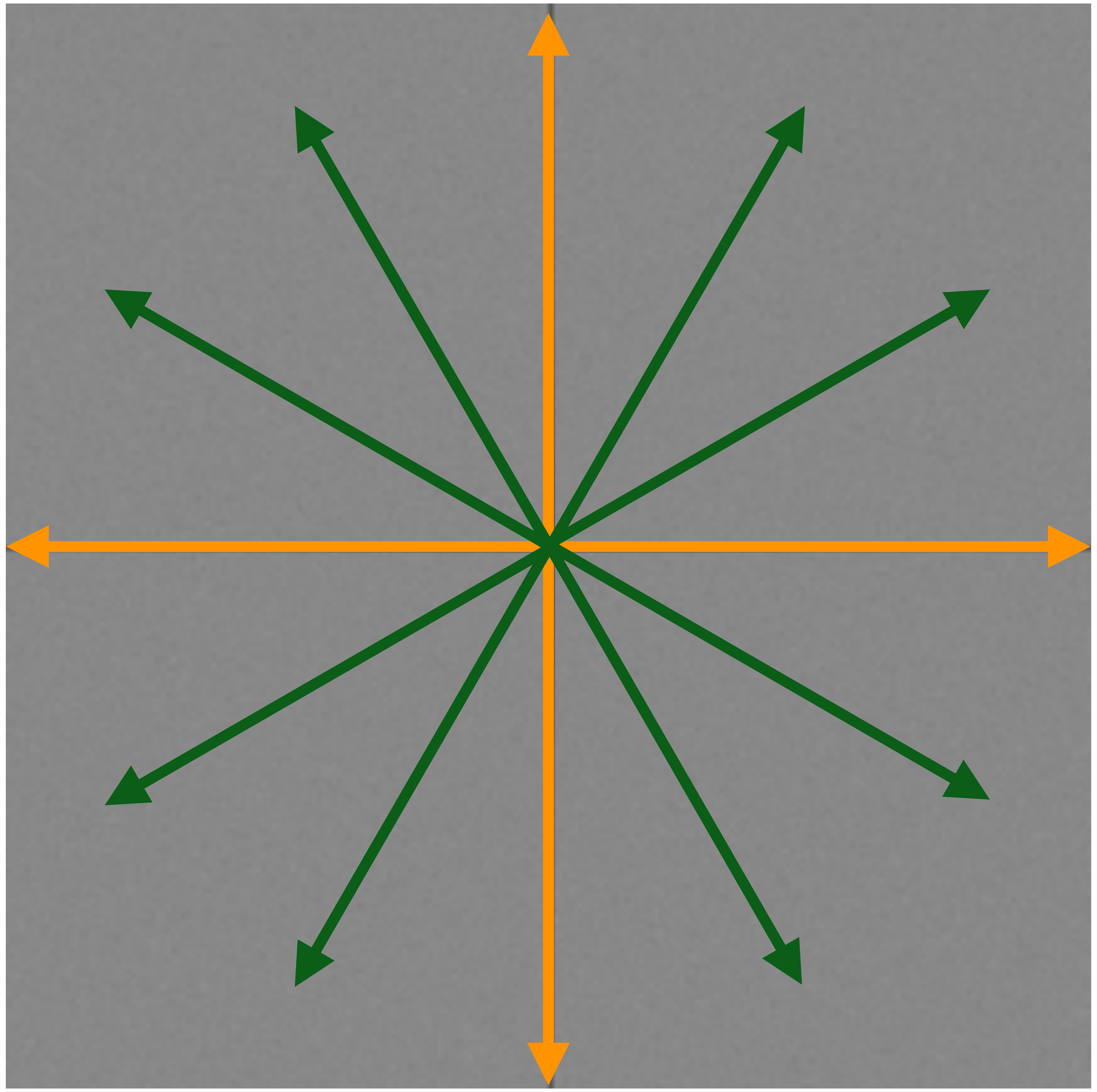


4D Sampling Spectra along Projections

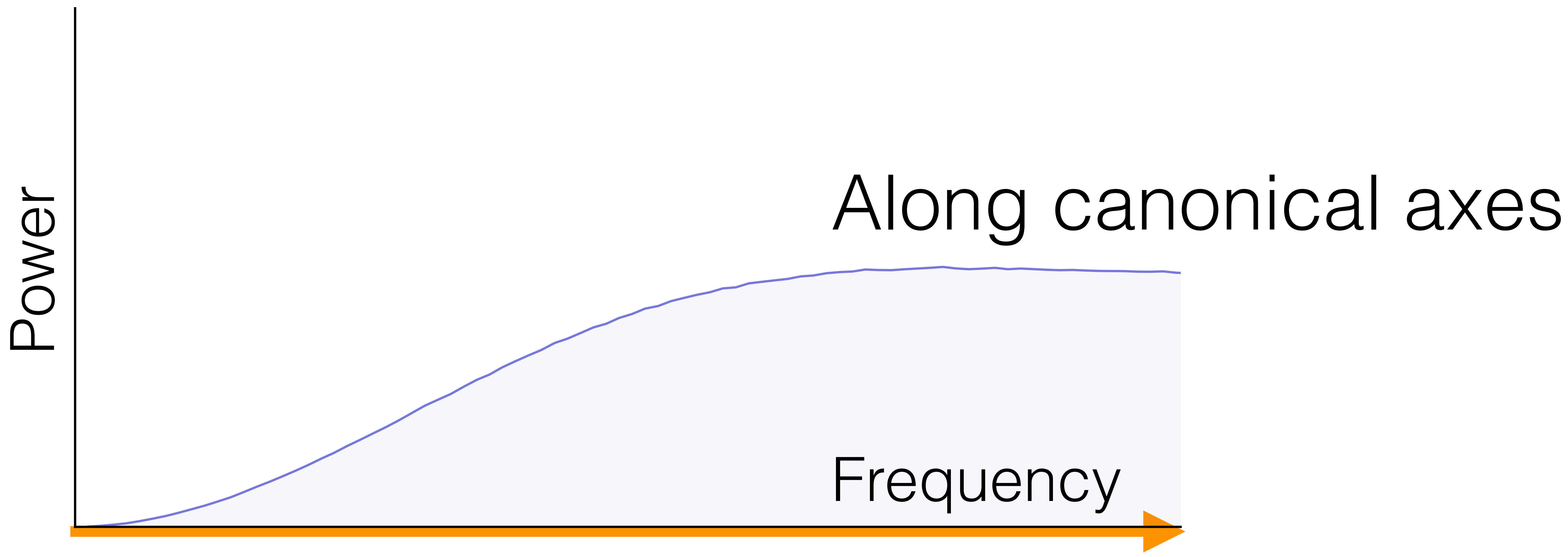


Convergence Analysis for Anisotropic Sampling Spectra

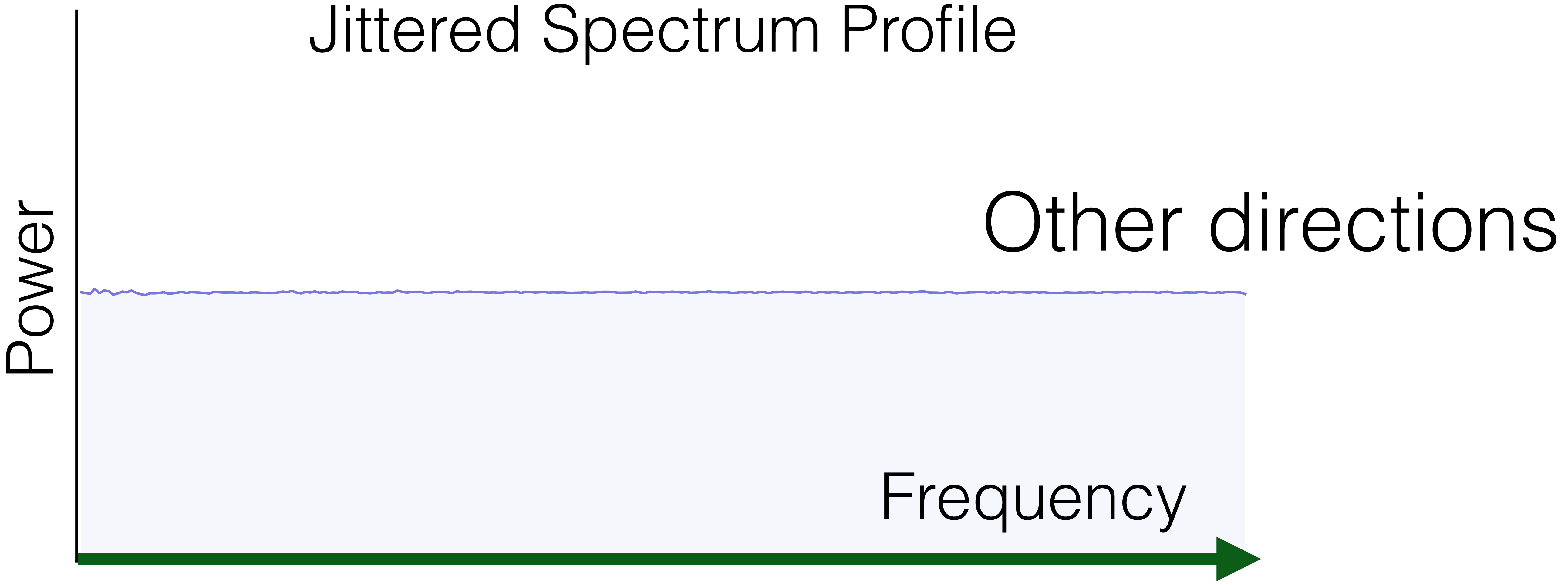
Power Spectrum



Radial Power Spectrum



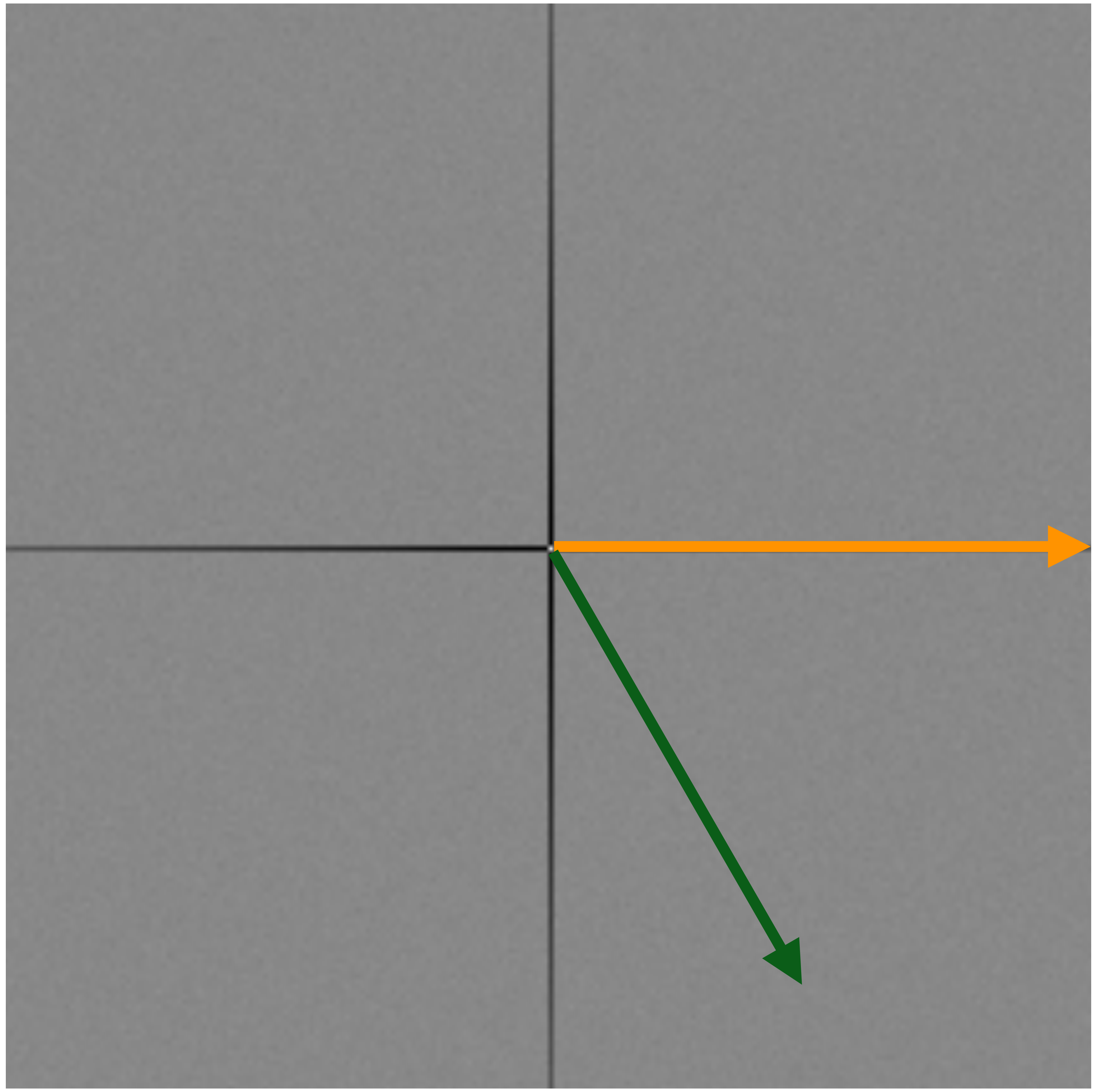
Jittered Spectrum Profile



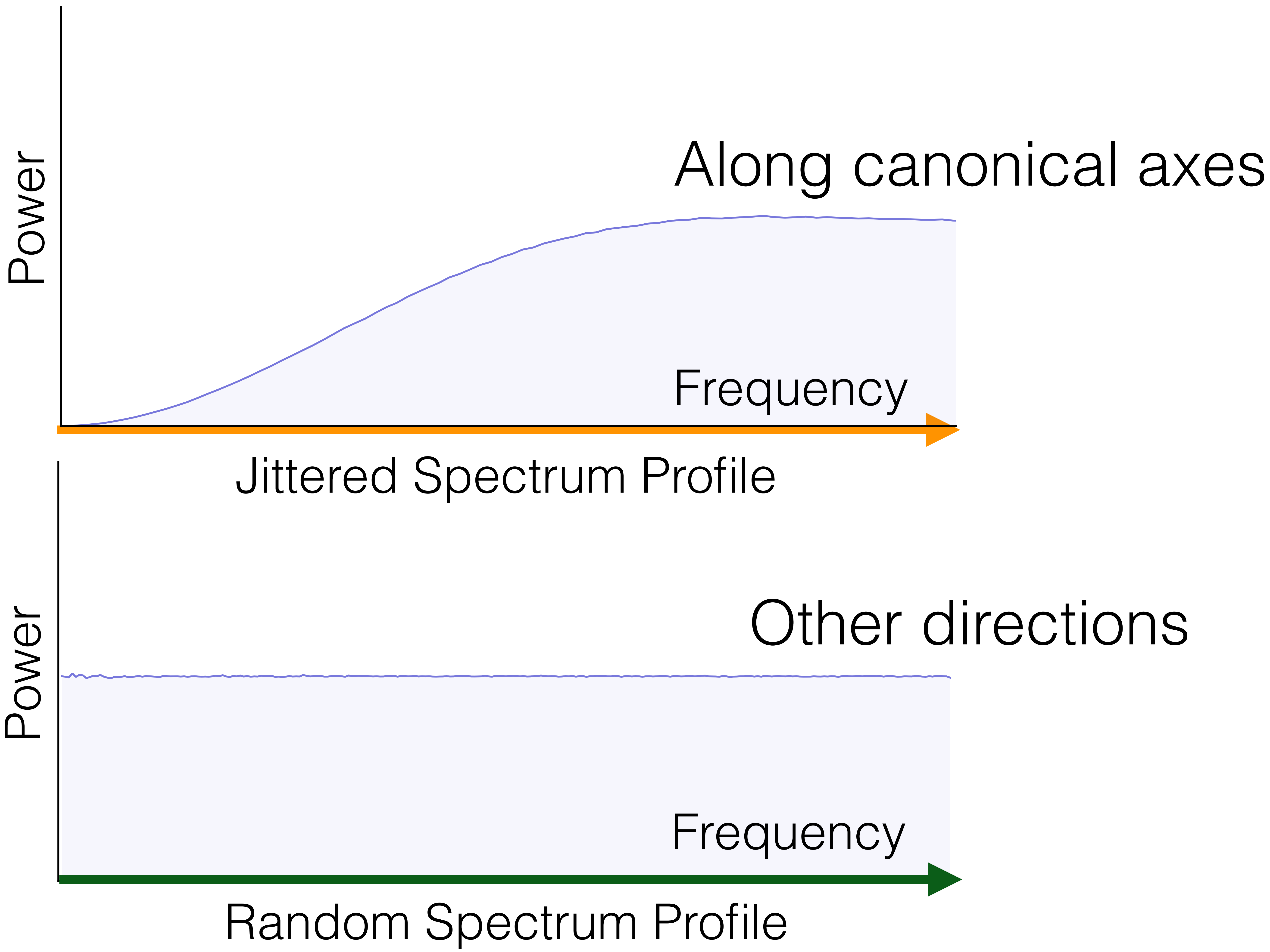
Random Spectrum Profile

Convergence Analysis for Anisotropic Sampling Spectra

Power Spectrum

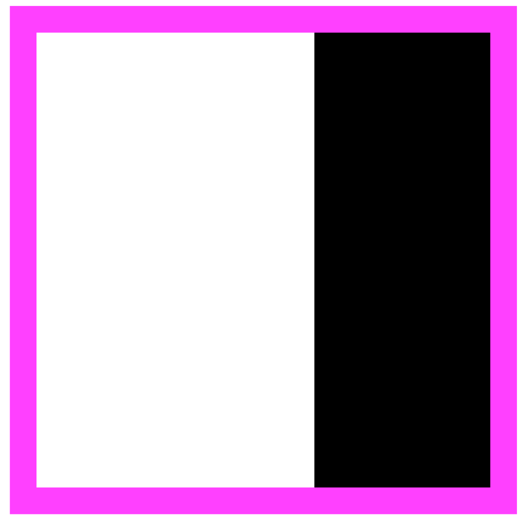


Radial Power Spectrum



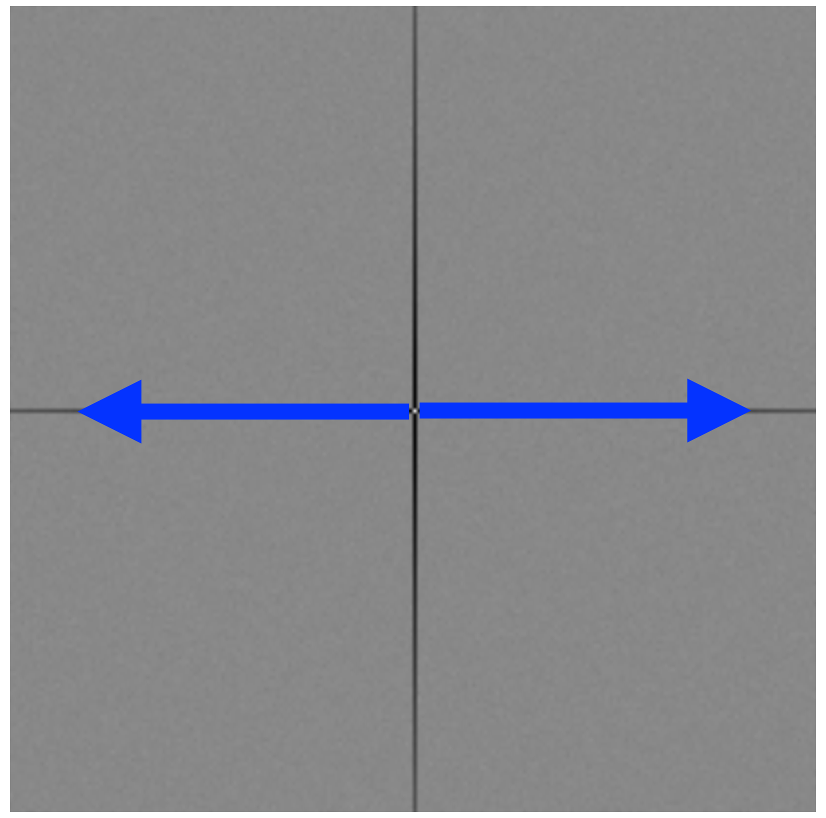
Variance due to N-rooks Sampler

$f(\vec{x})$



$\mathrm{Var}(I_N) = \sum_{\Omega}$

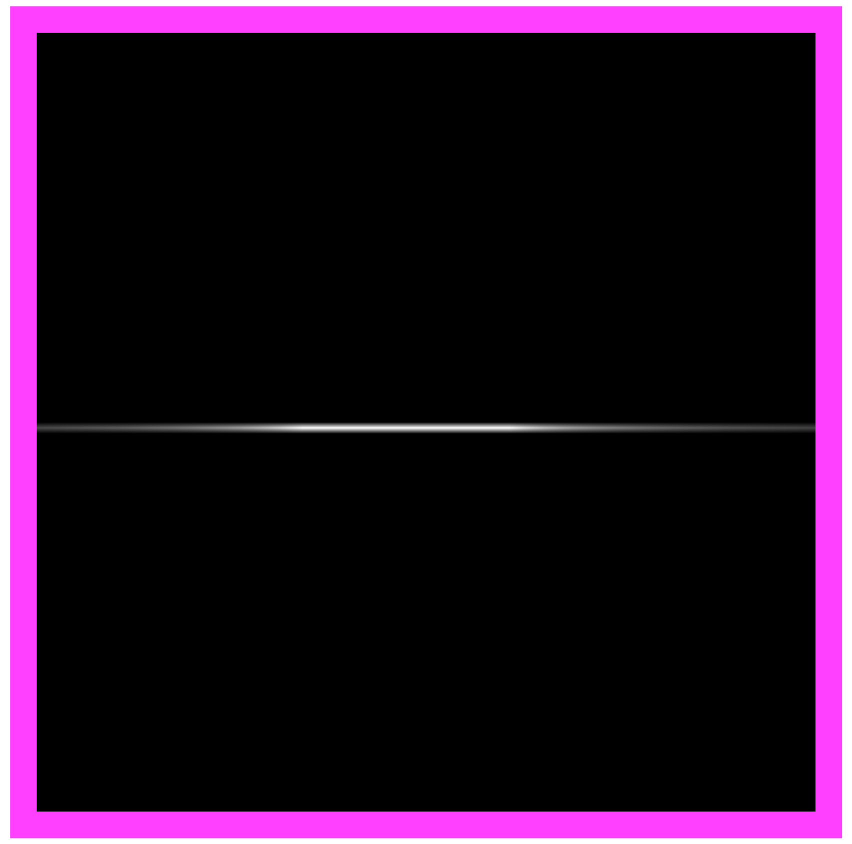
$\langle \mathcal{P}_{S_N}(\nu) \rangle$



N-rooks spectrum

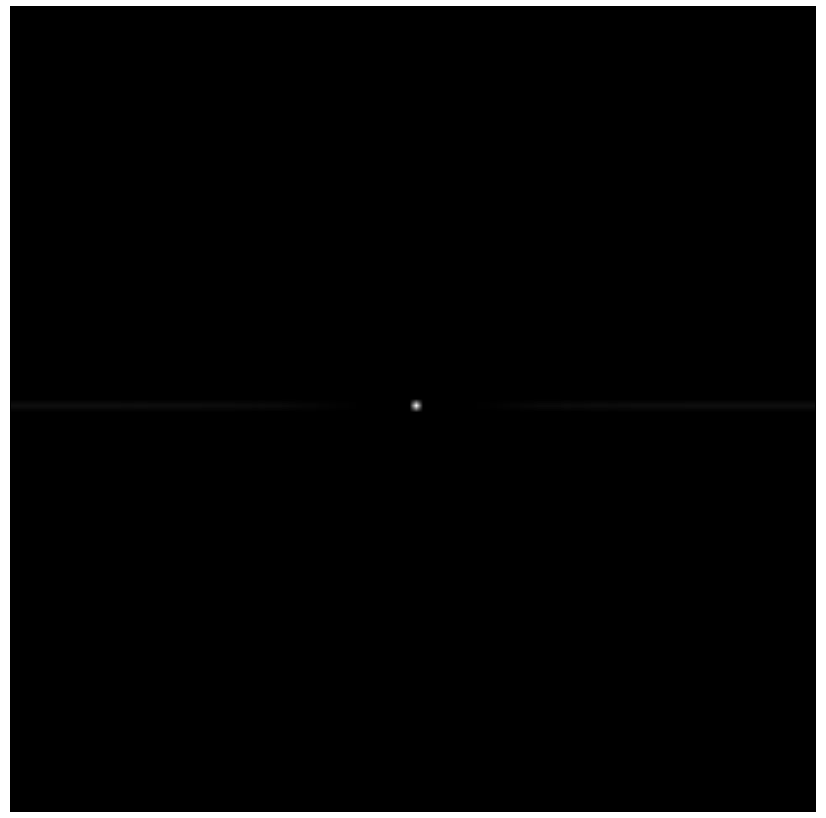
X

$\mathcal{P}_f(\nu)$



Integrand spectrum

$= \sum_{\Omega}$



Variance due to N-rooks Sampler

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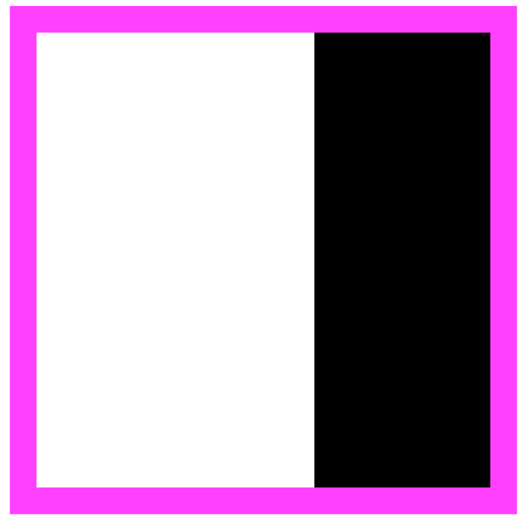

$\langle \mathcal{P}_{S_N}(\nu) \rangle$ $\mathcal{P}_f(\nu)$

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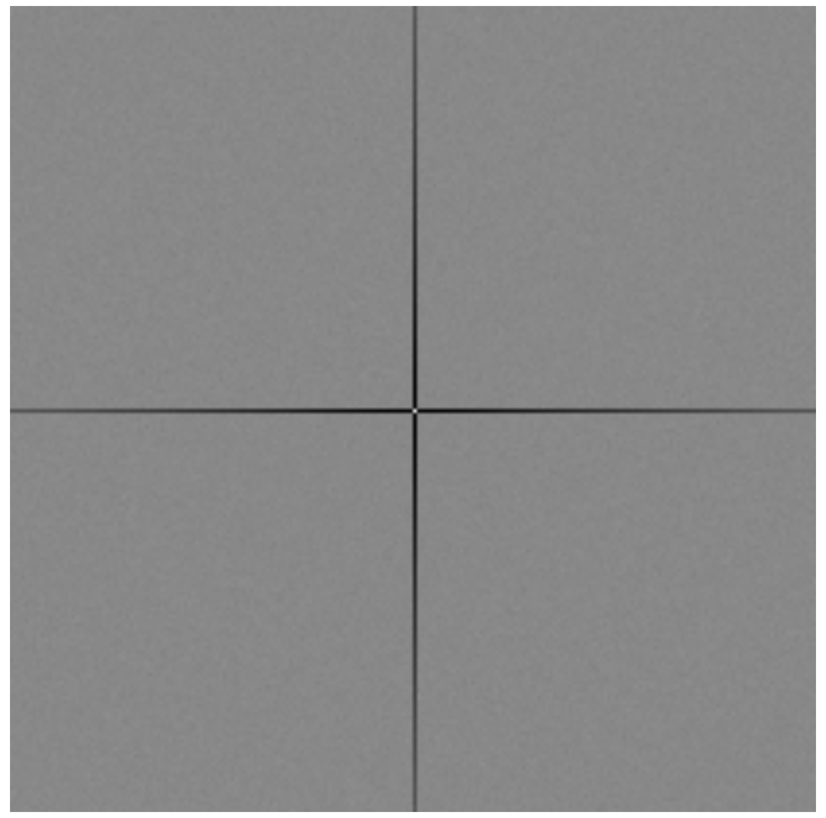
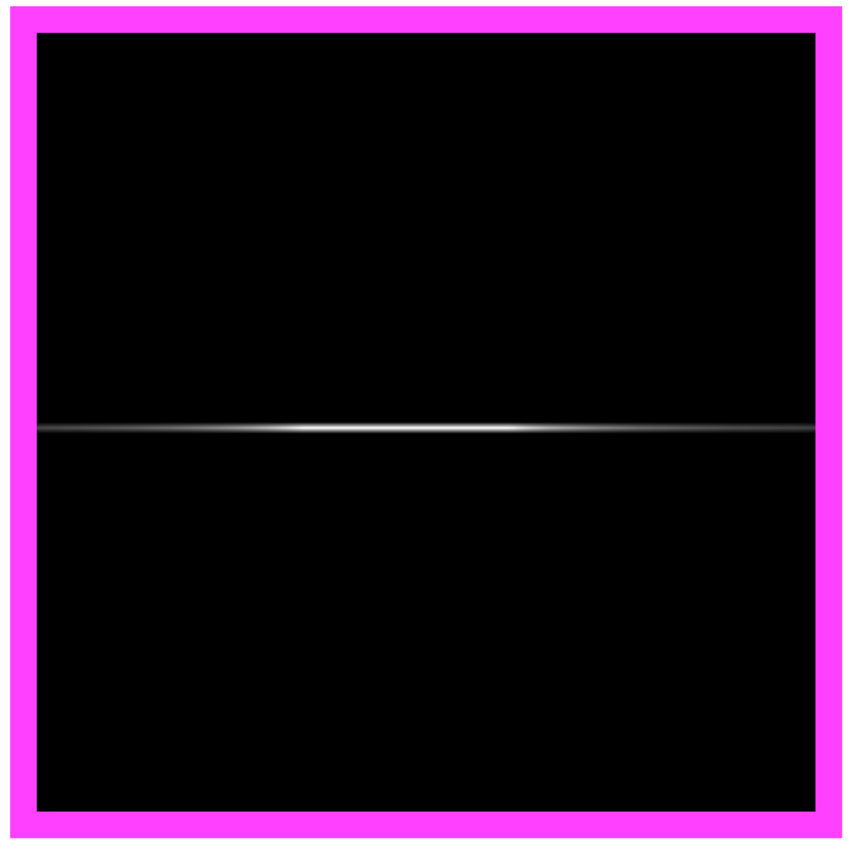


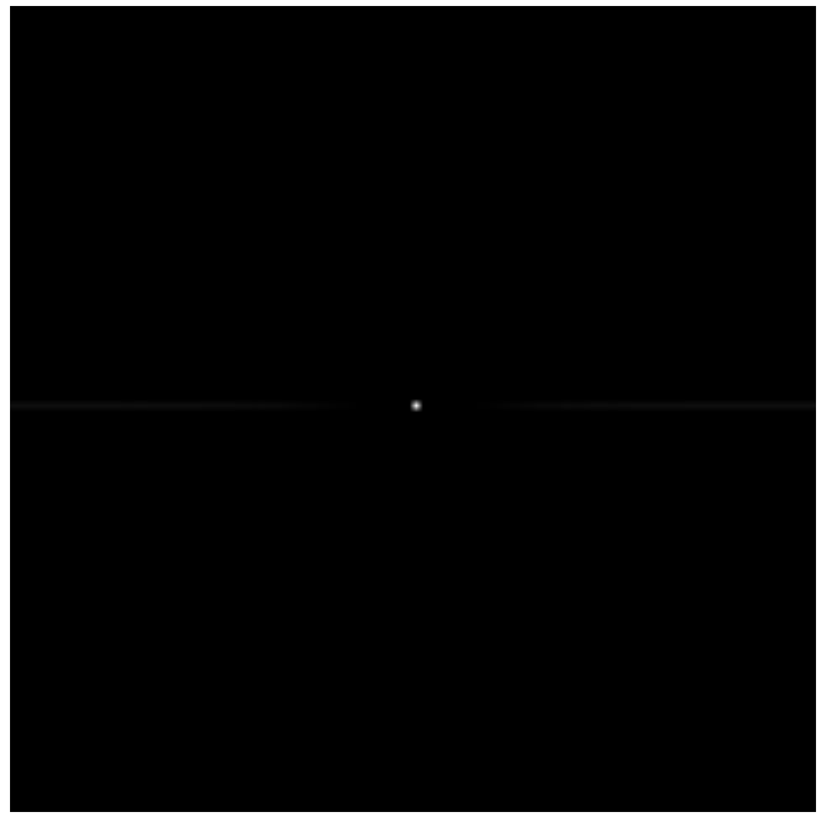
N-rooks spectrum

X

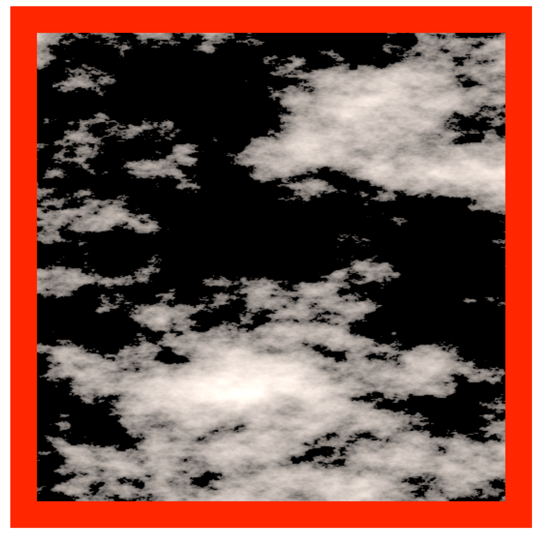


Integrand spectrum

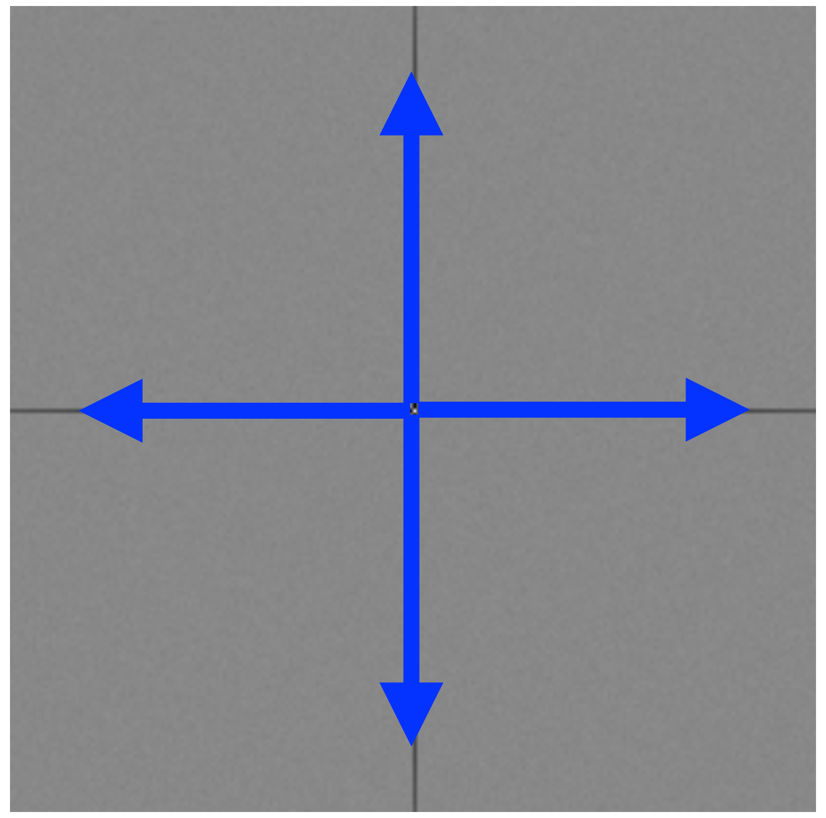
$= \sum_{\Omega}$



$f(\vec{x})$

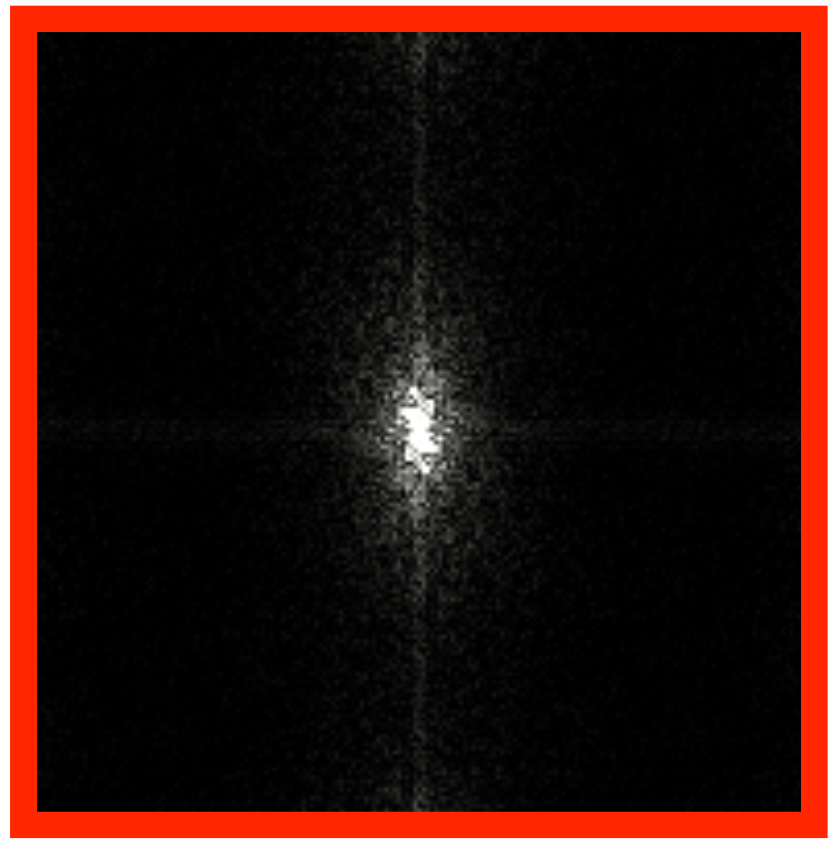


$\mathrm{Var}(I_N) = \sum_{\Omega}$



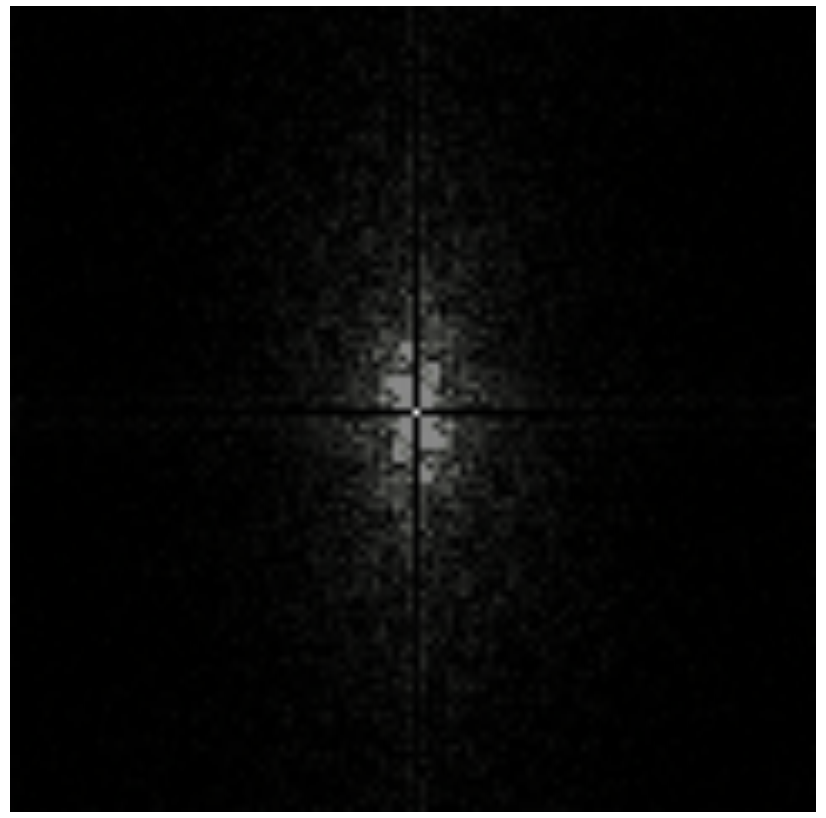
N-rooks spectrum

X

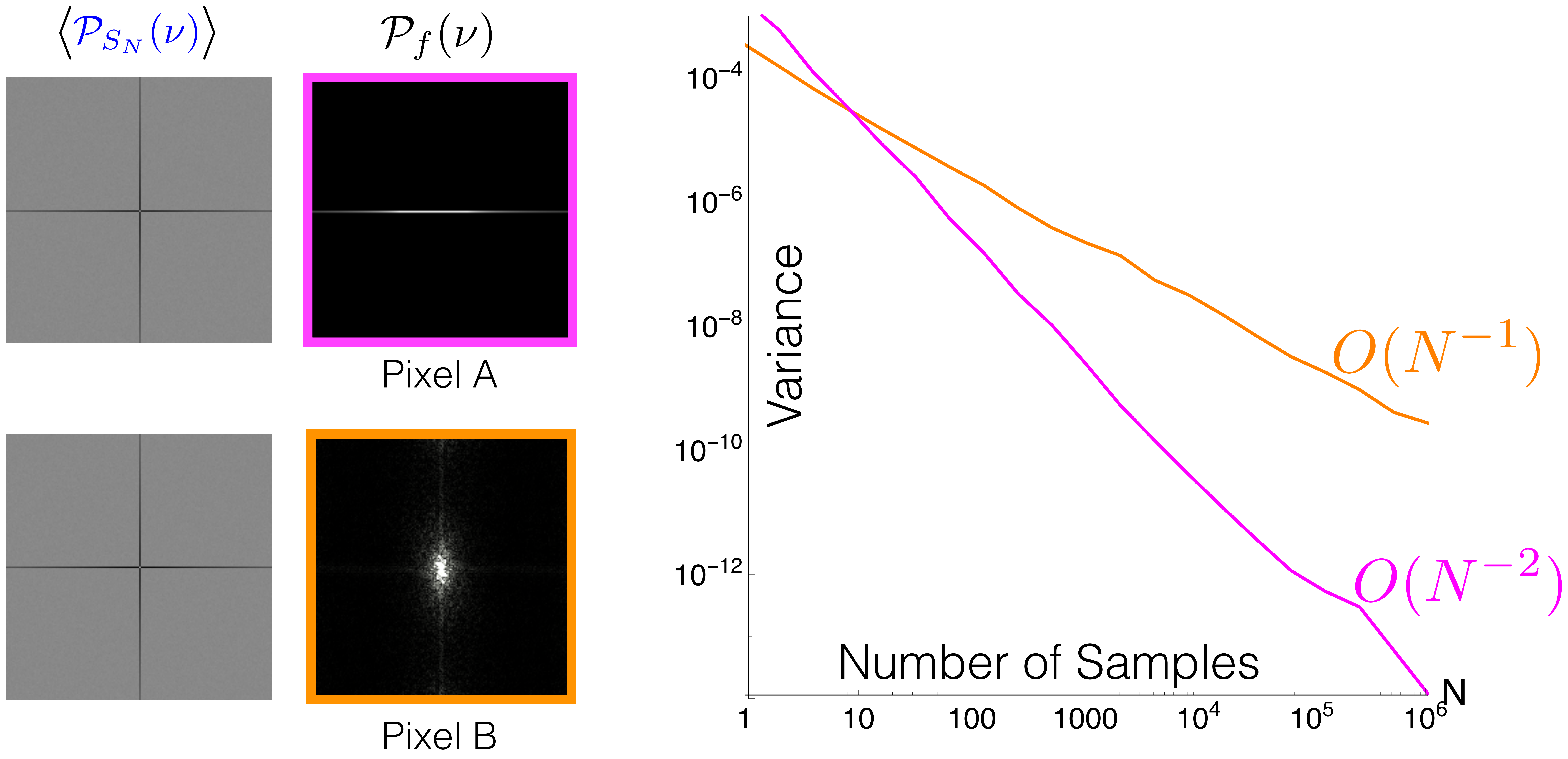


Integrand spectrum

$= \sum_{\Omega}$



Variance Convergence of Latin Hypercube (N-rooks)

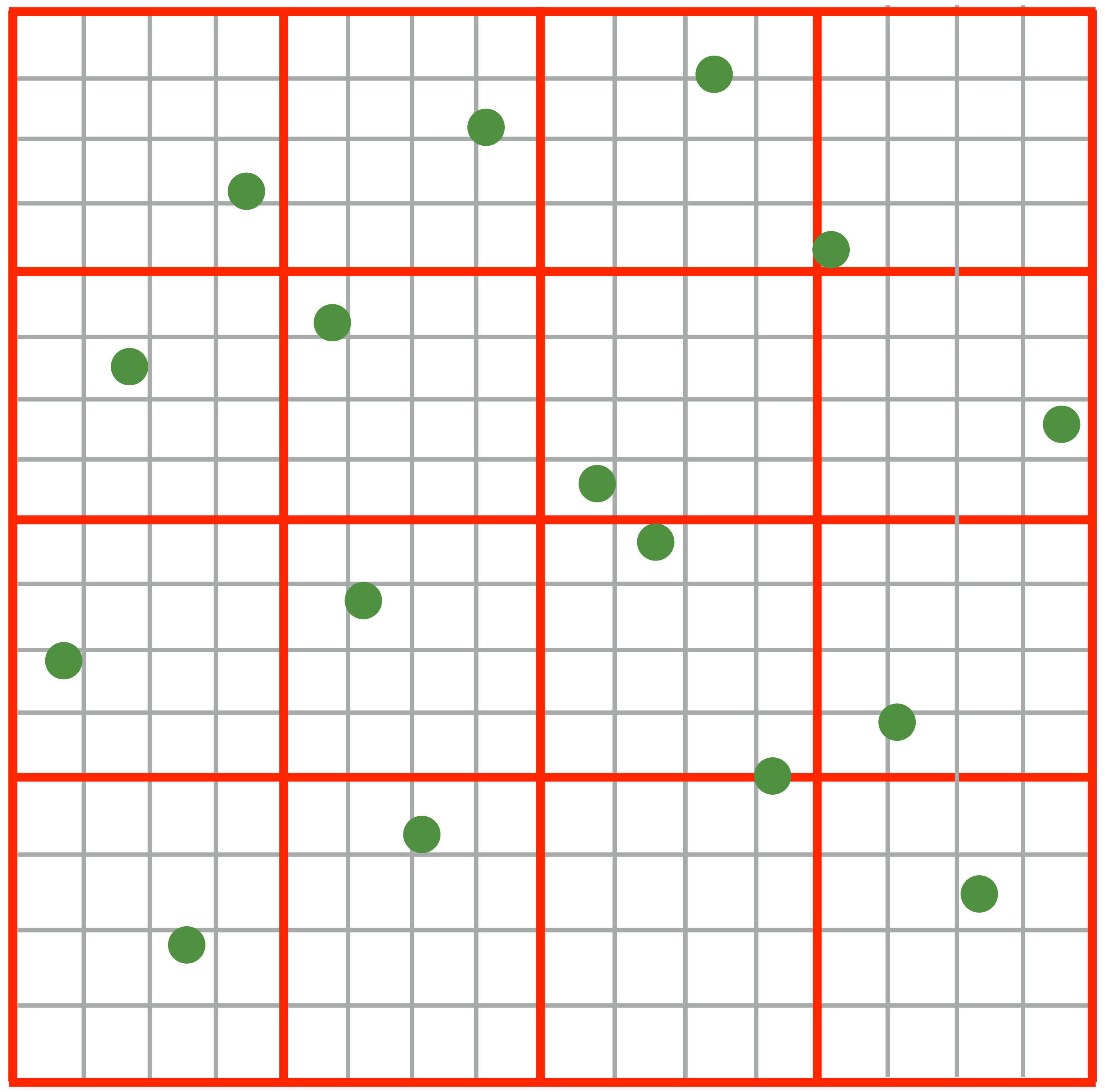


Non-Axis Aligned Integrand Spectra

$\mathcal{P}_f(\nu)$

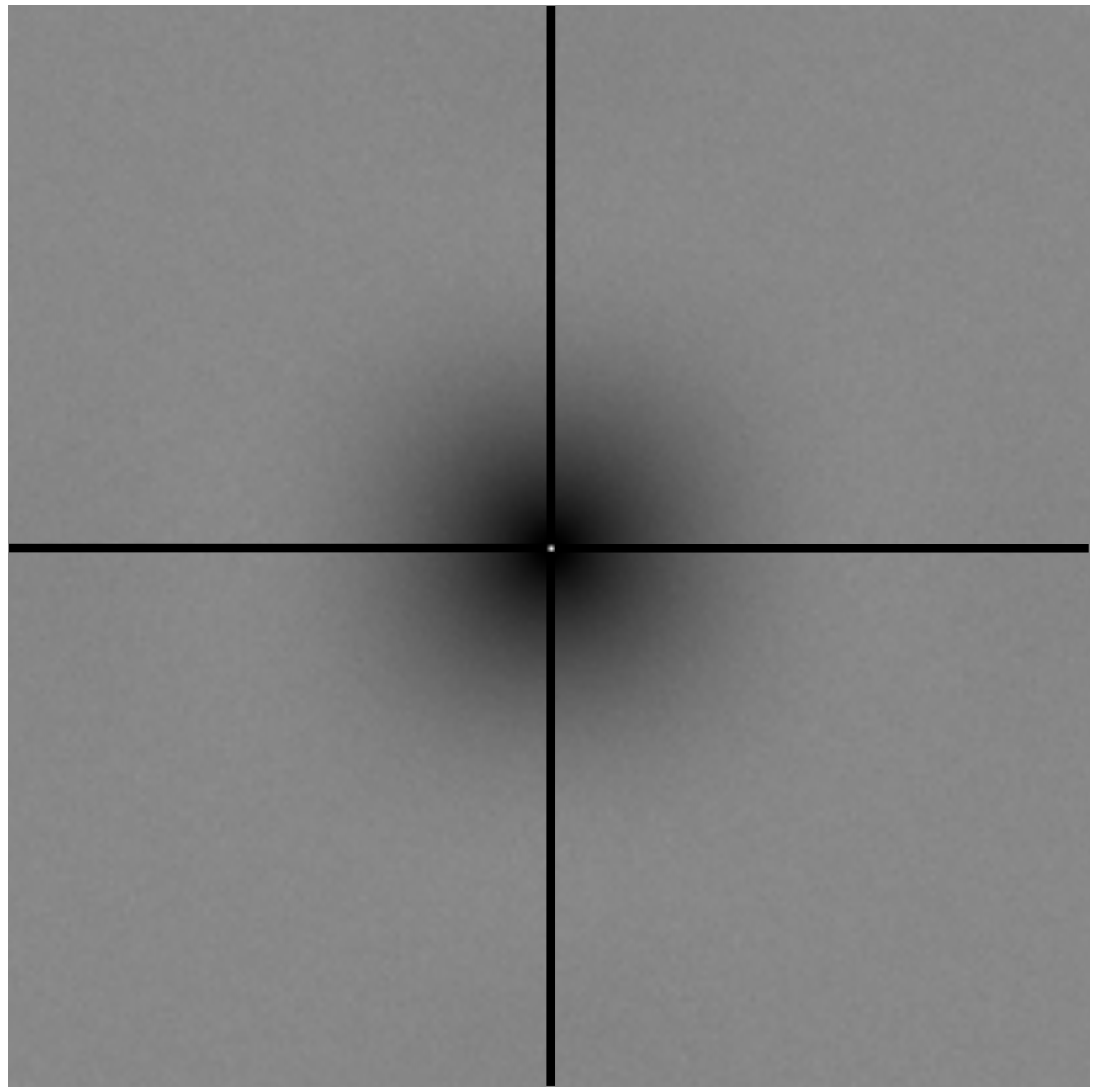
Integrand Spectrum

Non-Axis Aligned Integrand Spectra



Multi-jittered Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$

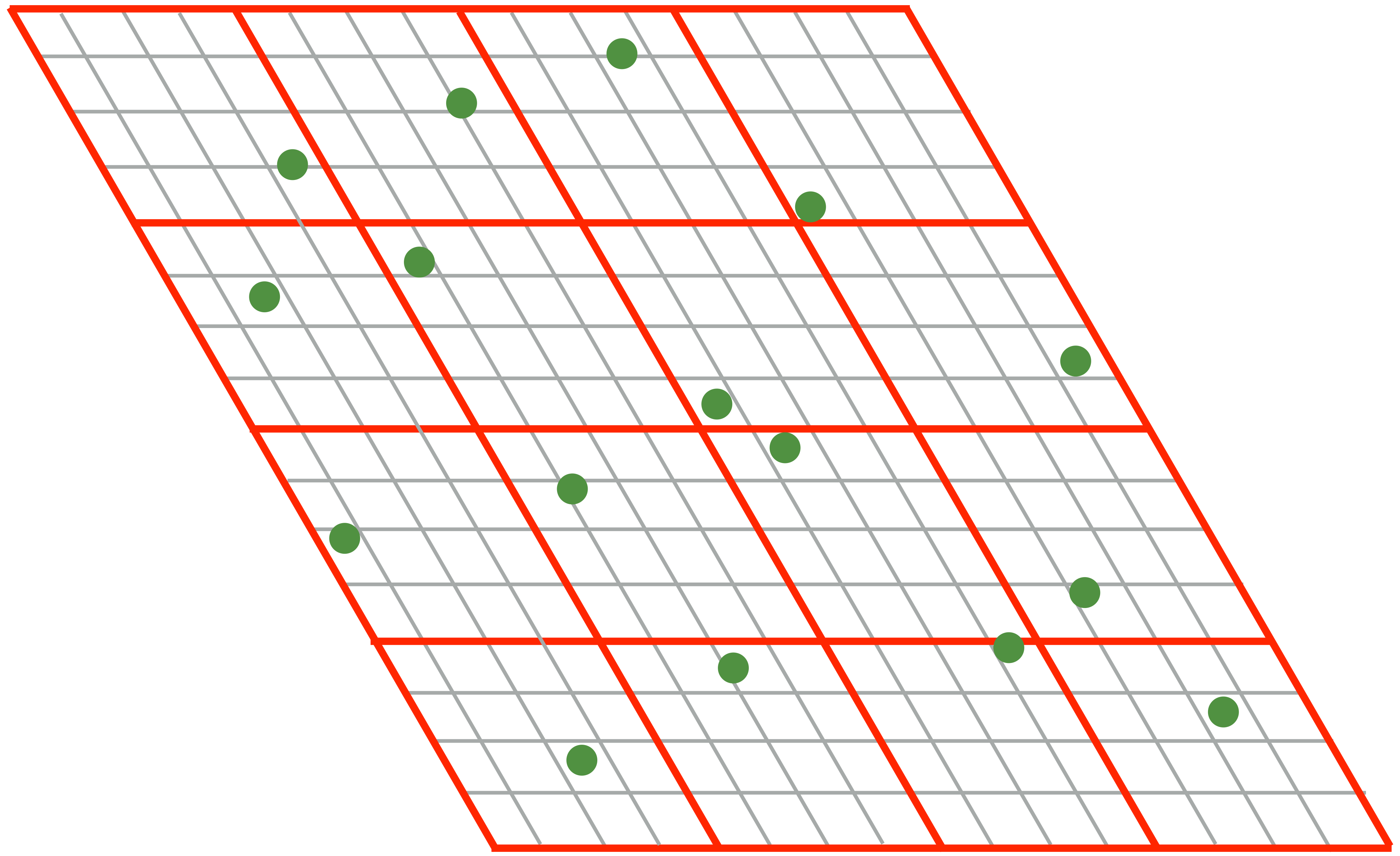


Sampling Spectrum

$\mathcal{P}_f(\nu)$

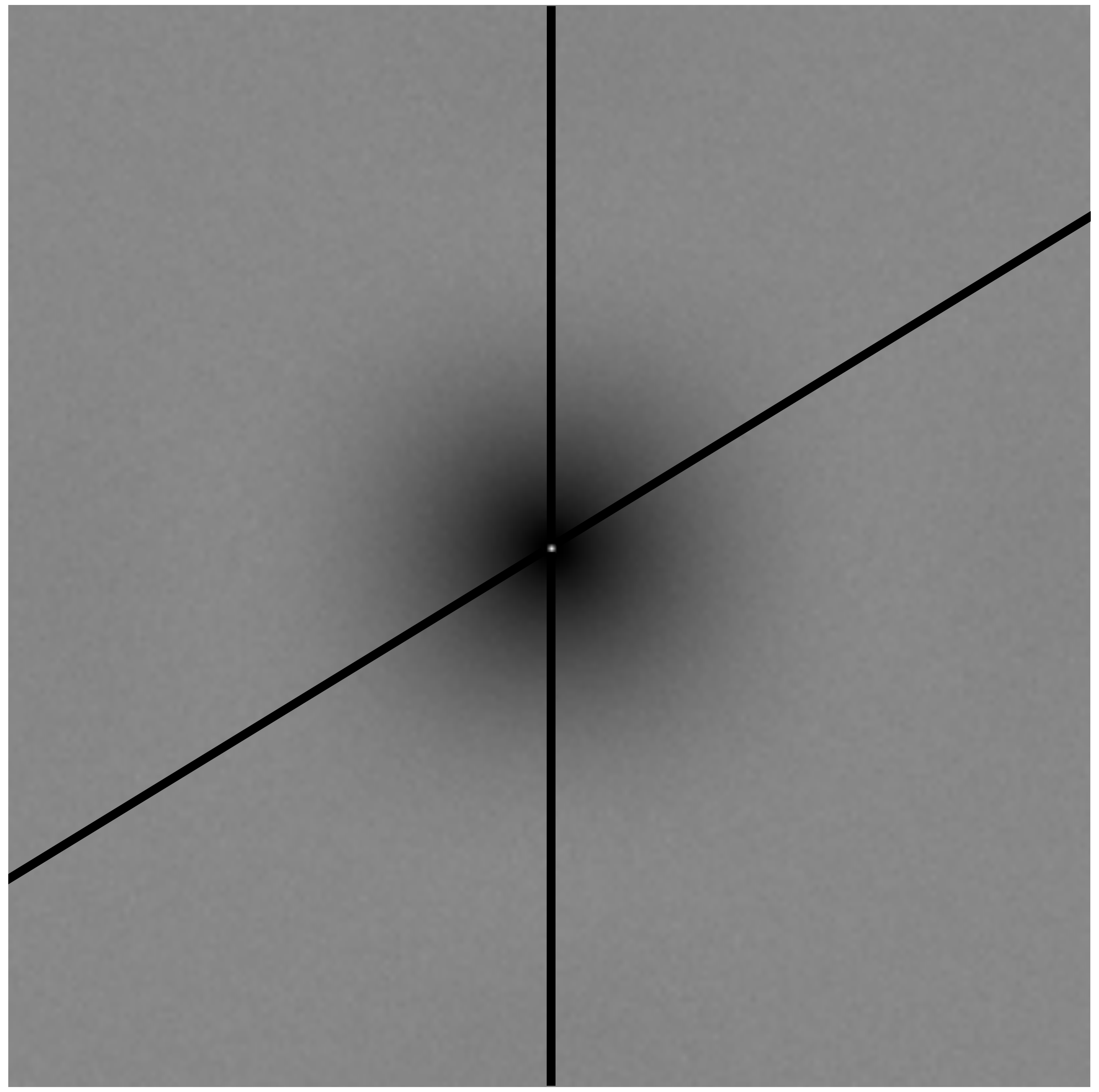
Integrand Spectrum

Shearing Multi-Jittered Samples



Sheared Samples

$\langle \mathcal{P}_{S_N}(\nu) \rangle$



Sheared Spectrum

$\mathcal{P}_f(\nu)$

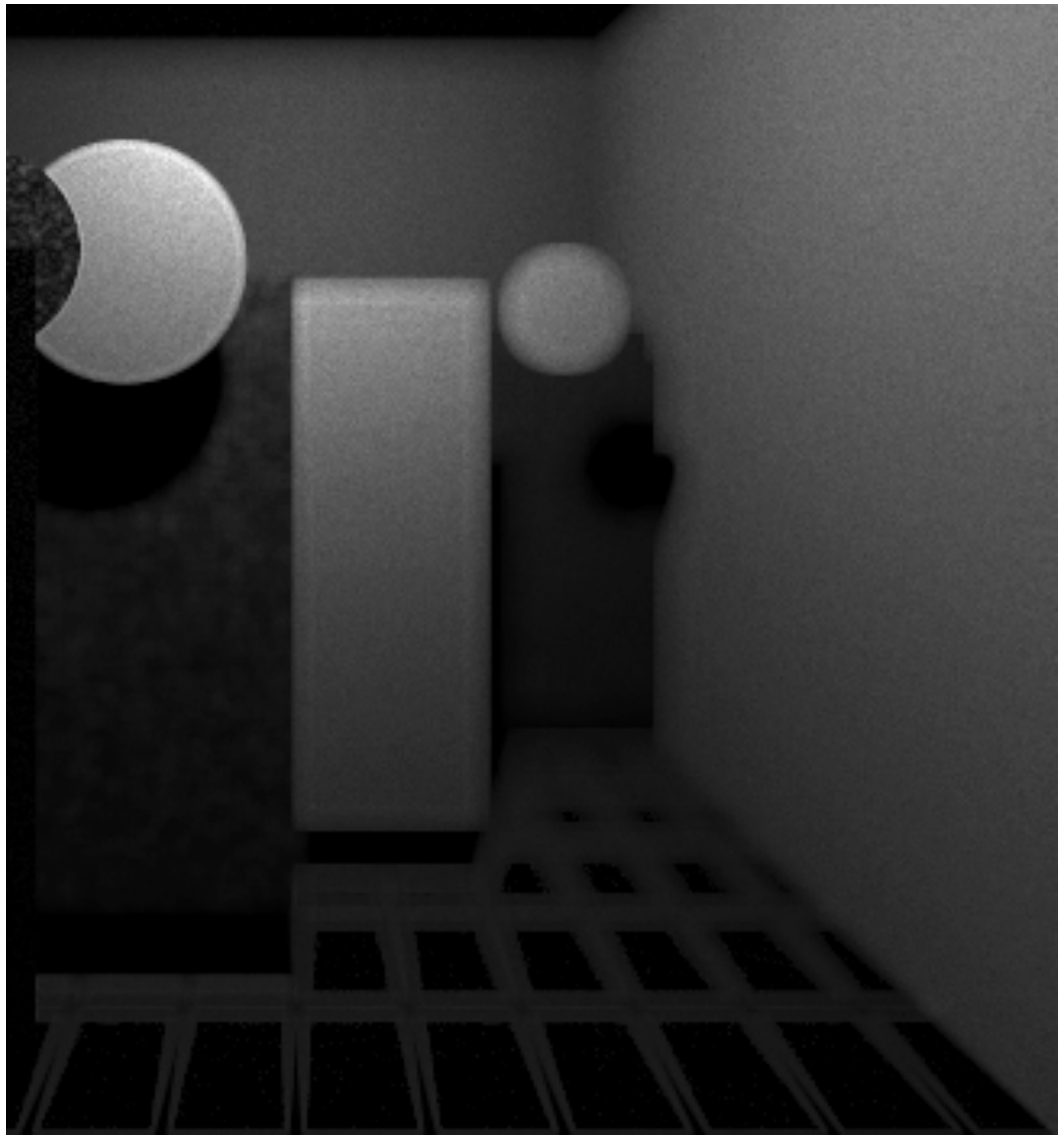
Integrand Spectrum

Singh and Jarosz [SIGGRAPH 2017]

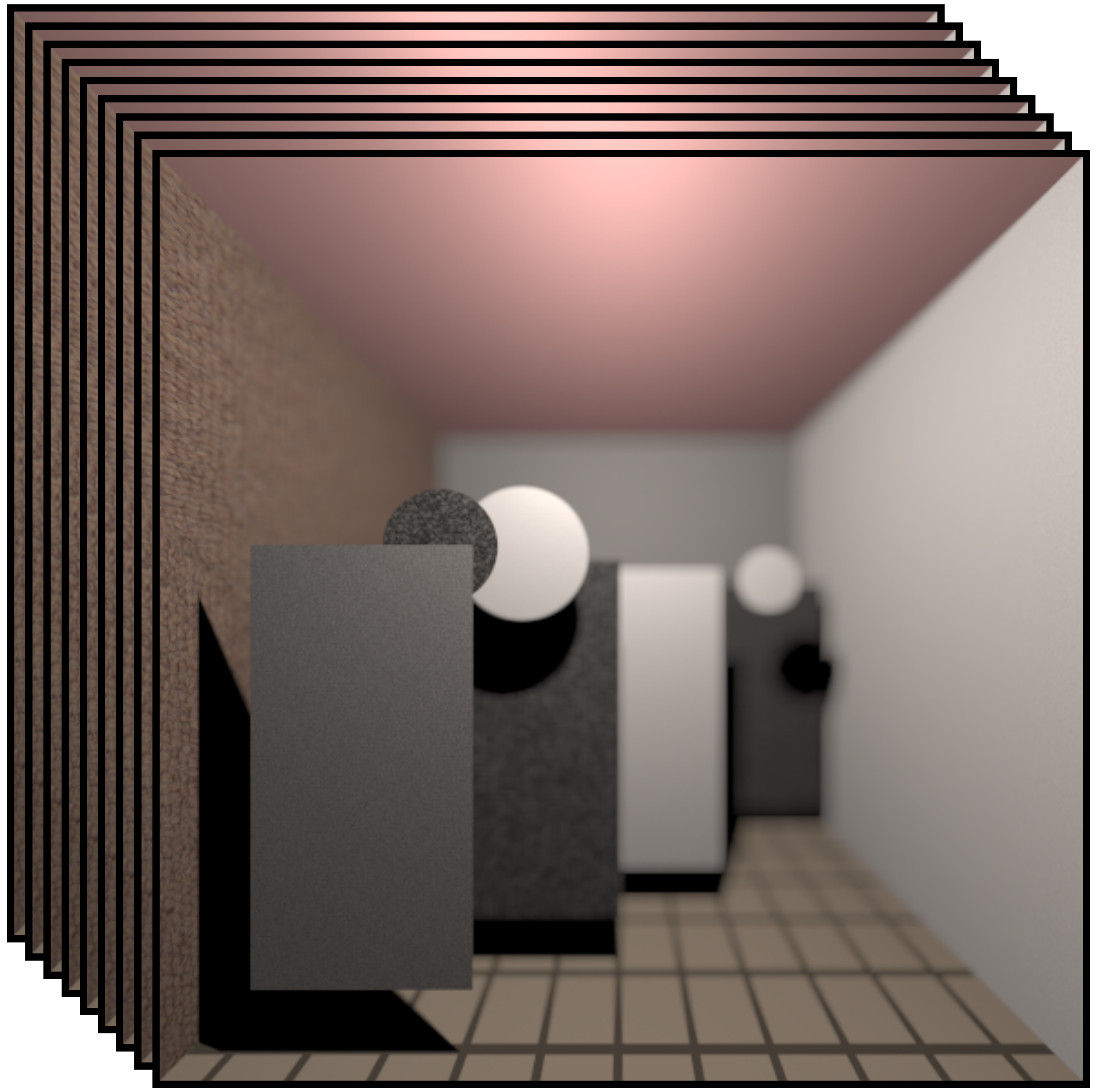
Variance Heatmap

With Original Samples

Uncorrelated Multi-jittered



Multiple images



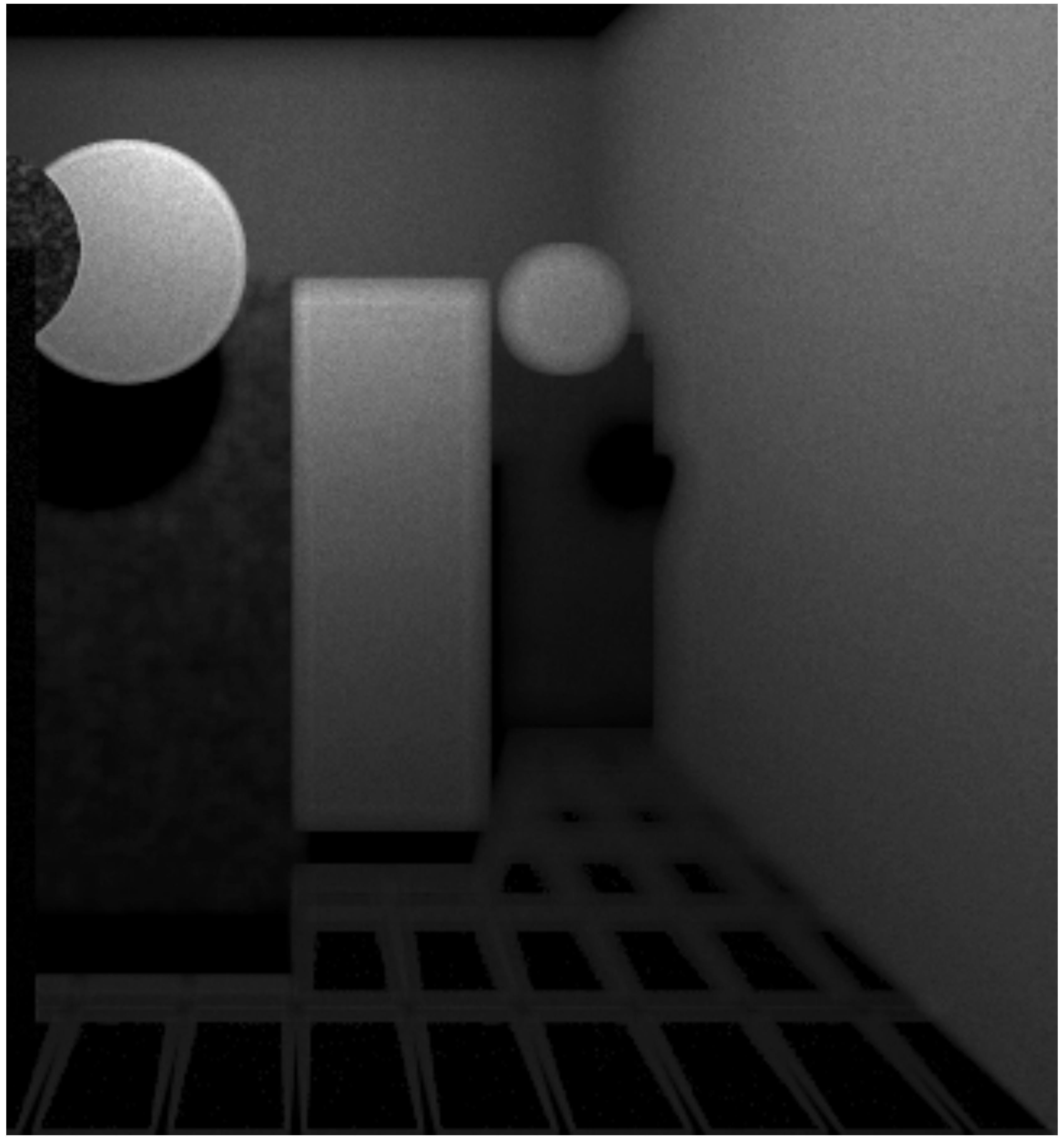
...

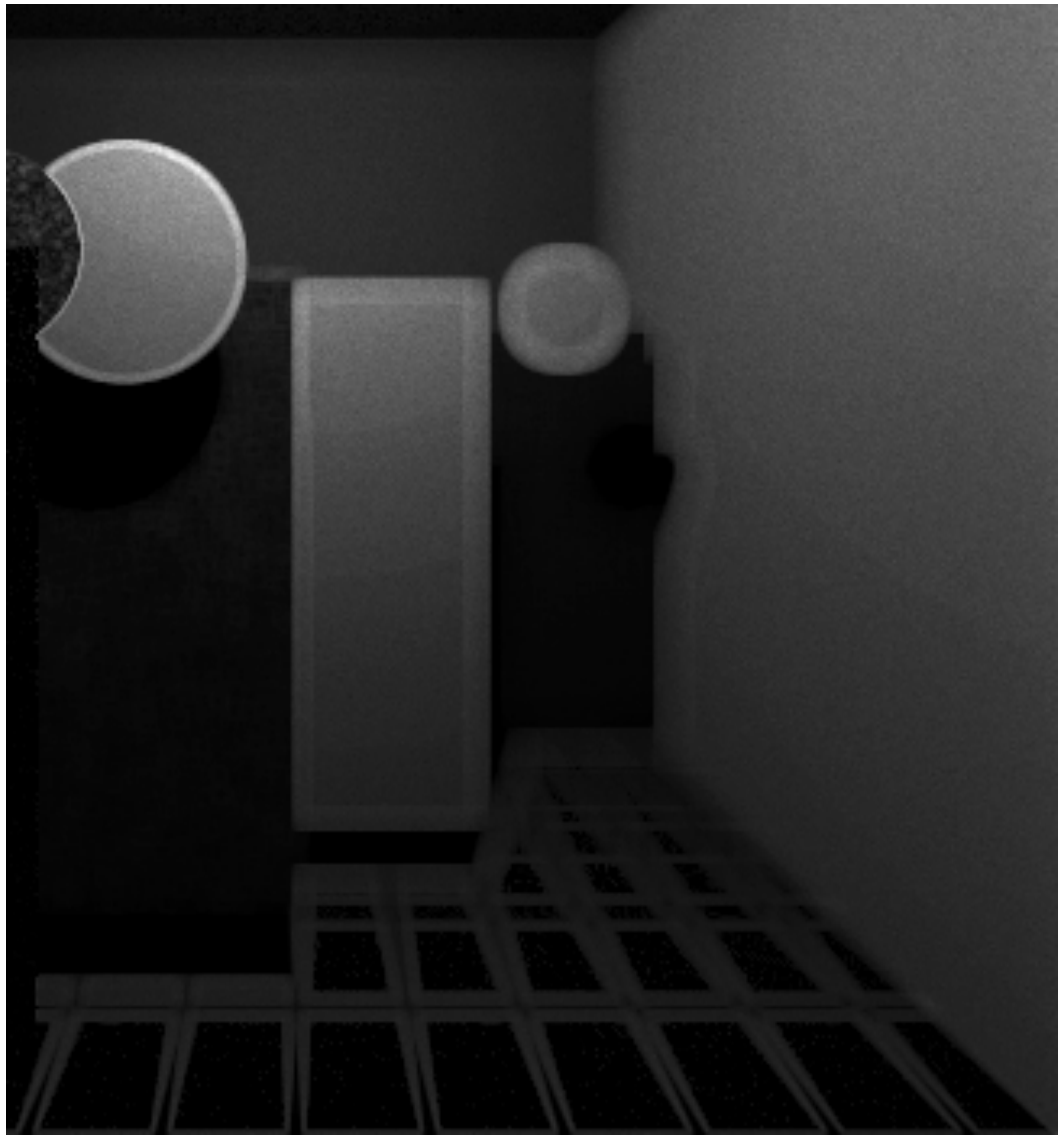
Variance Heatmap

With Original Samples

With Sheared Samples

Uncorrelated Multi-jittered





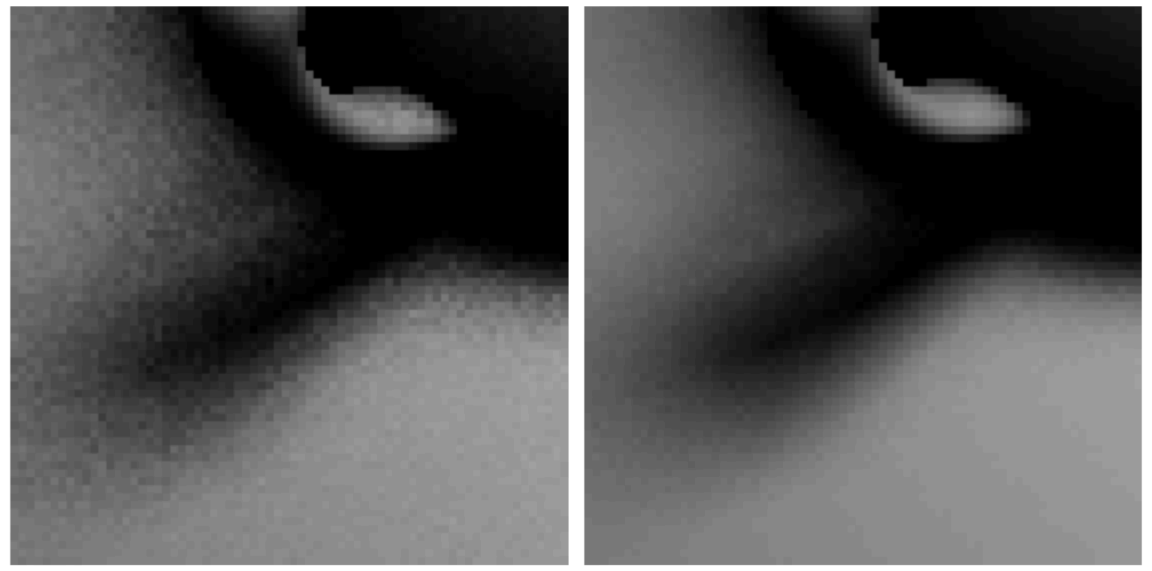
So far...

Blue noise samplers can have better convergence compared to stratified samples

Denser stratification can lead to anisotropic spectra which improves convergence

What properties we desire in a sampler?

Progressivity ✓ (Ahmed et al. [2017], Christensen et al. [2018])



non-progressive progressive

High speed (millions of samples per second) ✓

Extension to dimensions beyond 2D ✗
(Spoke dart throwing, Mitchell [2018])

Low-Discrepancy Sampling

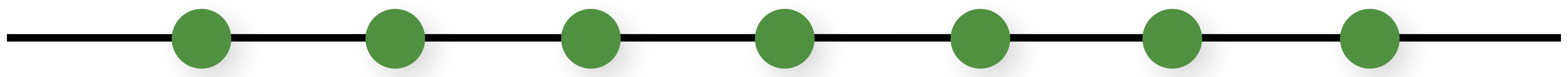
Deterministic sets of points specially crafted to be evenly distributed (have low discrepancy).

Entire field of study called Quasi-Monte Carlo (QMC)

The Van der Corput Sequence

- Radical Inverse Φ_b in base 2
- Subsequent points "fall into biggest holes"

k	Base 2	Φ_b
1	1	.1 = 1/2
2	10	.01 = 1/4
3	11	.11 = 3/4
4	100	.001 = 1/8
5	101	.101 = 5/8
6	110	.011 = 3/8
7	111	.111 = 7/8
...		



Halton and Hammersley Points

Halton: Radical inverse with different base for each dimension:

$$\vec{x}_k = (\Phi_2(k), \Phi_3(k), \Phi_5(k), \ \ldots \ , \Phi_{p_n}(k))$$

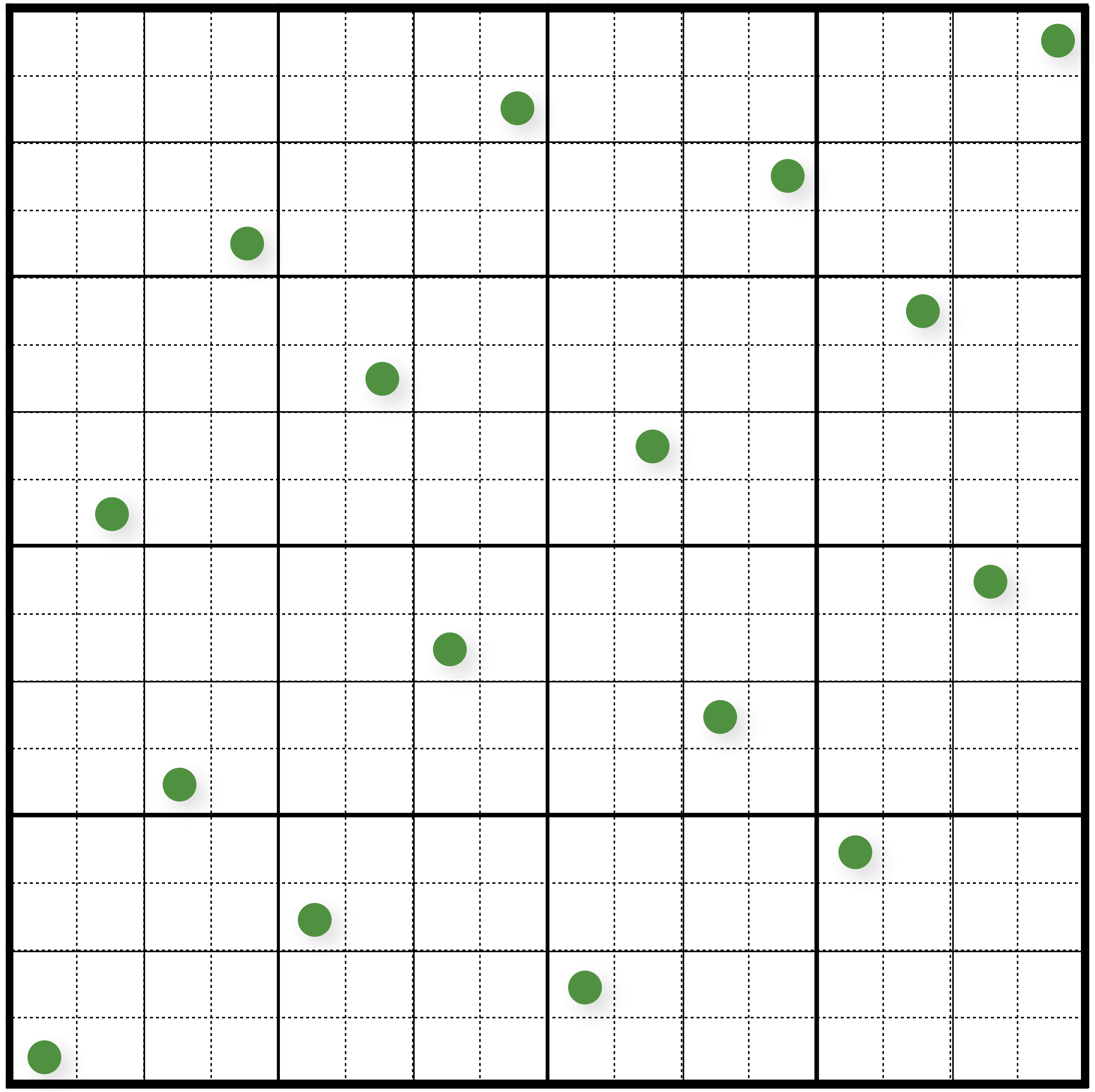
- The bases should all be relatively prime.
- Incremental/progressive generation of samples

Hammersley: Same as Halton, but first dimension is k/N:

$$\vec{x}_k = ({}^{k}/_{N}, \Phi_2(k), \Phi_3(k), \Phi_5(k), \ \ldots \ , \Phi_{p_n}(k))$$

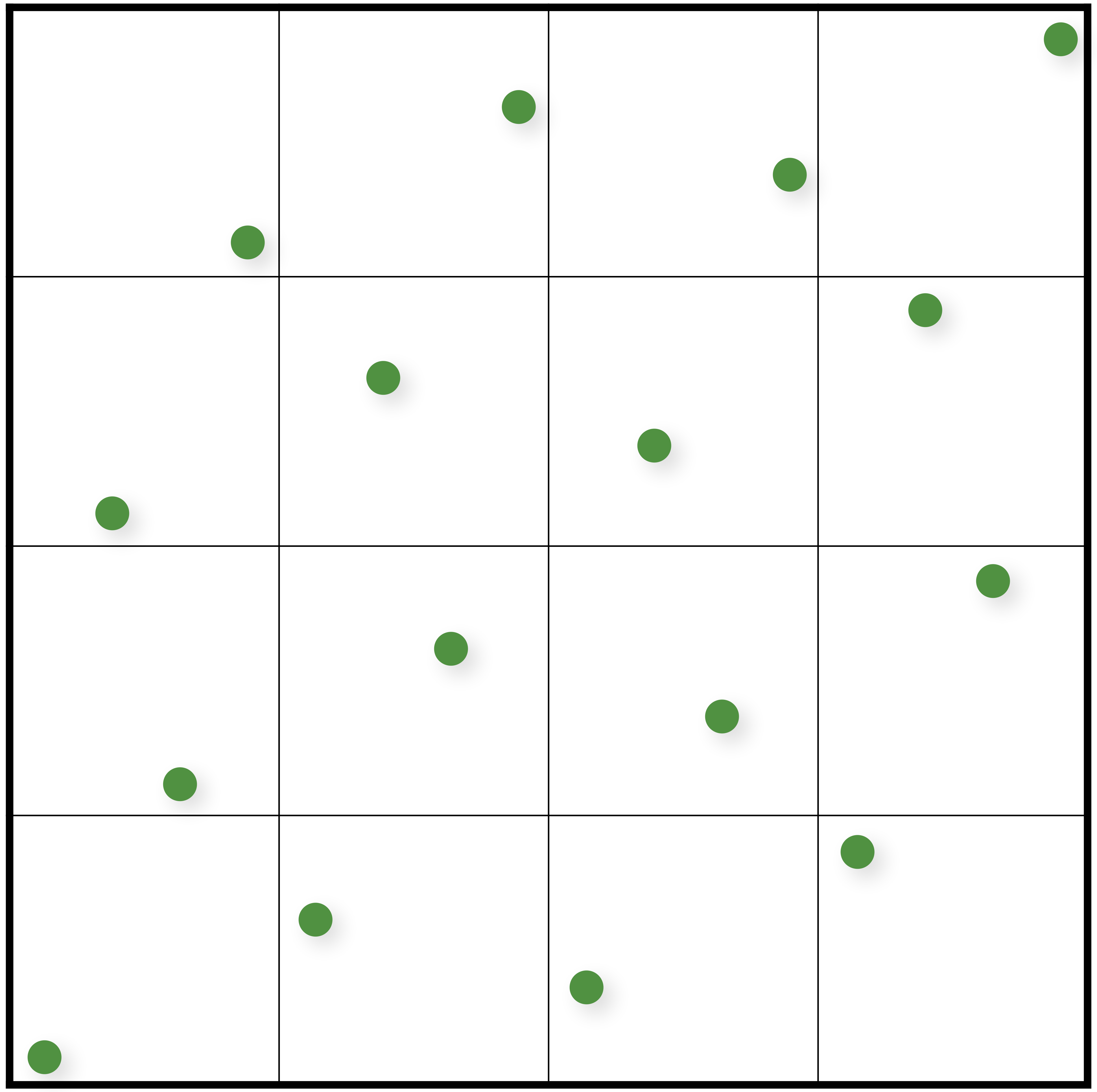
- Not incremental, need to know sample count, N, in advance

The Hammersley Sequence



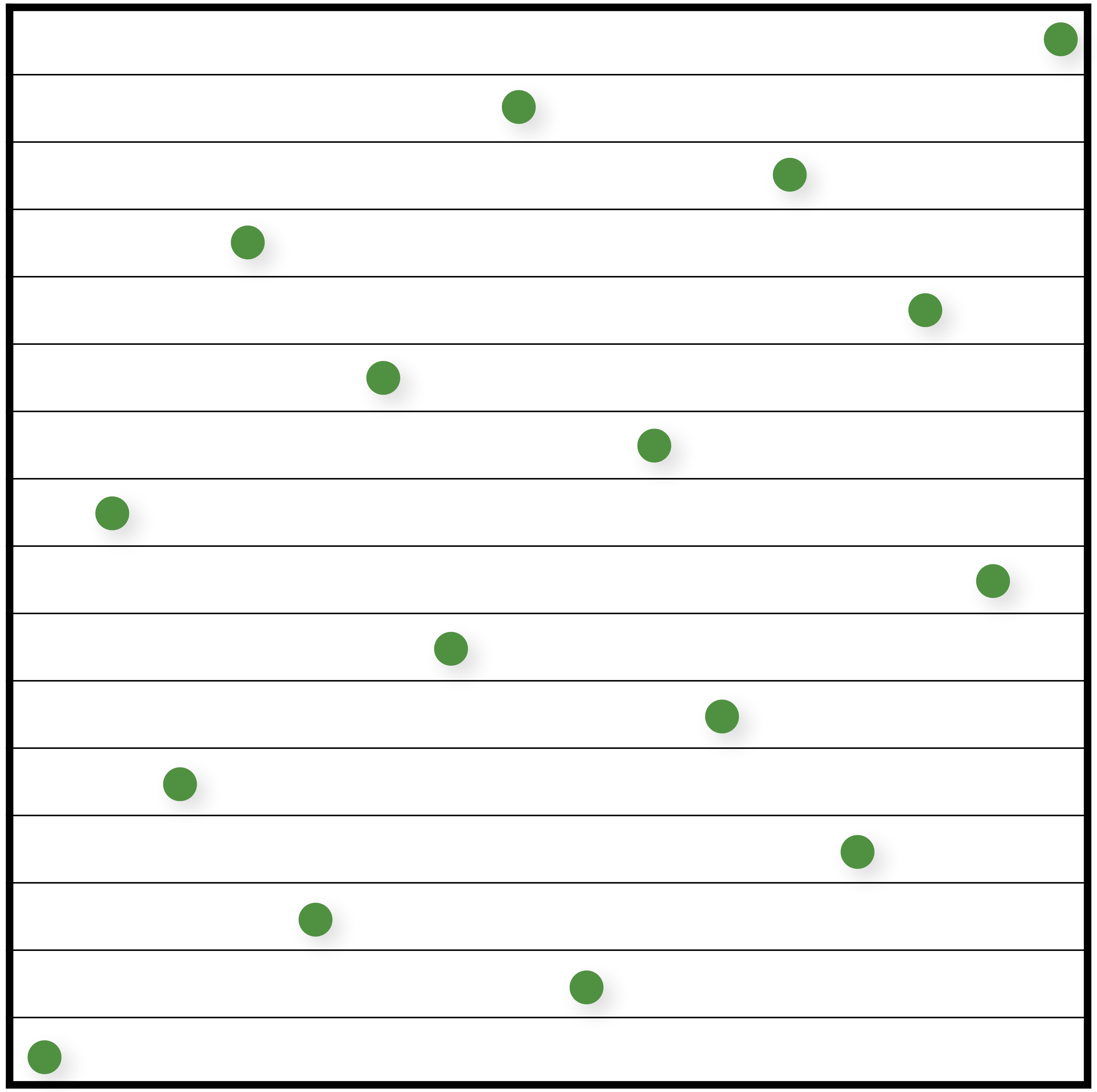
1 sample in each "elementary interval"

The Hammersley Sequence



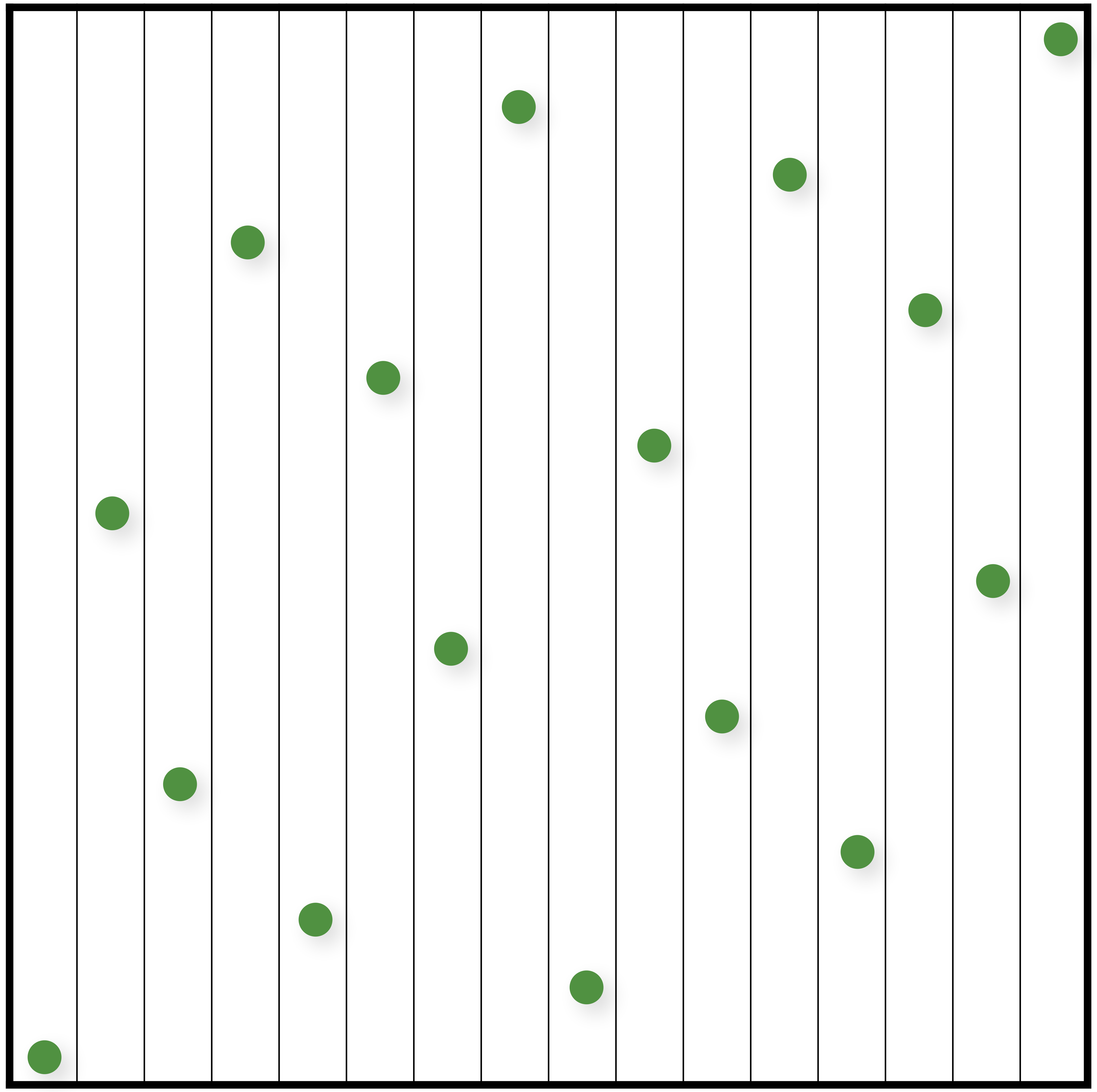
1 sample in each “elementary interval”

The Hammersley Sequence



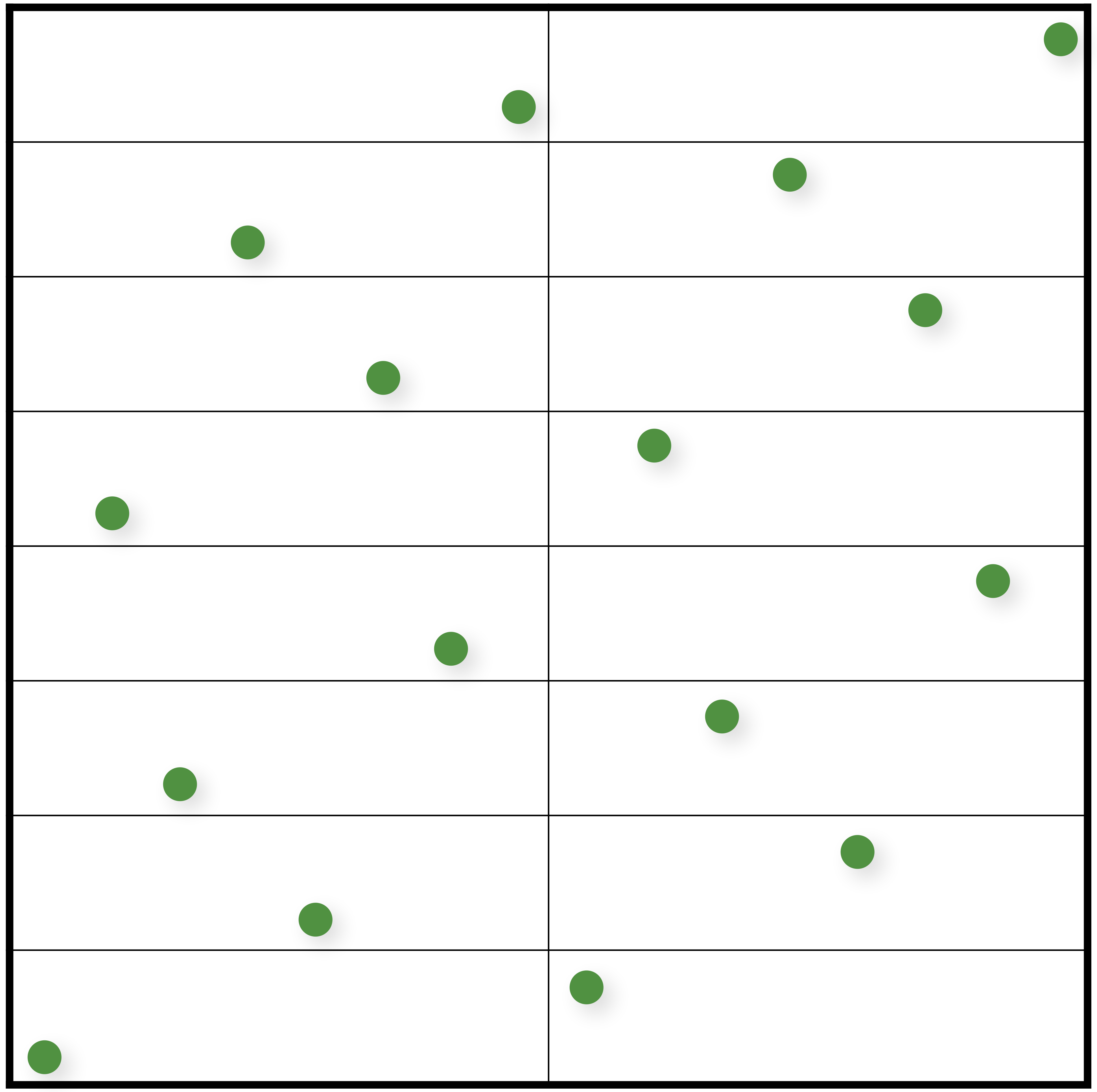
1 sample in each “elementary interval”

The Hammersley Sequence



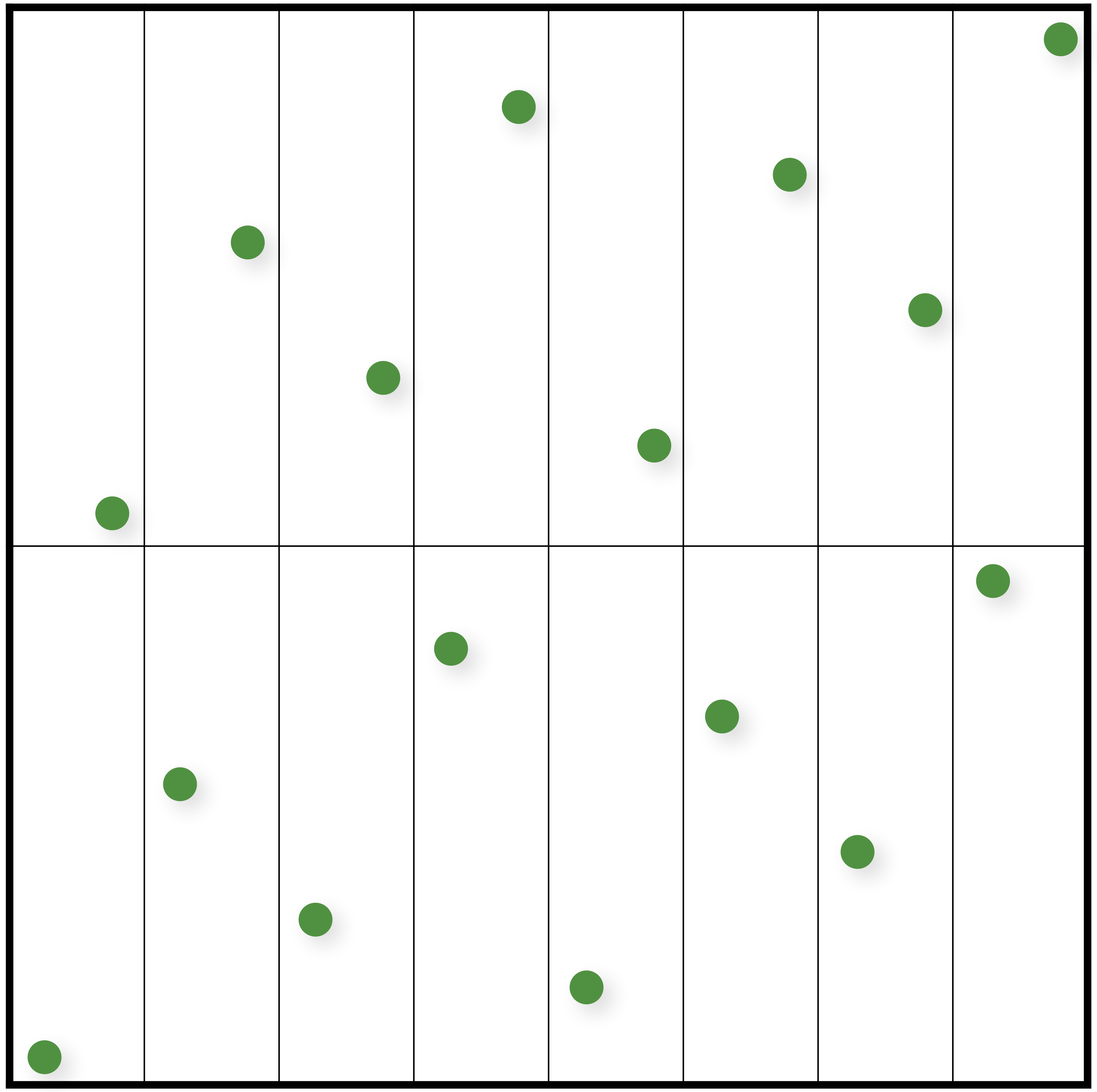
1 sample in each “elementary interval”

The Hammersley Sequence



1 sample in each “elementary interval”

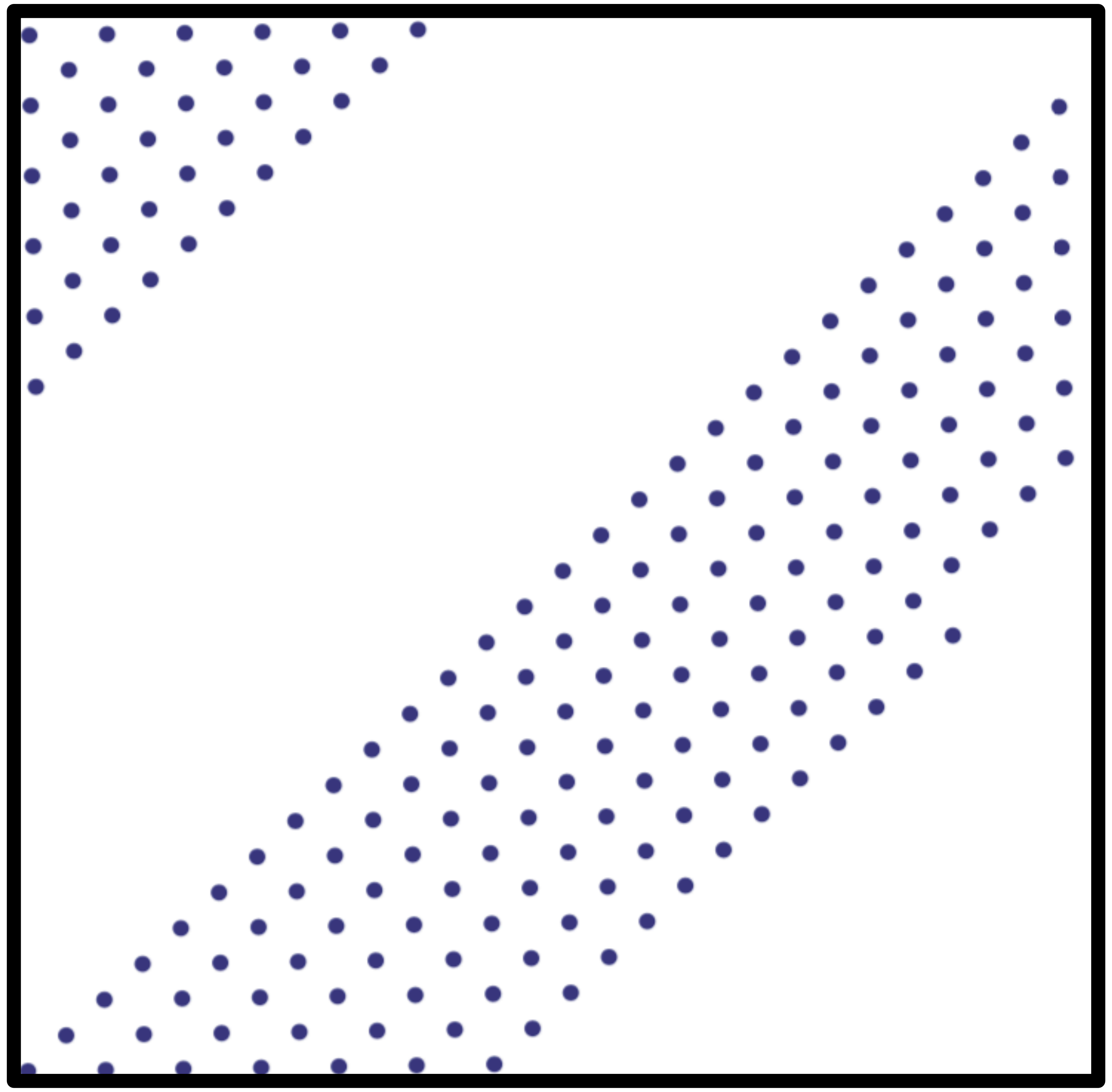
The Hammersley Sequence



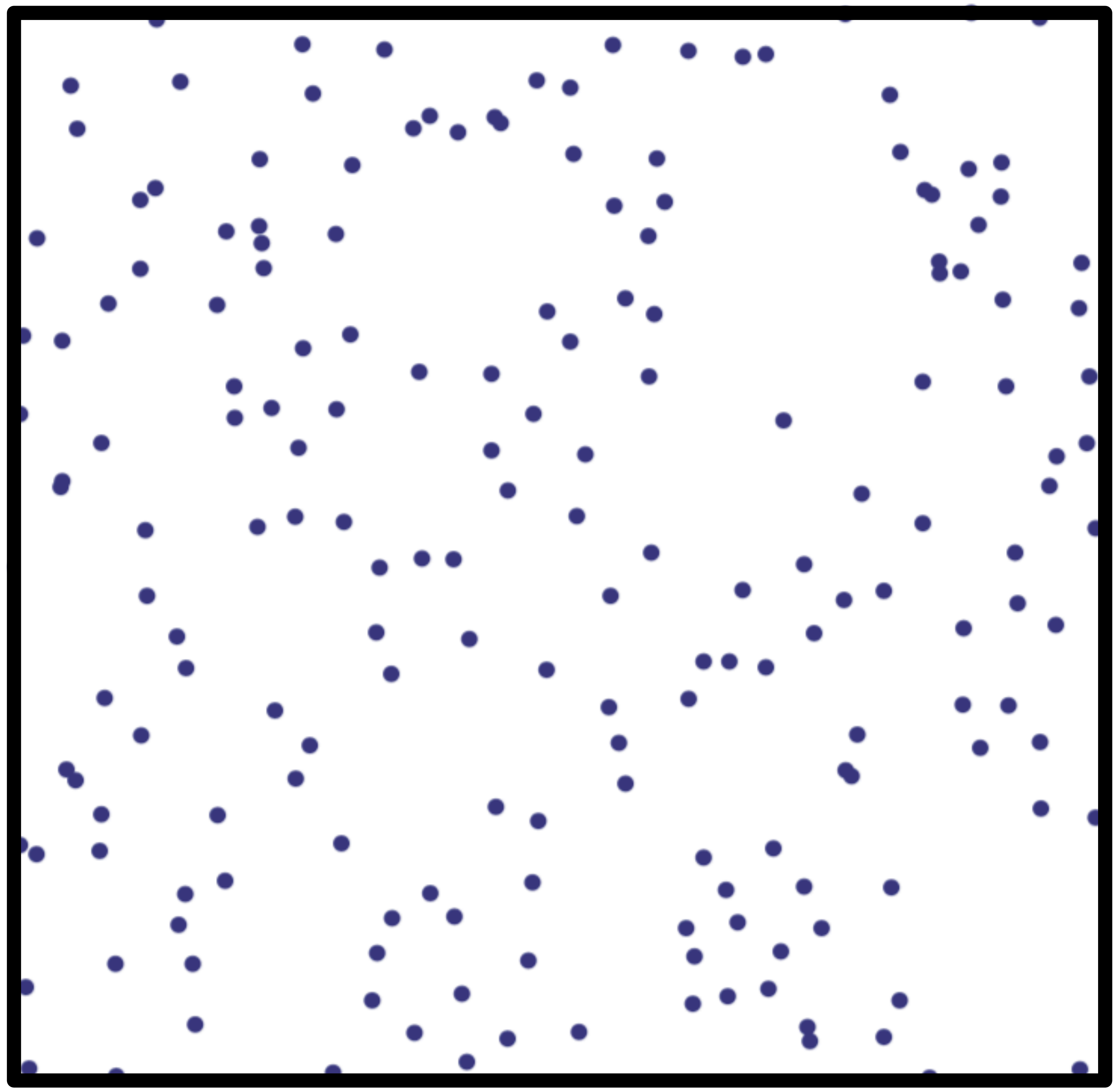
1 sample in each “elementary interval”

Why do we need to scramble?

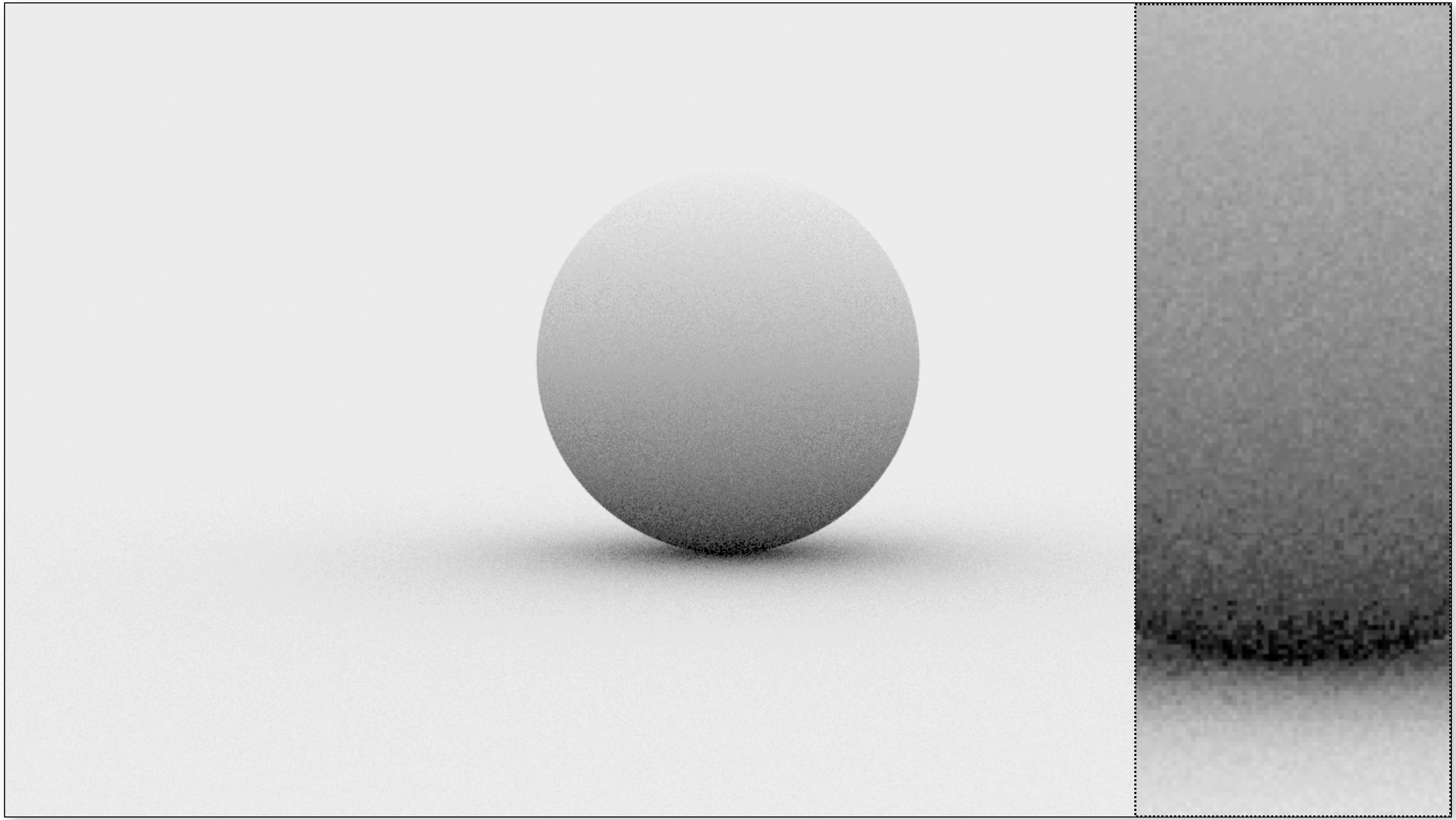
Halton Projection (29, 31)



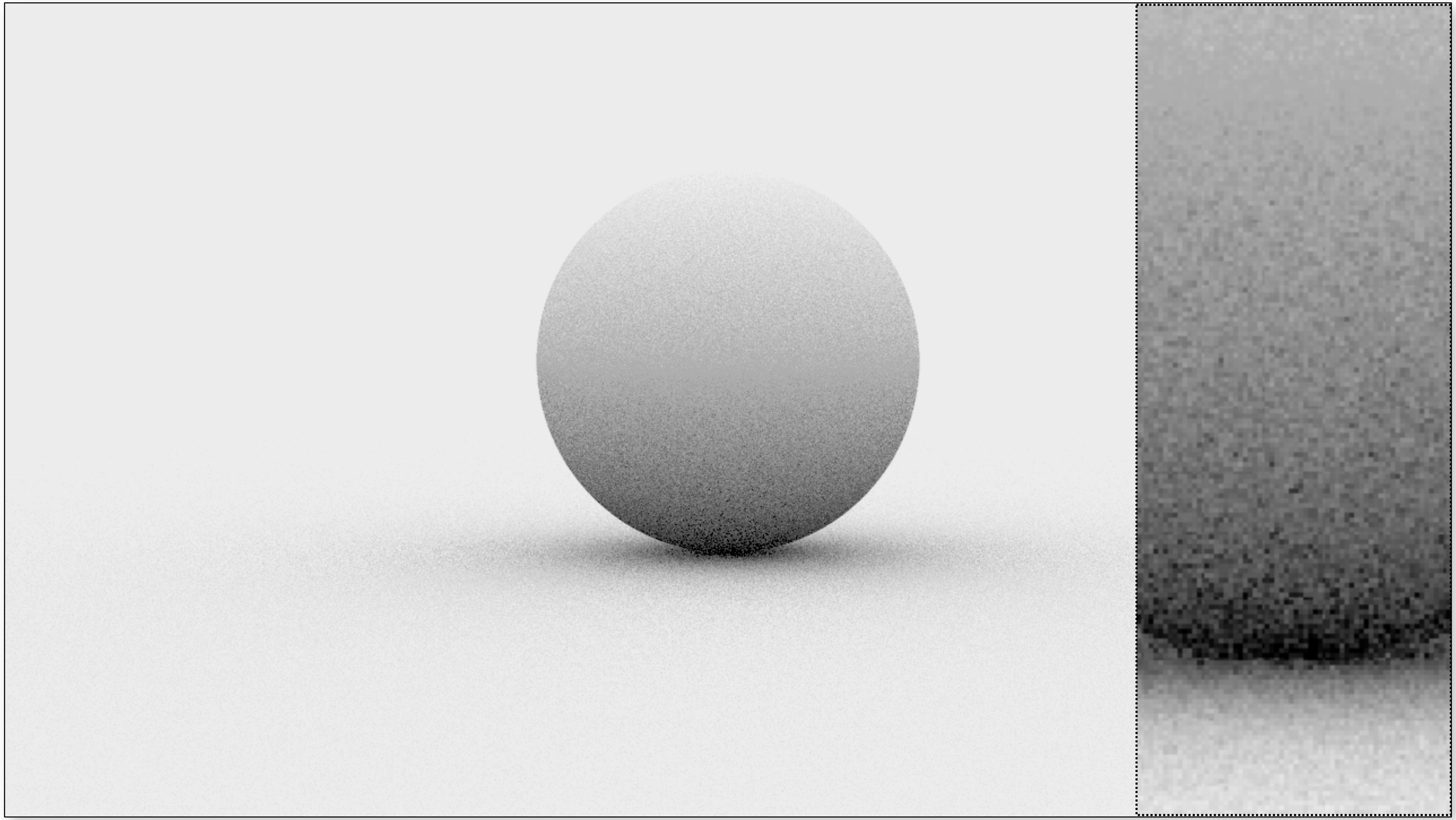
Scrambled Halton Projection (29, 31)



Scrambled Low-Discrepancy Sampling



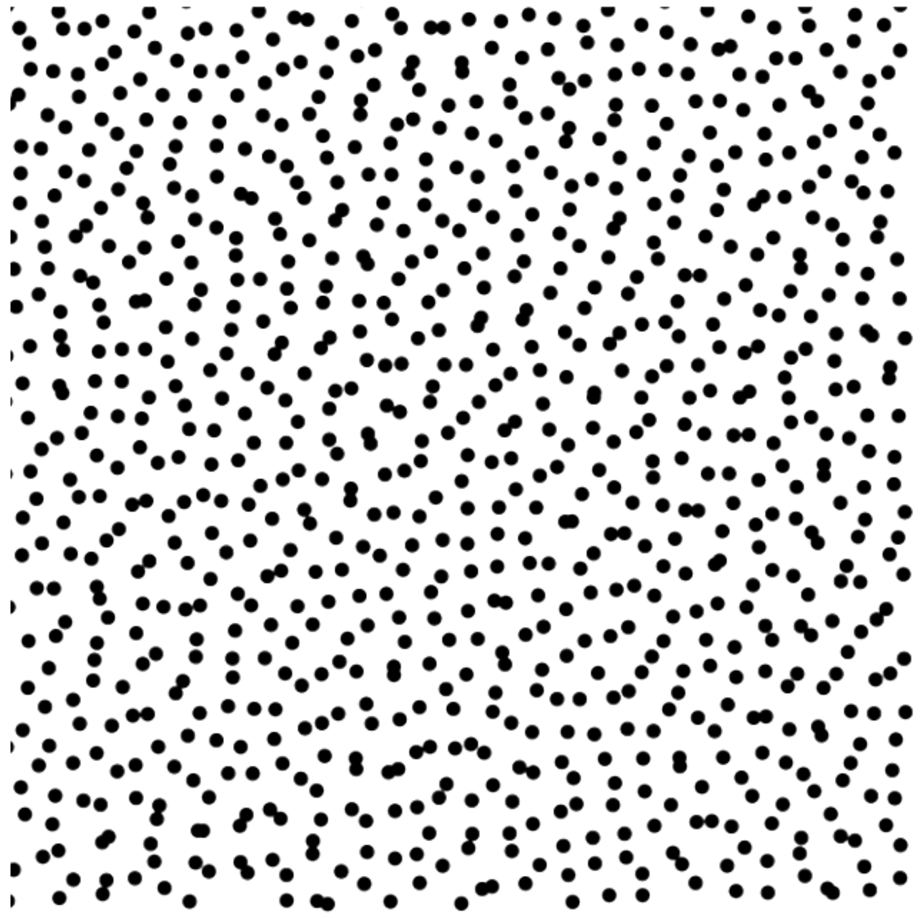
Monte Carlo (16 jittered samples)

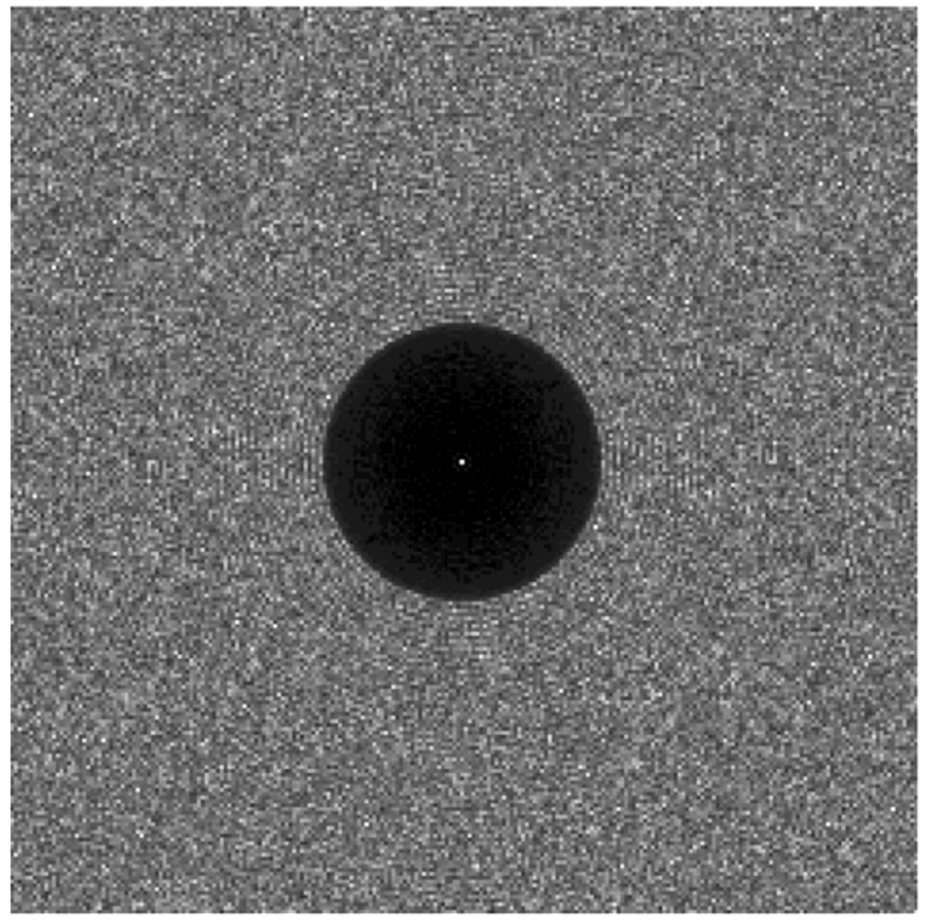


Can we combine blue noise properties with low discrepancy?

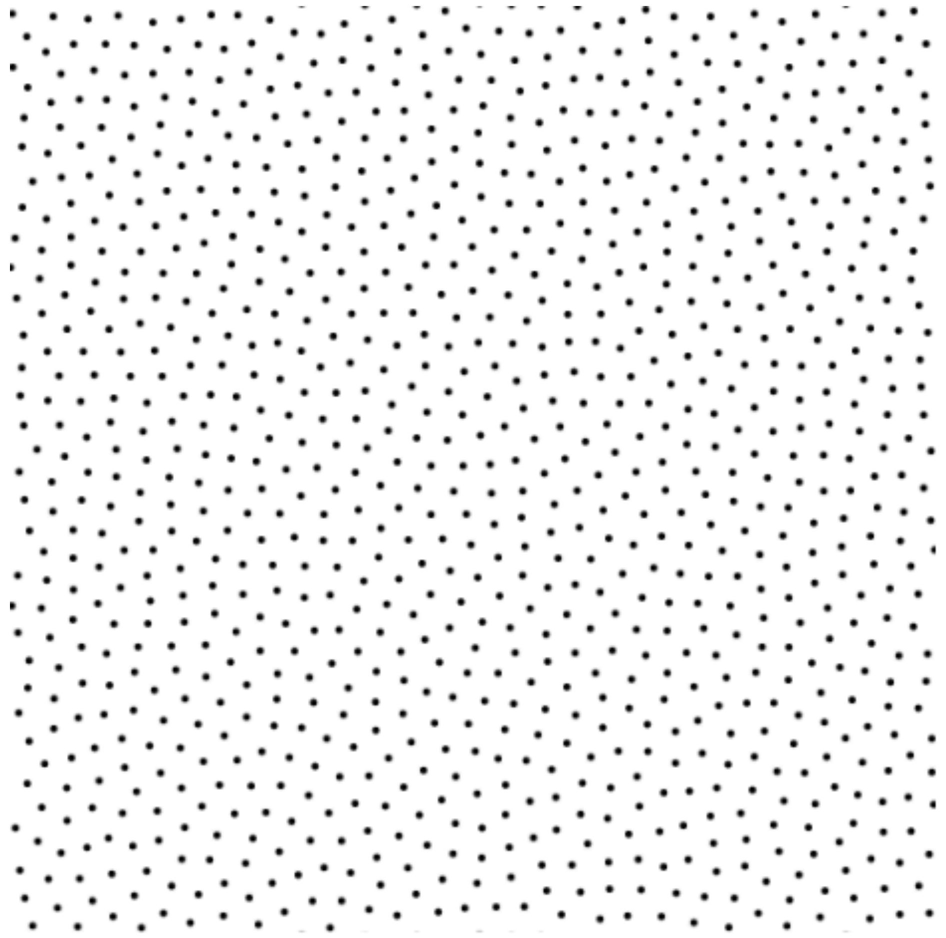
Low-Discrepancy Blue Noise

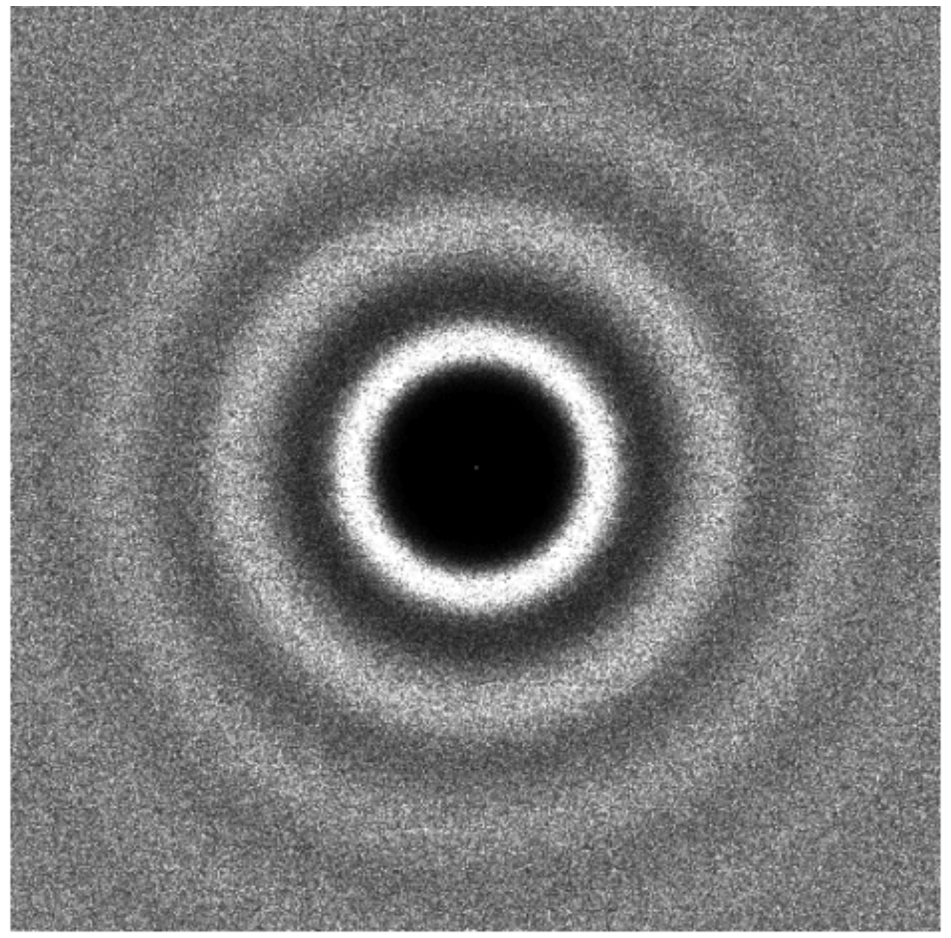
Step spectrum



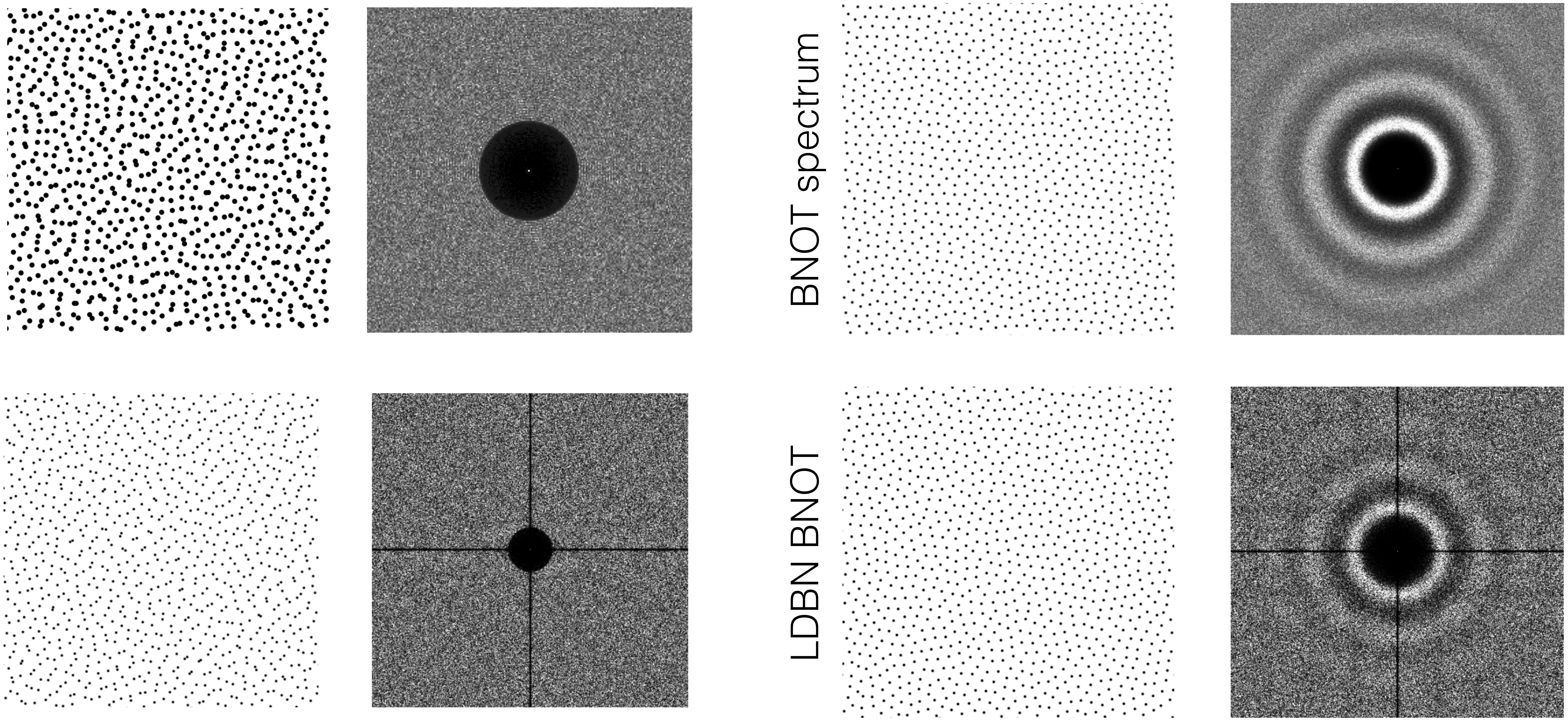


BNOT spectrum



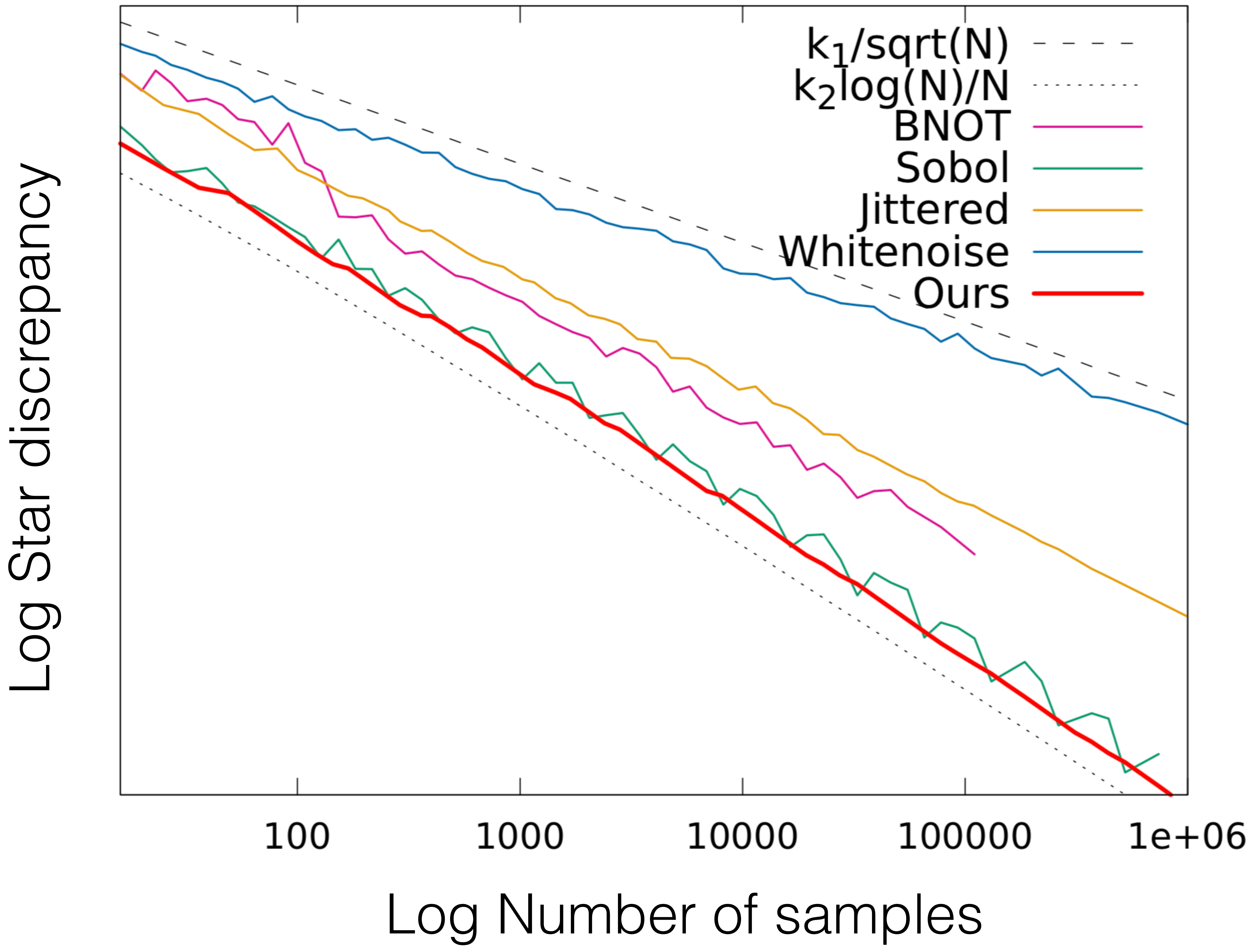


Low-Discrepancy Blue Noise



Ahmed et al. [2016]

Low-Discrepancy Blue Noise



Ahmed et al. [2016]

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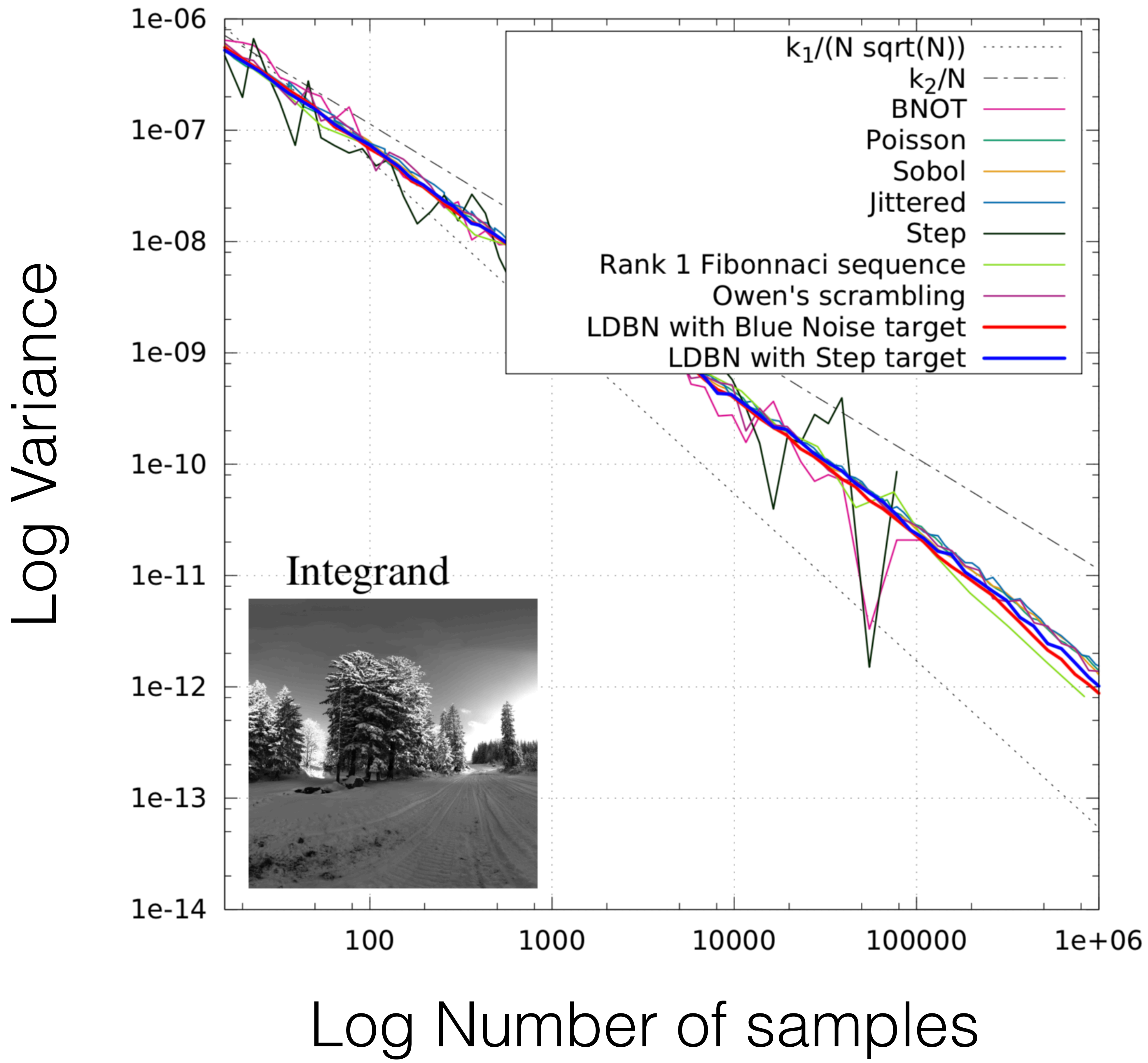
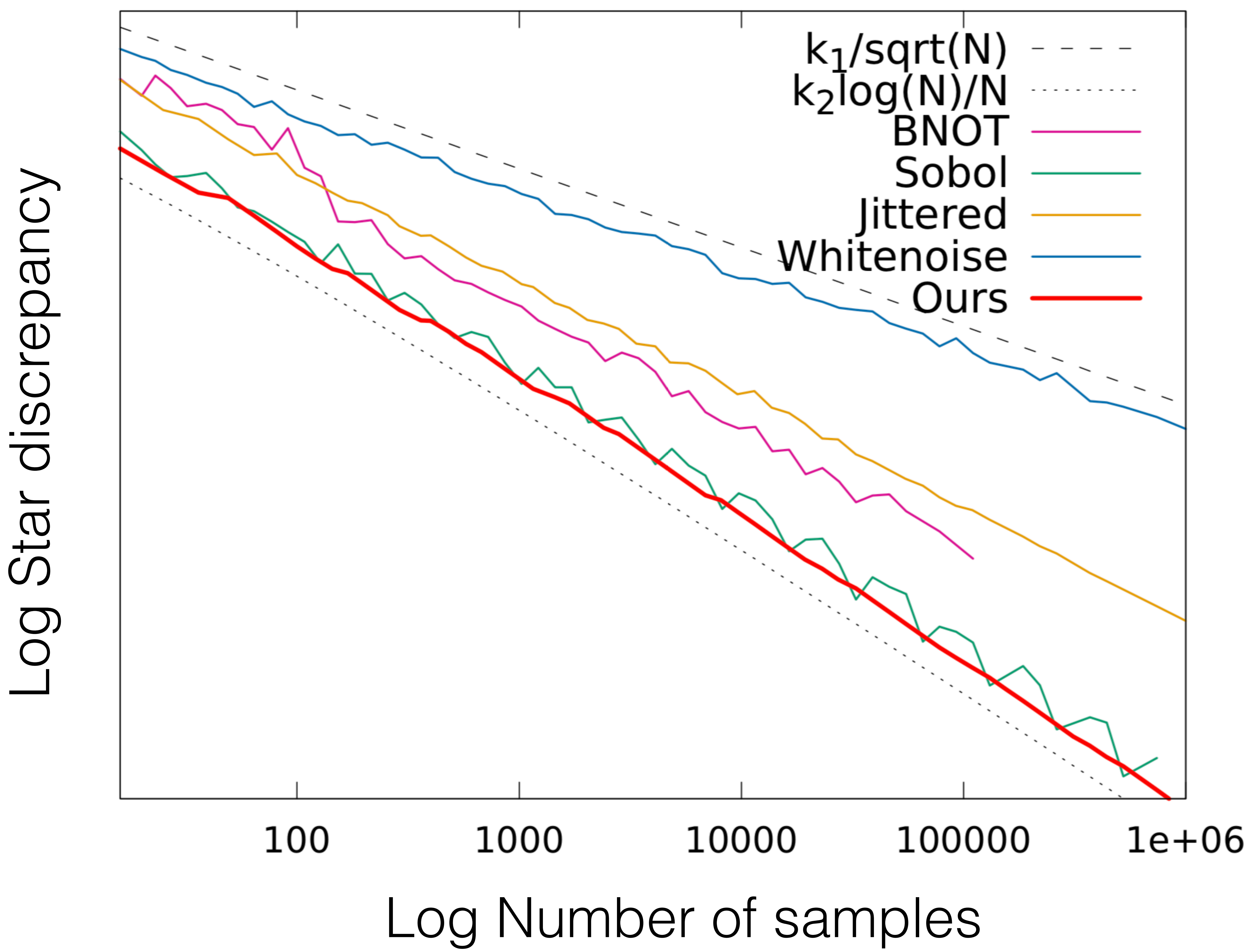
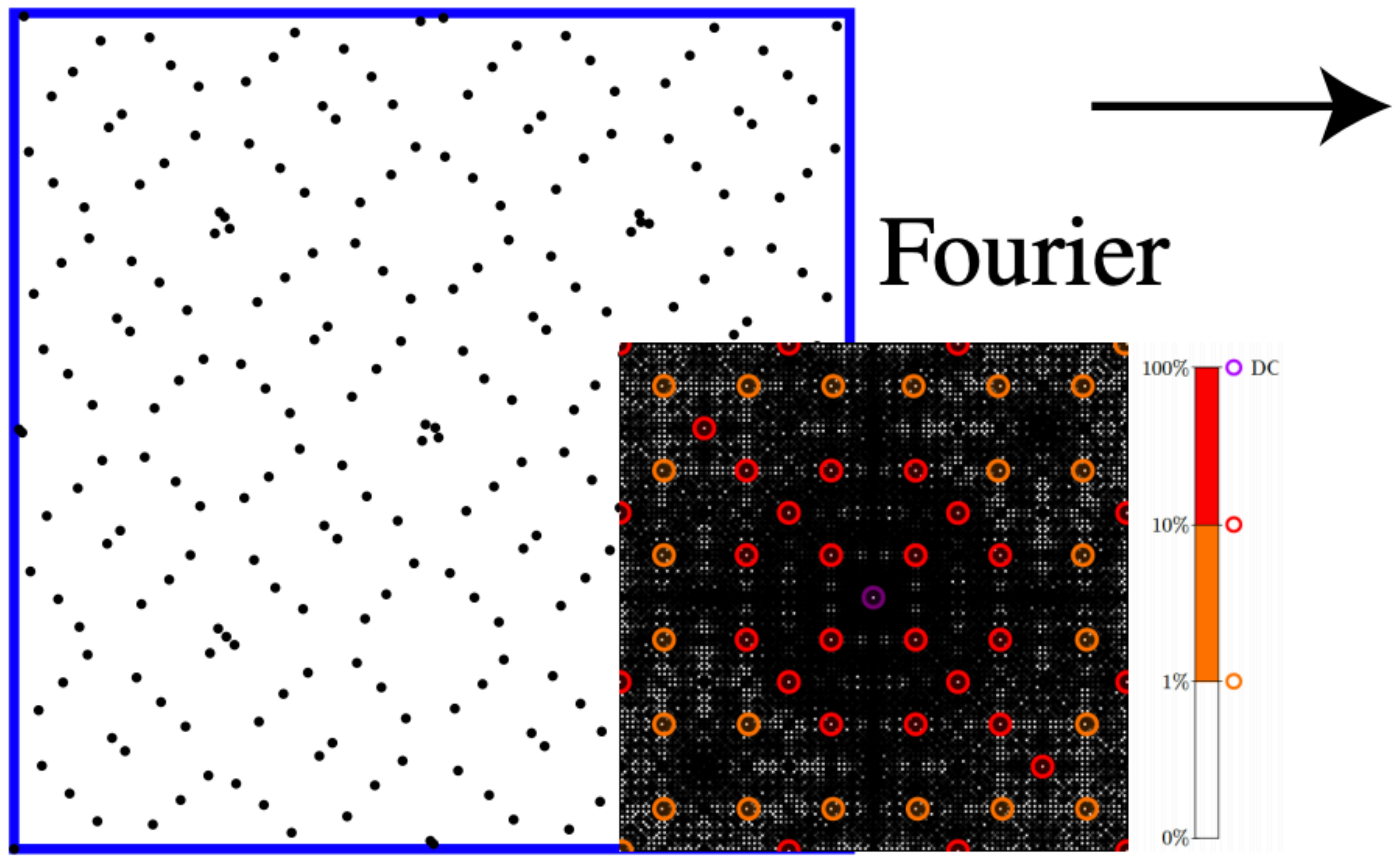




Ahmed et al. [2016]

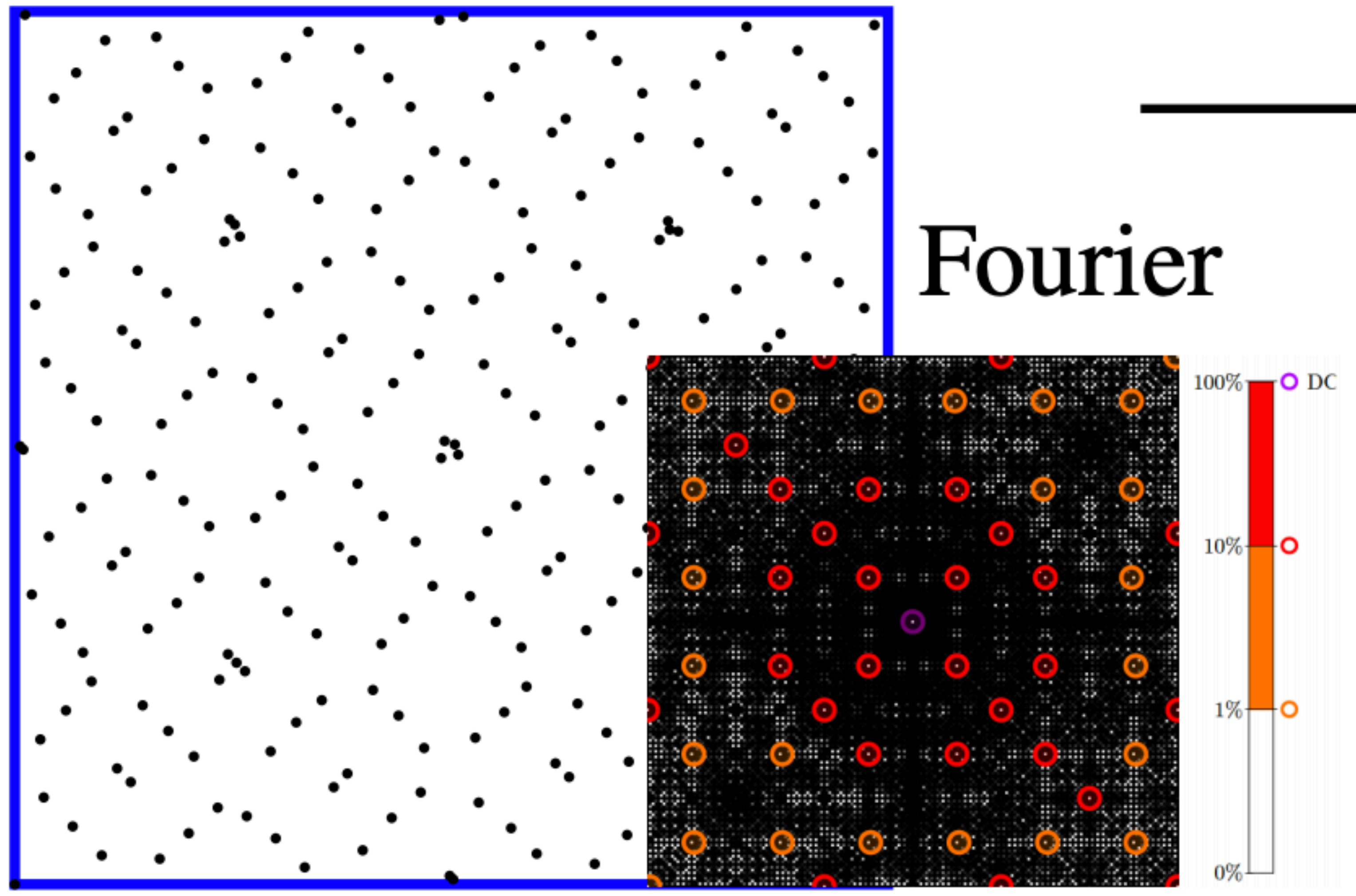
Low-Discrepancy Blue Noise 2D-Projections



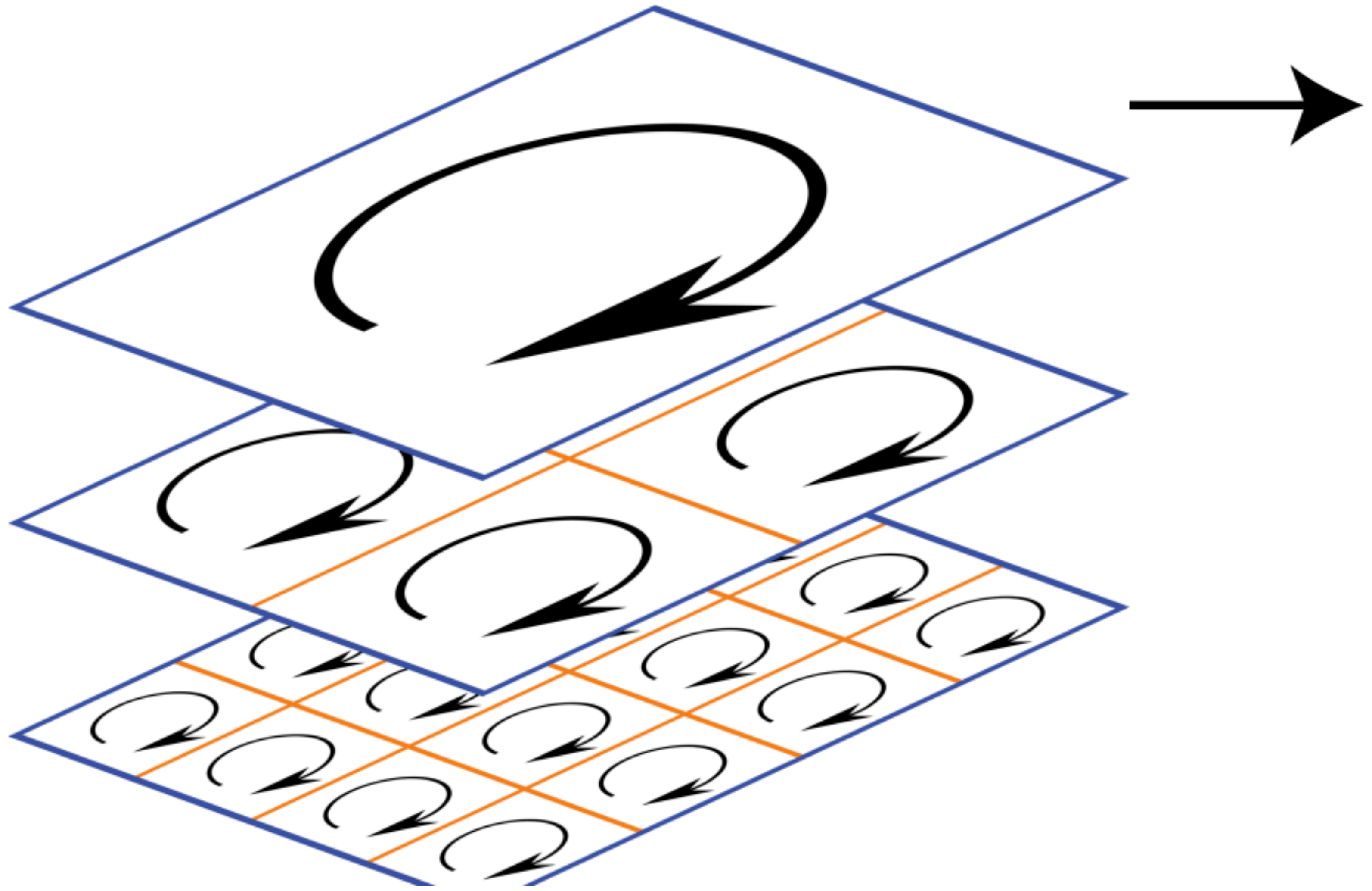
Sobol

Perrier et al. [2018]

Low-Discrepancy Blue Noise 2D-Projections



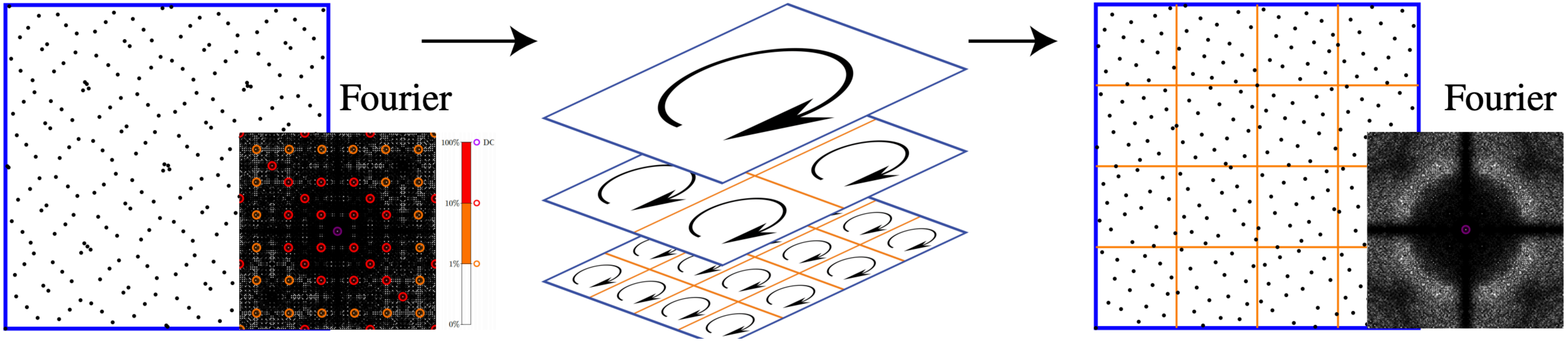
Sobol



Special scrambling

Perrier et al. [2018]

Low-Discrepancy Blue Noise 2D-Projections



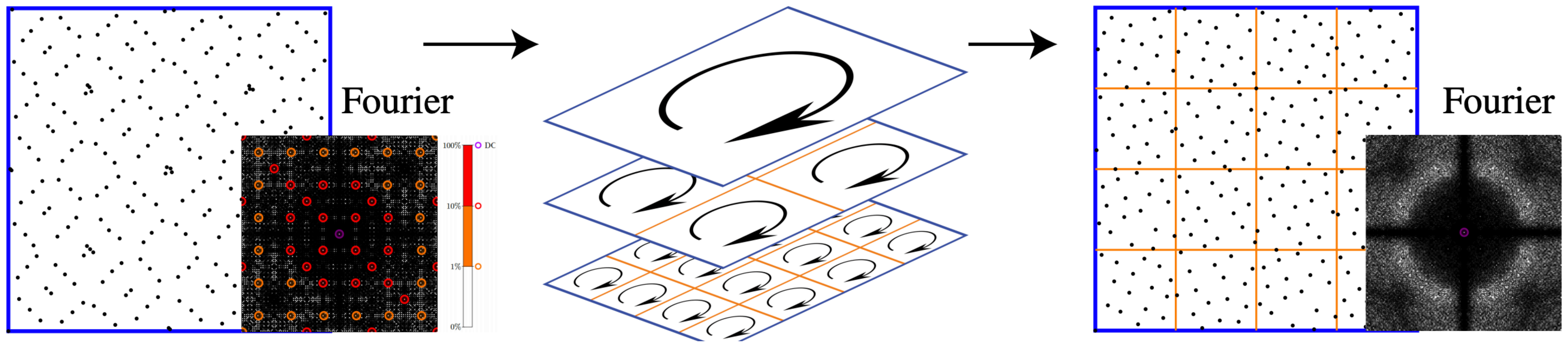
Sobol

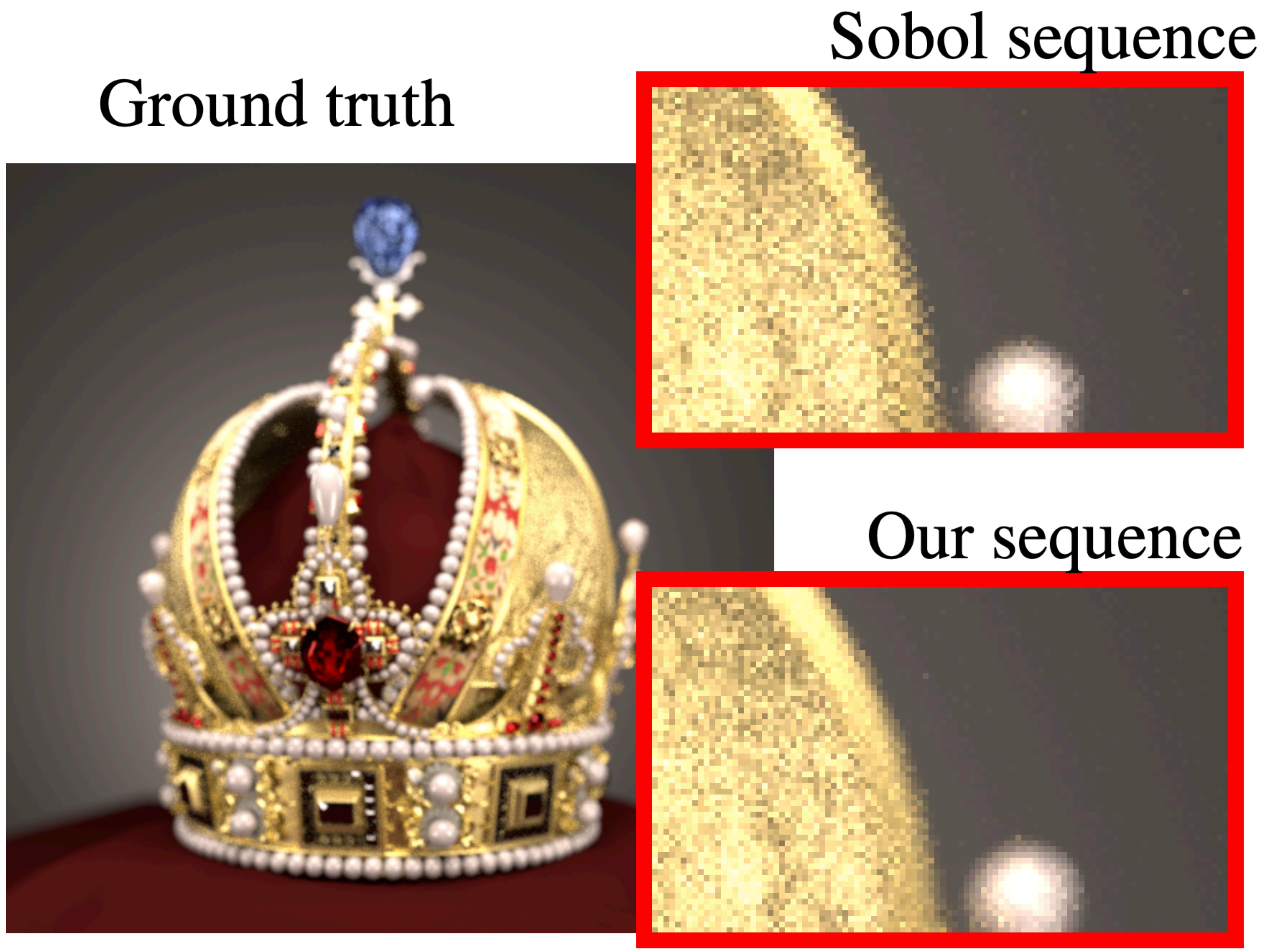
Special scrambling

Blue noise characteristics

Perrier et al. [2018]

Low-Discrepancy Blue Noise 2D Sobol Projections





Perrier et al. [2018]

Stratification strategies

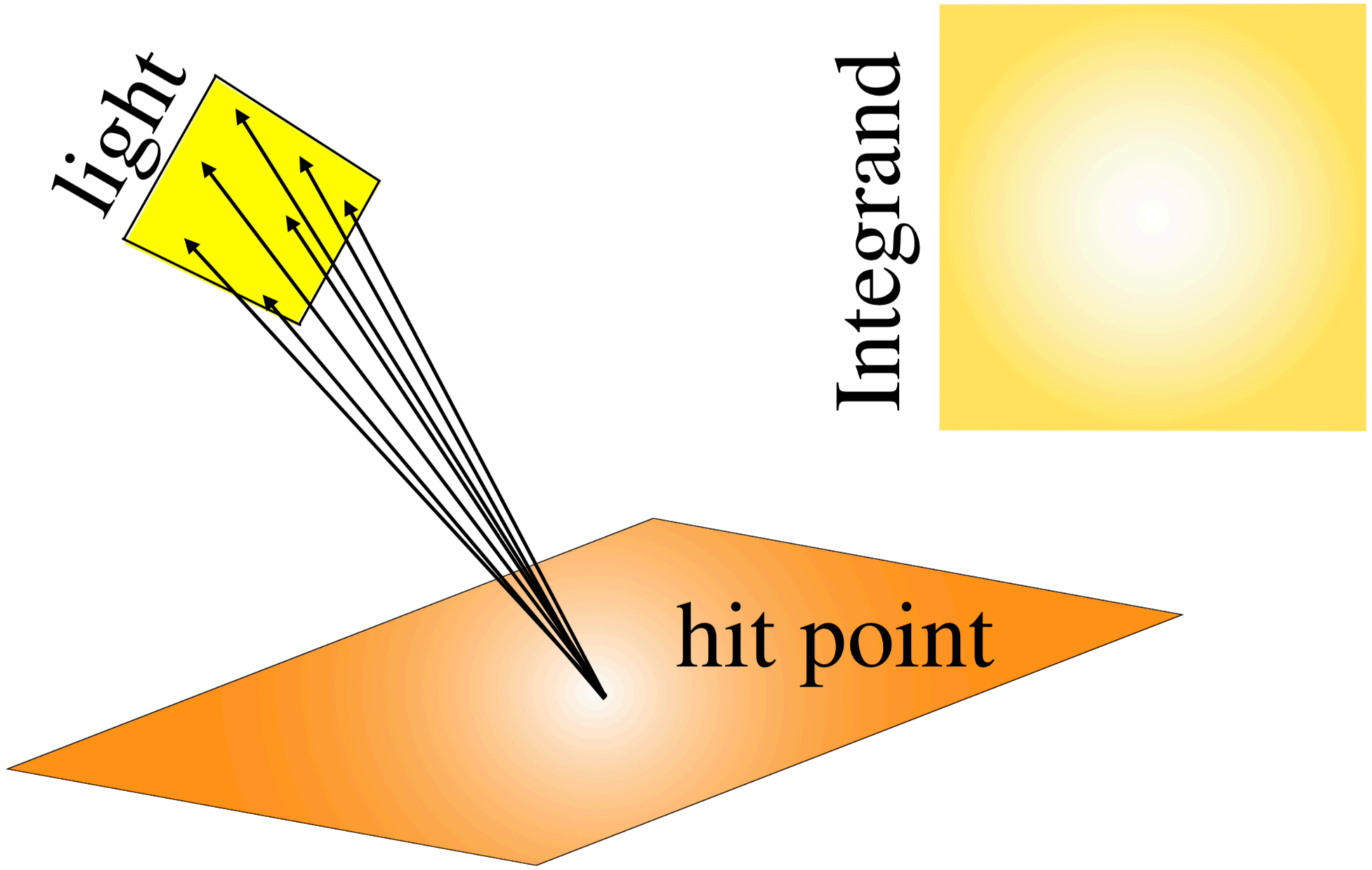
Blue noise sampling and beyond

Importance sampling

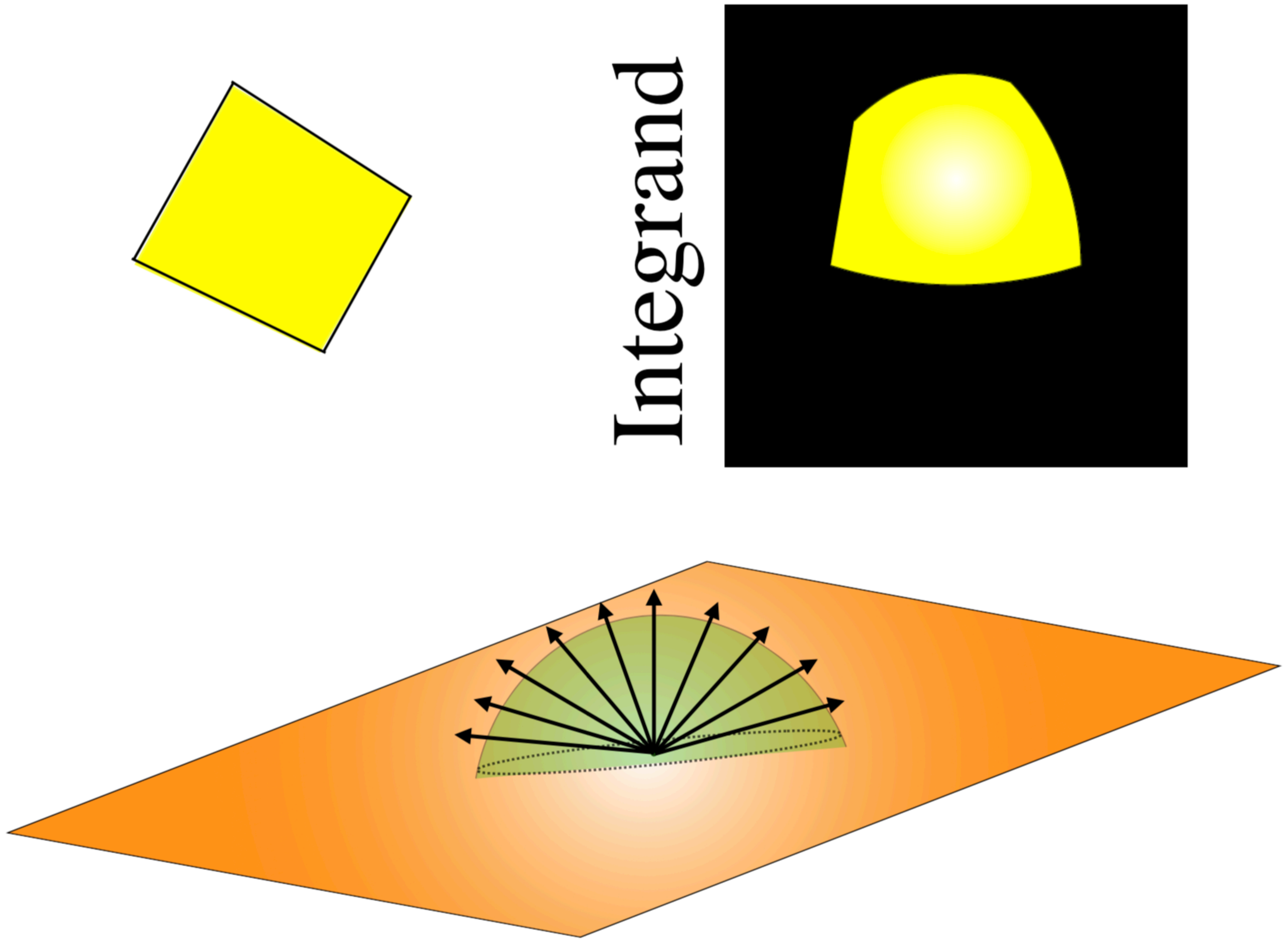
with correlated samples

EGSR 2019 EUROGRAPHICS SYMPOSIUM ON RENDERING

Light IS vs BSDF IS



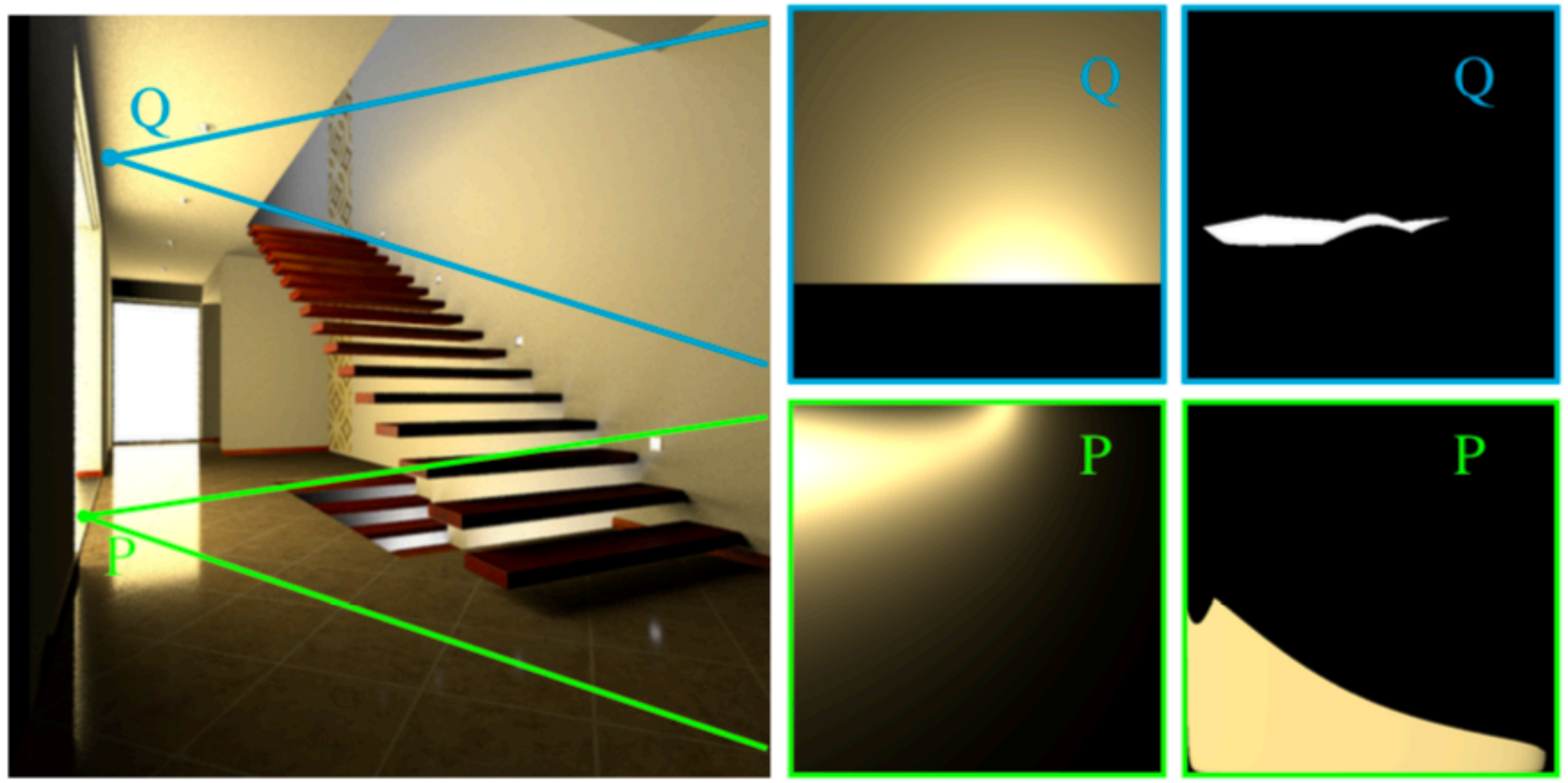
Light Importance Sampling



BSDF Importance Sampling

Singh et al. [2019]

Scene illuminated by area direct lighting

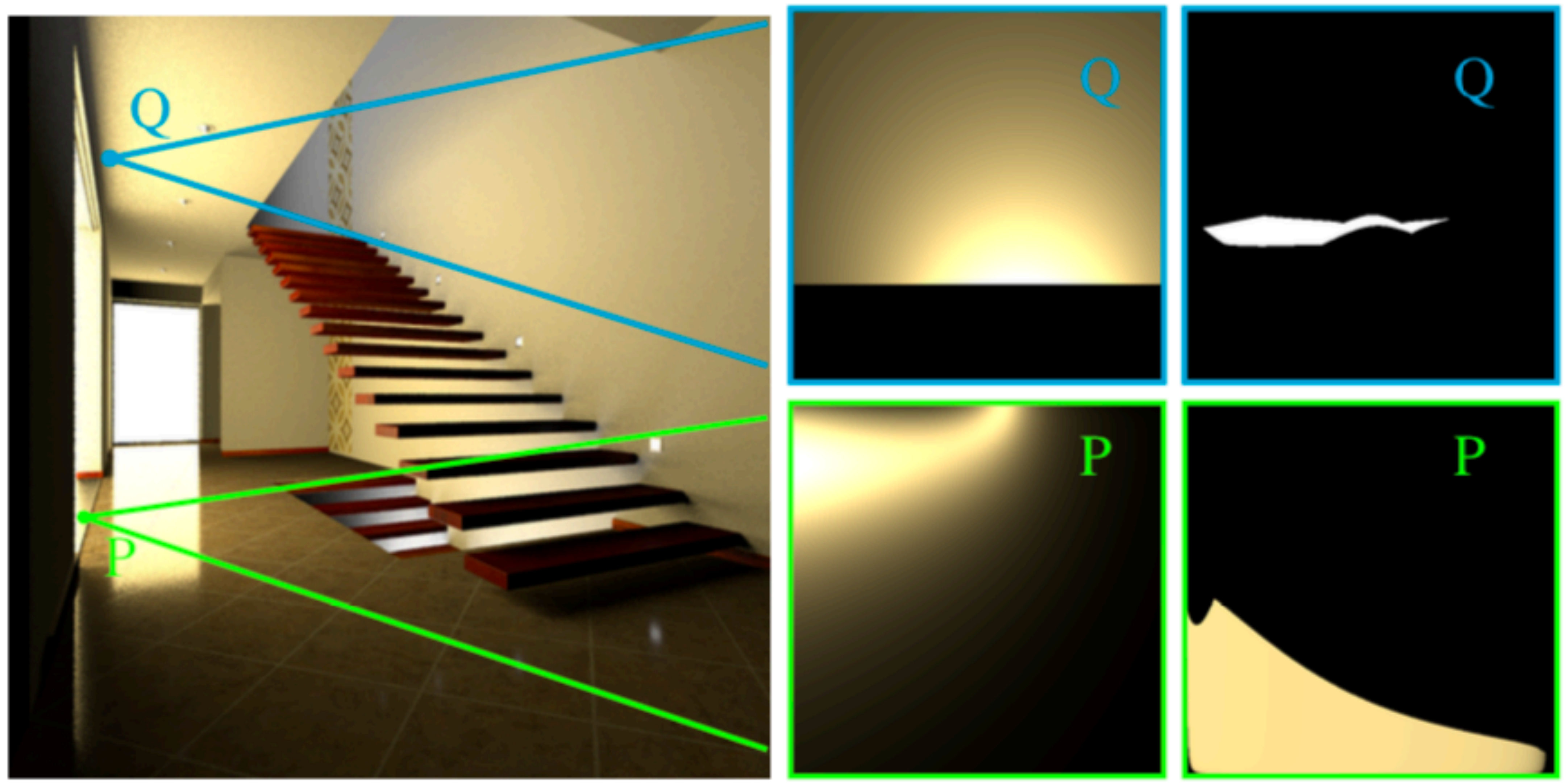


Reference

Underlying
pixel functions

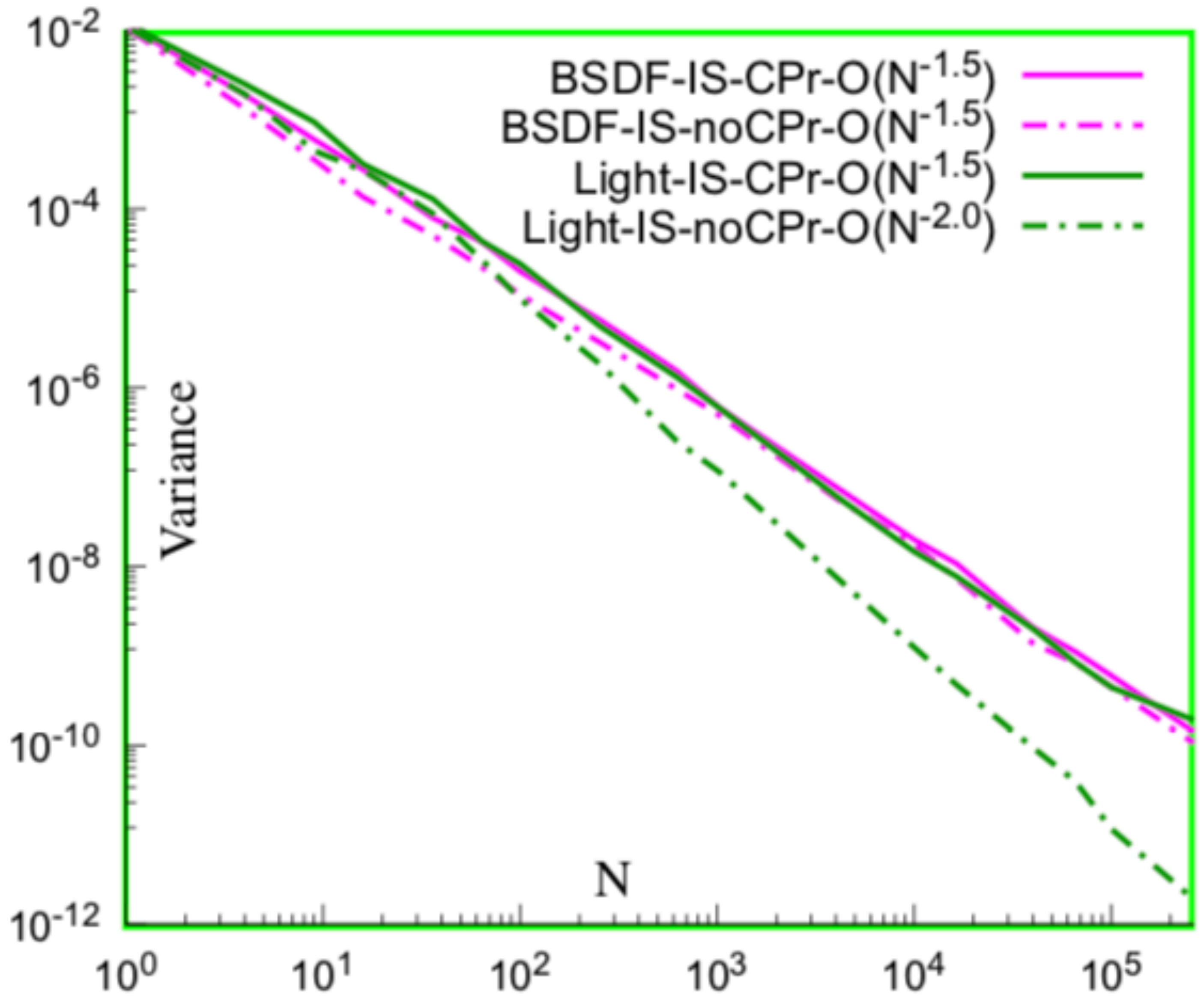
Singh et al. [2019]

Unoccluded pixels' convergence benefit from Light IS



Reference

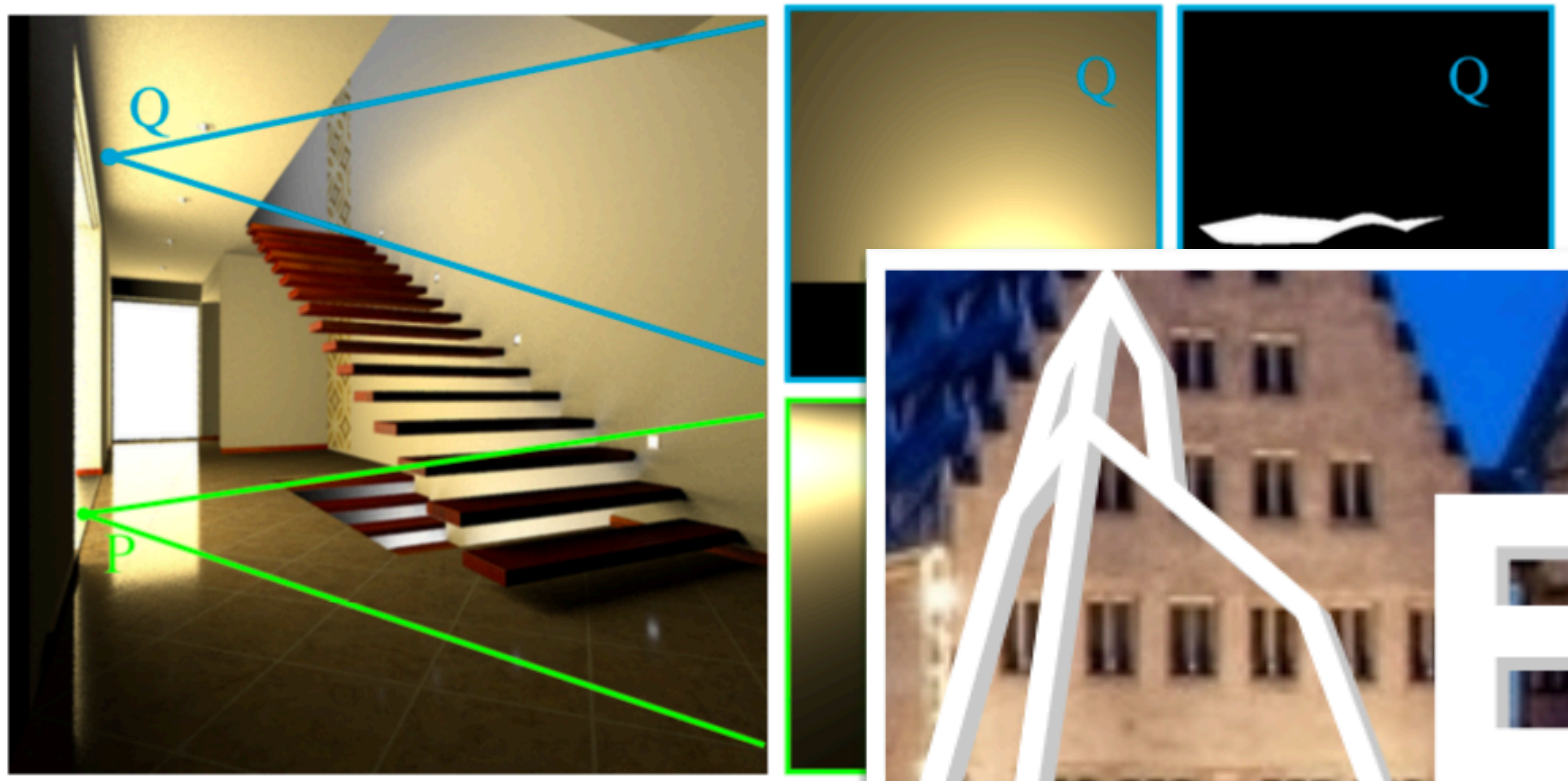
Underlying
pixel functions

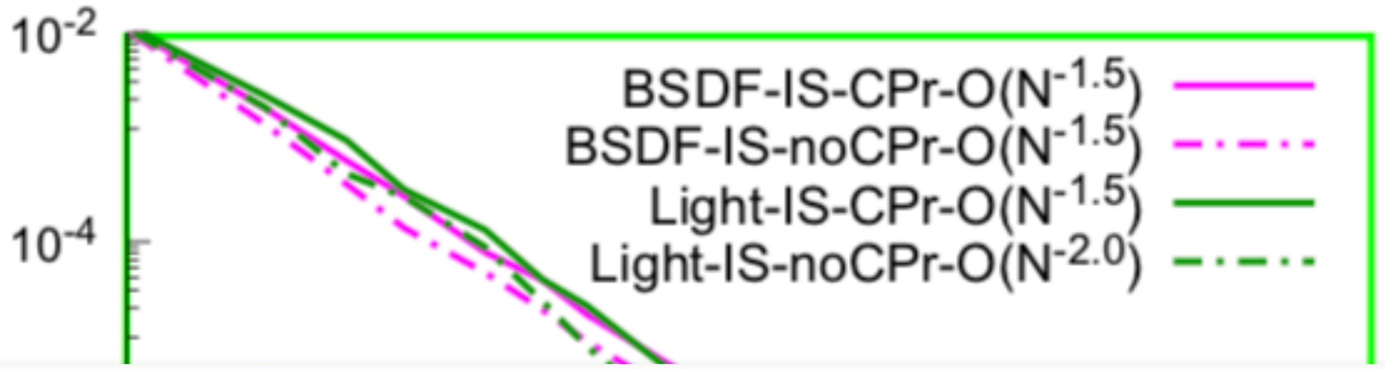


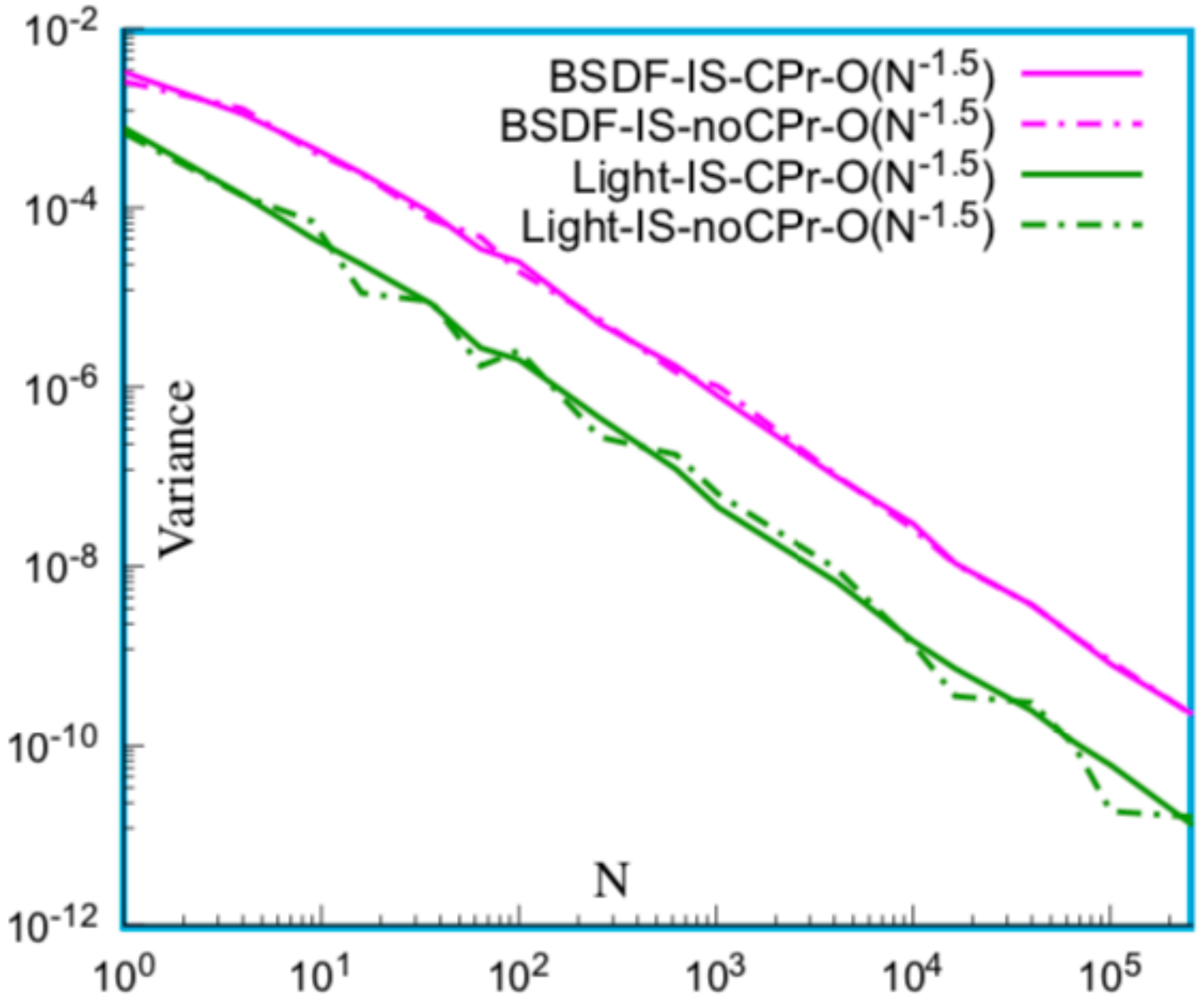
Pixel P

Singh et al. [2019]

Occluded pixels (no improvement in convergence)







Reference

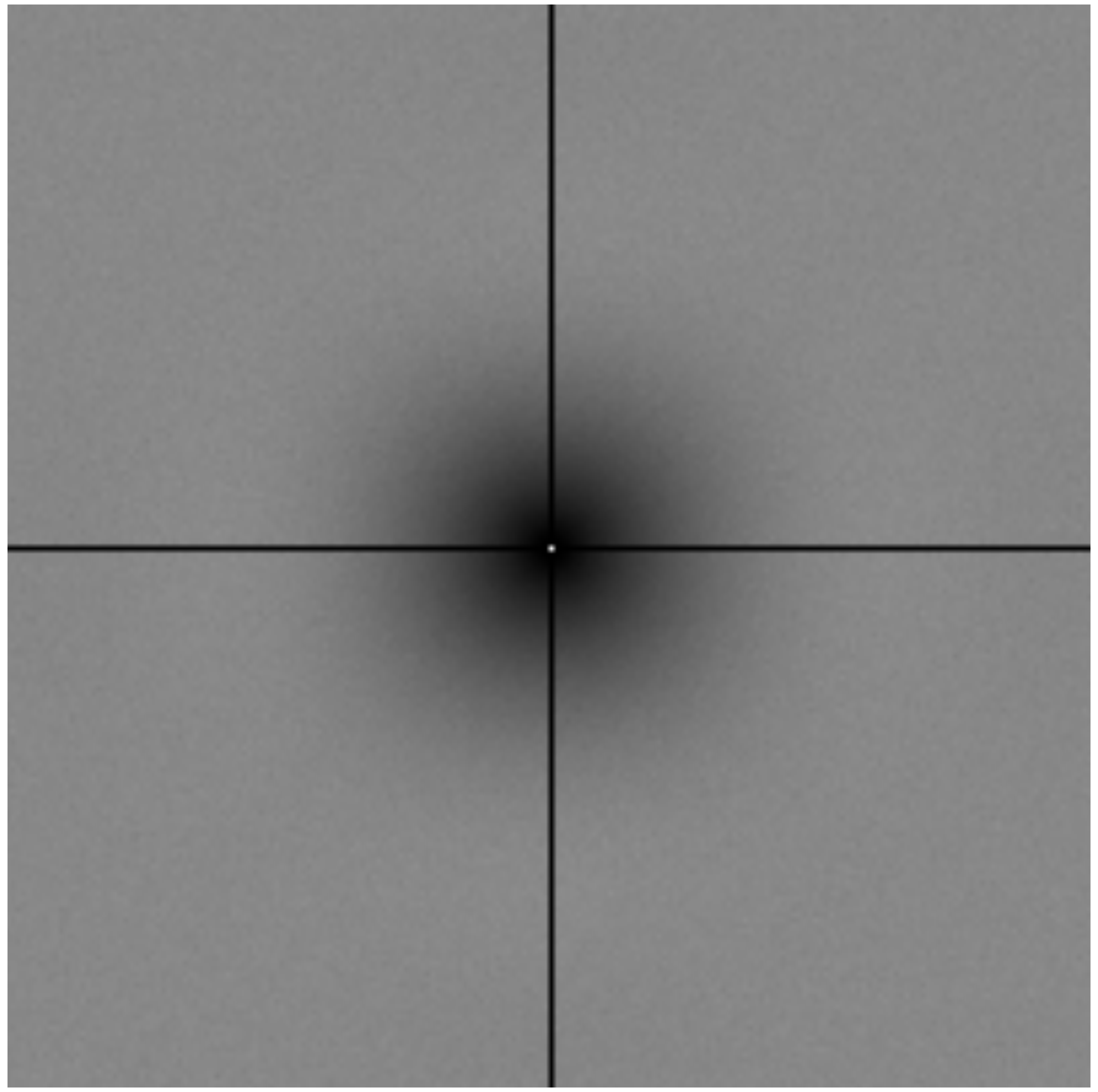
EGSR 2019 EUROGRAPHICS SYMPOSIUM ON RENDERING

Pixel Q

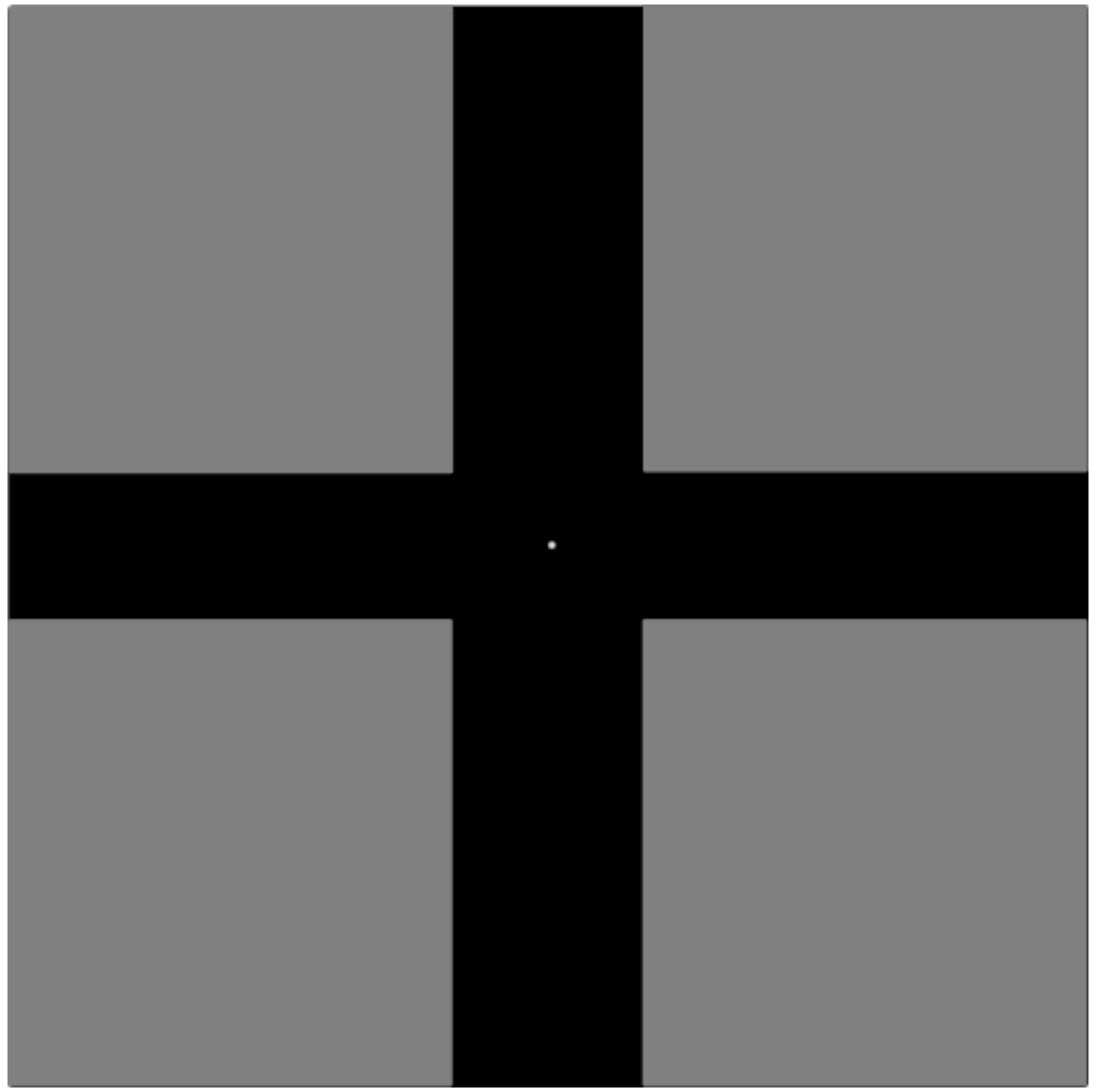
Singh et al. [2019]

Futuristic sampling target spectrum

Multi-jittered



Future design



Singh and Jarosz [2017]

Future research directions

Direct link between spatial and Fourier statistics needs further investigation

Progressive samplers in higher dimensions

Adapting sample correlations w.r.t. the underlying integrand in high dimensions

Acknowledgments

Some slides borrowed from Wojciech Jarosz and Kartic Subr

All the anonymous reviewers who helped shape this survey paper into its final form