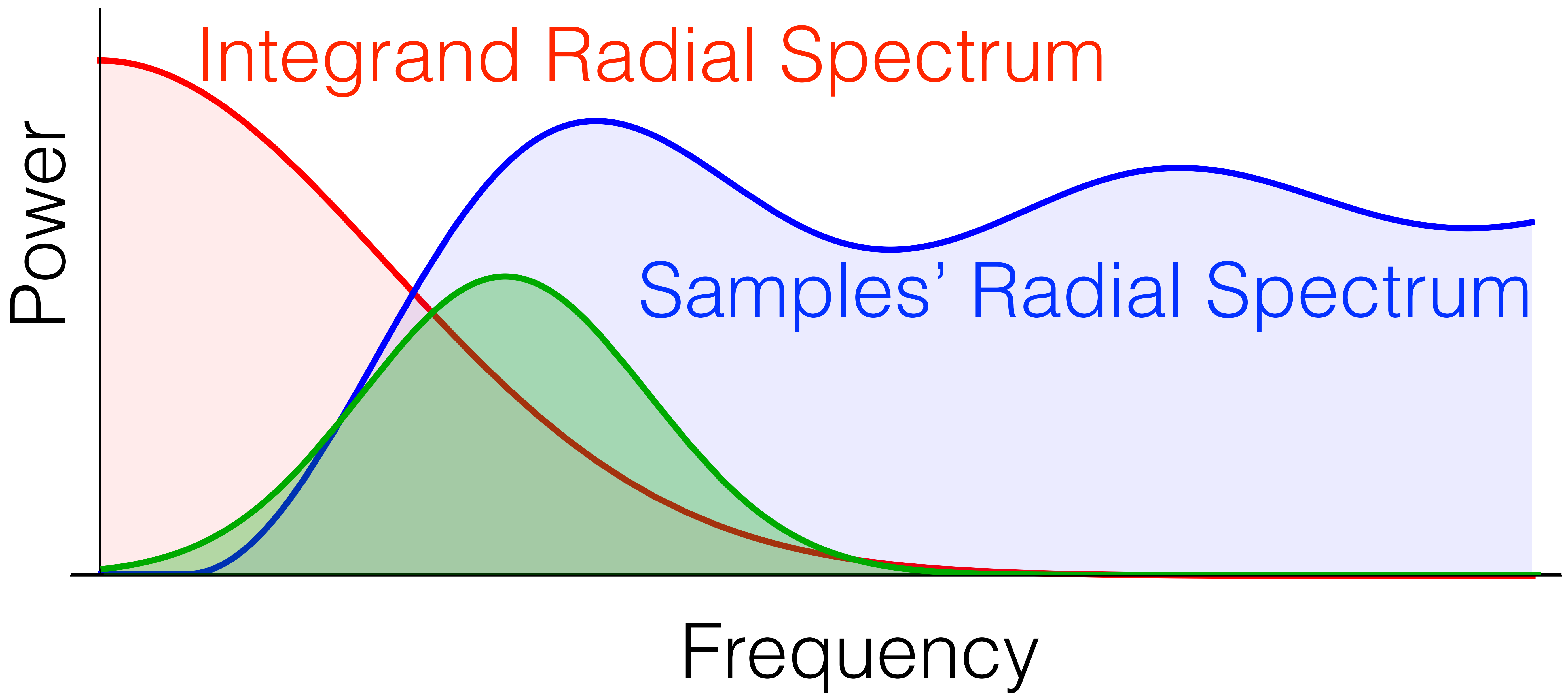
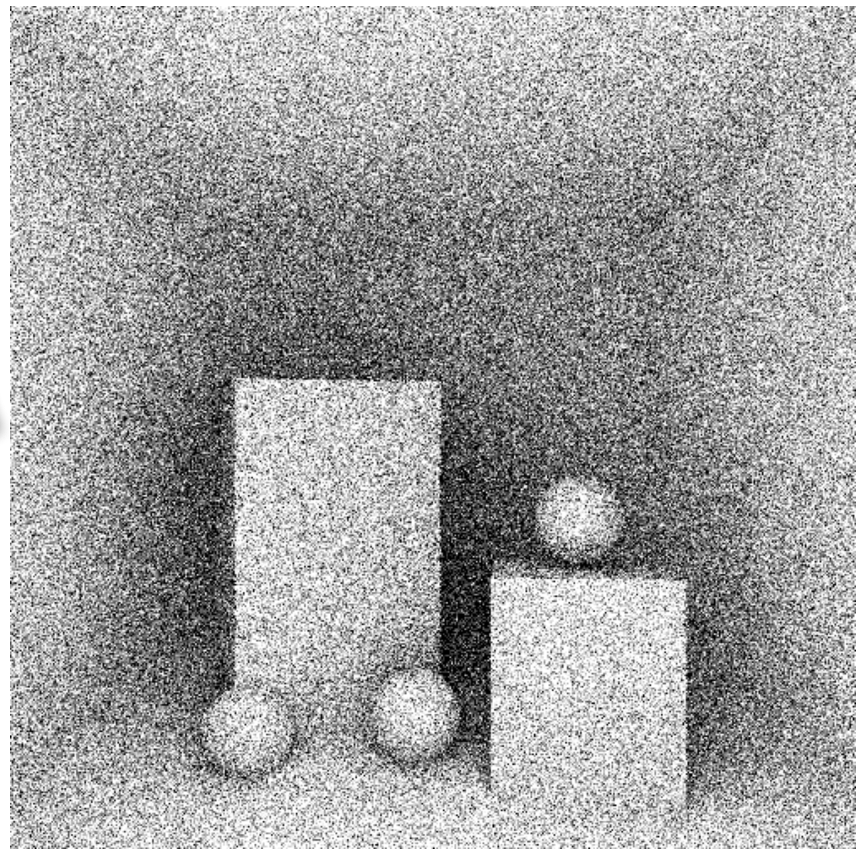
Part 3: Formal Treatment of MSE, Bias and Variance



Convergence rate for Random Samples



Variance

Increasing Samples

Render the Possibilities
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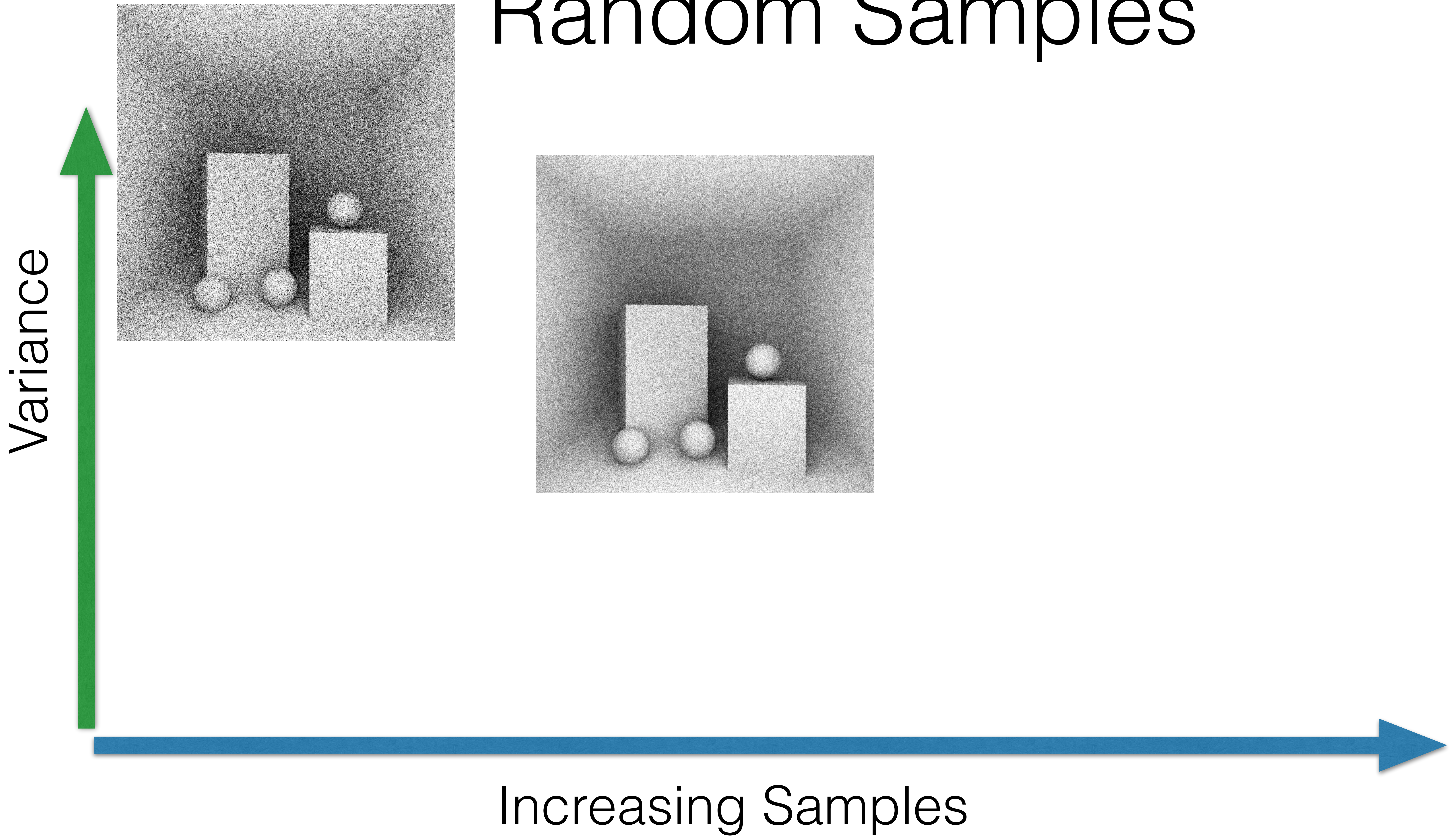
Convergence rate for Random Samples



Render the Possibilities
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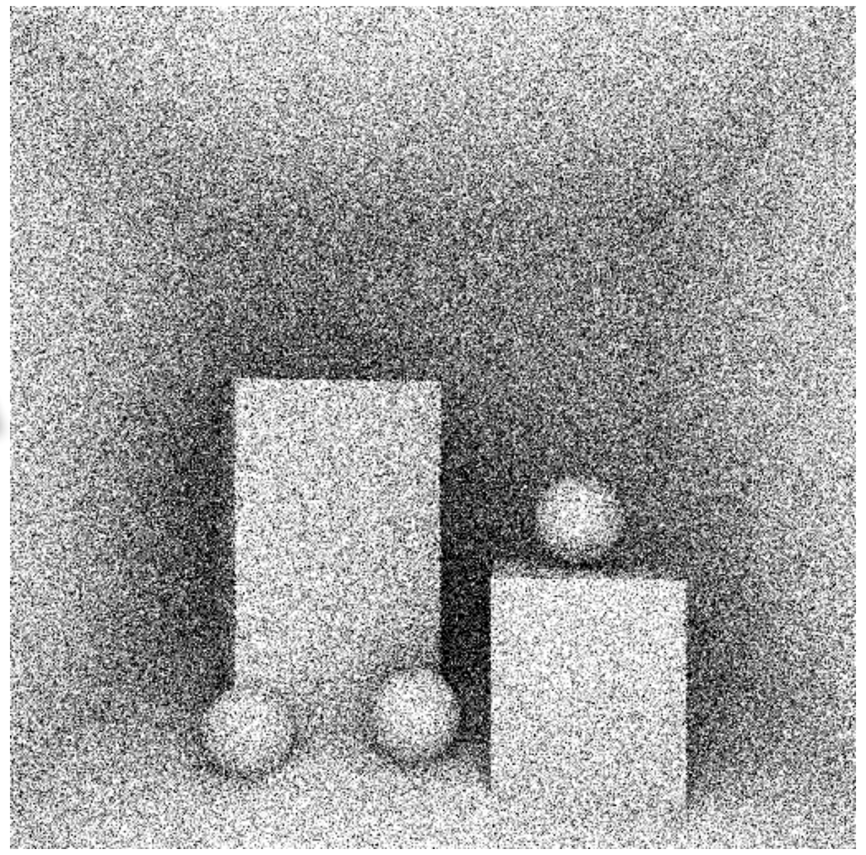
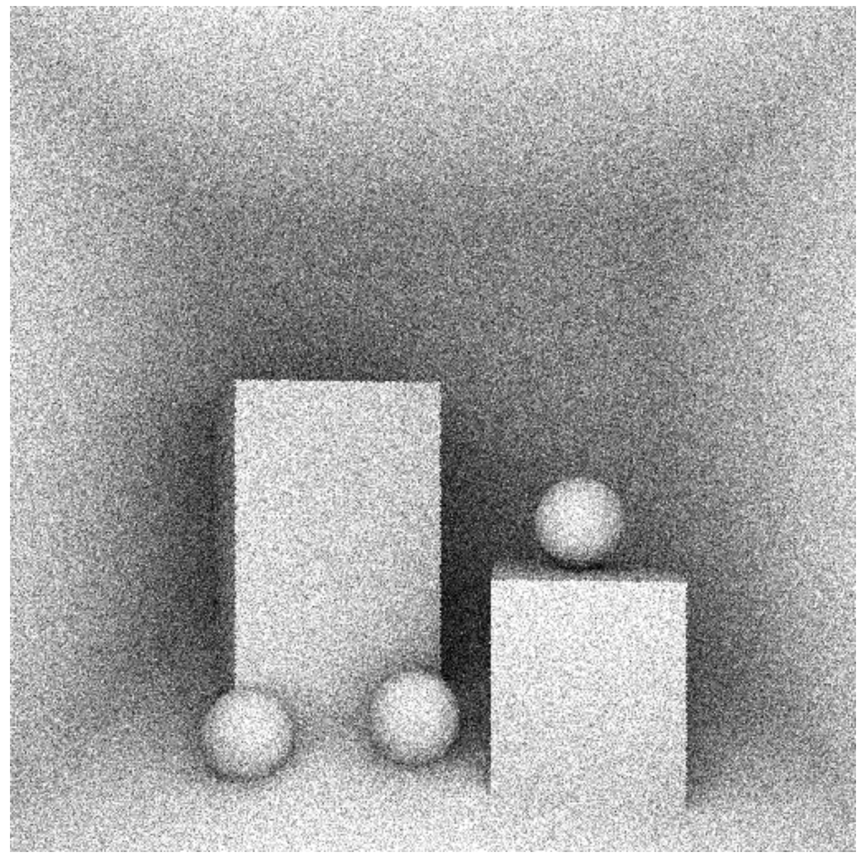
Convergence rate for Random Samples

Variance





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Increasing Samples

Render the Possibilities
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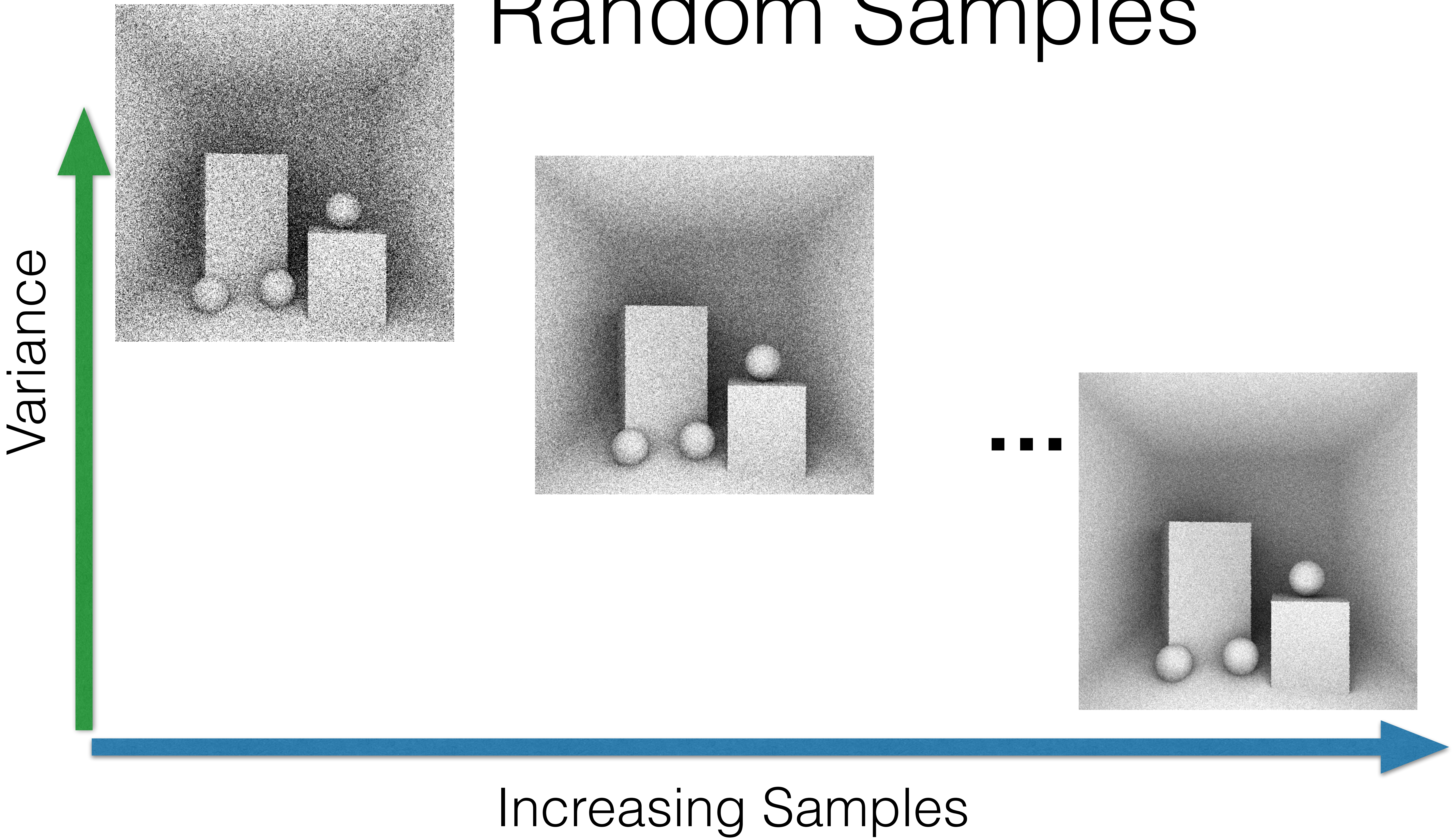
Convergence rate for Random Samples



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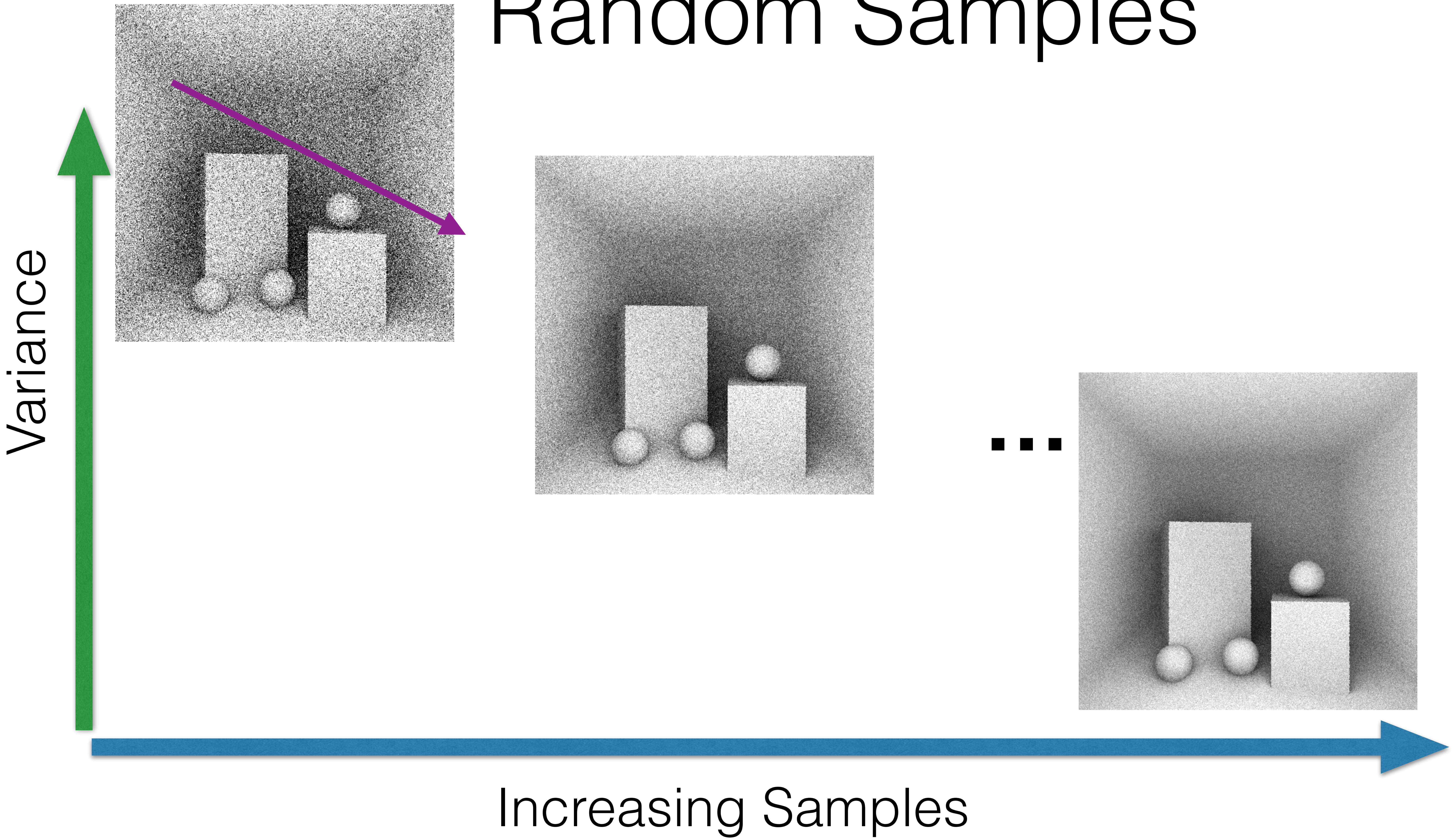
Convergence rate for Random Samples



Render the Possibilities
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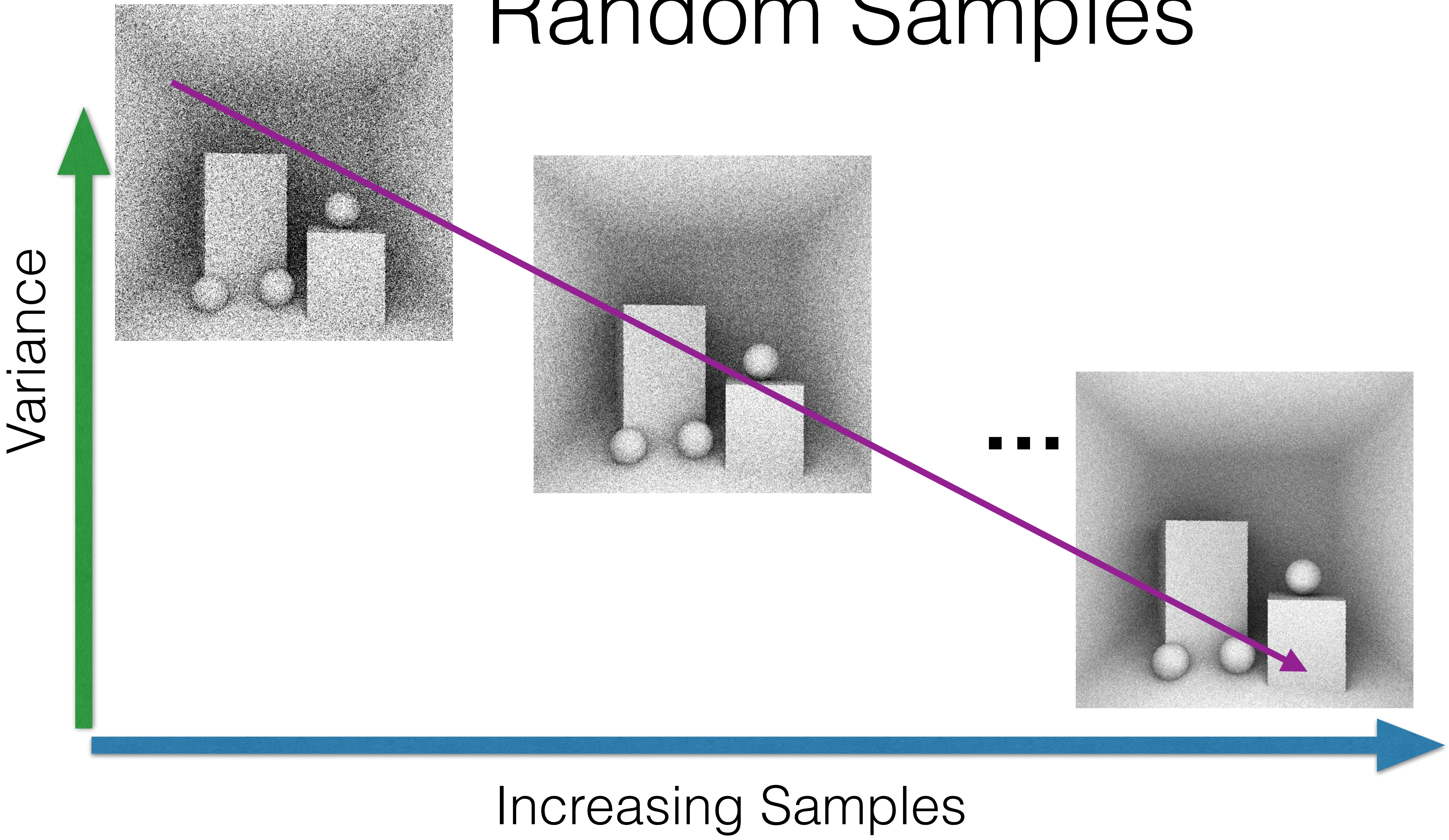
Convergence rate for Random Samples



Render the Possibilities
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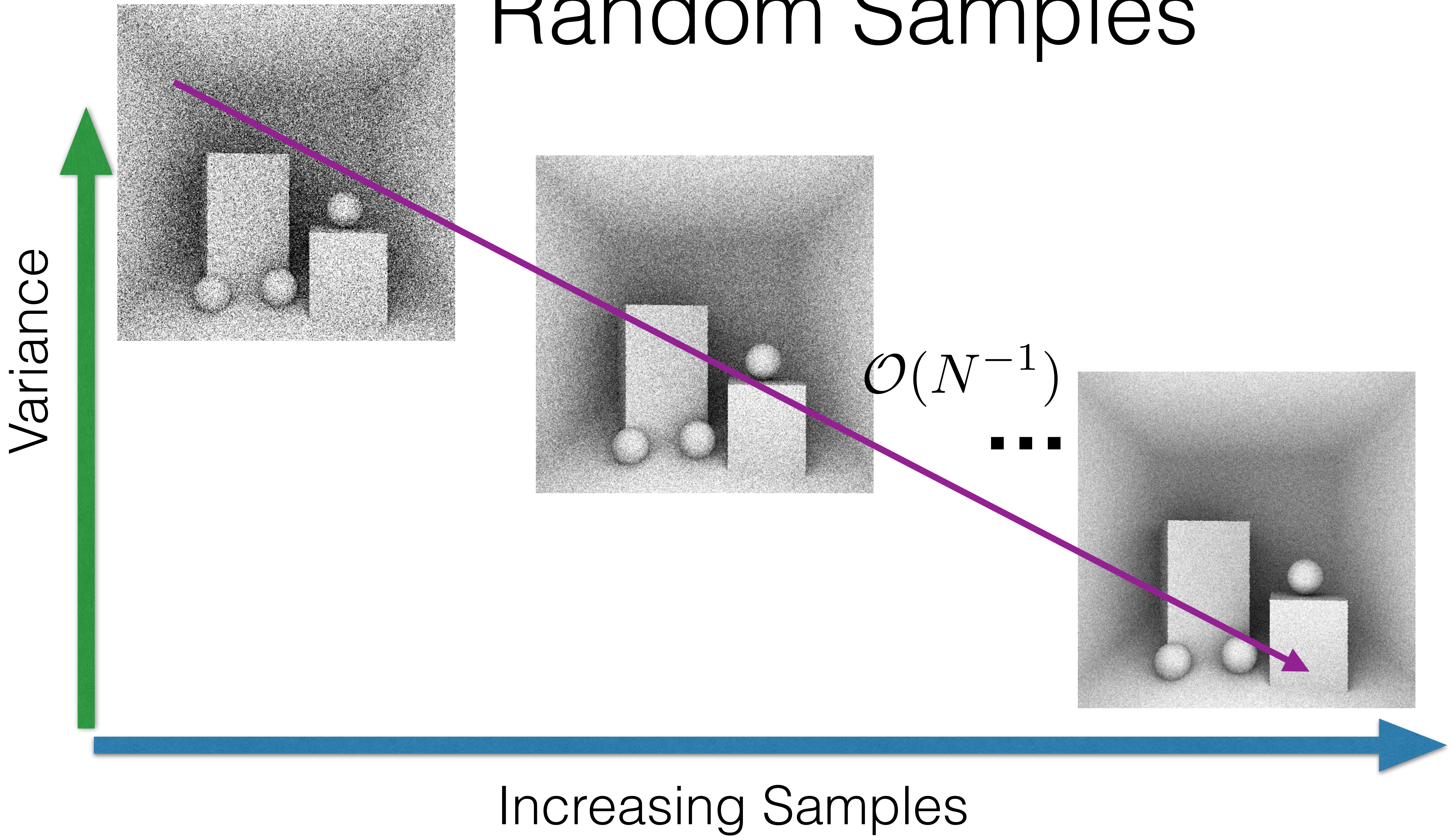
Convergence rate for Random Samples



Render the Possibilities
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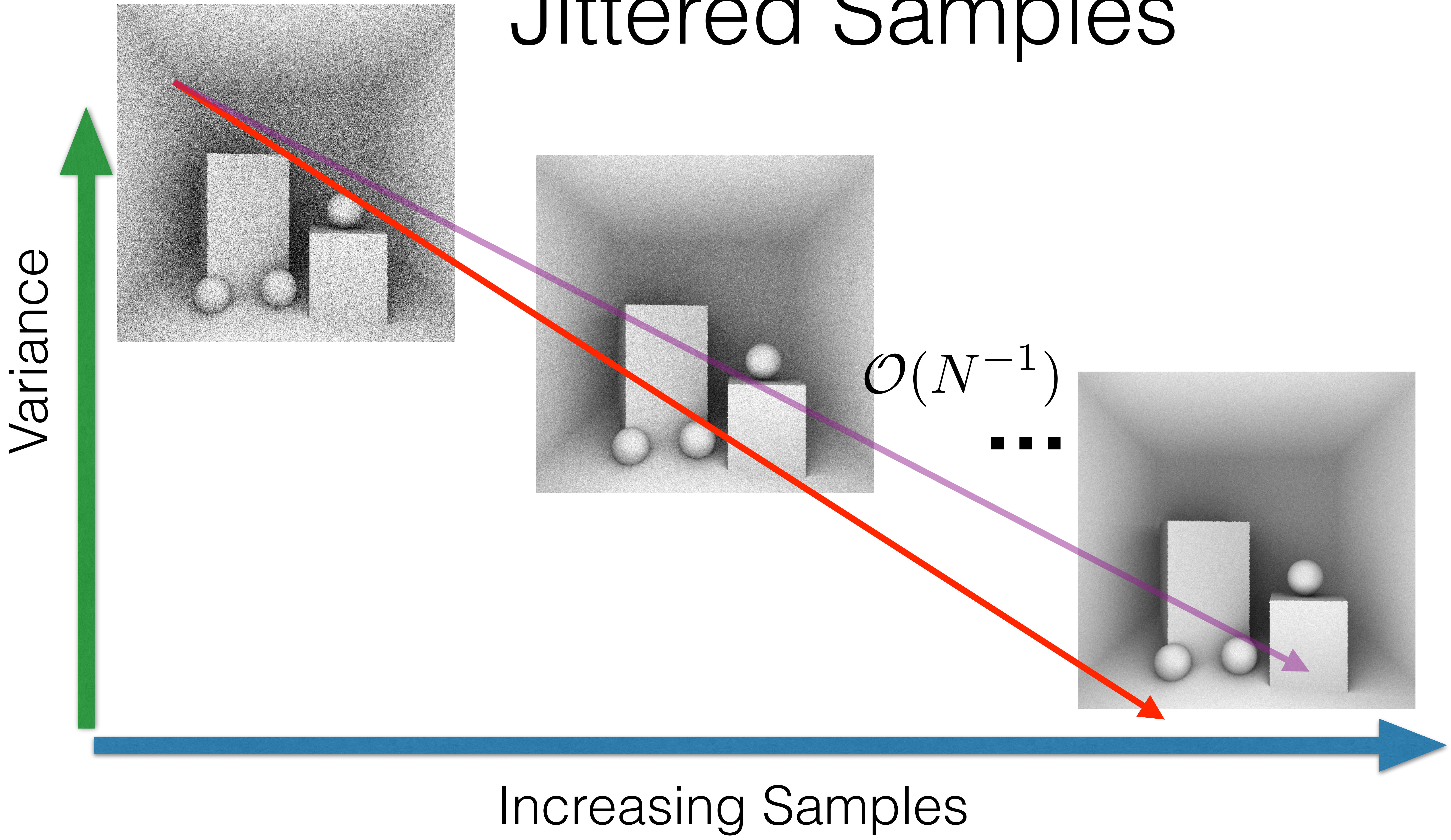
Convergence rate for Jittered Samples



Render the Possibilities
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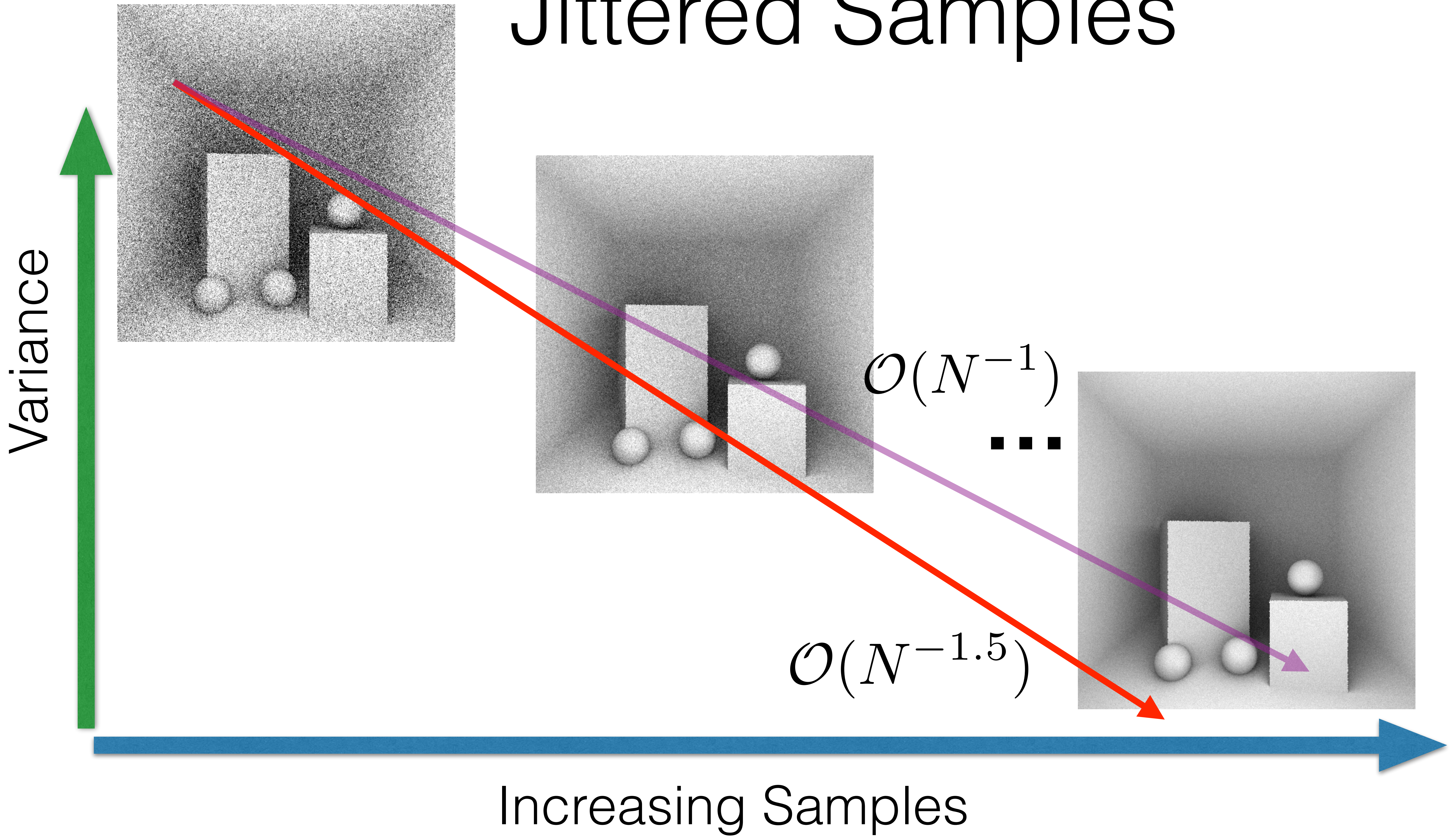
Convergence rate for Jittered Samples



Render the Possibilities
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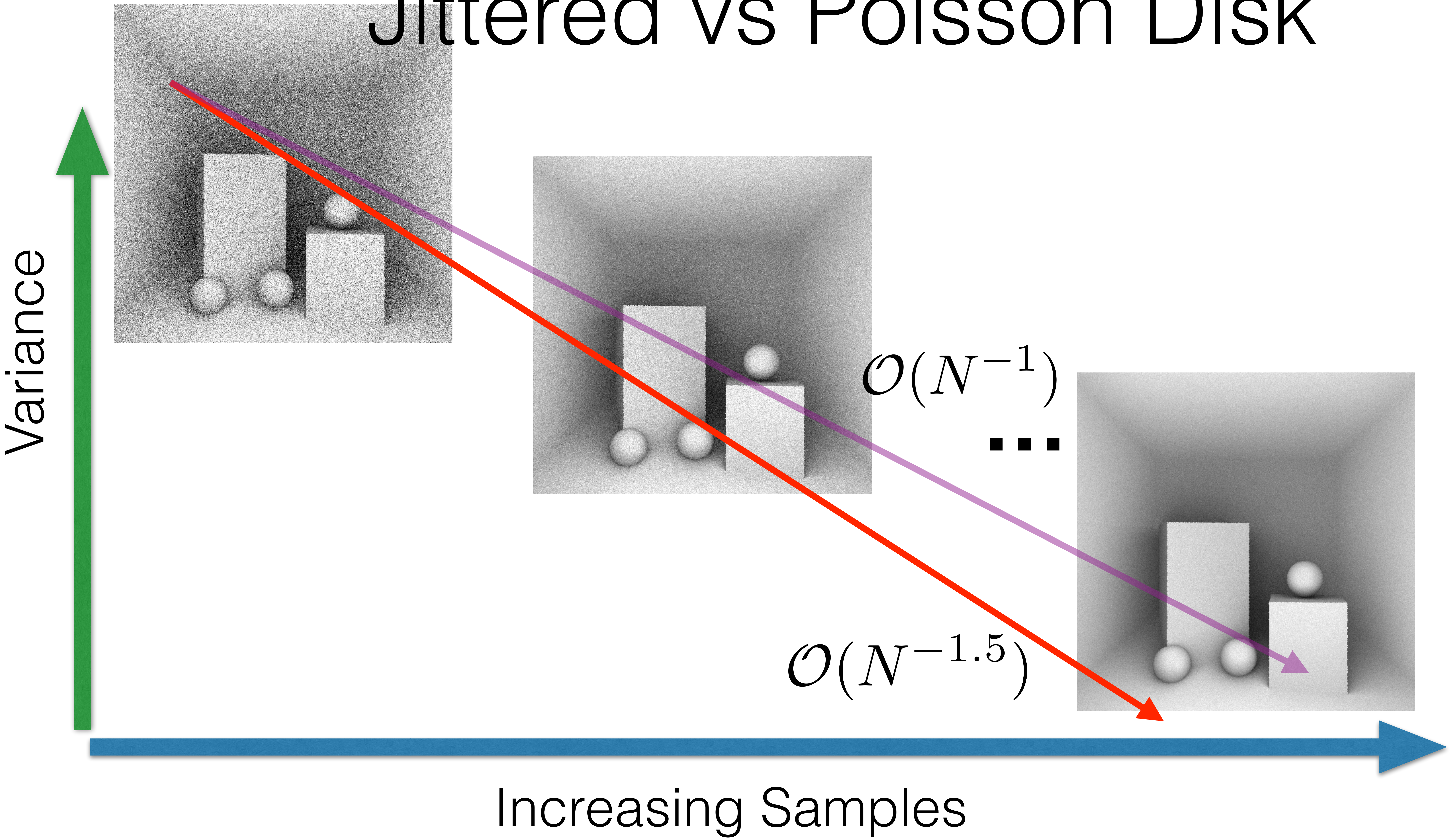
Convergence rate
Jittered vs Poisson Disk



Render the Possibilities
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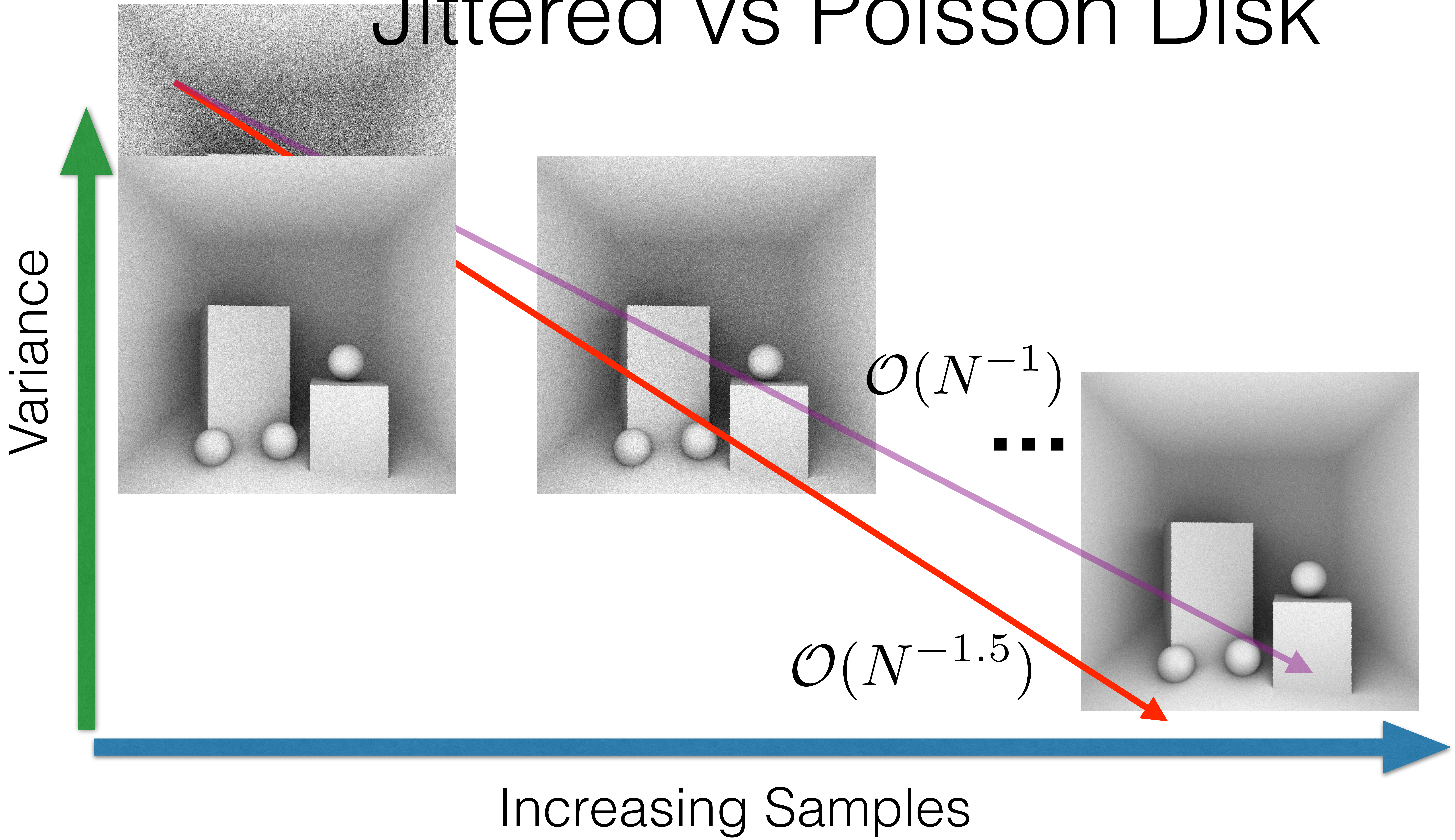
Convergence rate
Jittered vs Poisson Disk



Render the Possibilities
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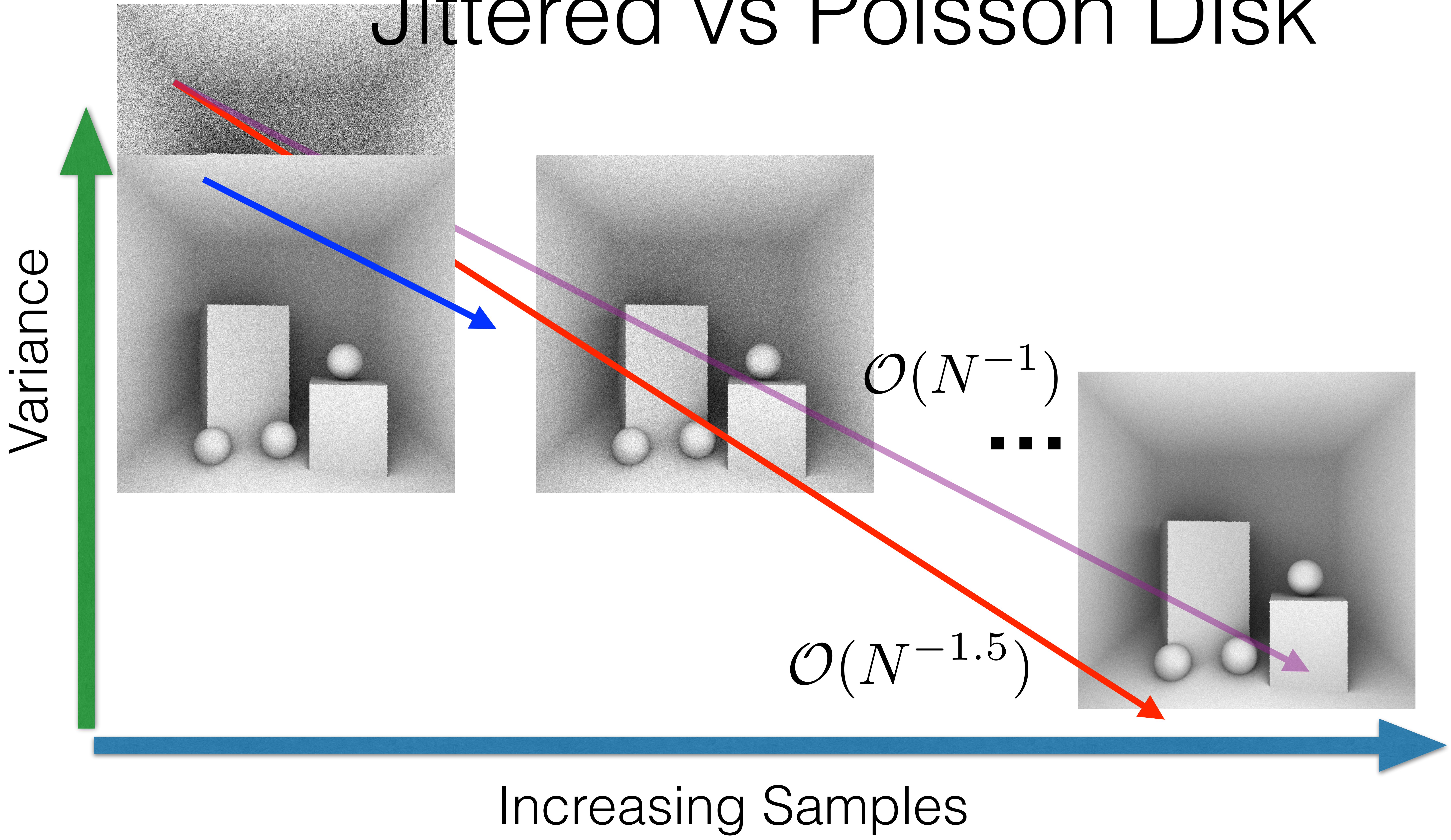
Convergence rate
Jittered vs Poisson Disk



Render the Possibilities
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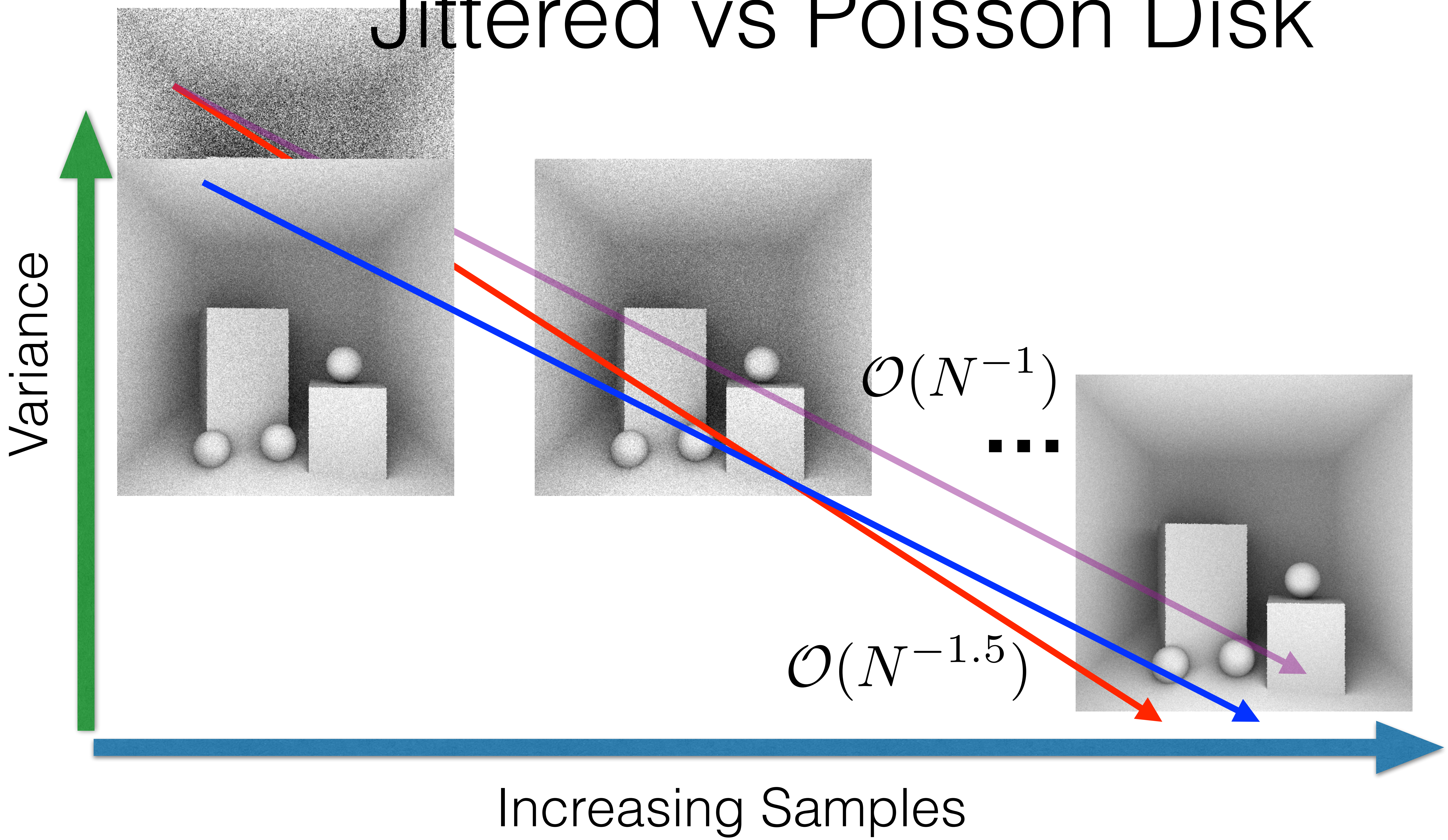
Convergence rate
Jittered vs Poisson Disk



Render the Possibilities
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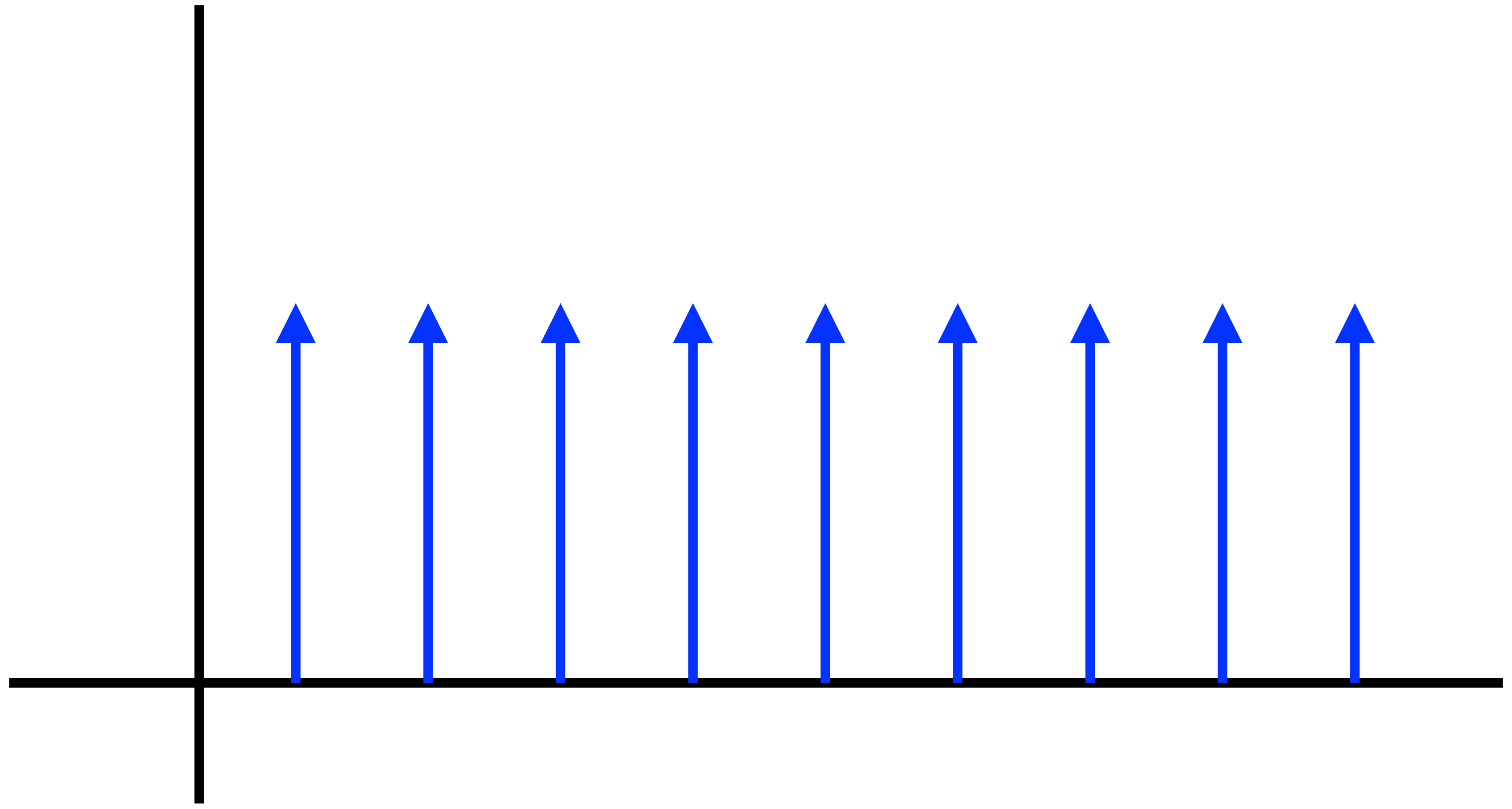
Samples and function in Fourier Domain

Spatial Domain



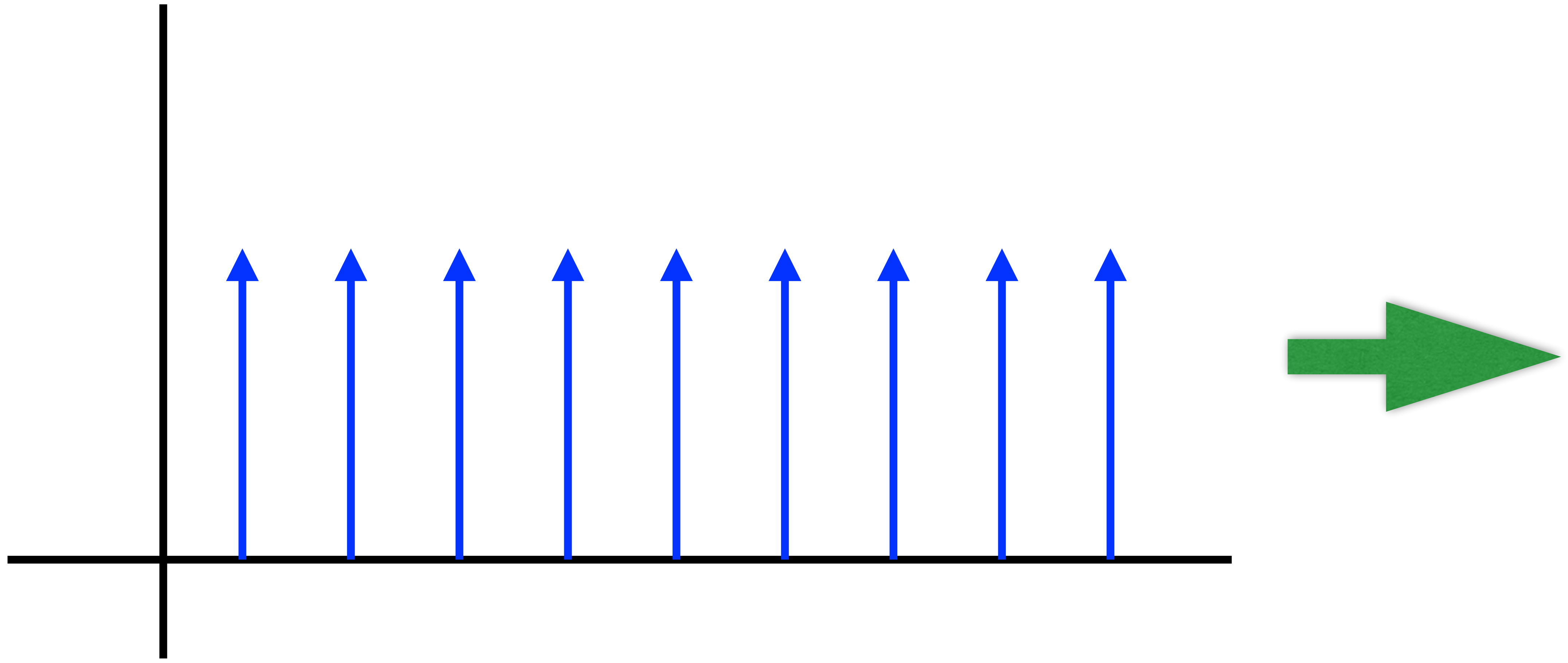
Fourier Domain

Render the Possibilities SIGGRAPH2016

Samples and function in Fourier Domain

Spatial Domain

Fourier Domain



Render the Possibilities
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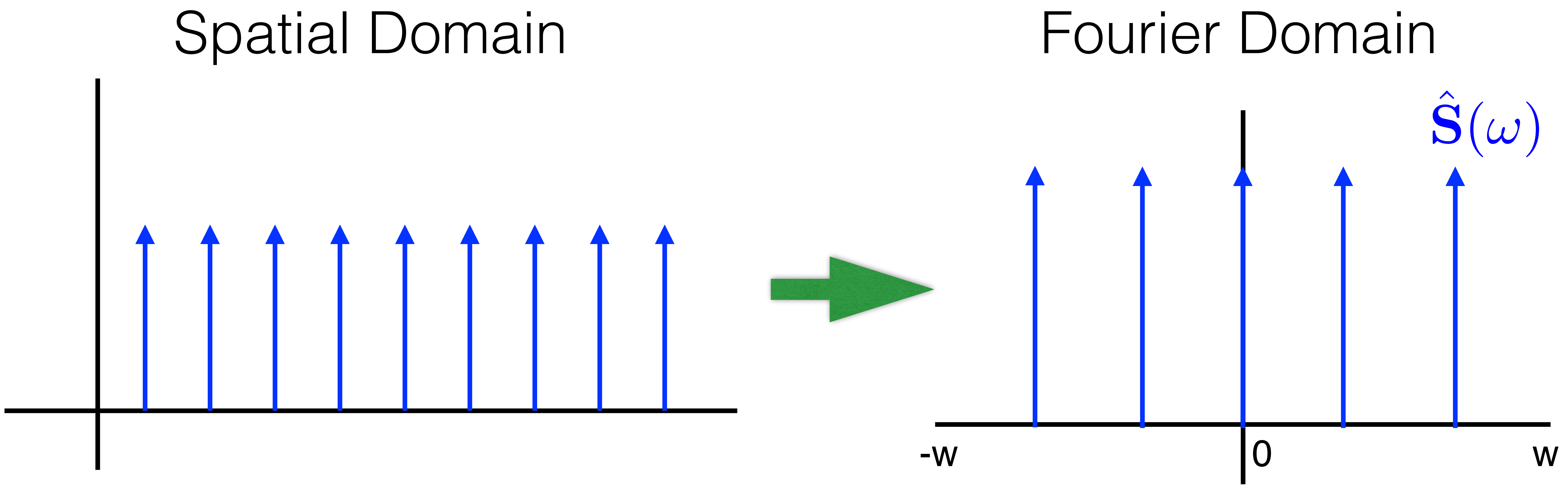
Samples and function in Fourier Domain



Render the Possibilities
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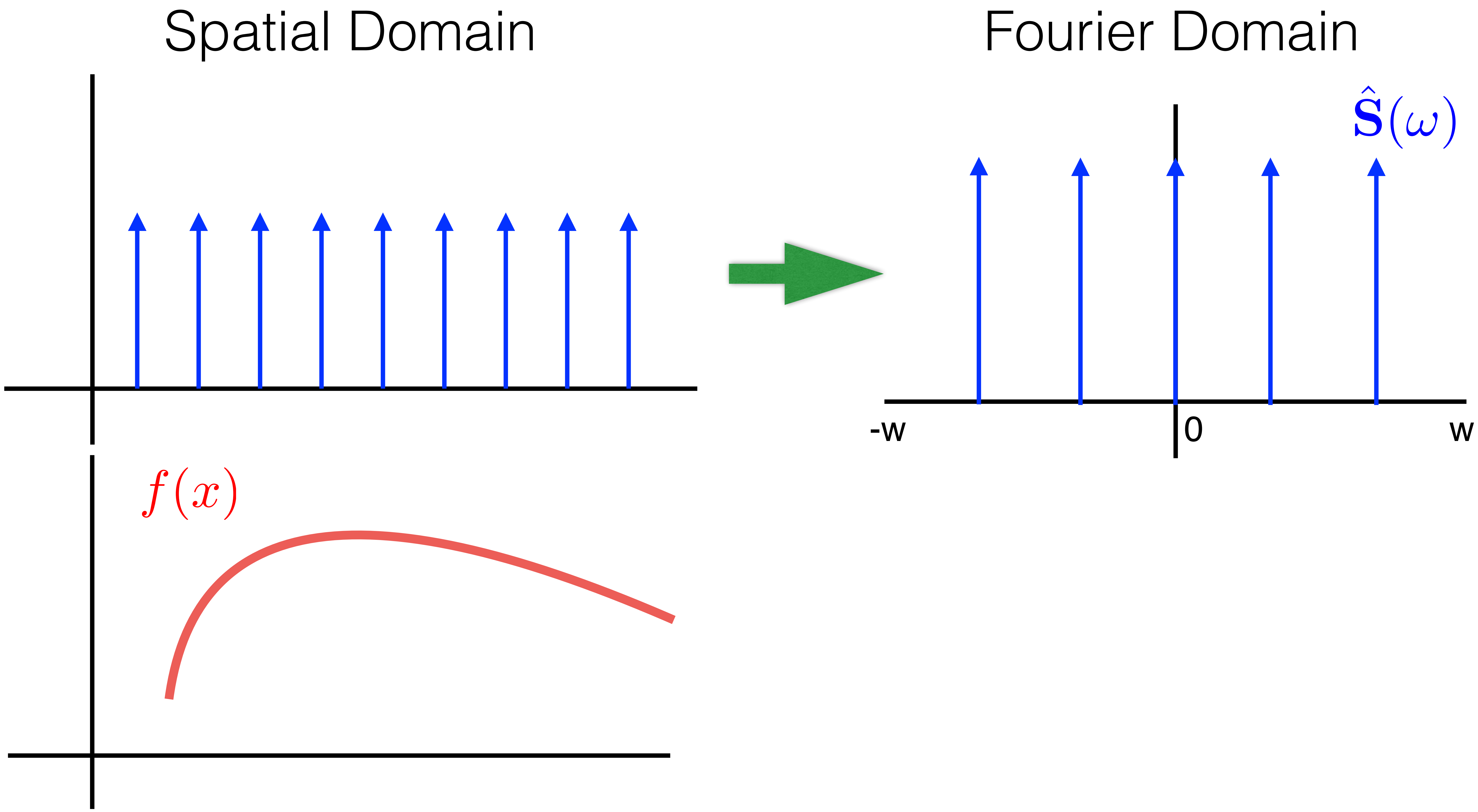
Samples and function in Fourier Domain



Render the Possibilities
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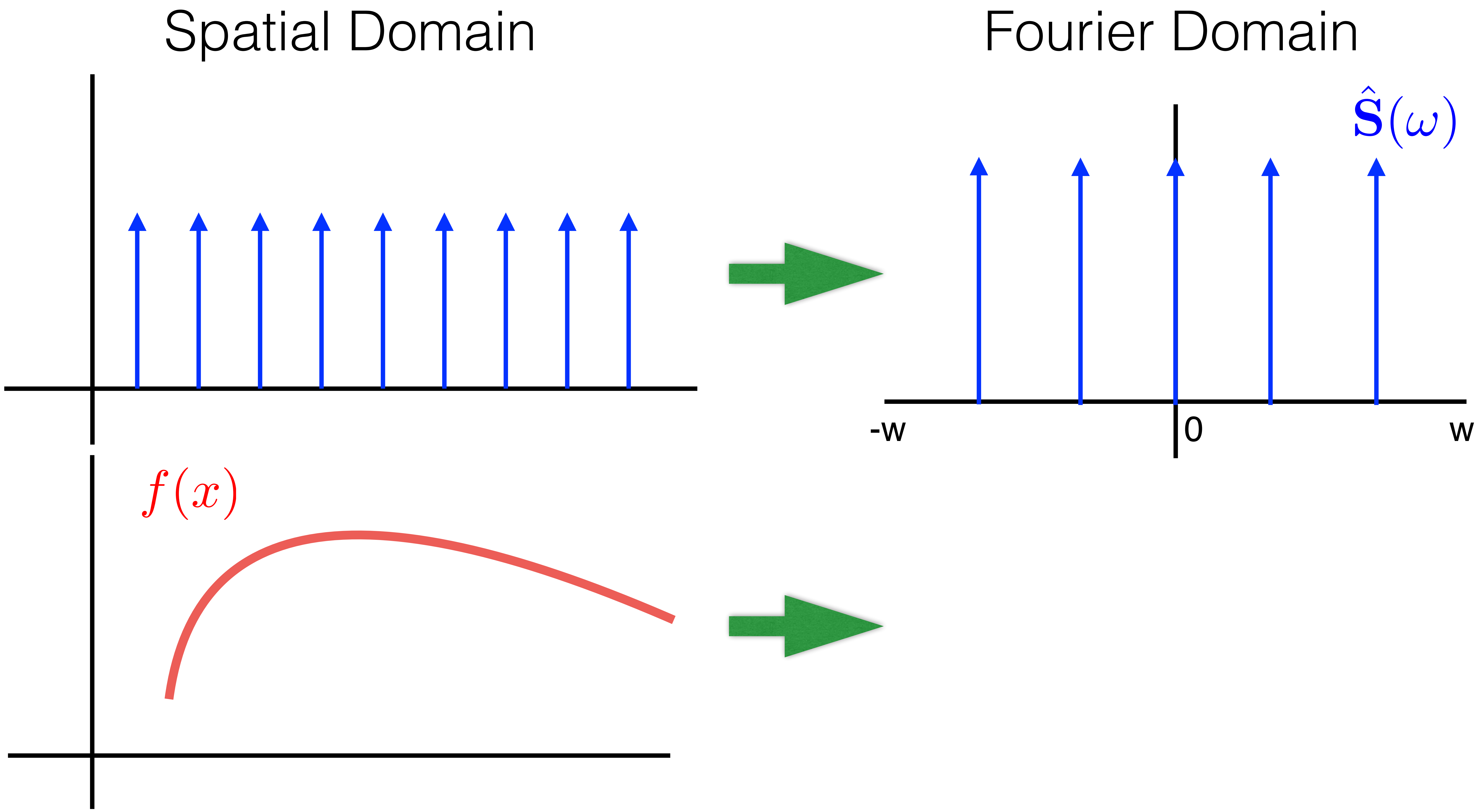
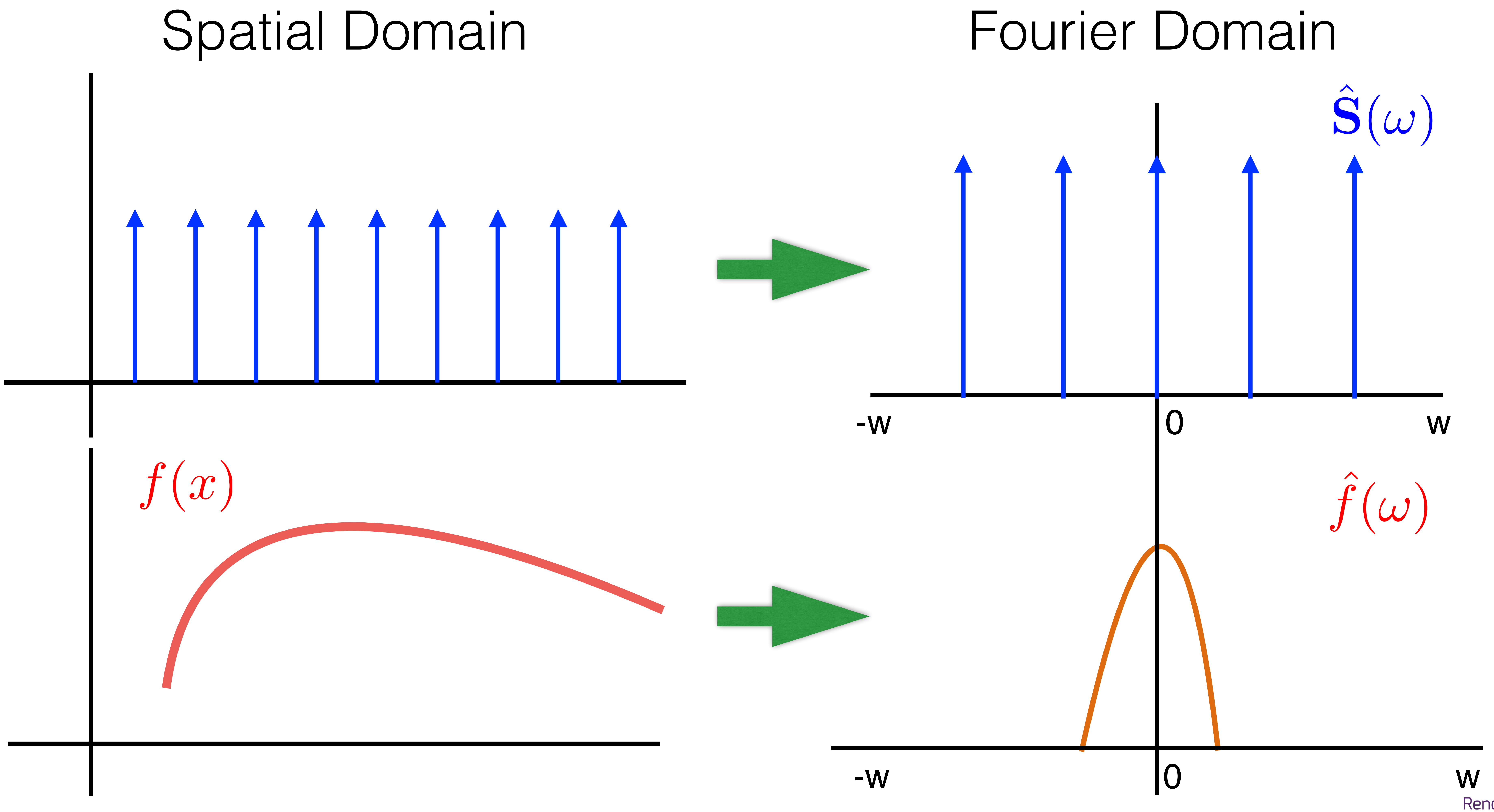
Samples and function in Fourier Domain



Render the Possibilities
SIGGRAPH2016

Samples and function in Fourier Domain



Render the Possibilities
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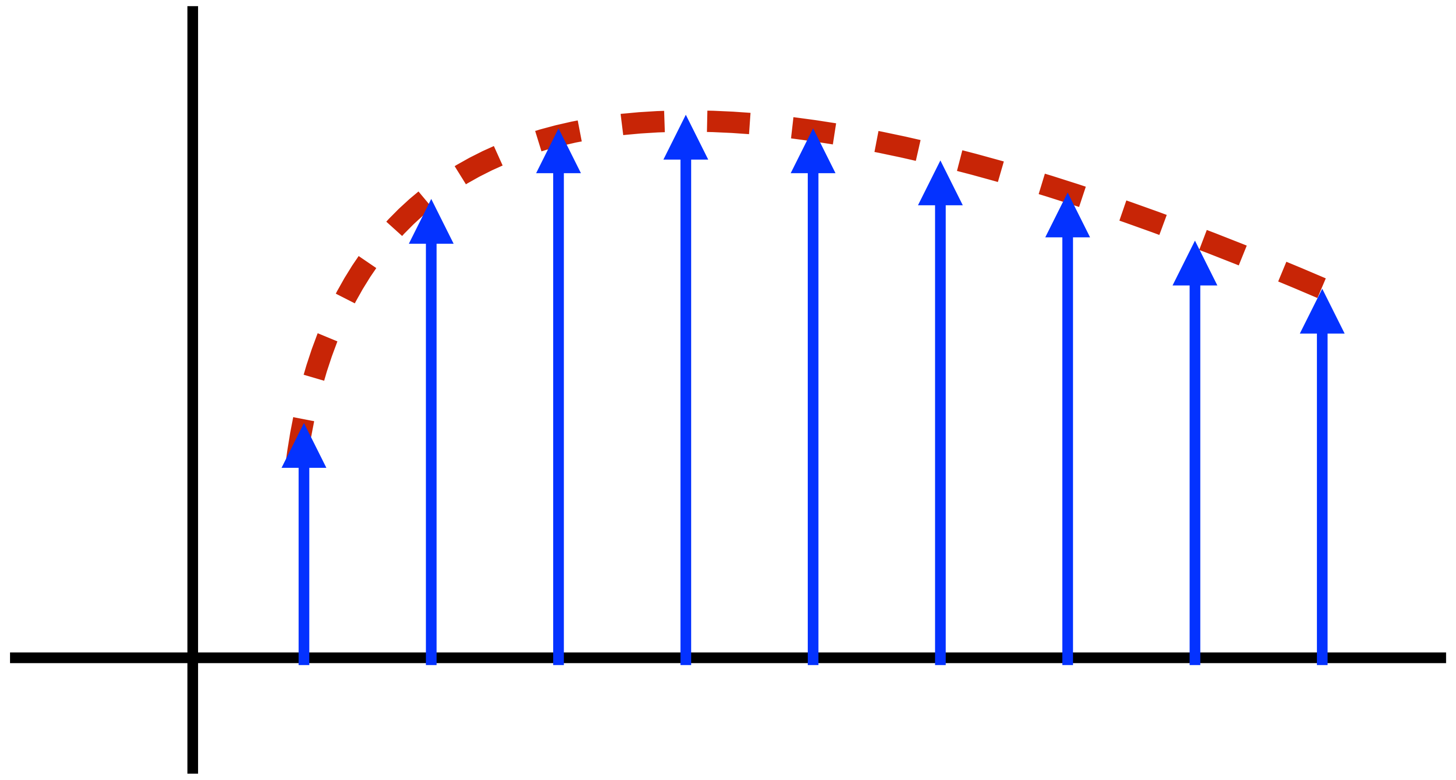
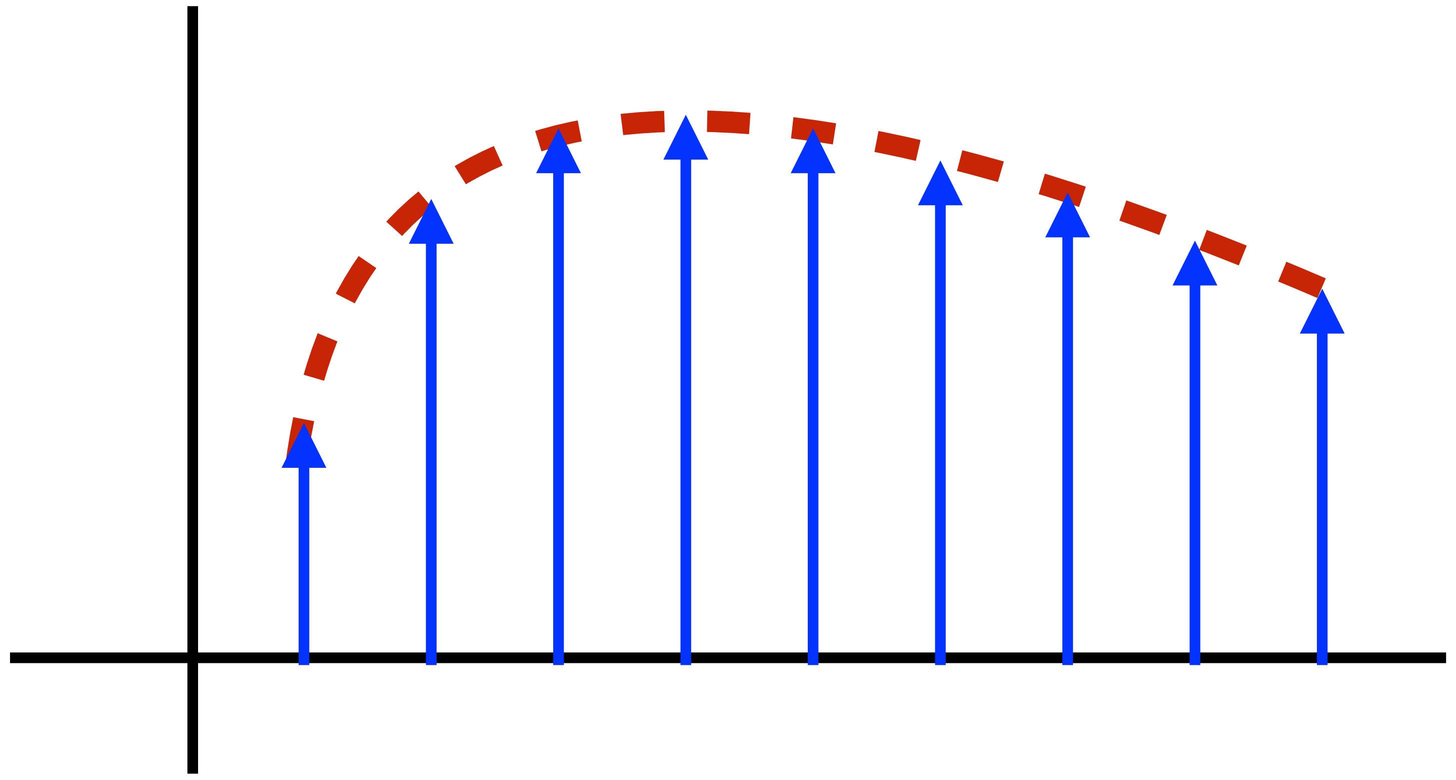
Sampling in Primal Domain is Convolution in Fourier Domain



$$f(x)\,\mathbf{S}(x)$$

Sampling in Primal Domain is Convolution in Fourier Domain

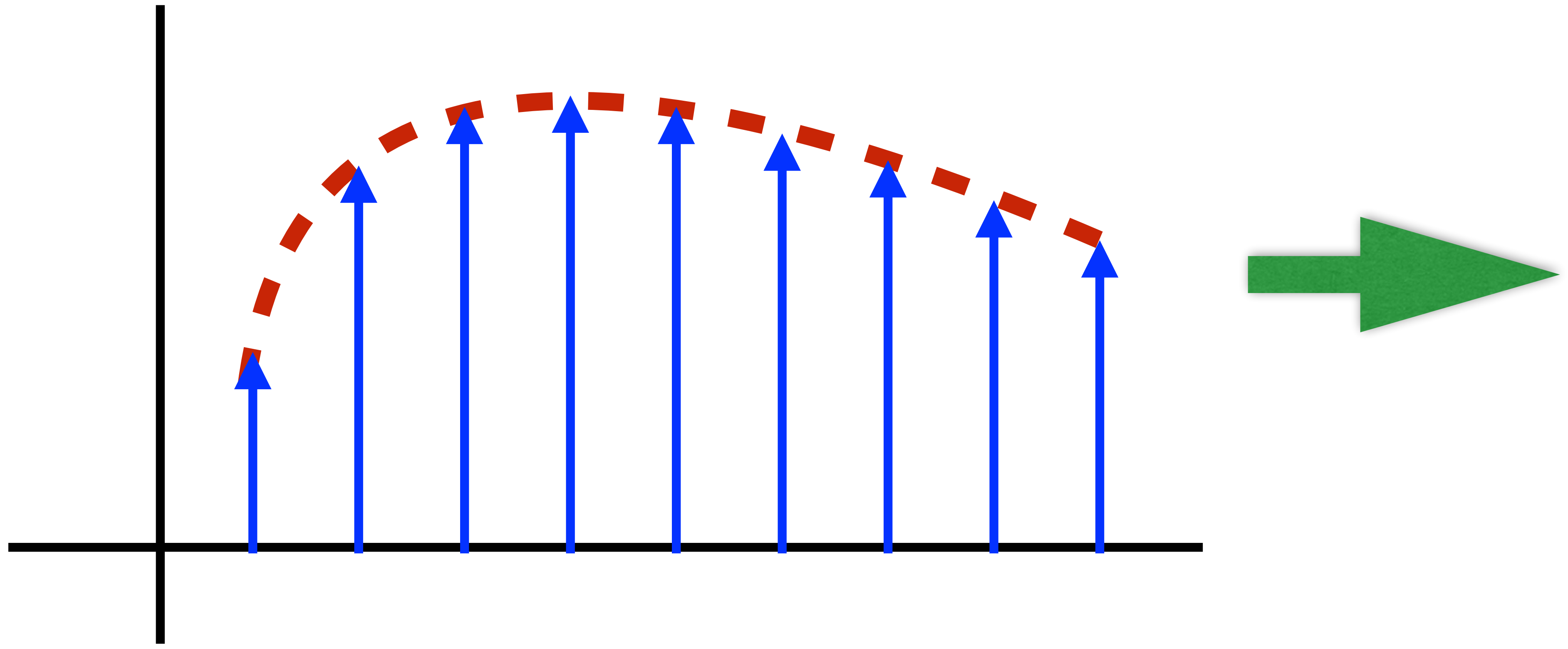


$f(x)\,\mathbf{S}(x)$

Fredo Durand [2011]

Render the Possibilities SIGGRAPH2016

Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$

Fredo Durand [2011]

Render the Possibilities
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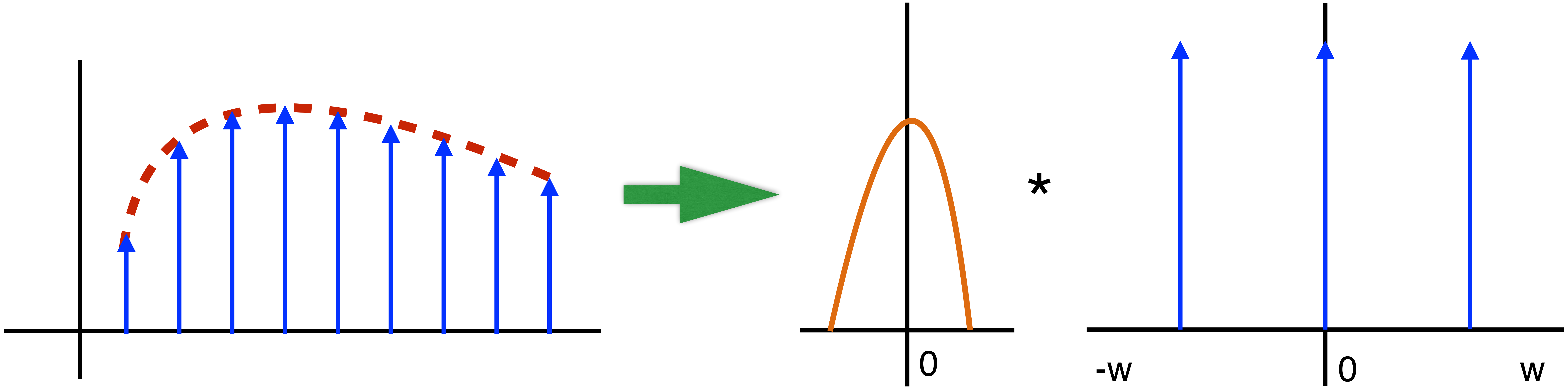
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Render the Possibilities
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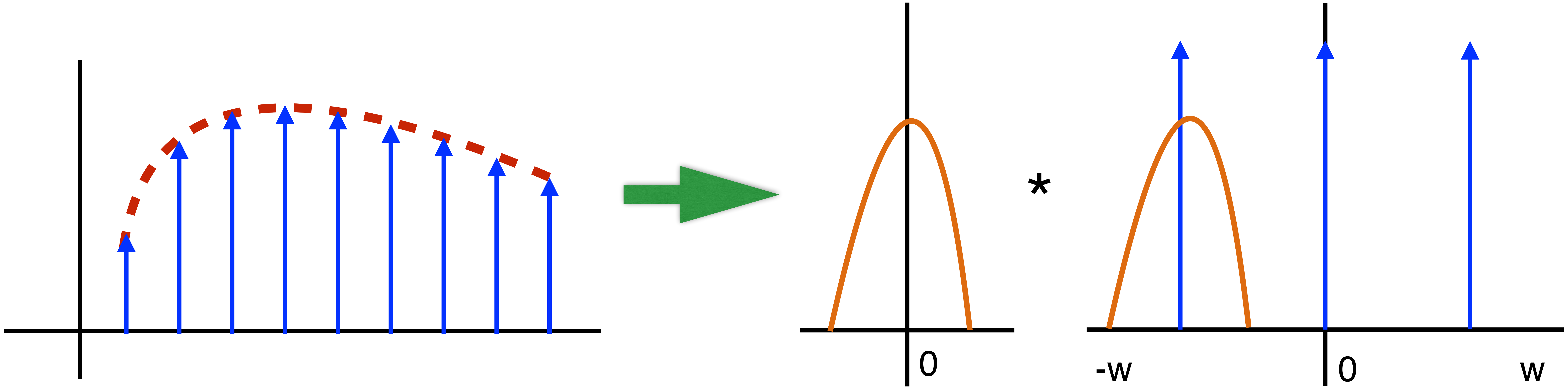
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Render the Possibilities
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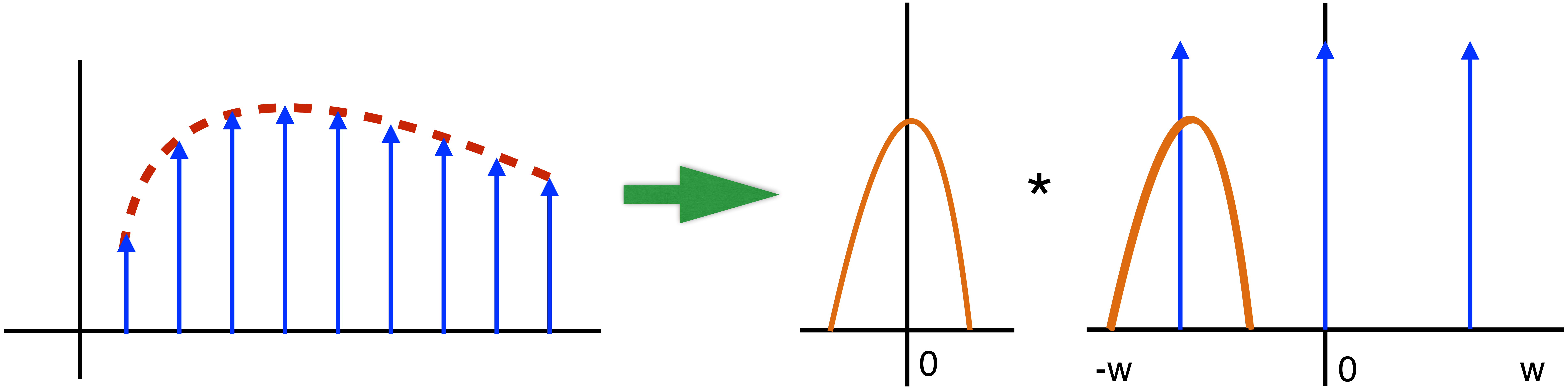
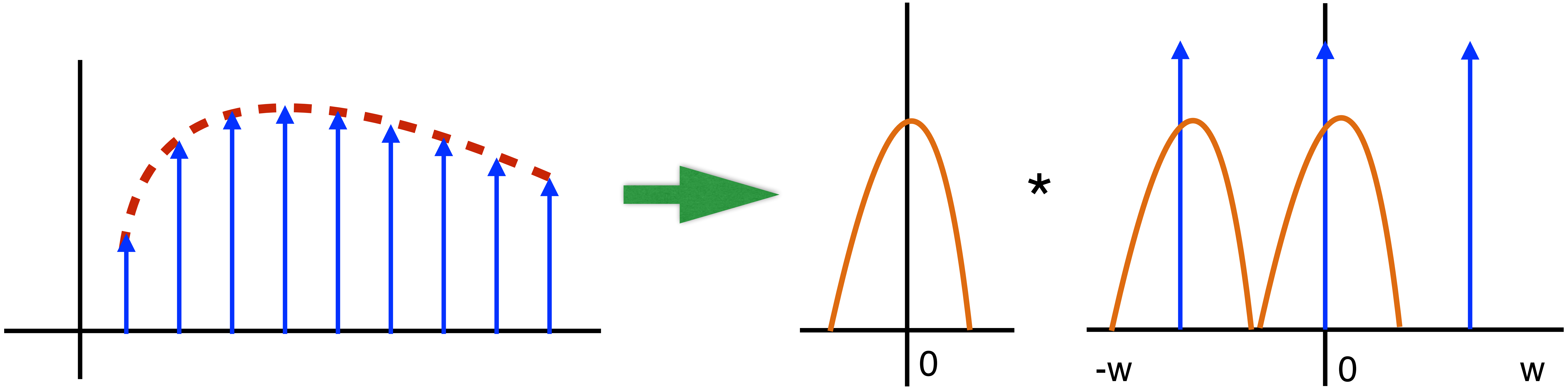


$f(x)\,\mathbf{S}(x)$ $\quad\quad$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$ $\qquad$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Render the Possibilities
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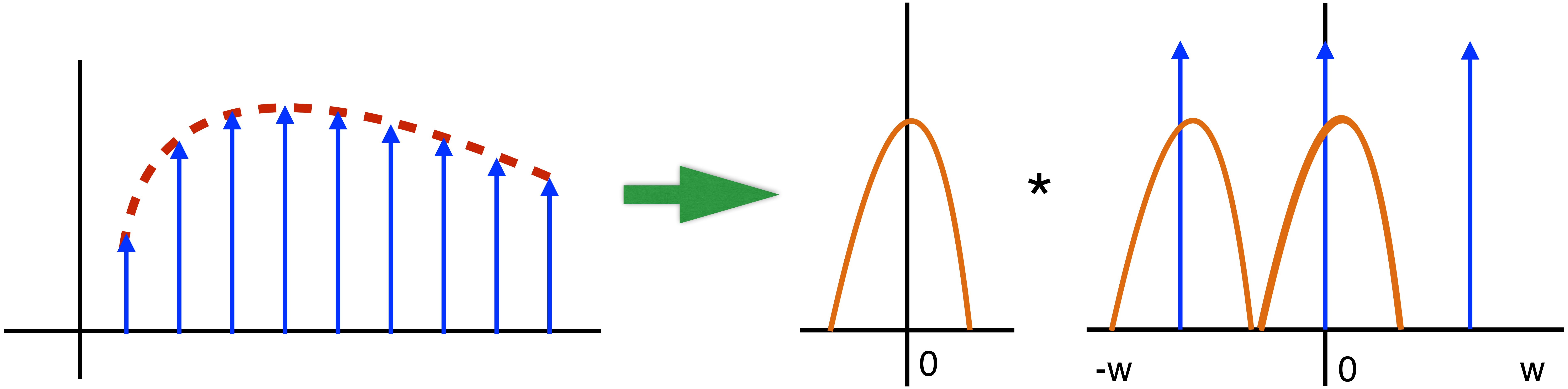
Sampling in Primal Domain is Convolution in Fourier Domain

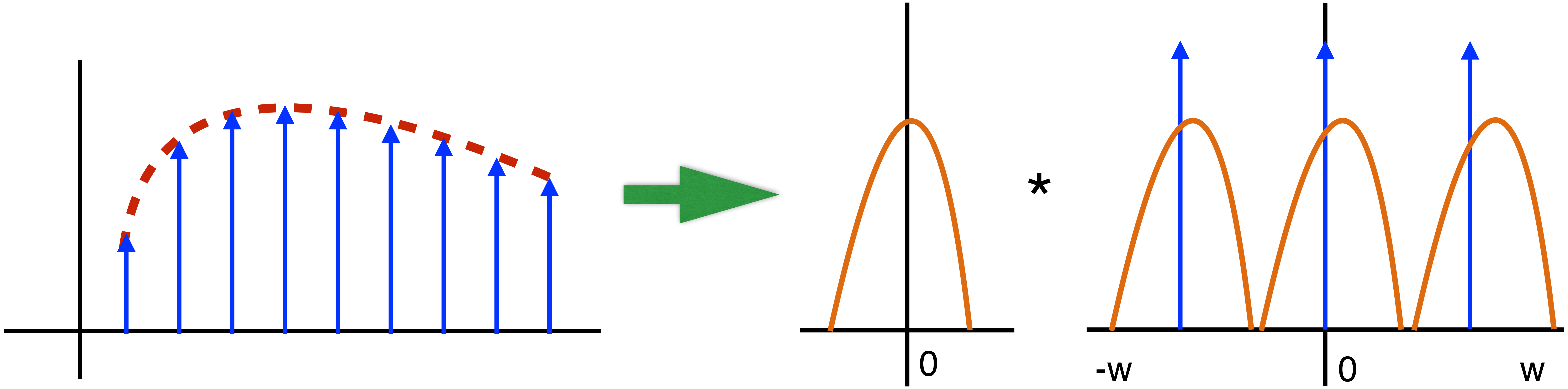


$$f(x)\,\mathbf{S}(x) \qquad \hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$$

Fredo Durand [2011]

Render the Possibilities SIGGRAPH2016

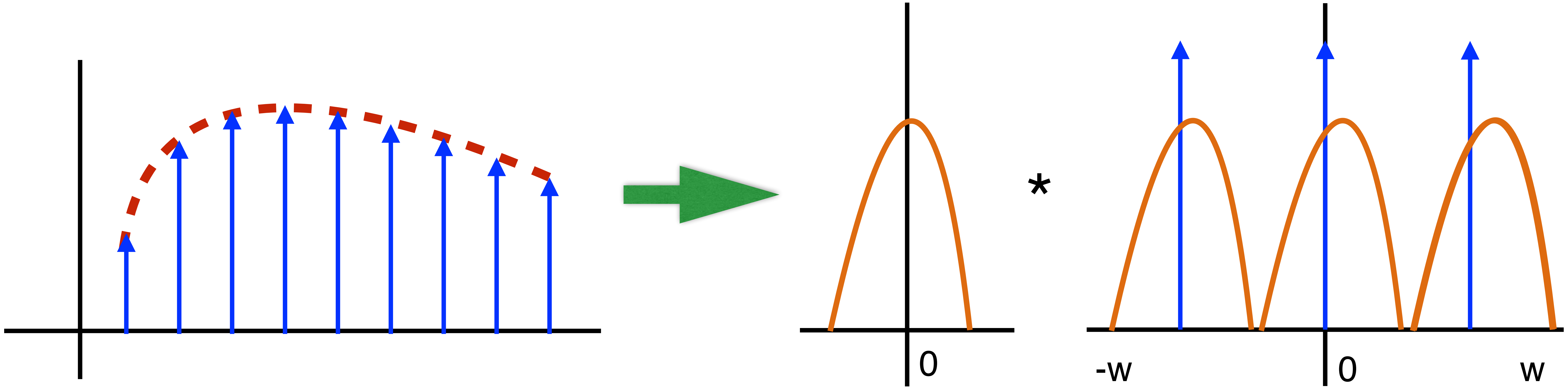
Sampling in Primal Domain is Convolution in Fourier Domain



$f(x)\,\mathbf{S}(x)$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

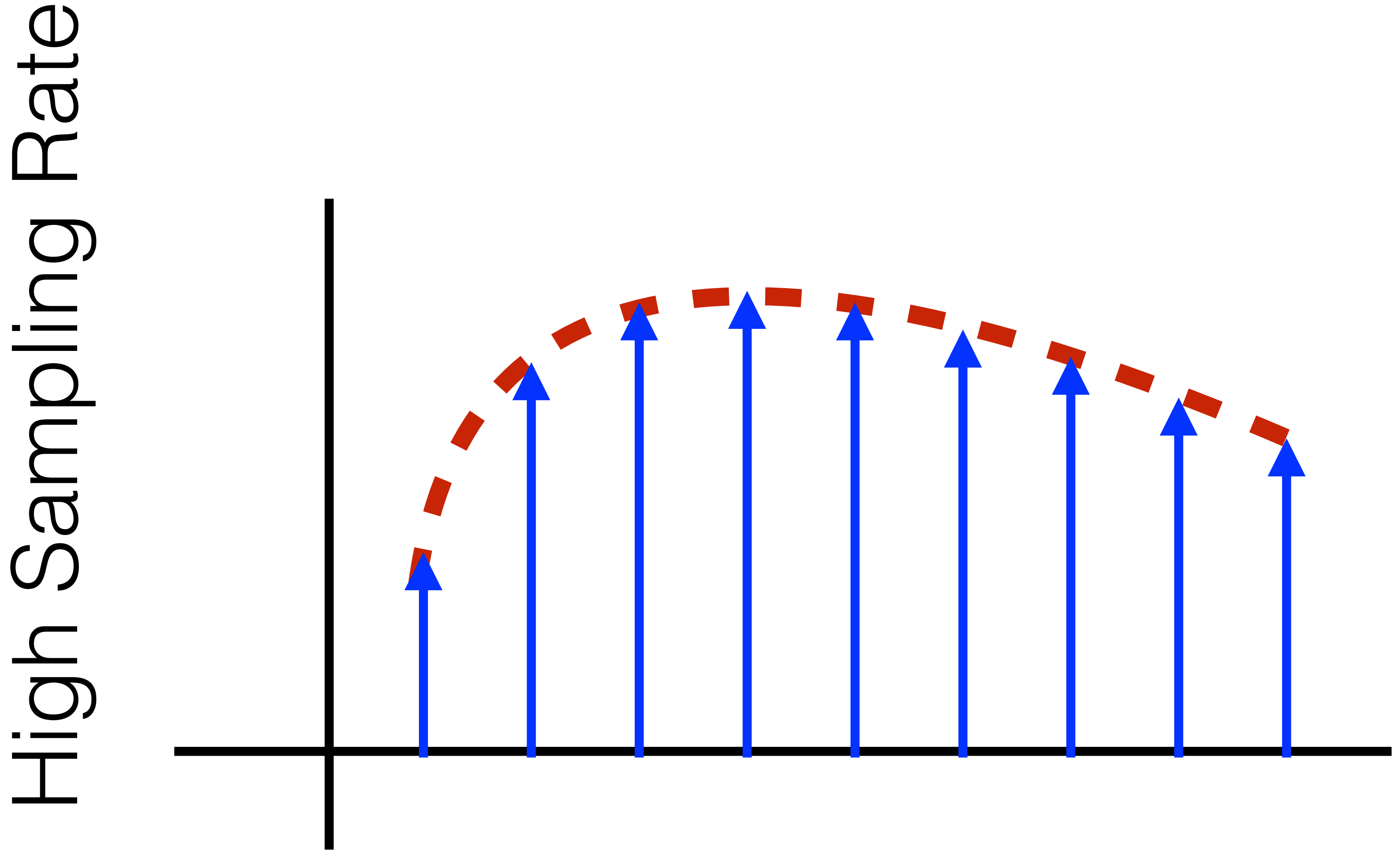
Sampling in Primal Domain is Convolution in Fourier Domain

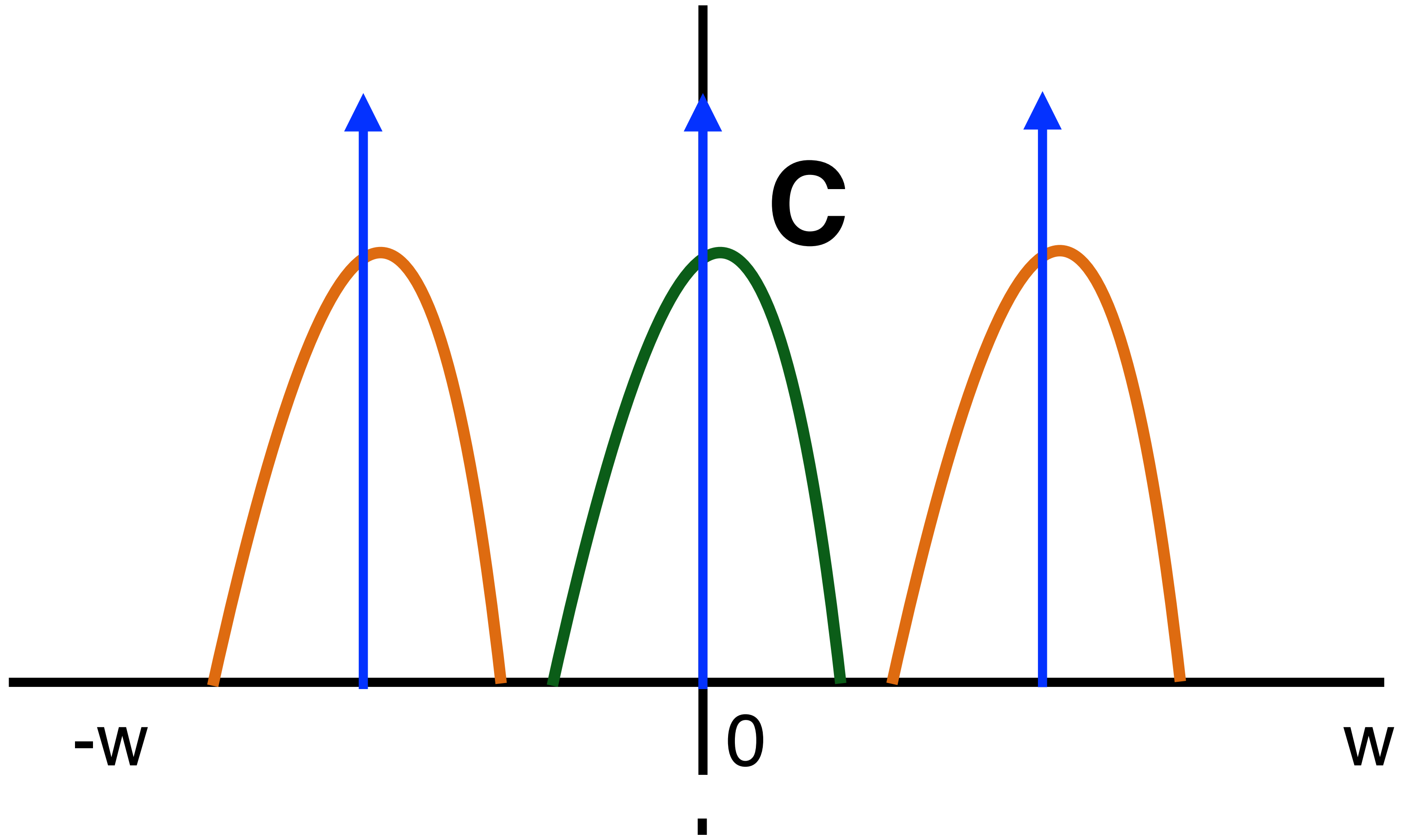


$f(x)\,\mathbf{S}(x)$ $\hat{f}(\omega) \otimes \hat{\mathbf{S}}(\omega)$

Fredo Durand [2011]

Aliasing in Reconstruction

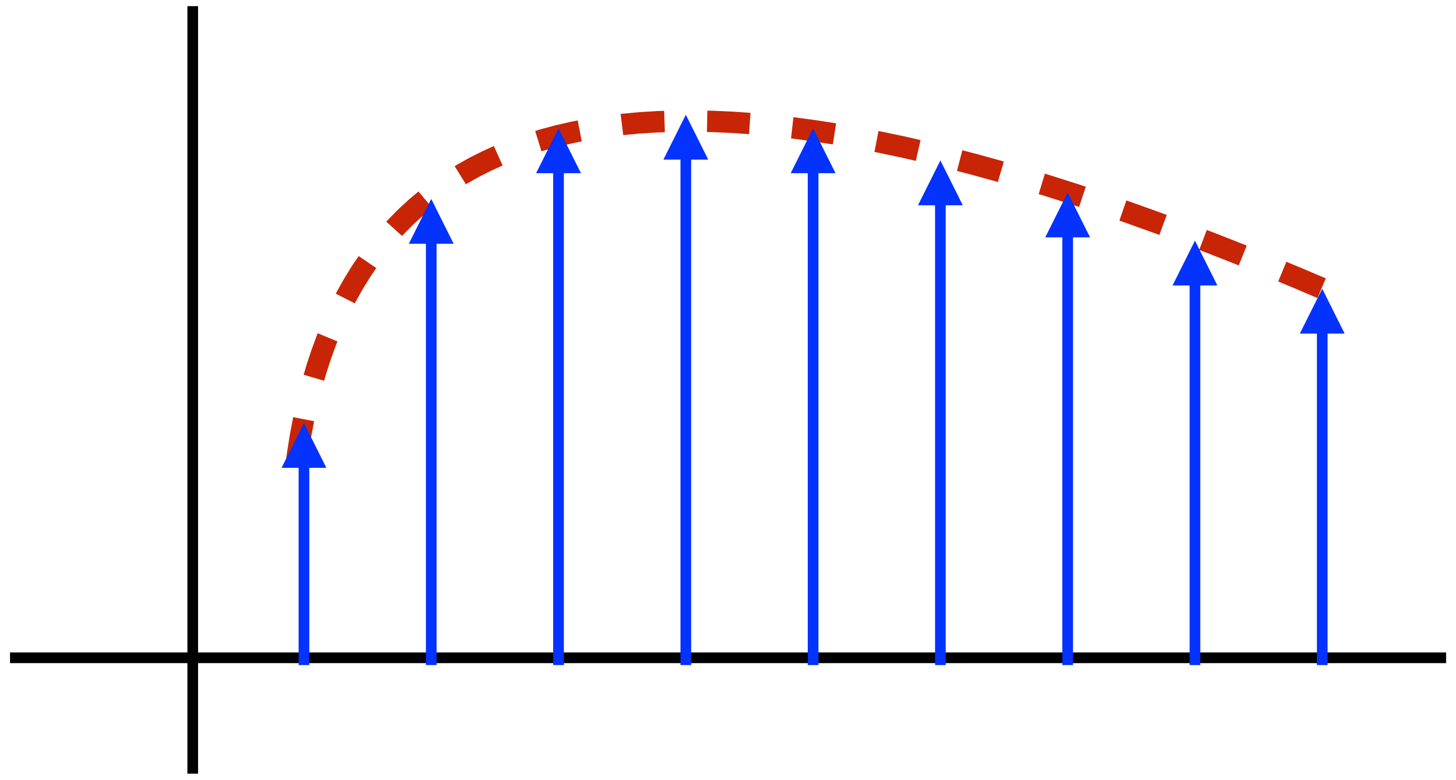


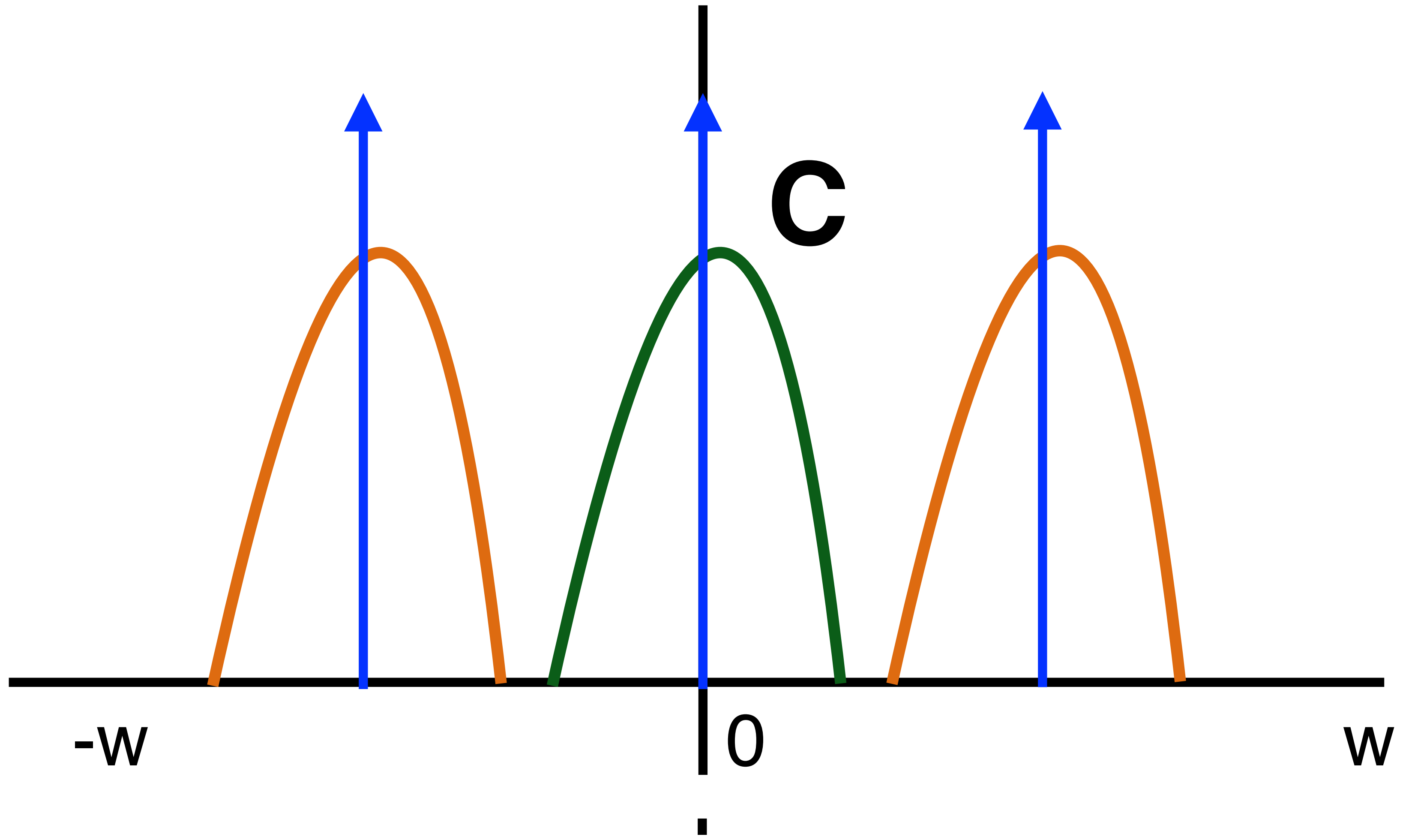


Render the Possibilities
SIGGRAPH2016

Aliasing in Reconstruction

High Sampling Rate

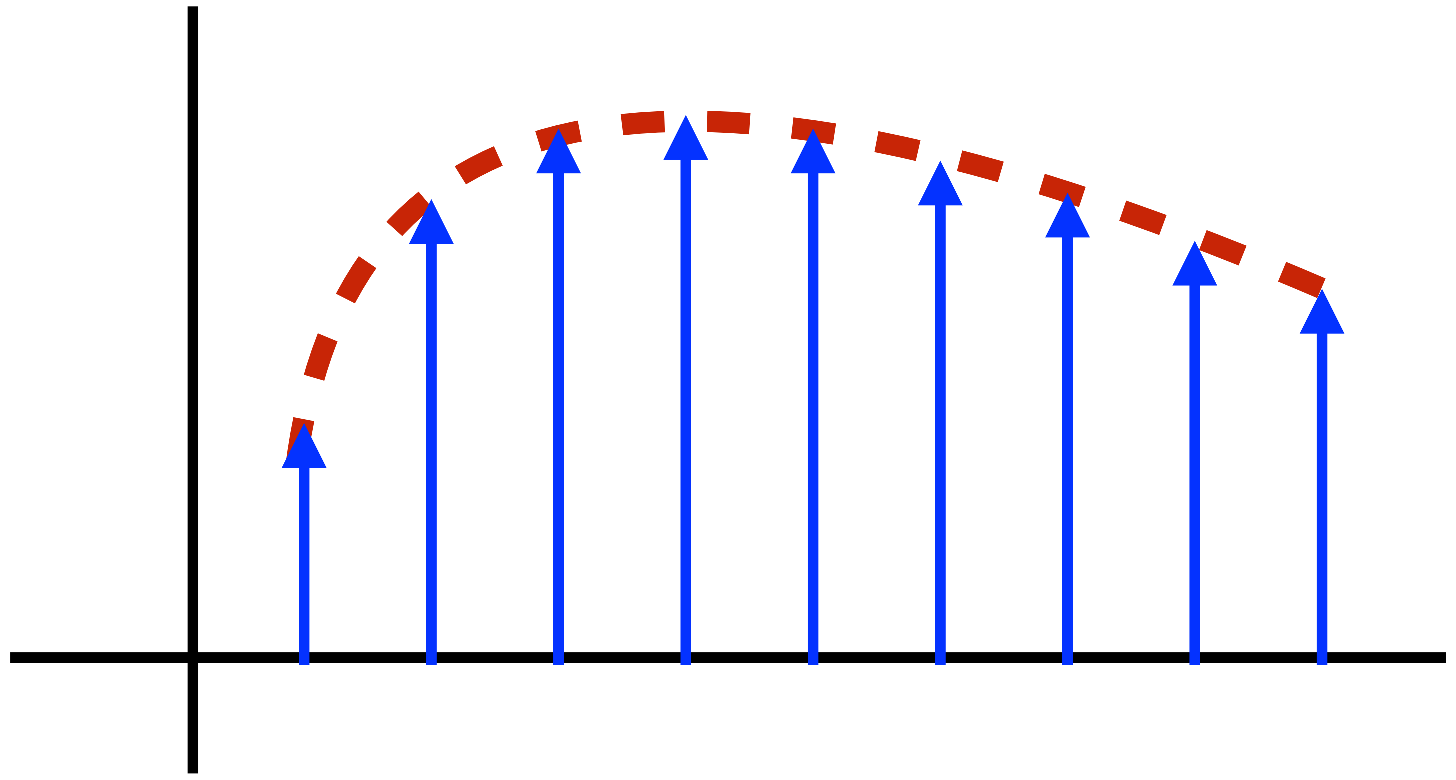


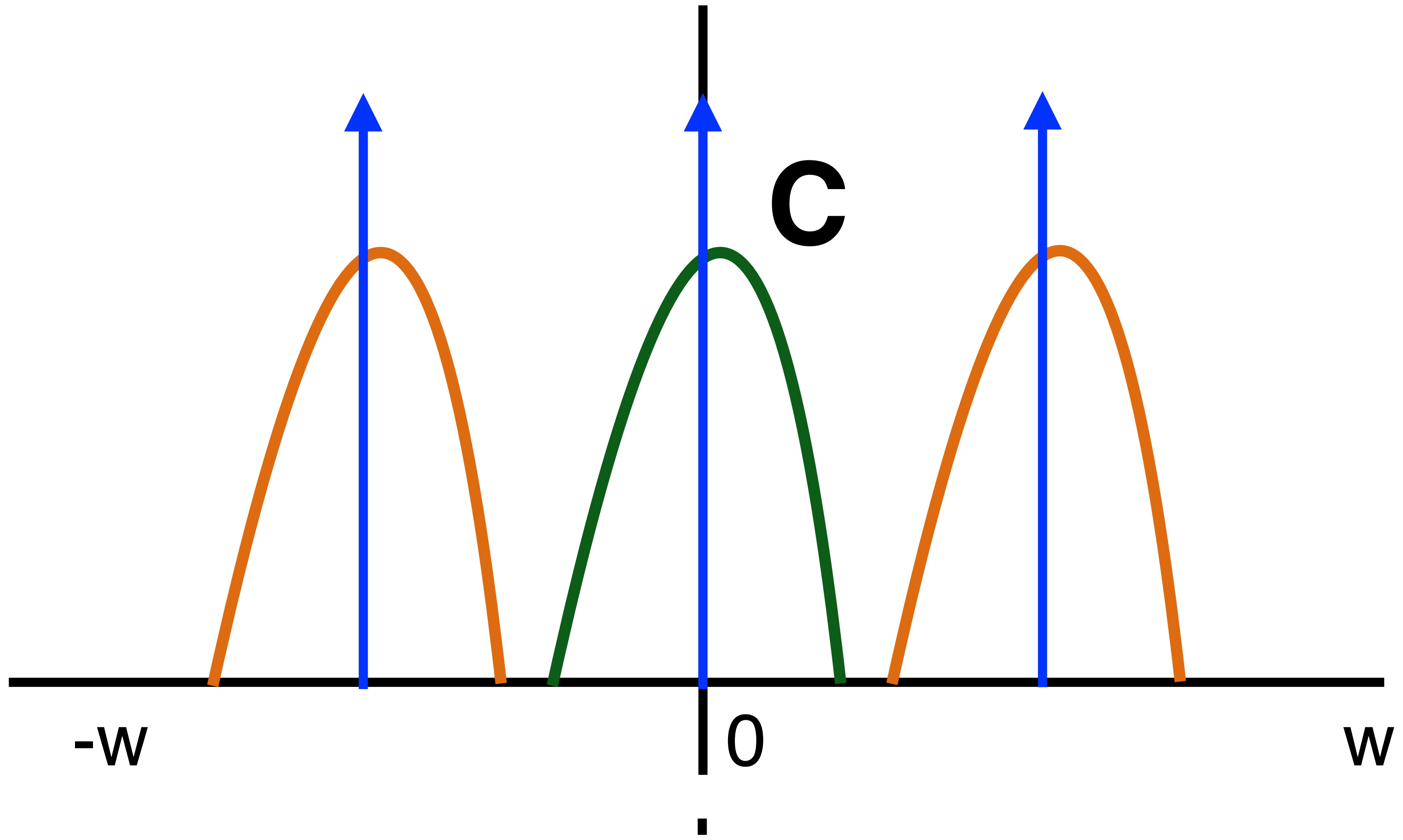


Render the Possibilities
SIGGRAPH2016

Aliasing in Reconstruction

High Sampling Rate

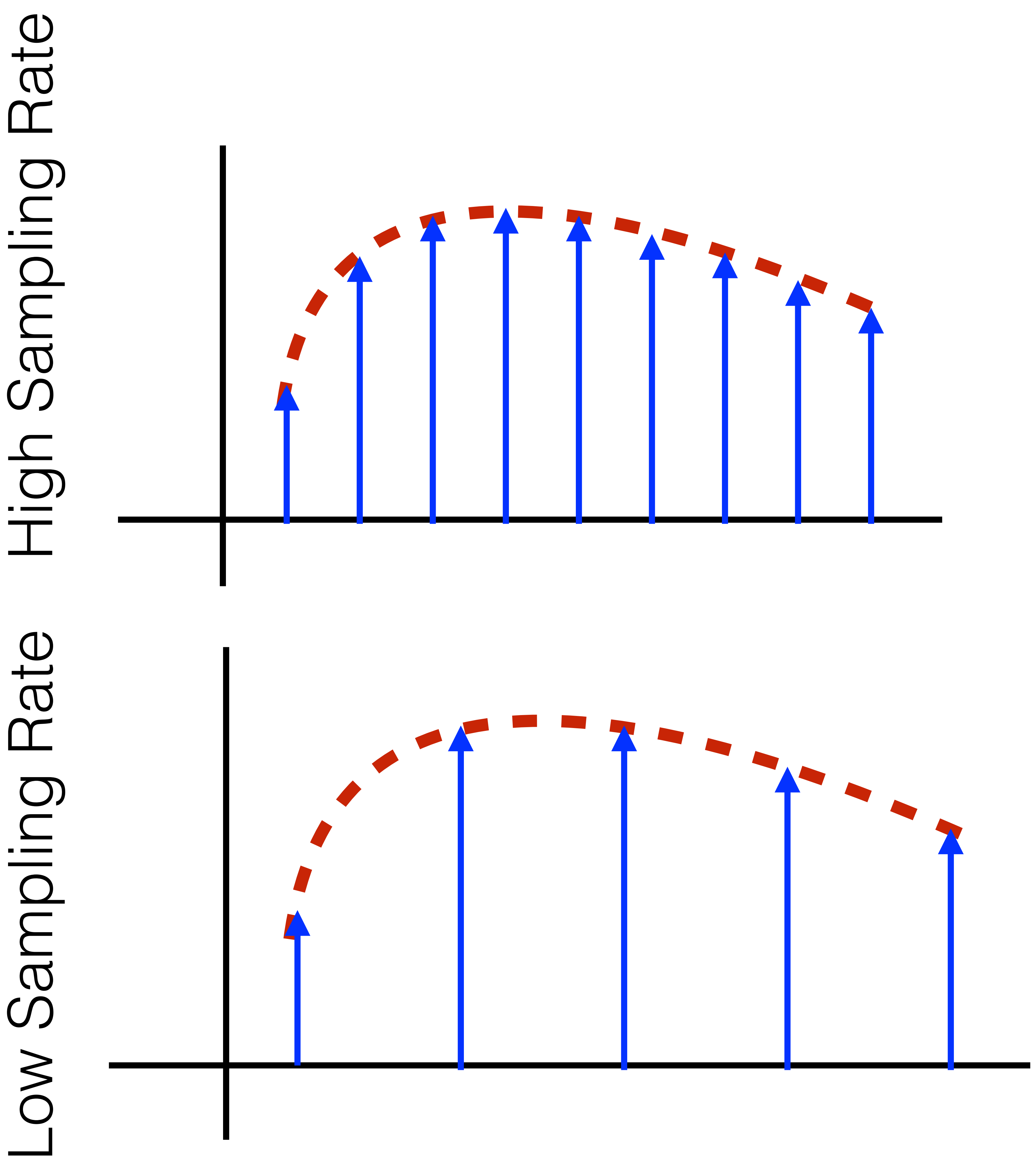


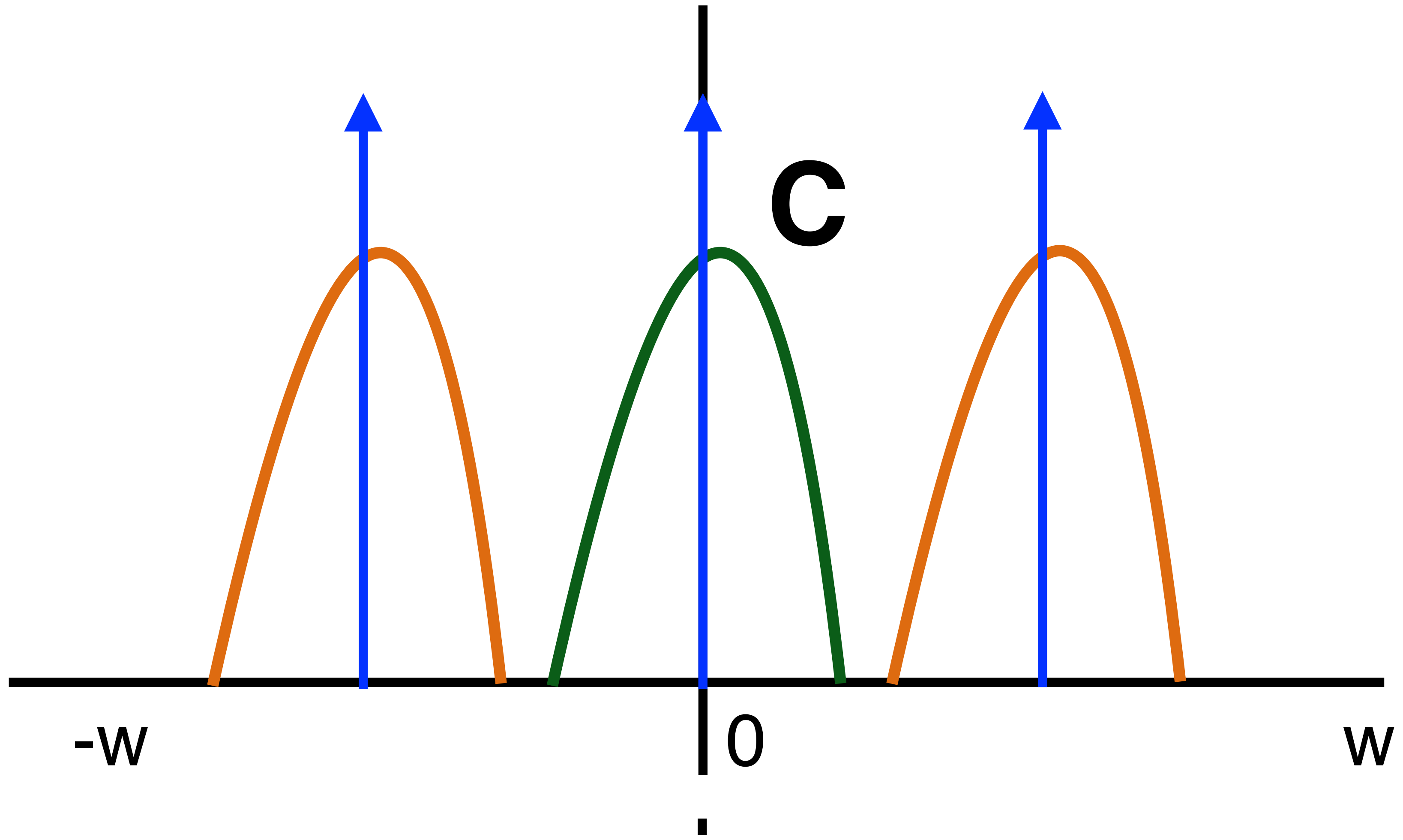


Low Sampling Rate

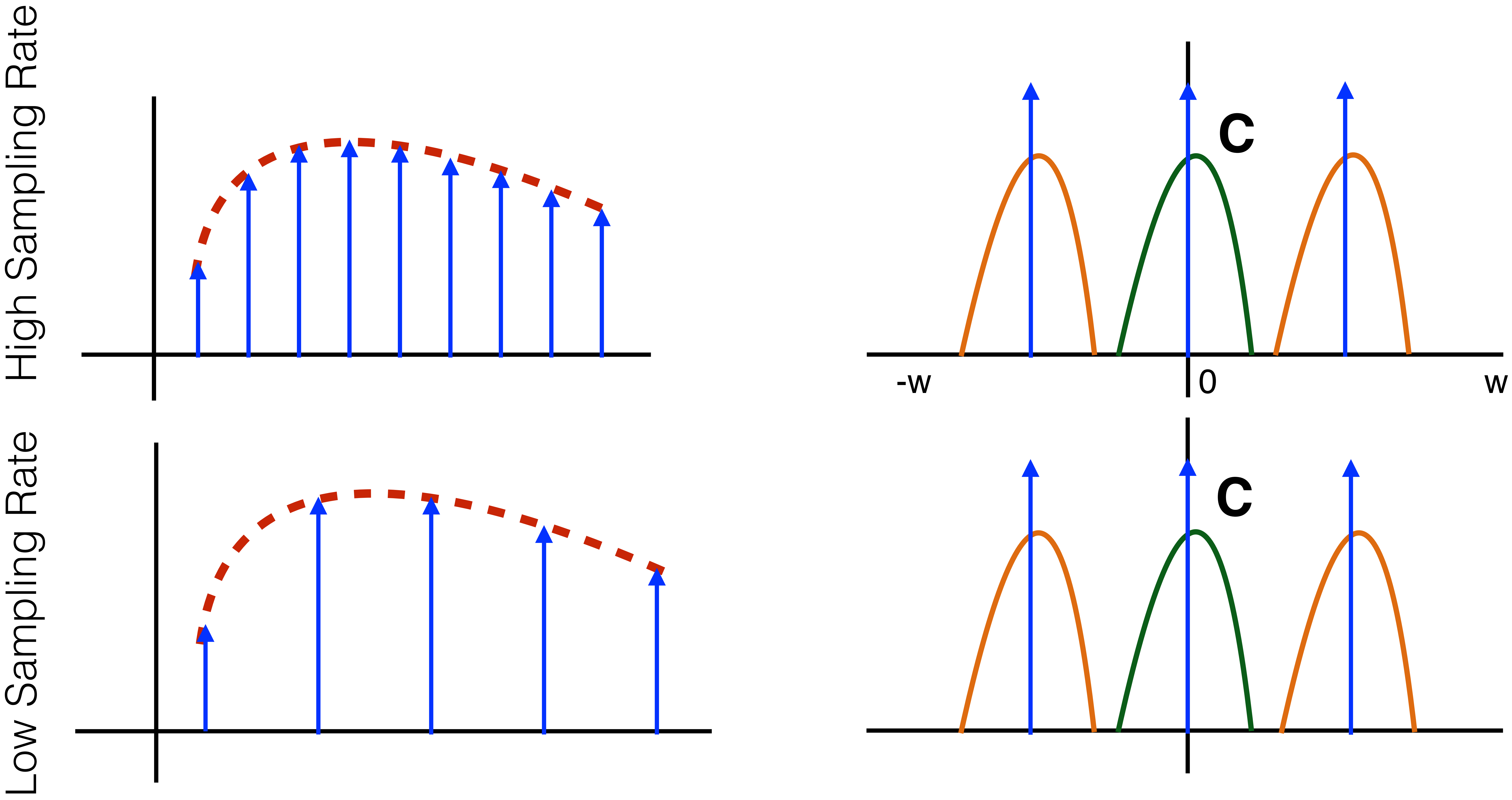
Render the Possibilities
SIGGRAPH2016

Aliasing in Reconstruction





Aliasing in Reconstruction



Render the Possibilities
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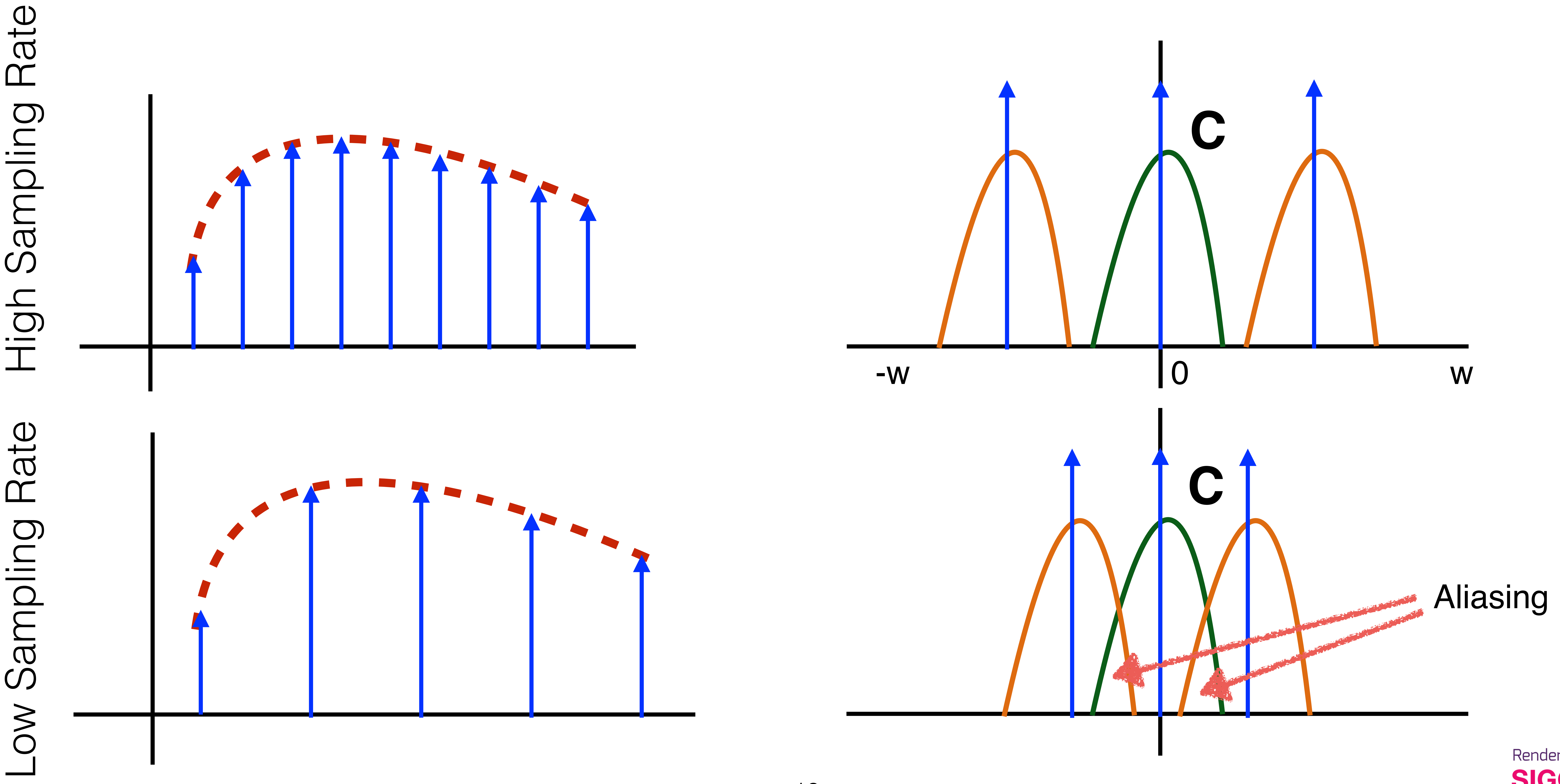
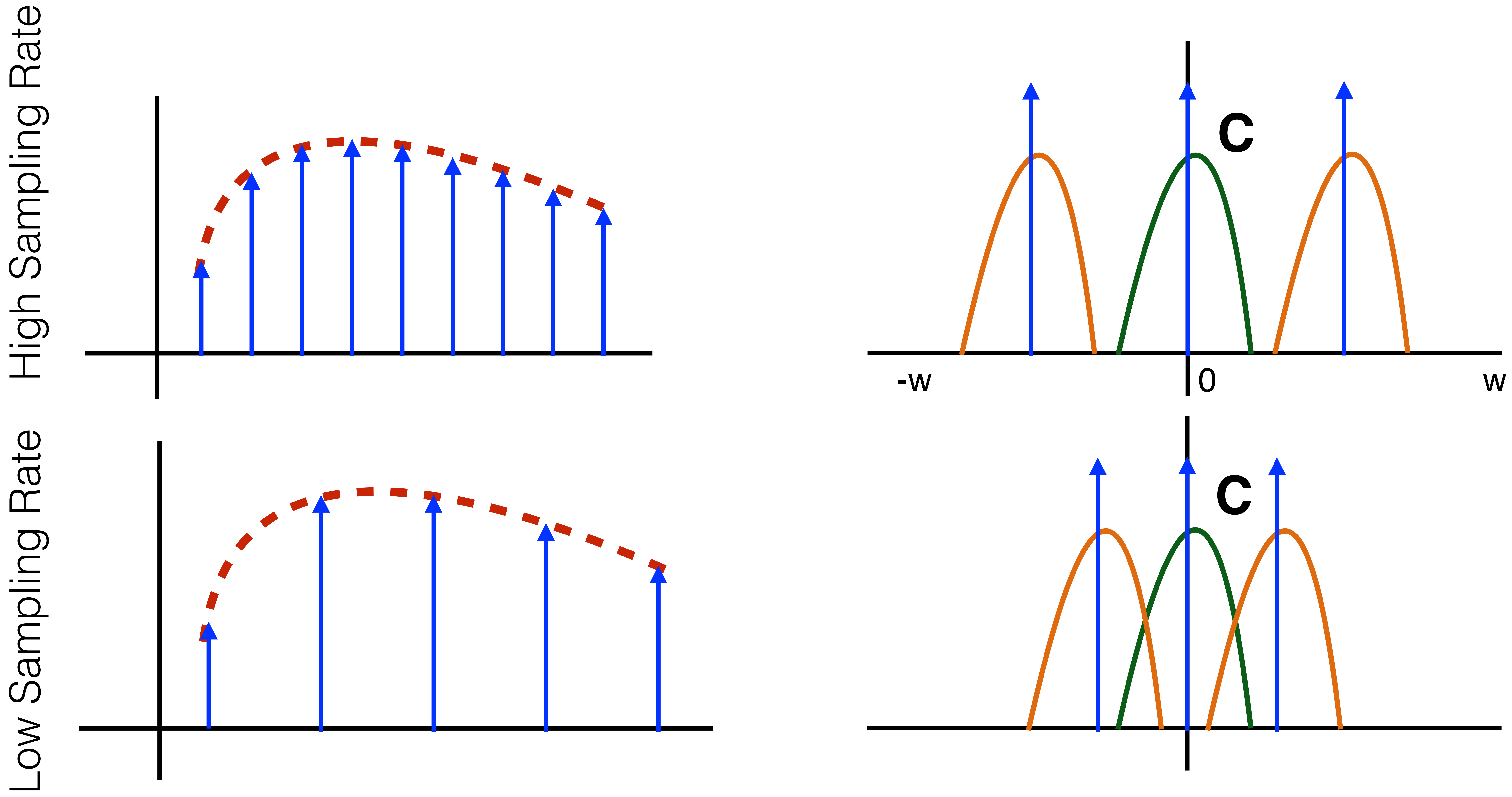
Aliasing in Reconstruction



Render the Possibilities
SIGGRAPH2016

Aliasing in Reconstruction



Render the Possibilities
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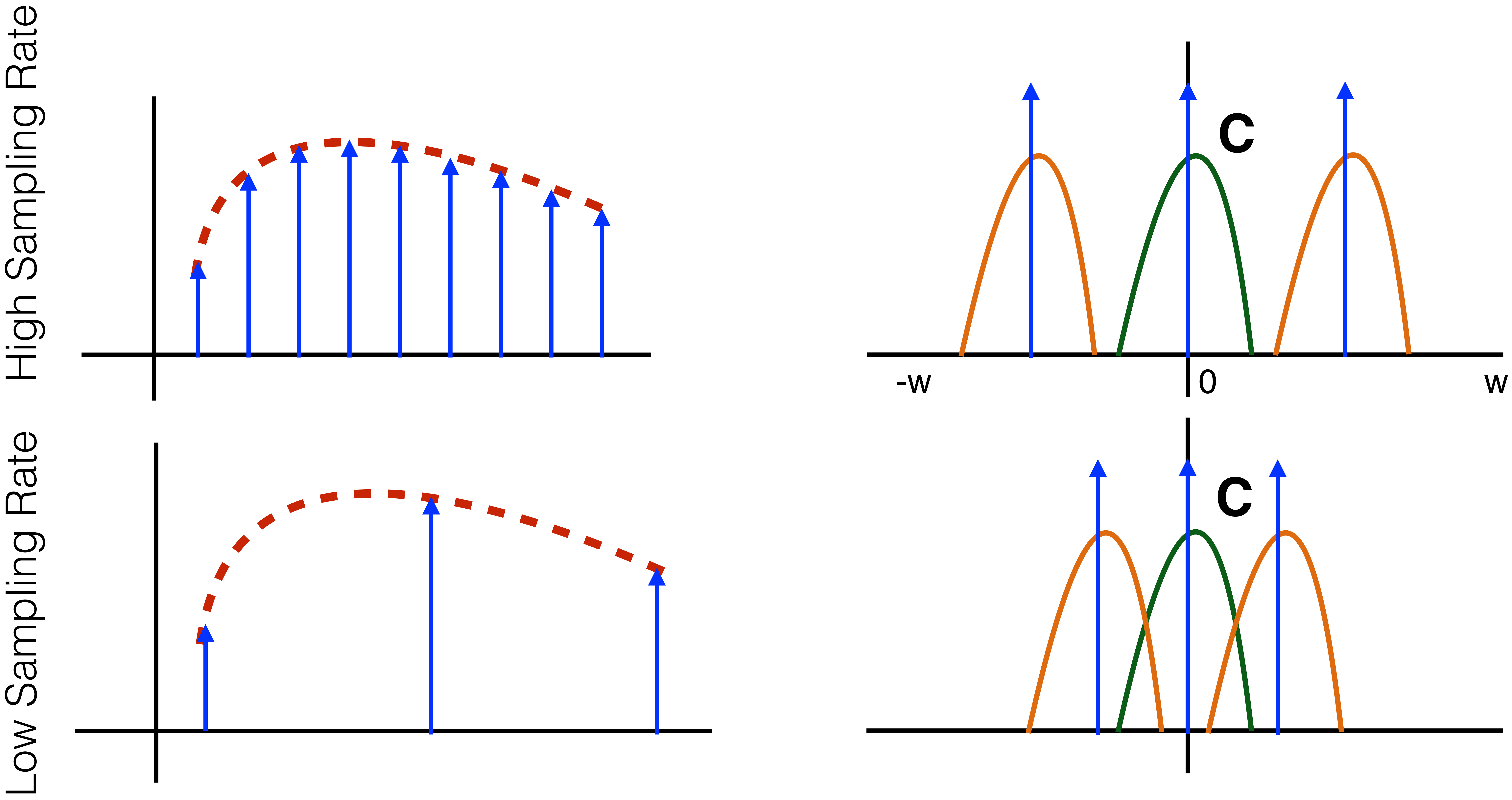
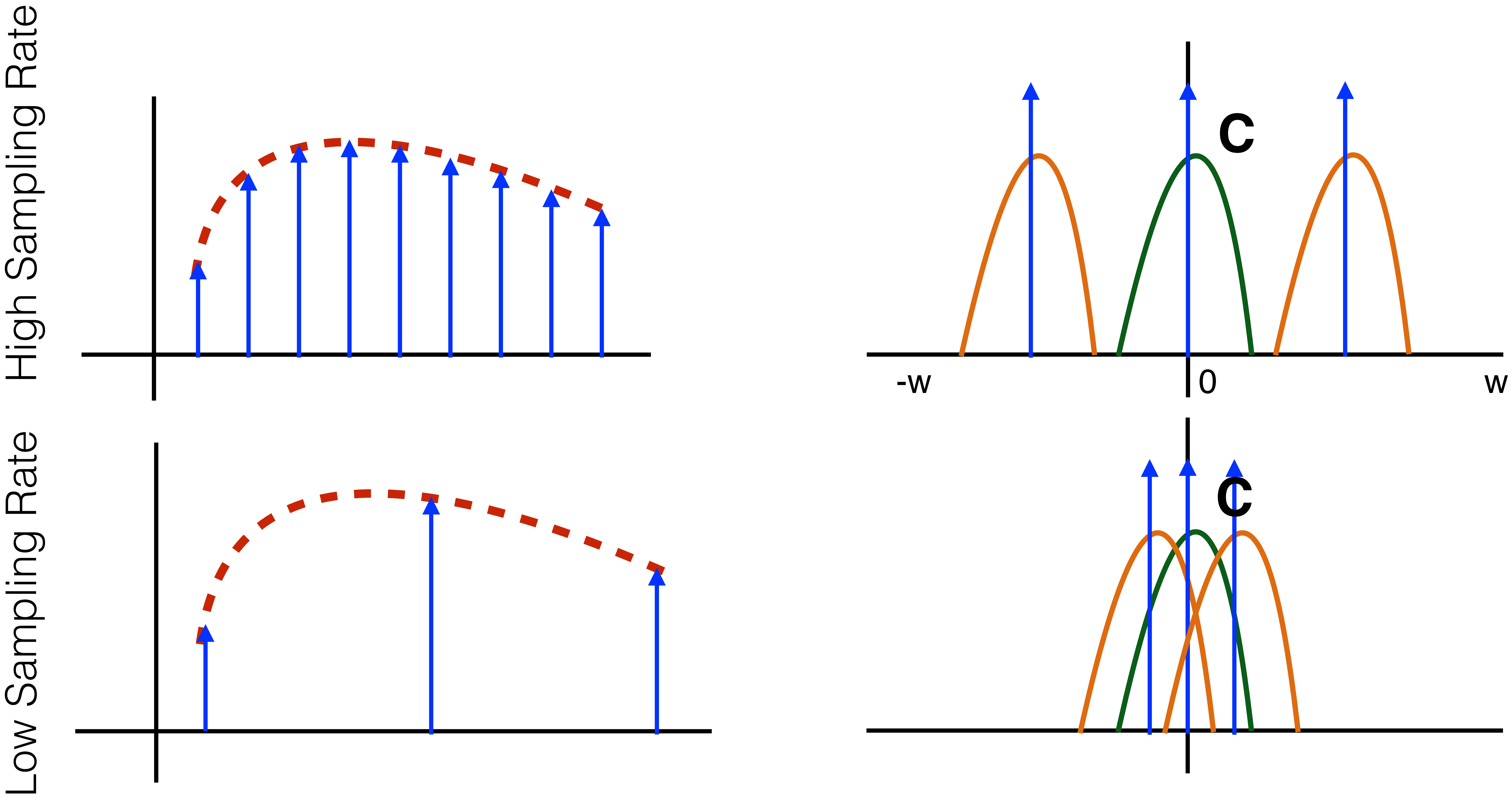
Error in Monte Carlo Integration



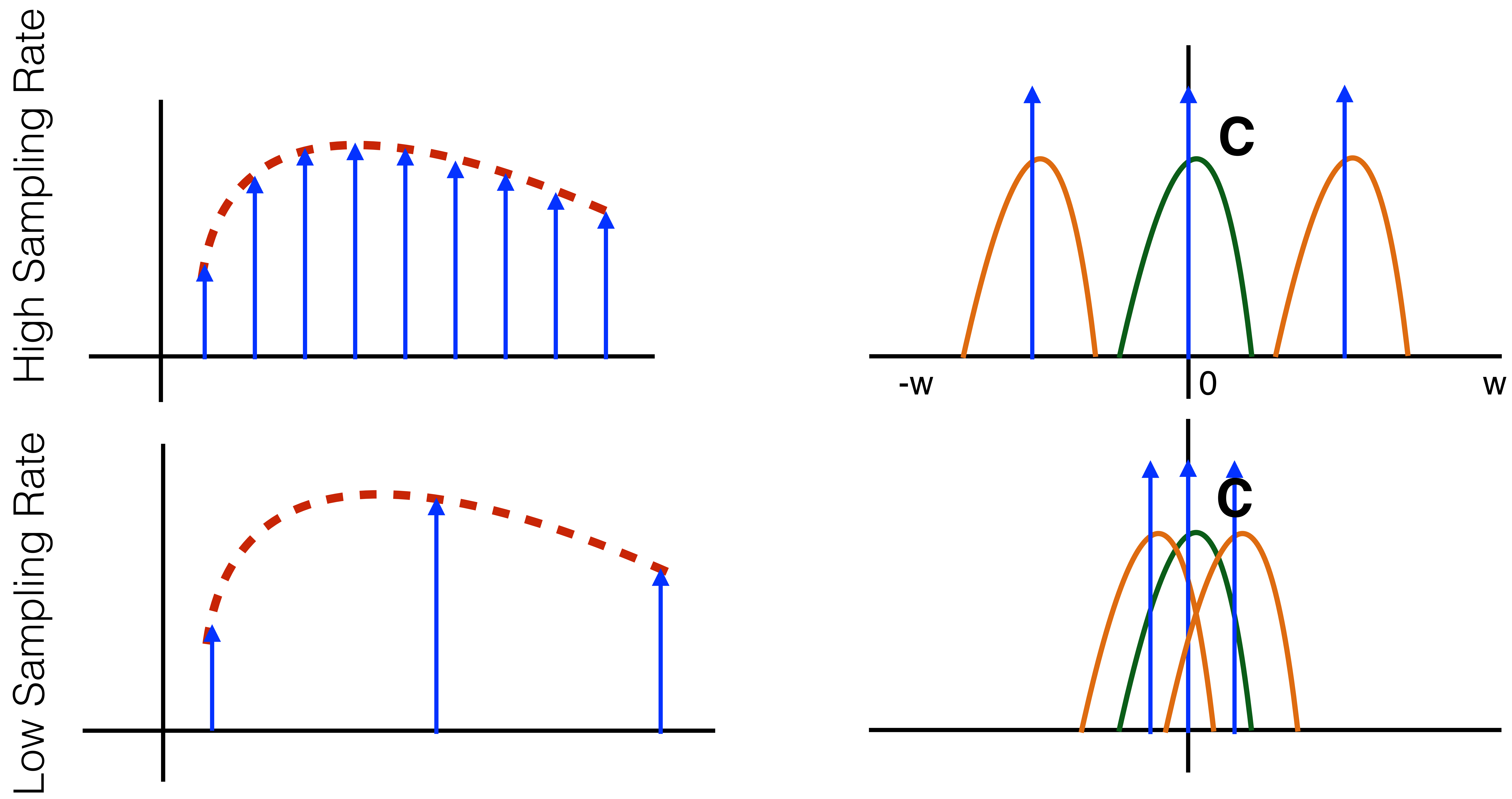
Render the Possibilities
SIGGRAPH2016

Error in Monte Carlo Integration



Render the Possibilities
SIGGRAPH2016

Error in Monte Carlo Integration



Render the Possibilities
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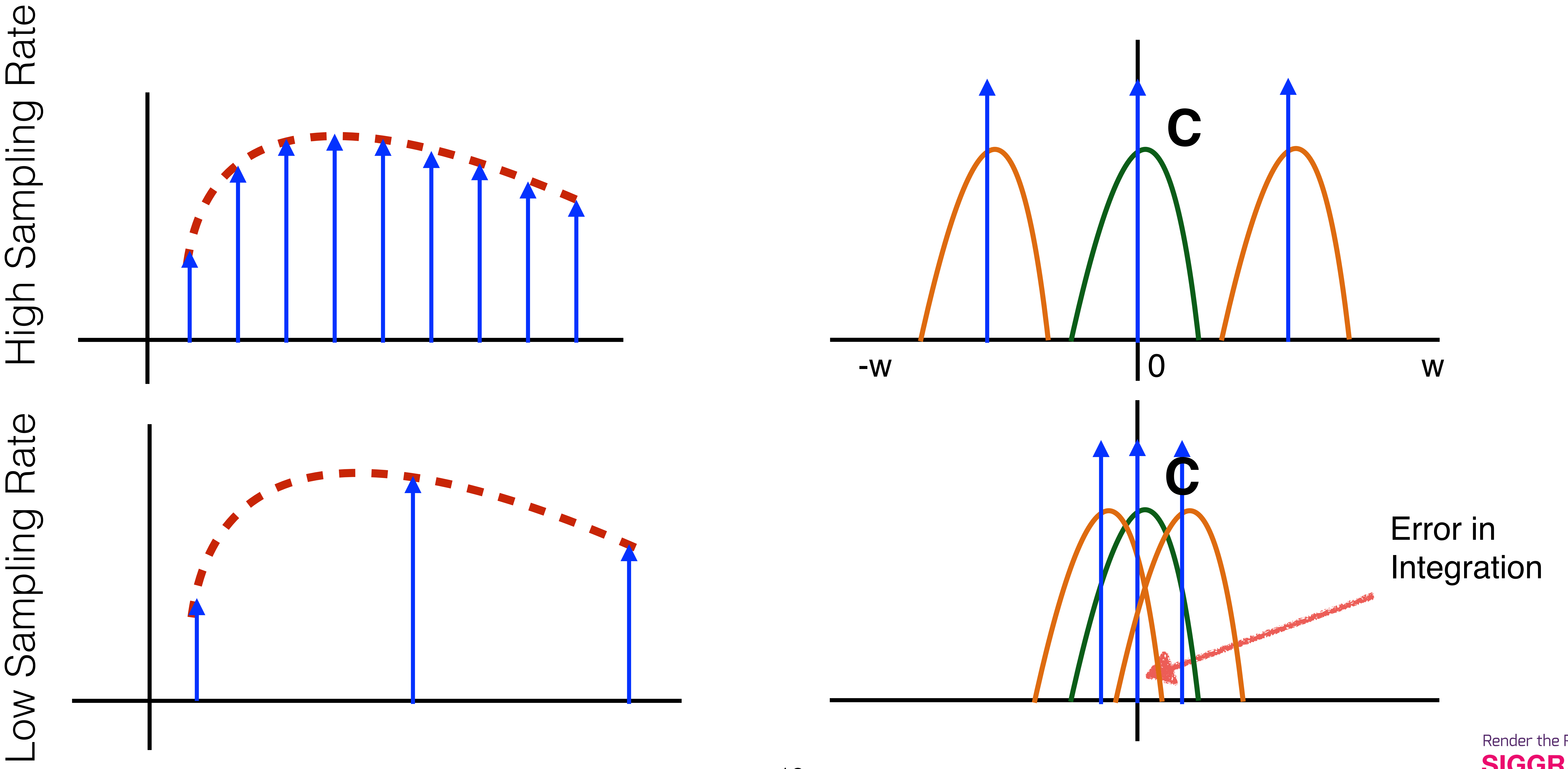
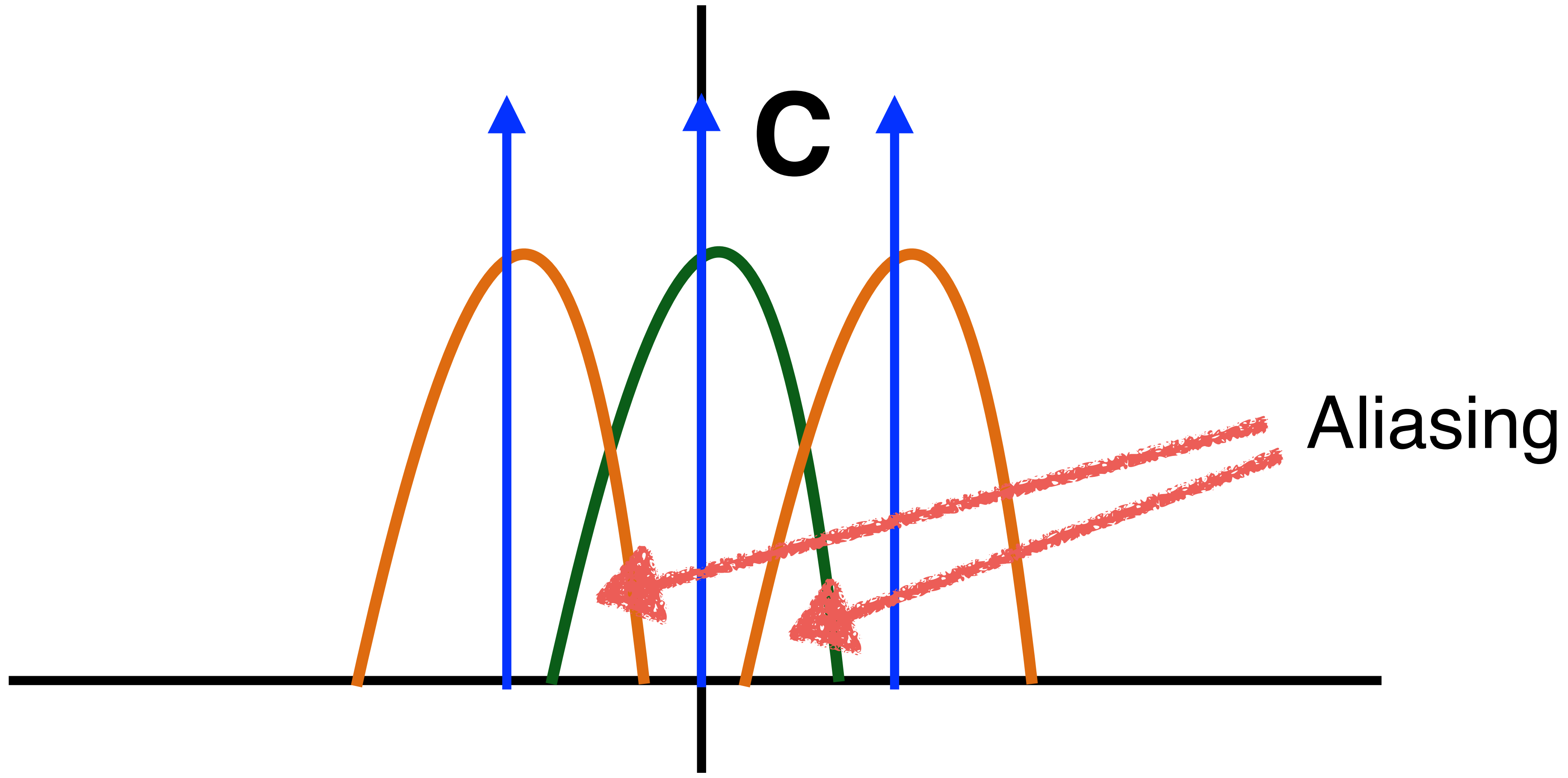
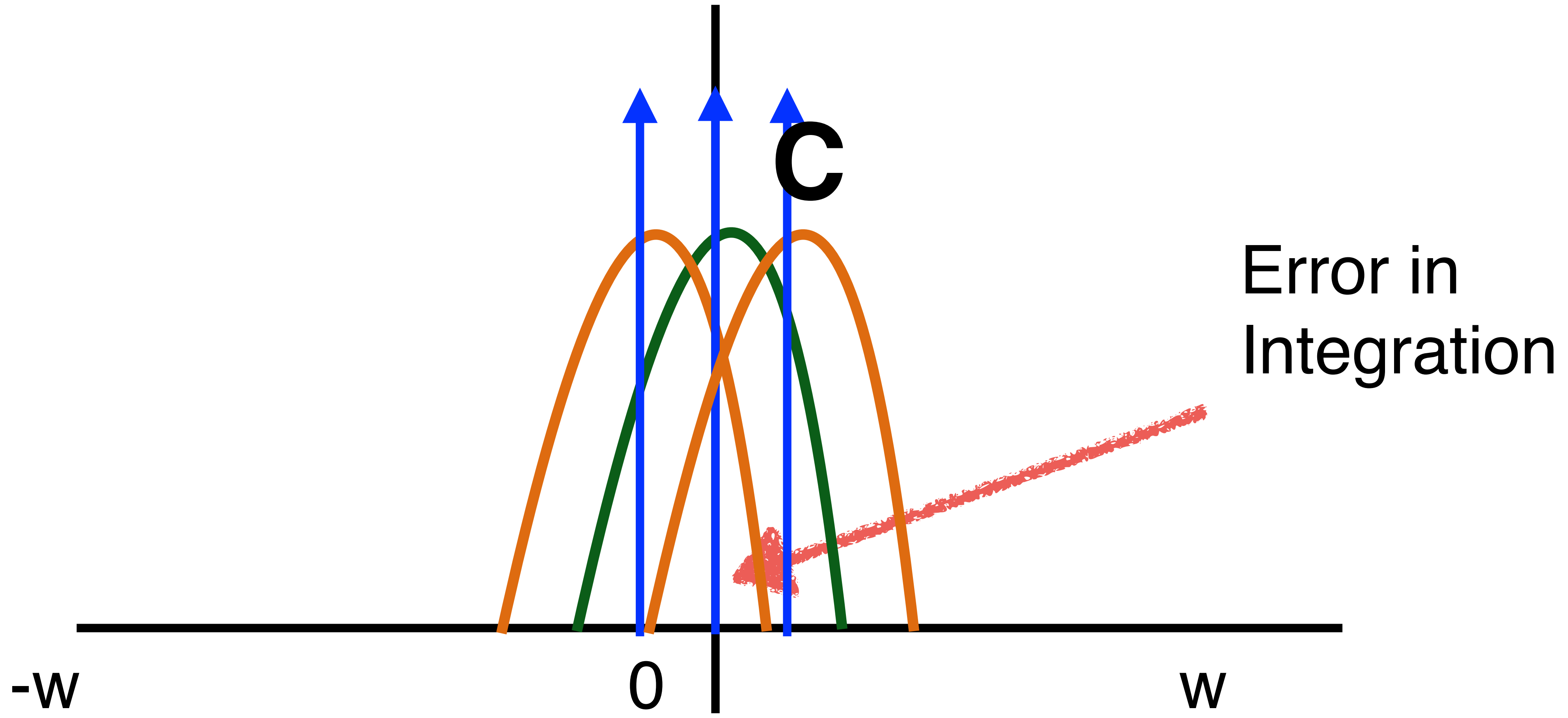
Error in Monte Carlo Integration



Render the Possibilities
SIGGRAPH2016

Aliasing (Reconstruction) vs. Error (Integration)





Render the Possibilities
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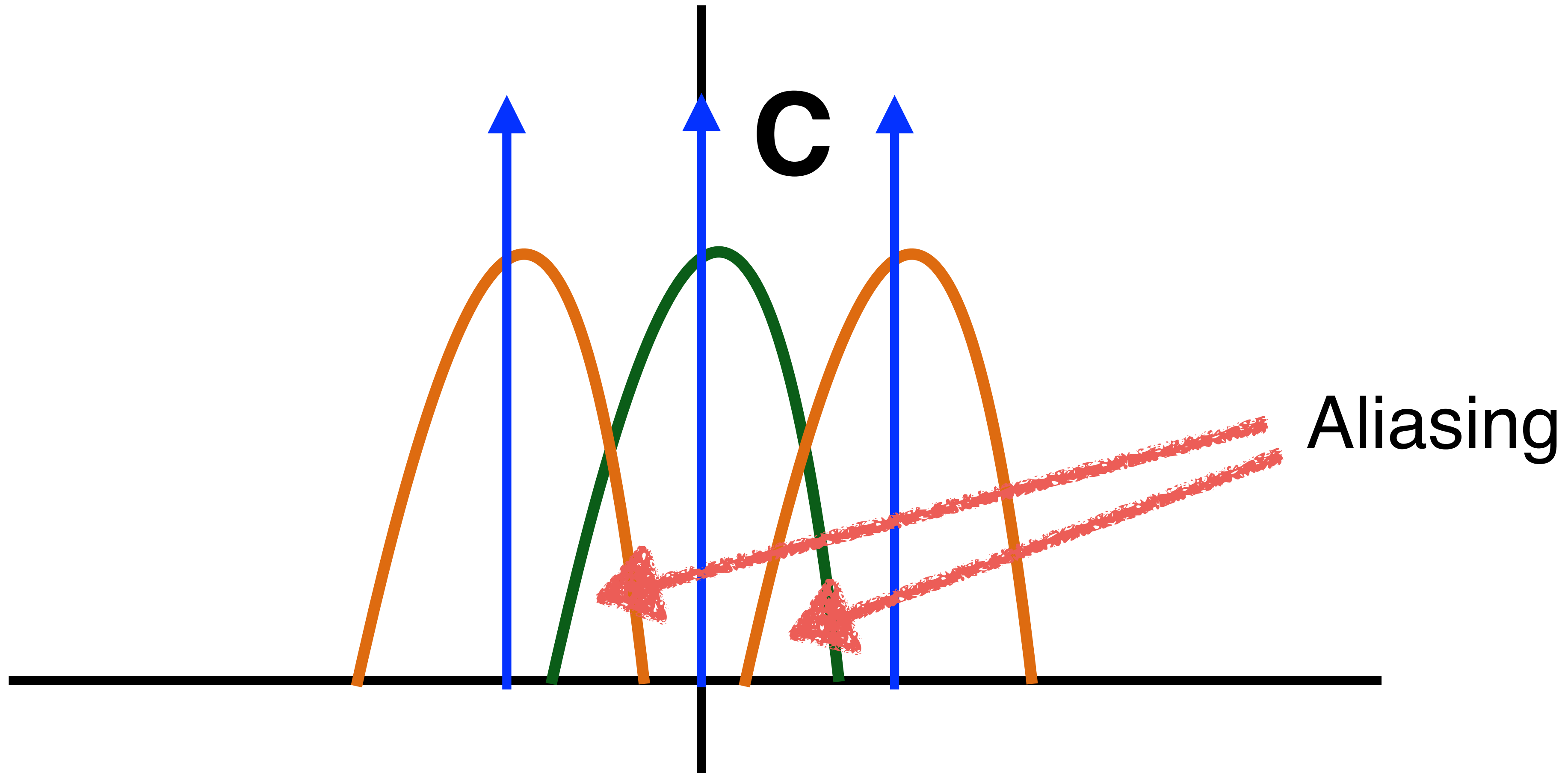
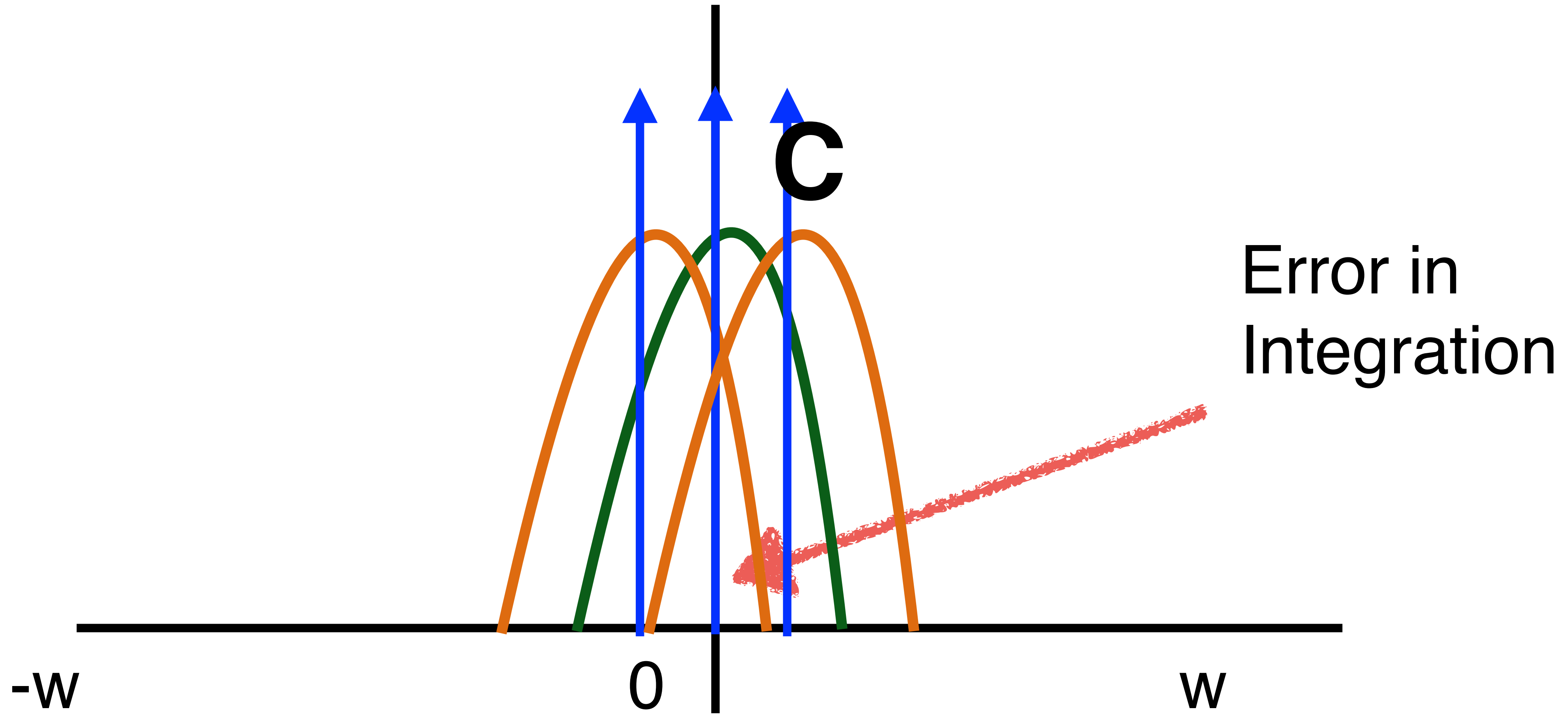
Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]





Render the Possibilities
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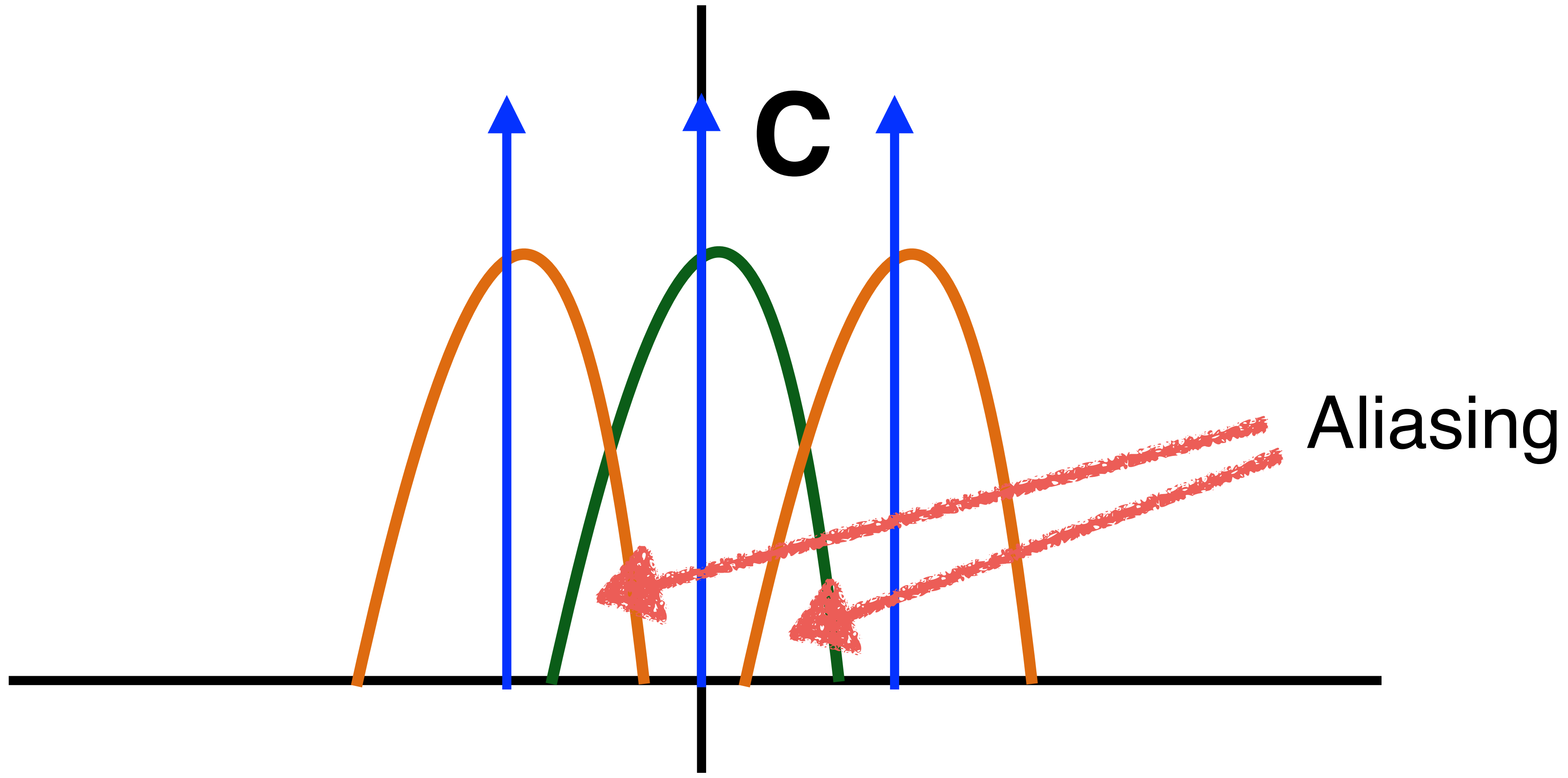
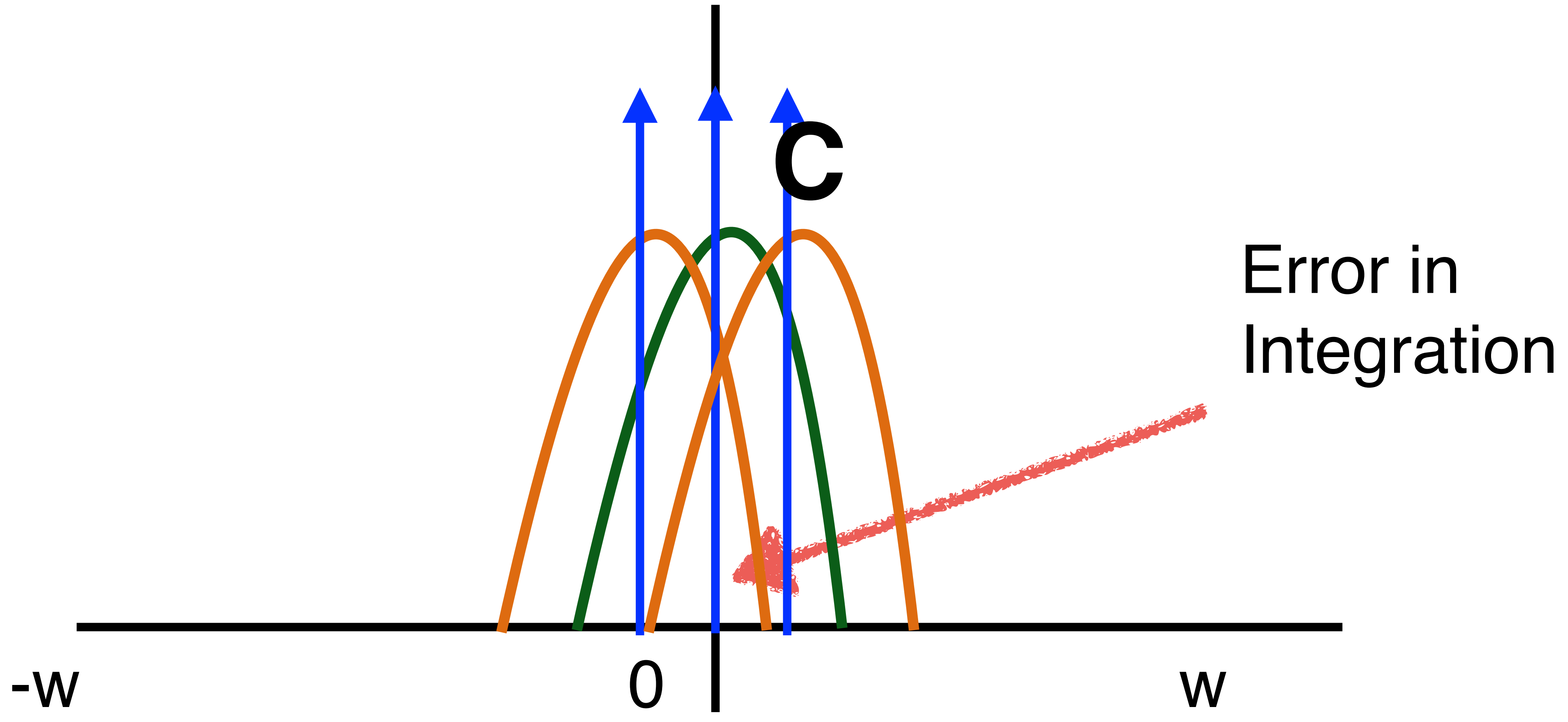
Aliasing (Reconstruction) vs. Error (Integration)

Fredo Durand [2011]
Belcour et al. [2013]





Render the Possibilities
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Integration in the Fourier Domain

Render the Possibilities
SIGGRAPH2016

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Render the Possibilities
SIGGRAPH2016

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x)dx$$

Fourier Domain:

Render the Possibilities
SIGGRAPH2016

Integration is the DC term in the Fourier Domain

Spatial Domain:

$$I = \int_D f(x) dx$$

Fourier Domain:

$$\hat{f}(0)$$

Render the Possibilities
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Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

Render the Possibilities
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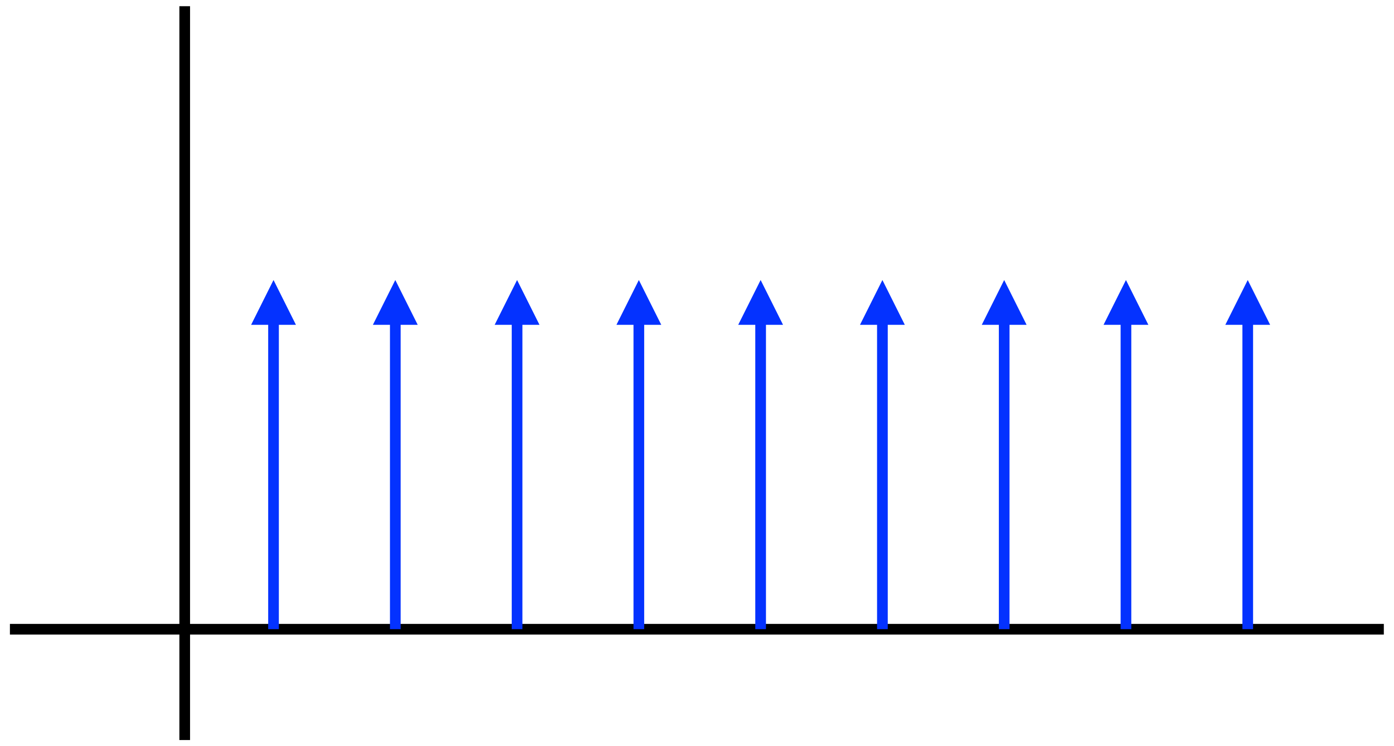
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Render the Possibilities
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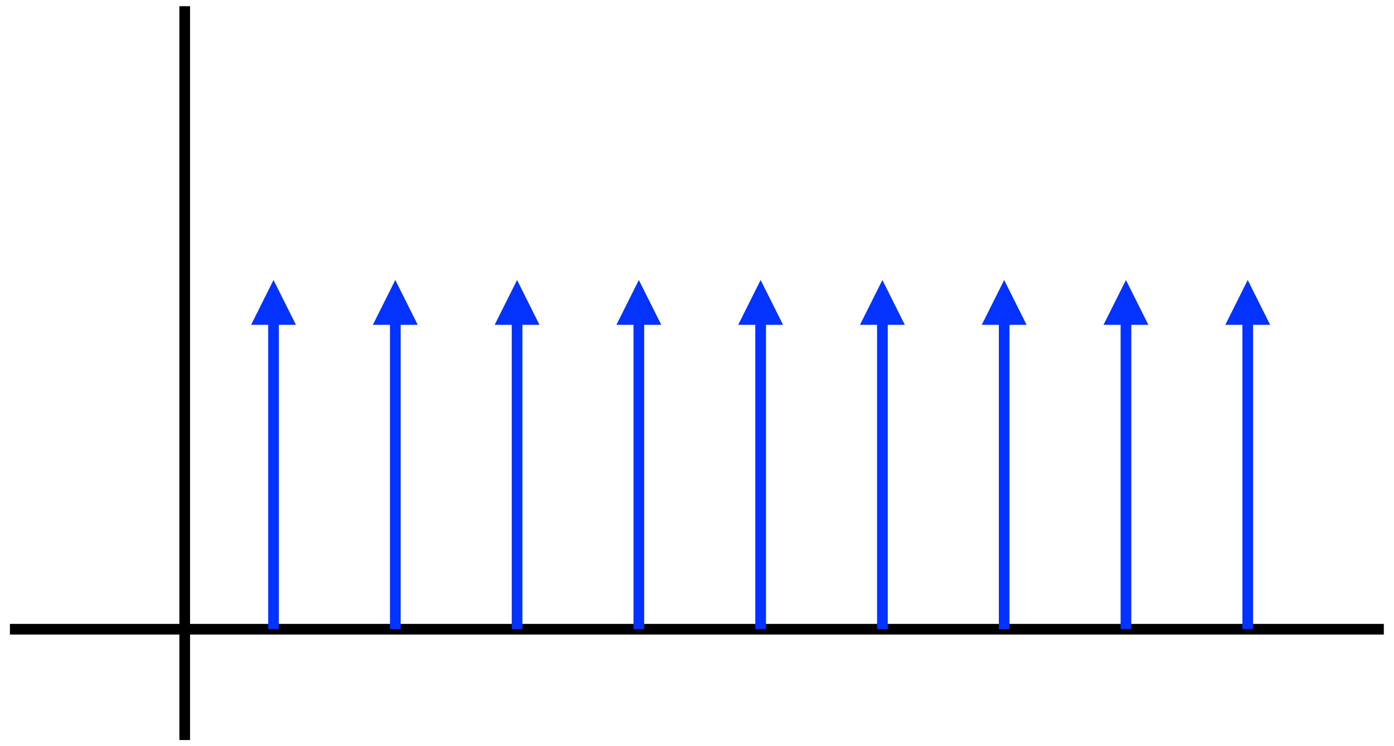
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Render the Possibilities SIGGRAPH2016

Monte Carlo Estimator in Spatial Domain

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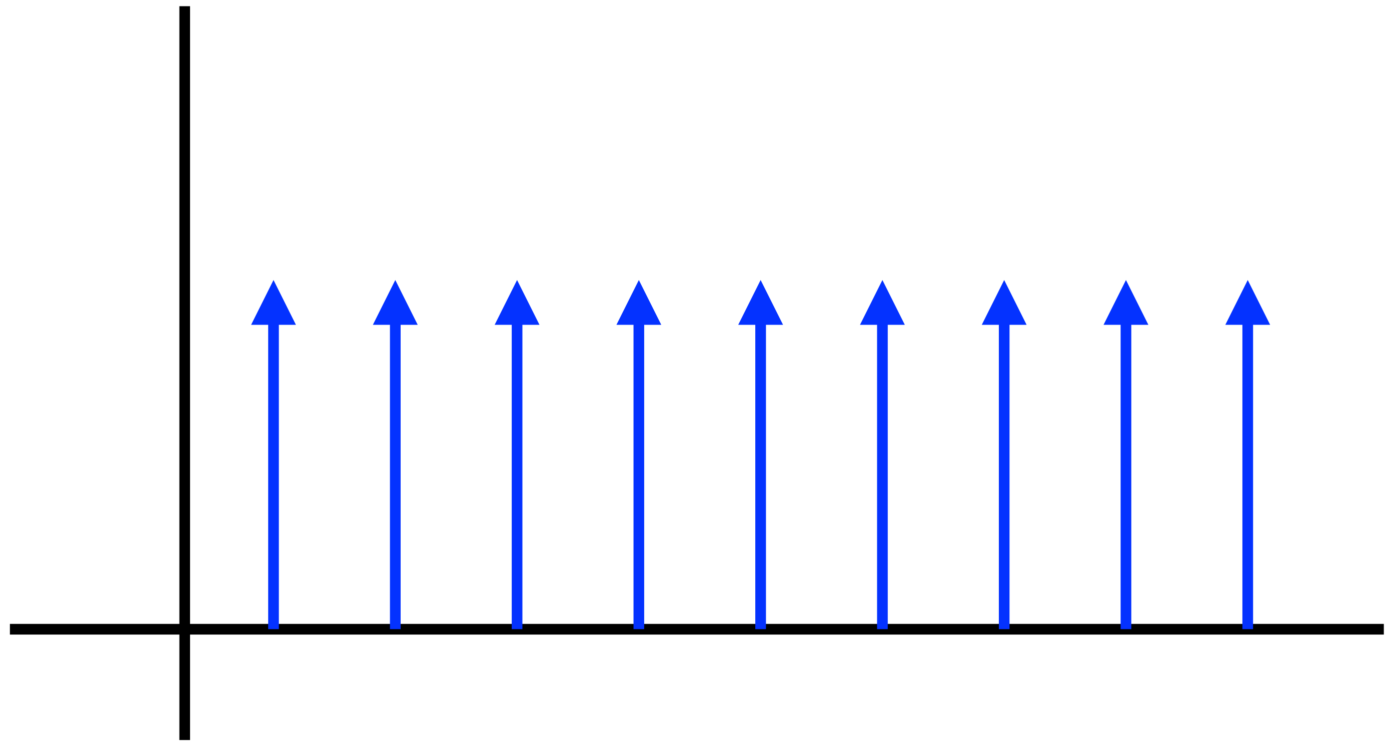
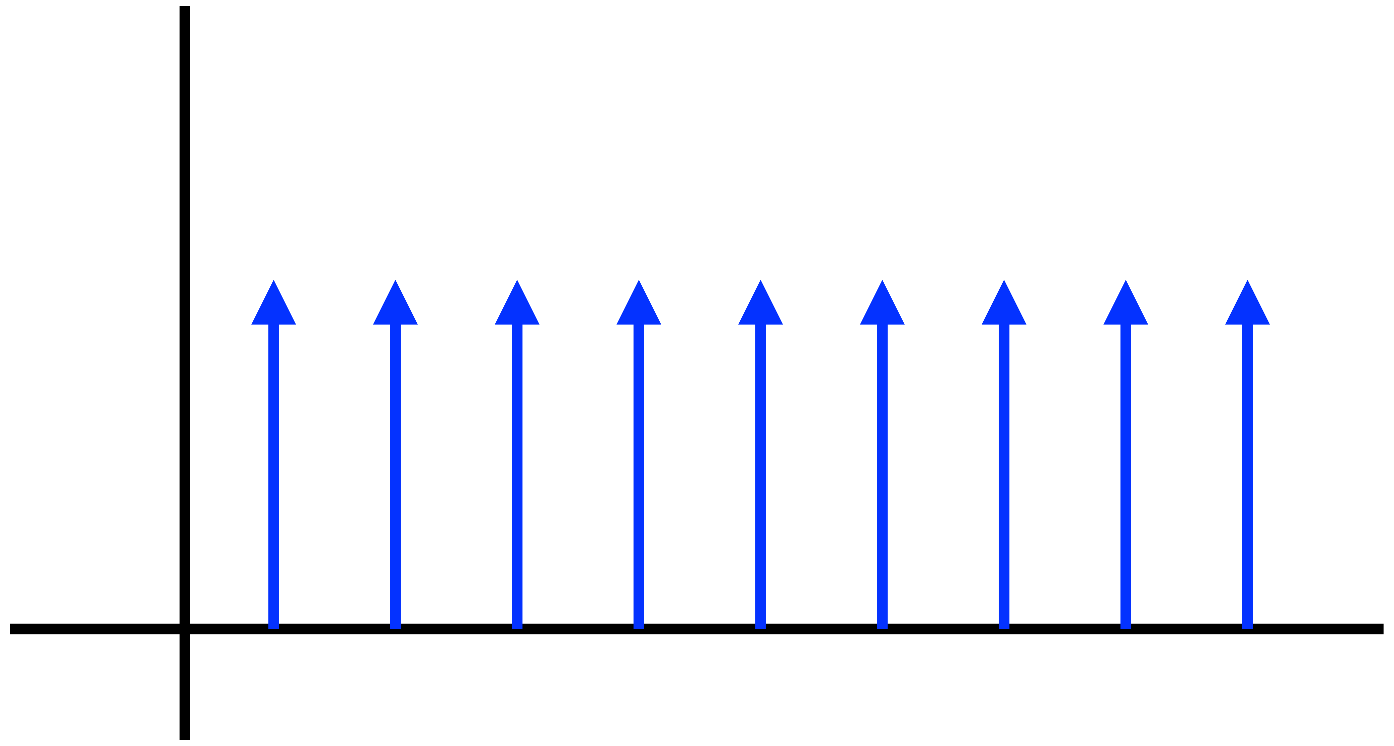
$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



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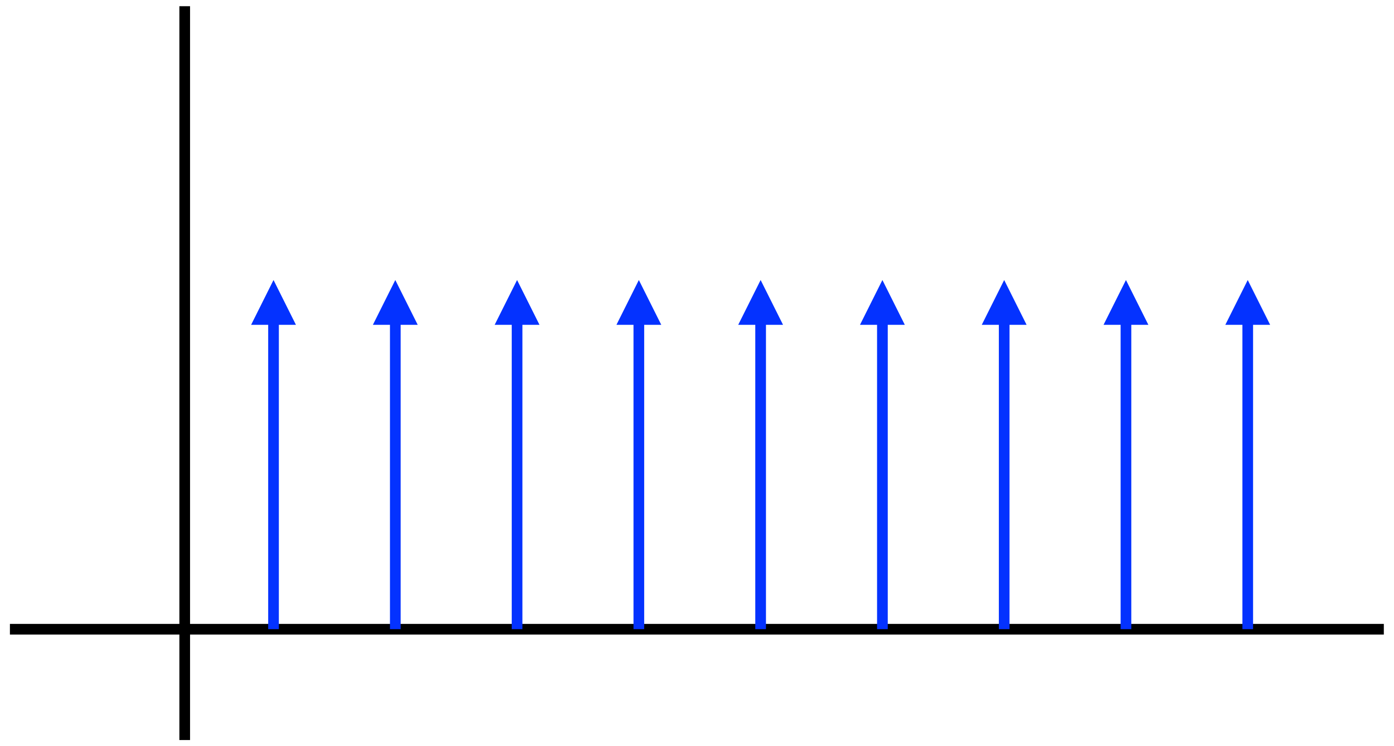
Monte Carlo Estimator in Spatial Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Render the Possibilities
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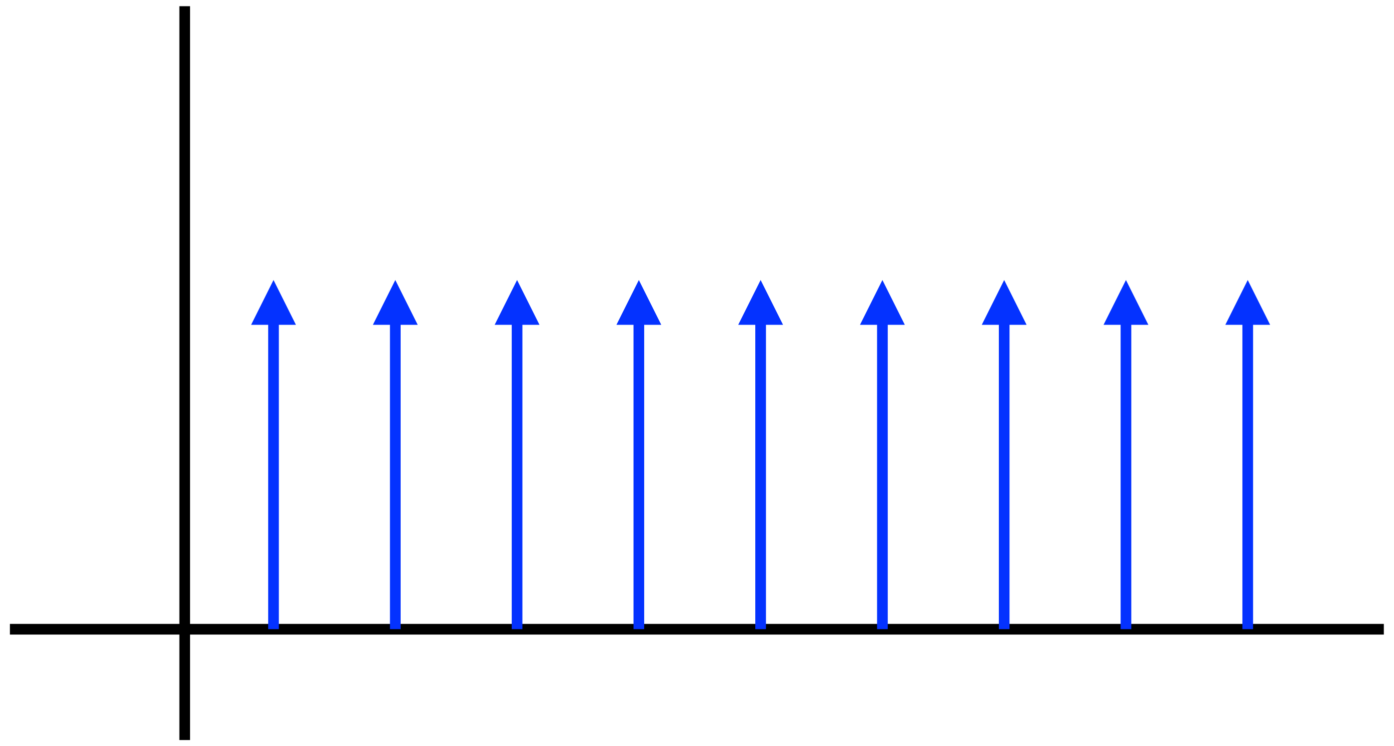
Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



Render the Possibilities
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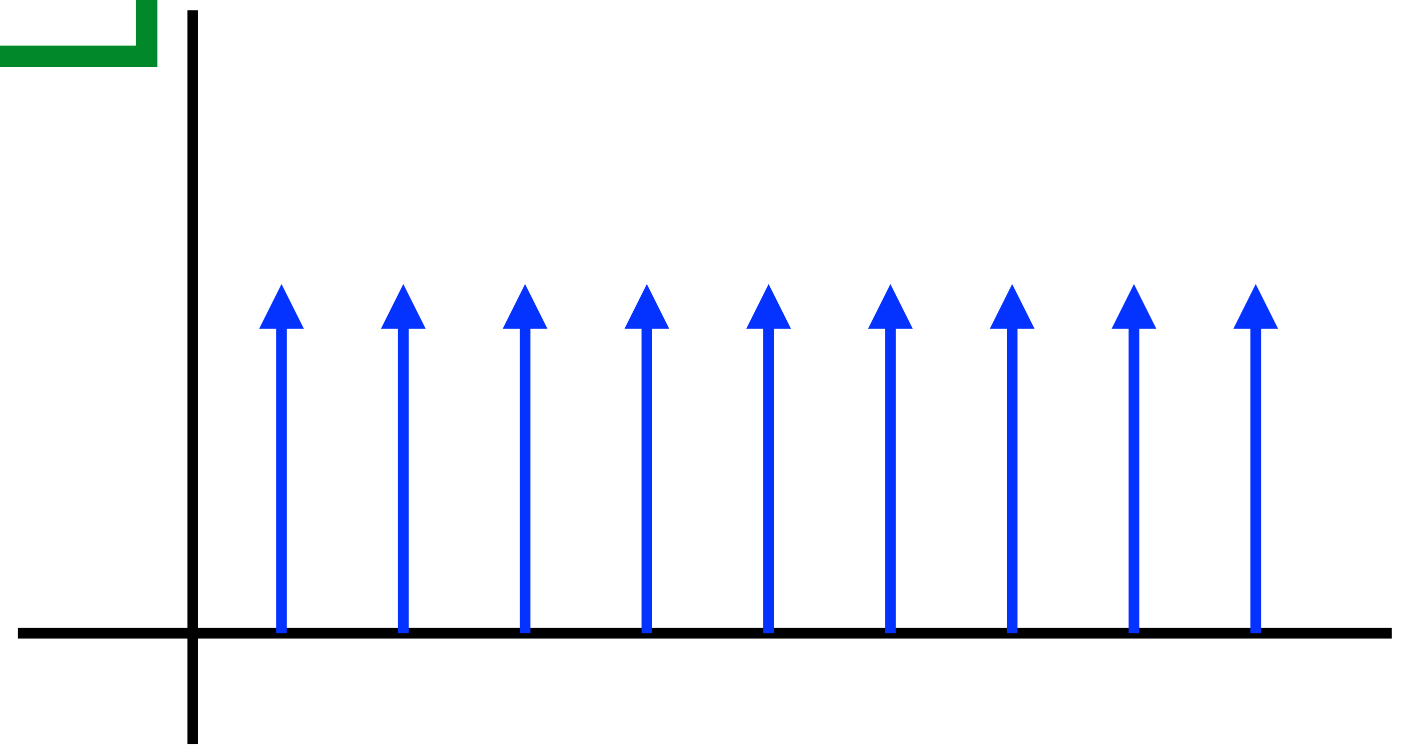
Monte Carlo Estimator in Fourier Domain

$$\tilde{\mu}_N = \int_D f(x)\mathbf{S}(x)dx = \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$\mathbf{S}(x) = \frac{1}{N}\sum_{k=1}^{N}\delta(x - x_k)$$



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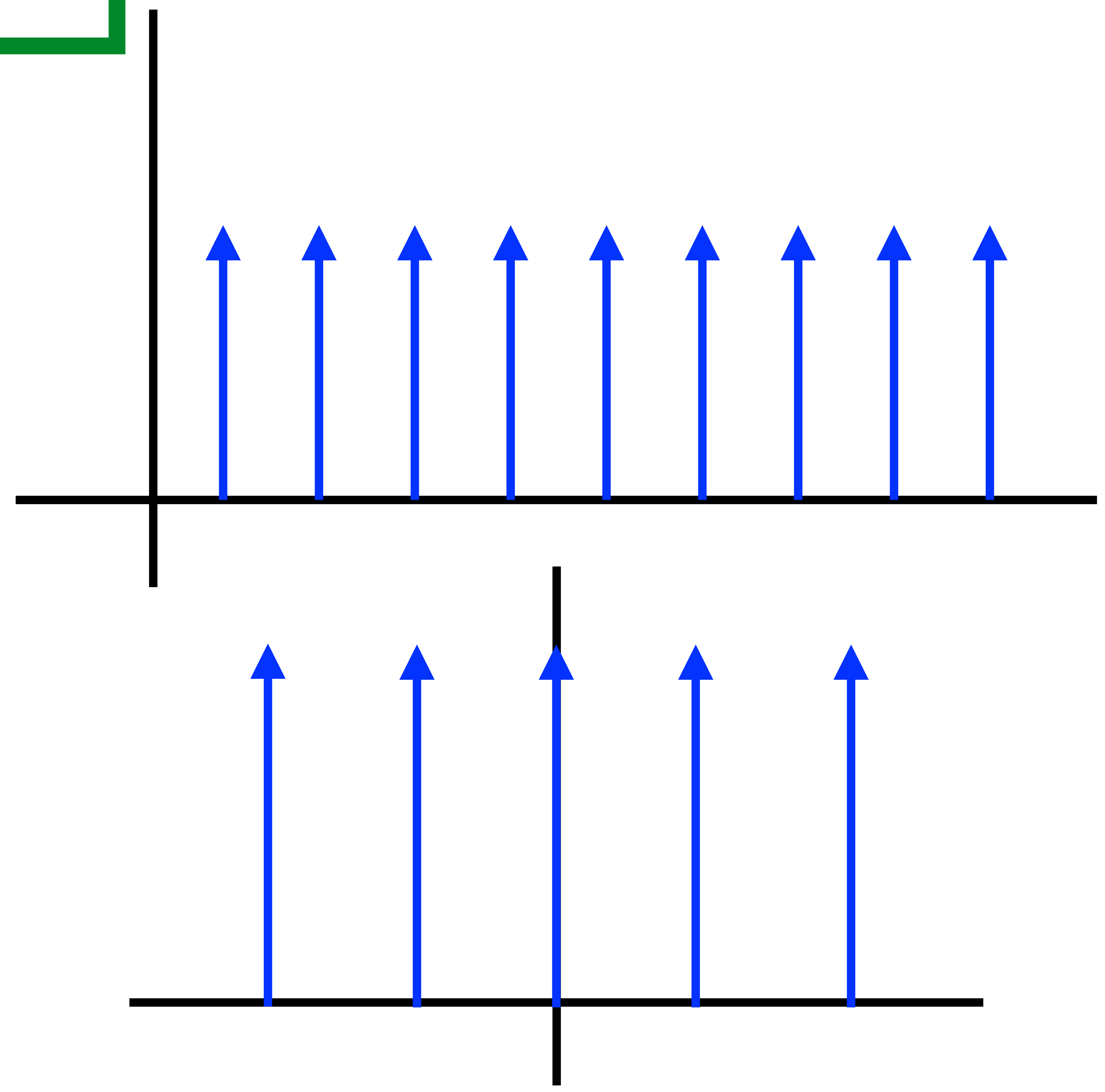

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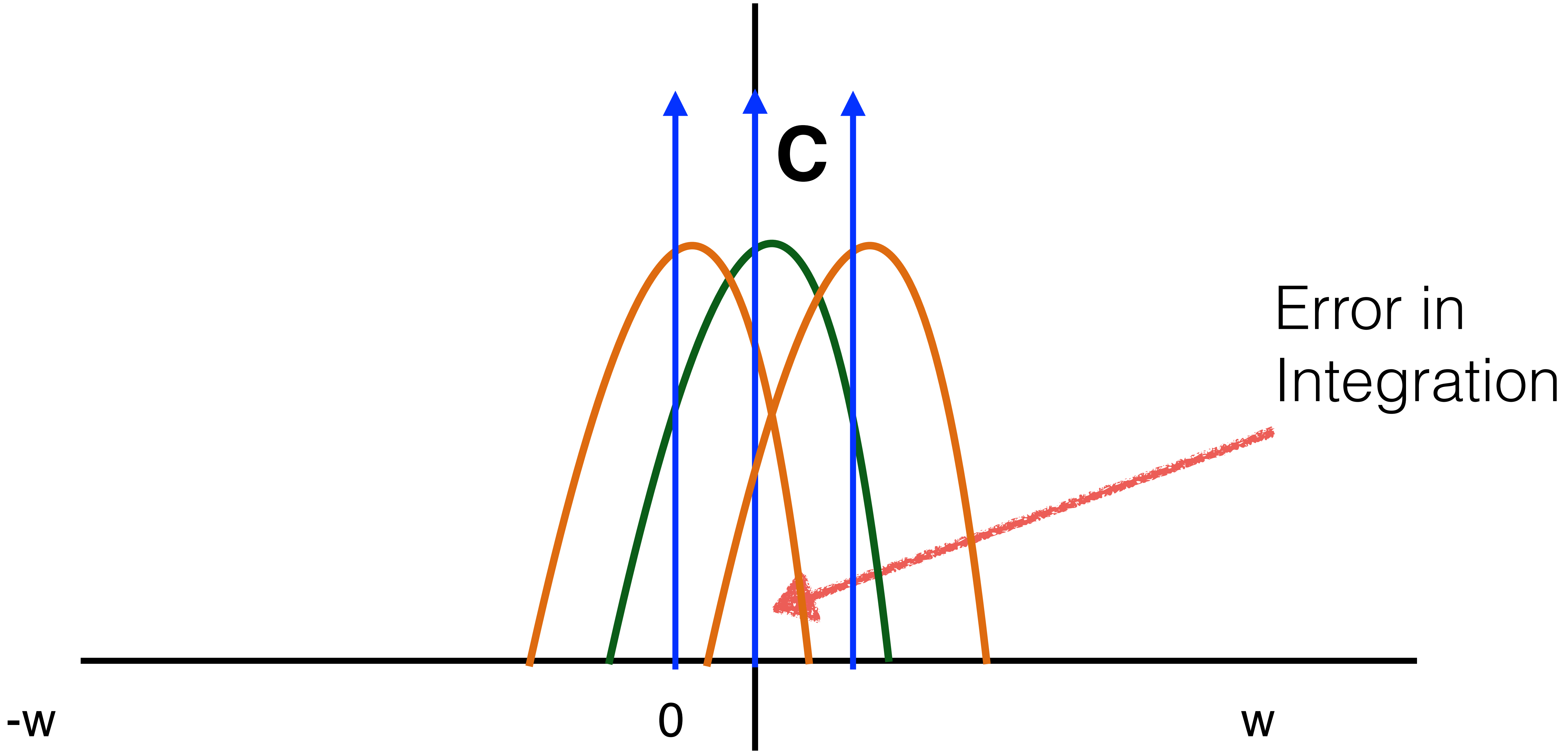
How to Formulate Error in Fourier Domain ?

$$I = \hat{f}(0)$$

$$\tilde{\mu}_N = \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$



Fredo Durand [2011]

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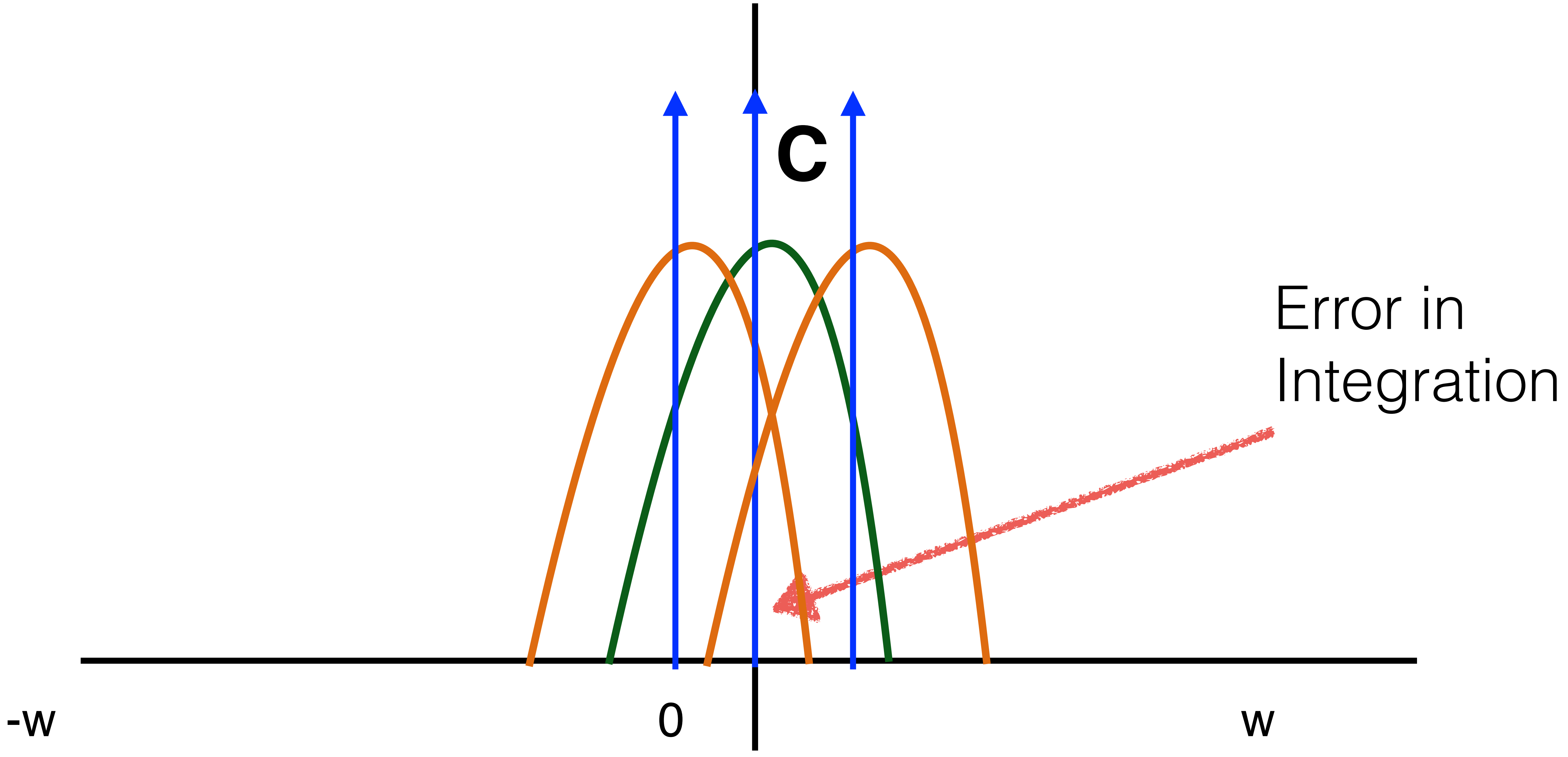

Fredo Durand [2011]

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Error in Spatial Domain

$$I = \hat{f}(0)$$

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Error in Spatial Domain

$$I = \hat{f}(0)$$

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True Integral

$$I - \tilde{\mu}_N = \boxed{\int_D f(x)dx} - \int_D f(x)\mathbf{S}(x)dx$$

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Monte Carlo Estimator

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Fredo Durand [2011]

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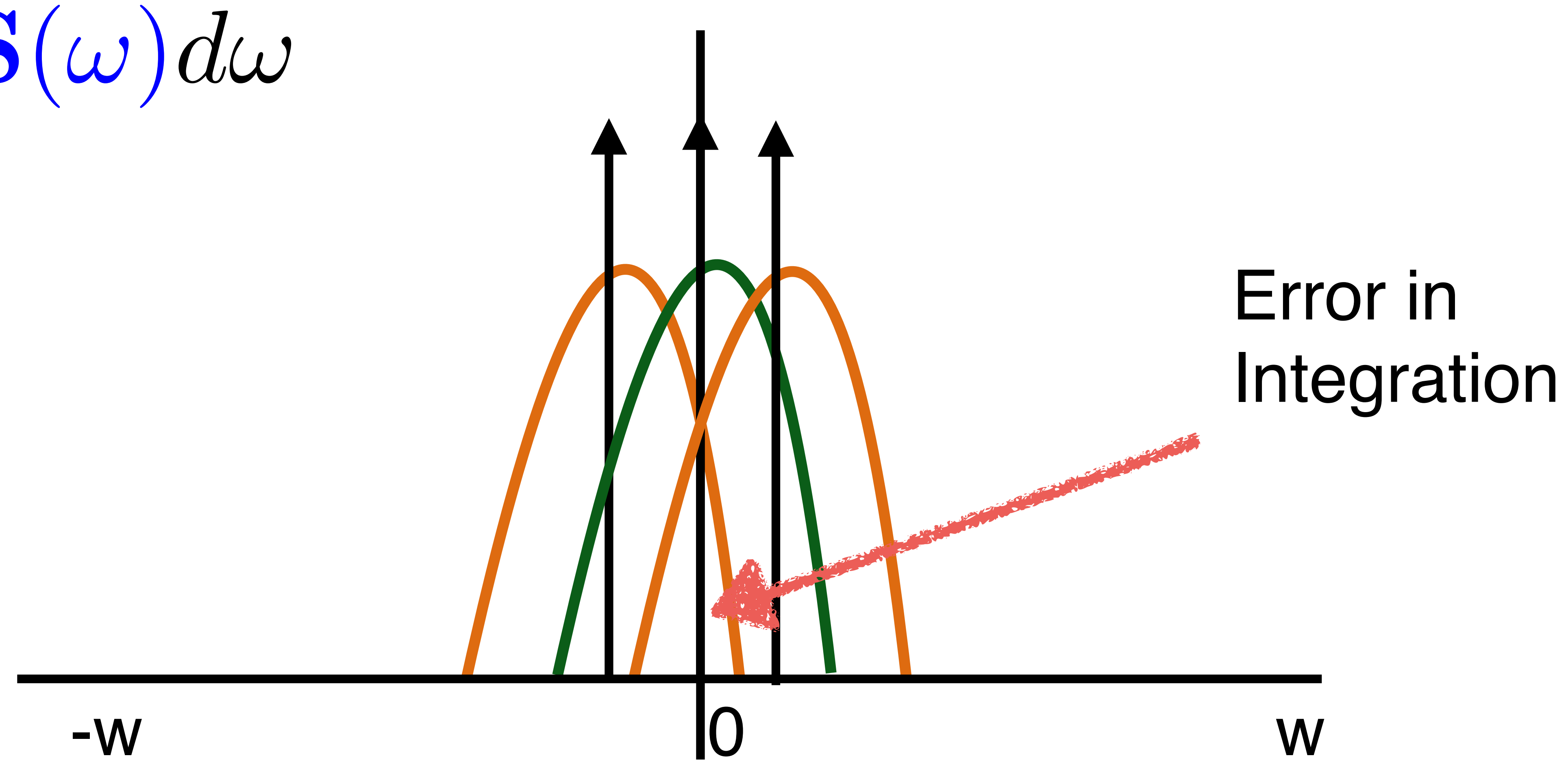

Fredo Durand [2011]

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$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

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Properties of Error

- Bias
- Variance

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Properties of Error

- Bias: Expected value of the Error
- Variance

Properties of Error

- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
- Variance

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Properties of Error

- Bias: Expected value of the Error $\langle I - \tilde{\mu}_N \rangle$
- Variance: $\mathrm{Var}(I - \mu_N)$

Subr and Kautz [2013]

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Bias in the
Monte Carlo Estimator

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Bias in Fourier Domain

Error:

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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Bias in Fourier Domain

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Bias: $$\langle I - \tilde{\mu}_N \rangle$$

Bias in Fourier Domain

Bias: $$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \left\langle \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega \right\rangle$$

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$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega) \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega$$

Subr and Kautz [2013]

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Subr and Kautz [2013]

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Subr and Kautz [2013]

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$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \boxed{\int_{\Omega} \hat{f}^*(\omega) \, \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega}$$

To obtain an unbiased estimator:

Subr and Kautz [2013]

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Bias in Fourier Domain

$$\langle I - \tilde{\mu}_N \rangle = \hat{f}(0) - \boxed{\int_\Omega \hat{f}^*(\omega) \, \langle \hat{\mathbf{S}}(\omega) \rangle \, d\omega}$$

To obtain an unbiased estimator:

Subr and Kautz [2013]

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0$$

for frequencies other than zero

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How to obtain $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$?

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Complex form in Amplitude and Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = |\langle \hat{\mathbf{S}}(\omega) \rangle| \, e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

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Complex form in Amplitude and Phase

Amplitude

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} e^{-\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}$$

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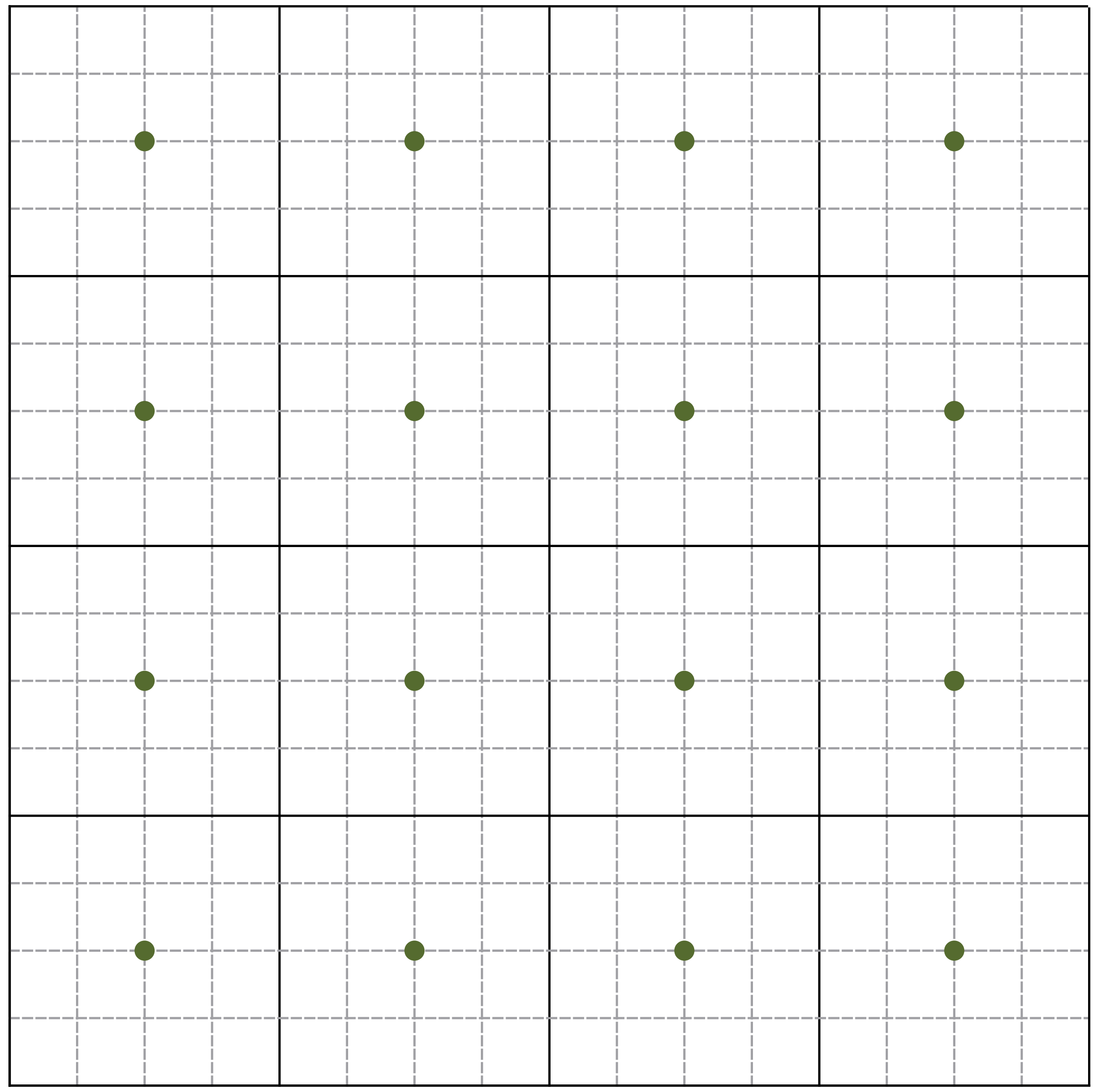
Complex form in Amplitude and Phase

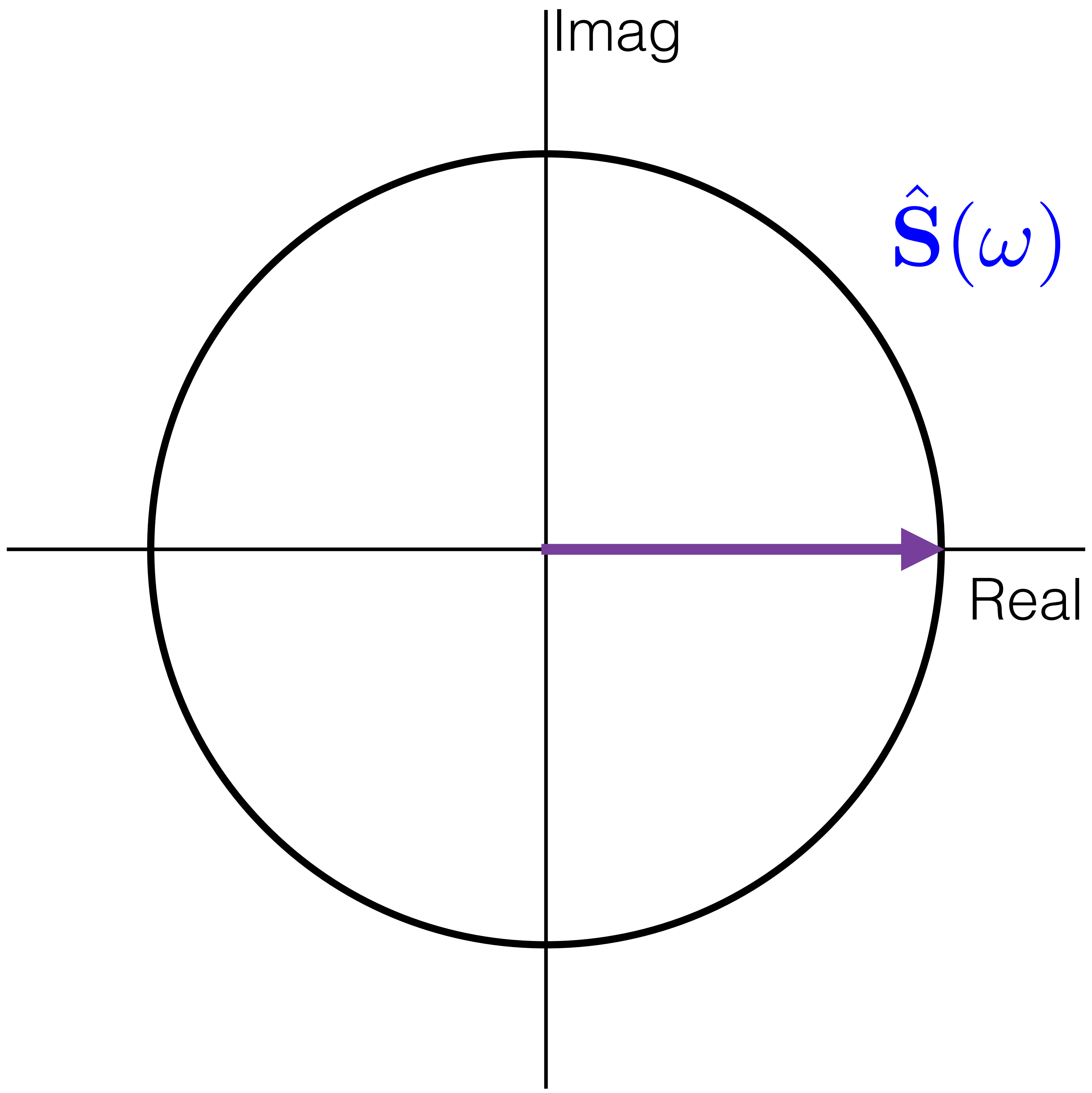
Amplitude Phase

$$\langle \hat{\mathbf{S}}(\omega) \rangle = \boxed{|\langle \hat{\mathbf{S}}(\omega) \rangle|} e^{-\boxed{\Phi(\langle \hat{\mathbf{S}}(\omega) \rangle)}}$$

Phase change due to Random Shift



For a given frequency ω



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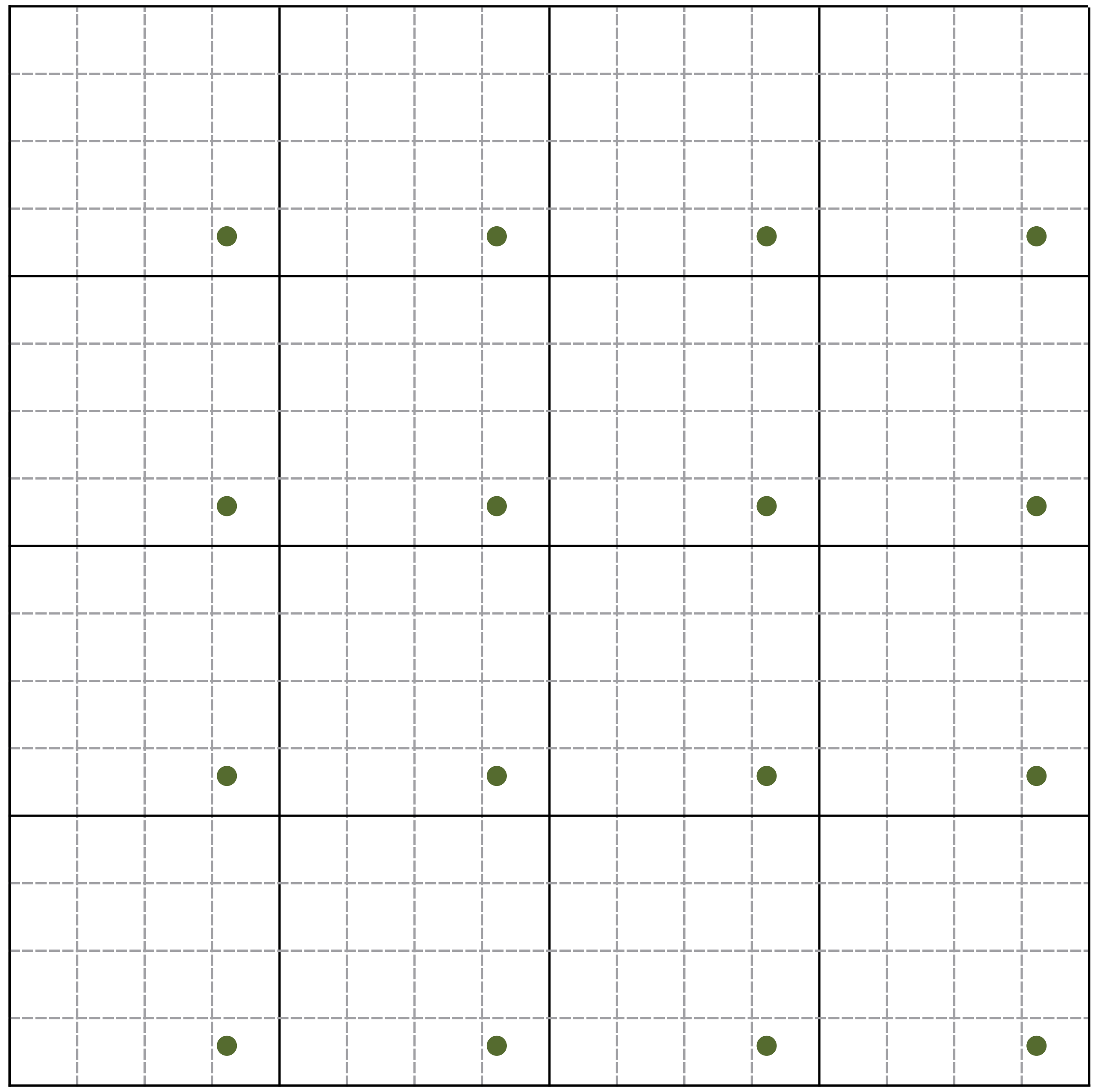
Phase change due to Random Shift



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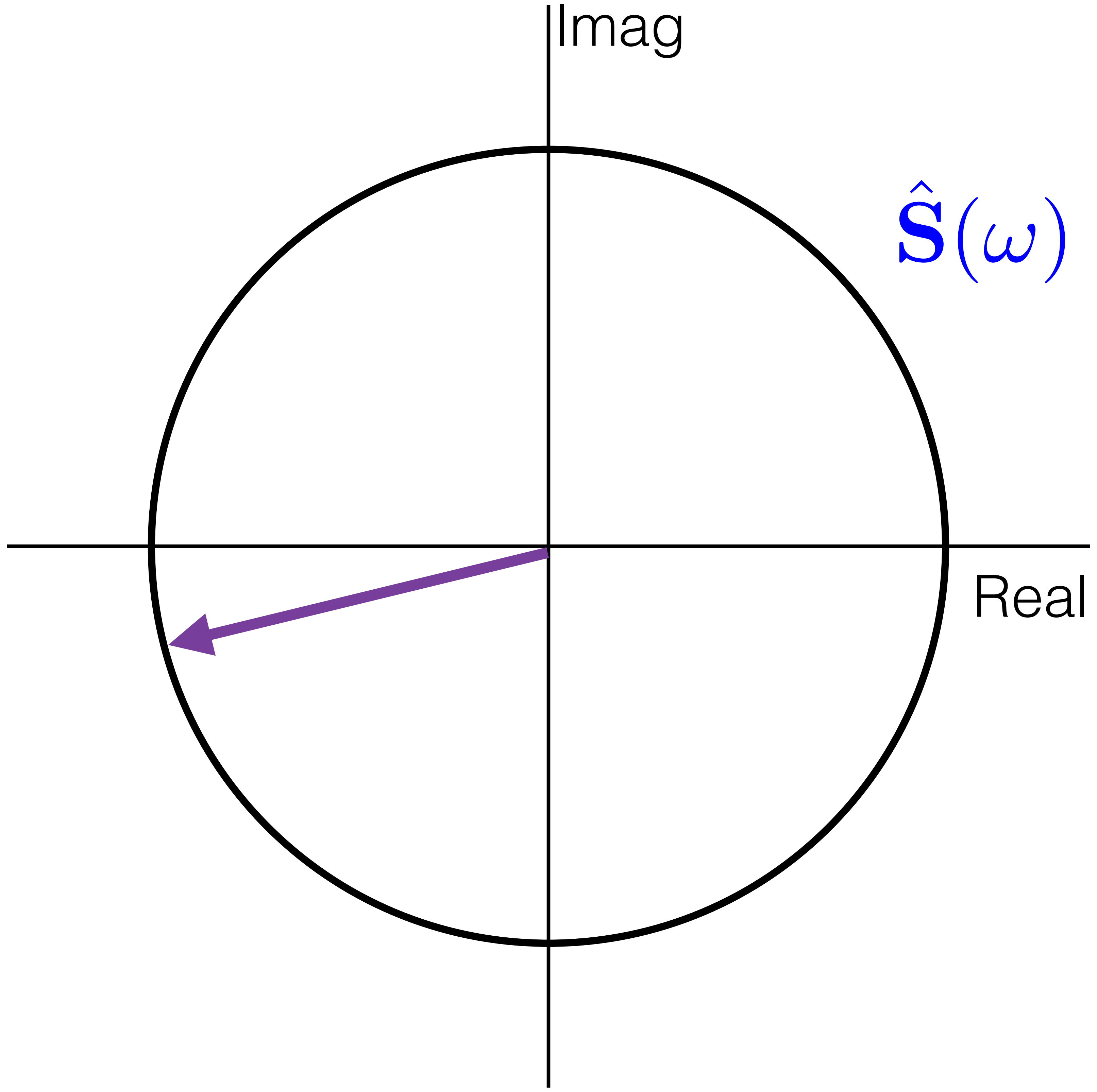

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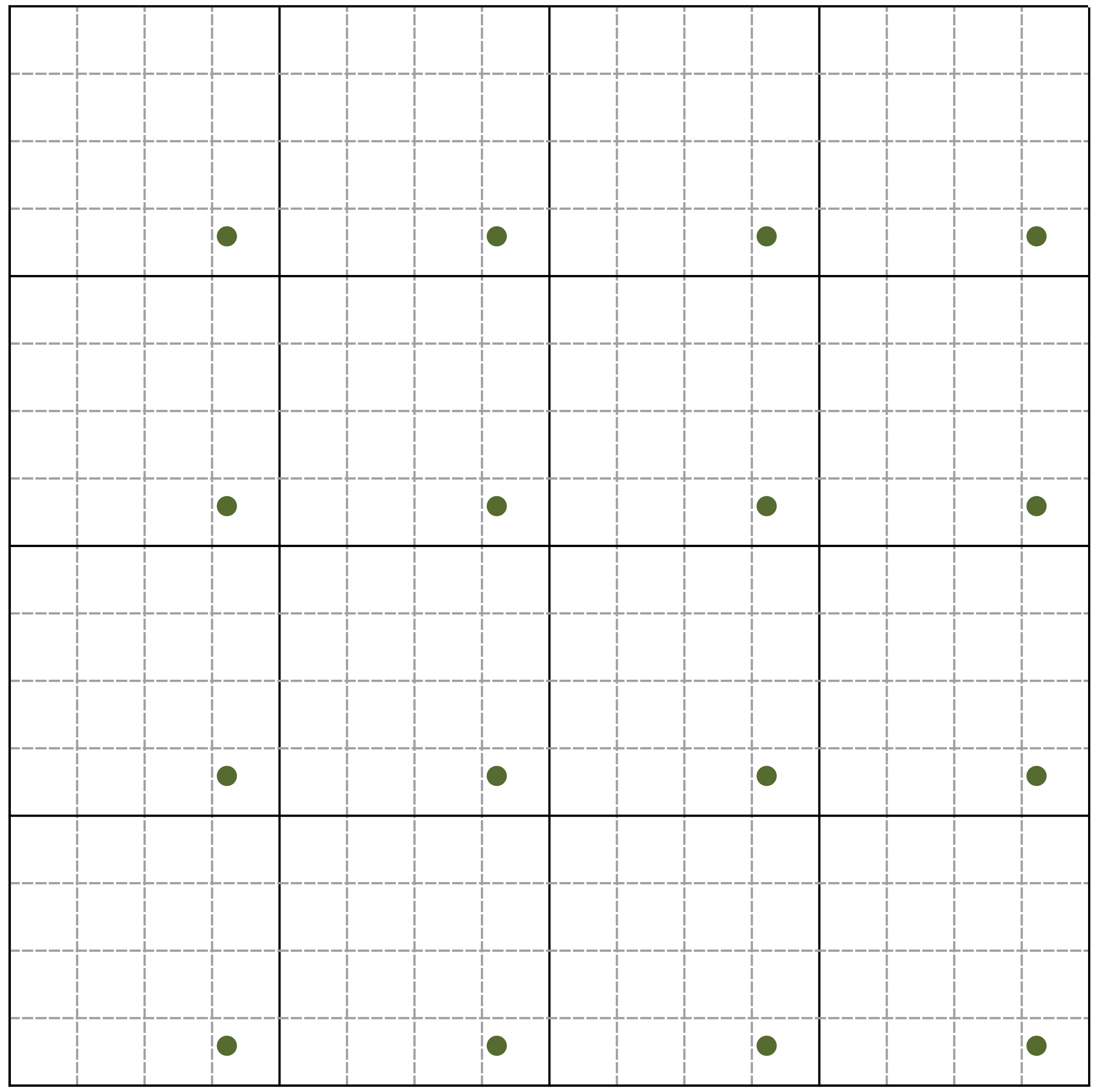
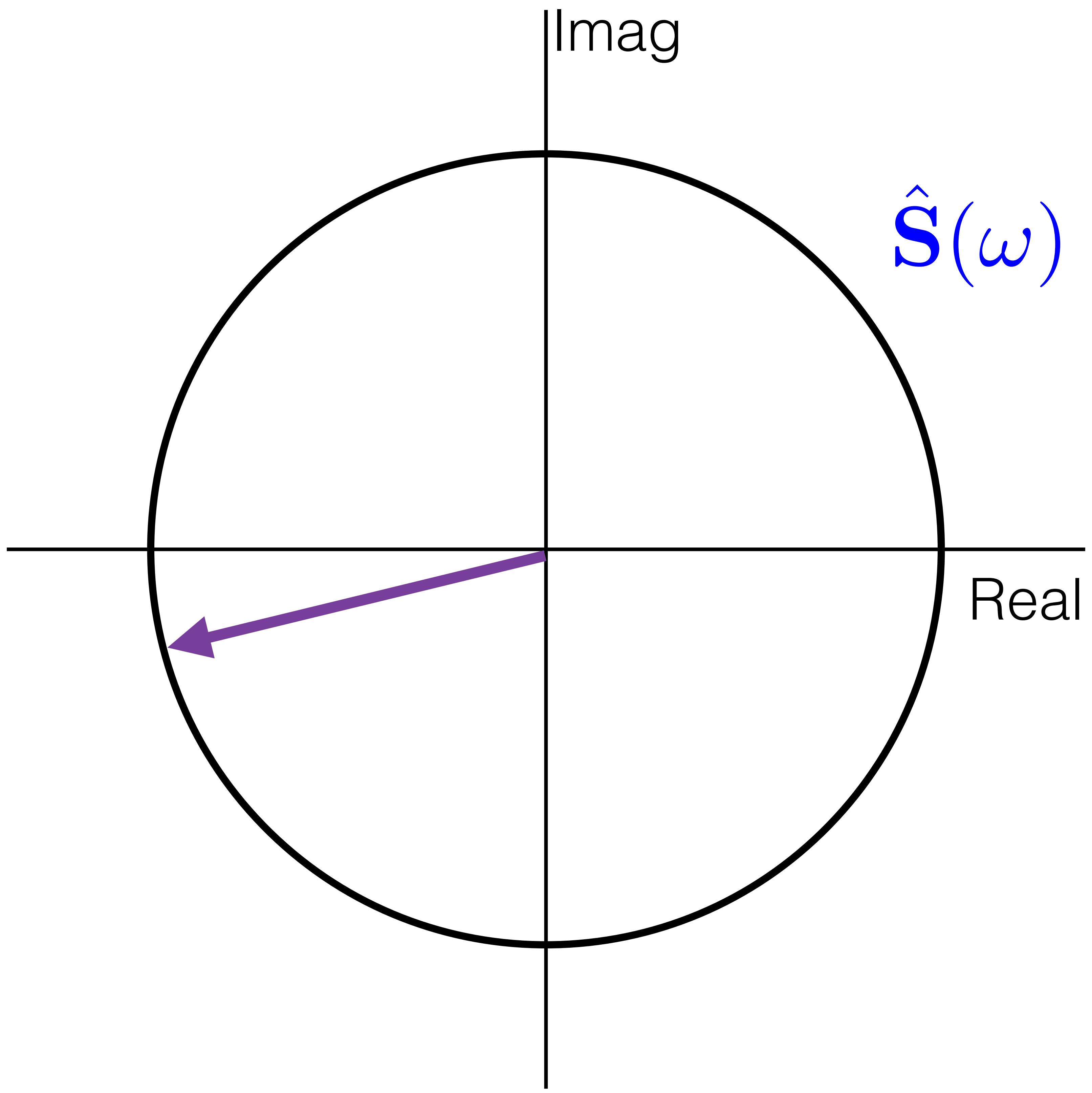
Phase change due to Random Shift



Pauly et al. [2000]
Ramamoorthi et al. [2012]

For a given frequency ω



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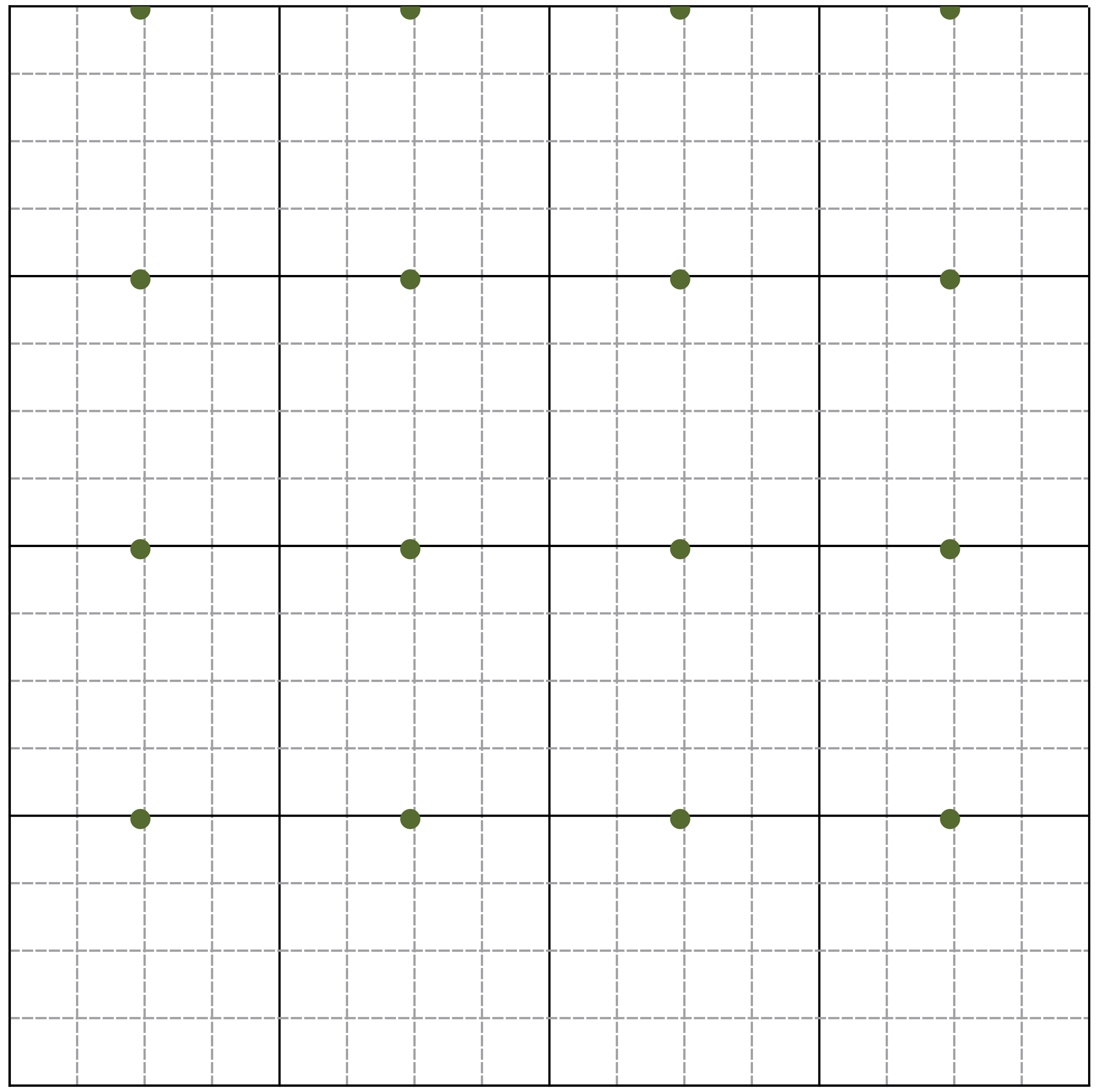
Phase change due to Random Shift



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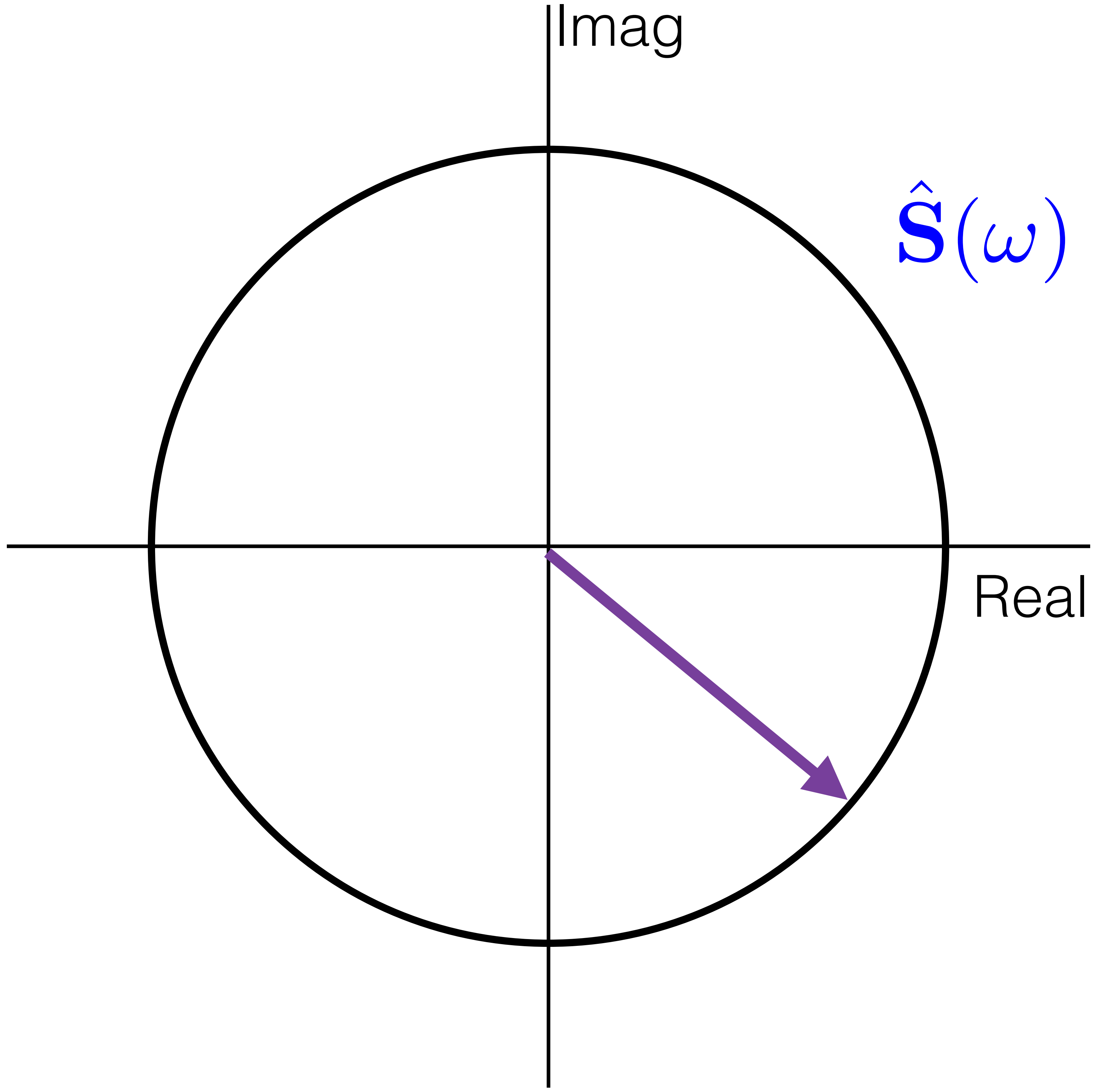

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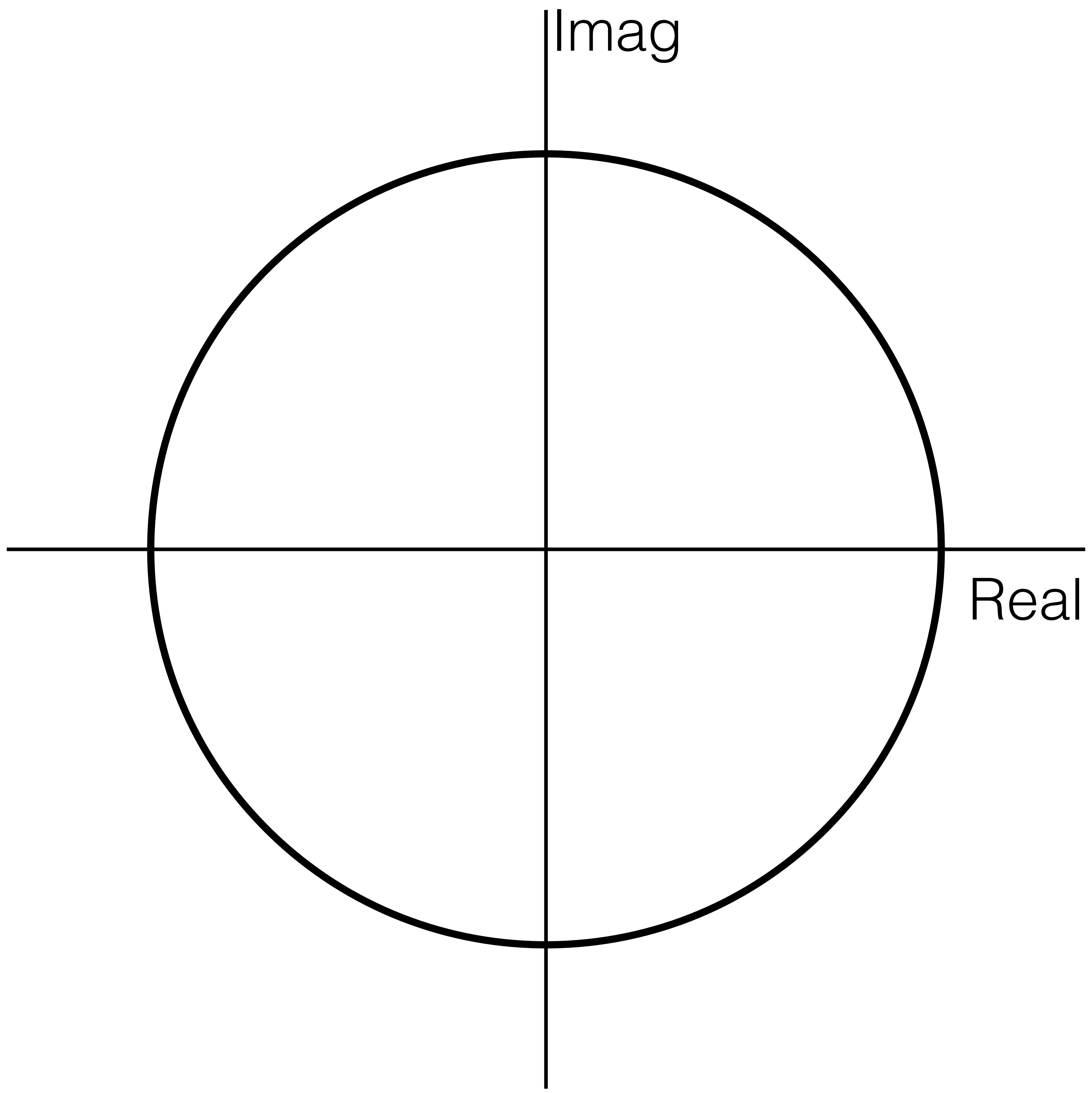
Phase change due to Random Shift

Multiple realizations

For a given frequency ω



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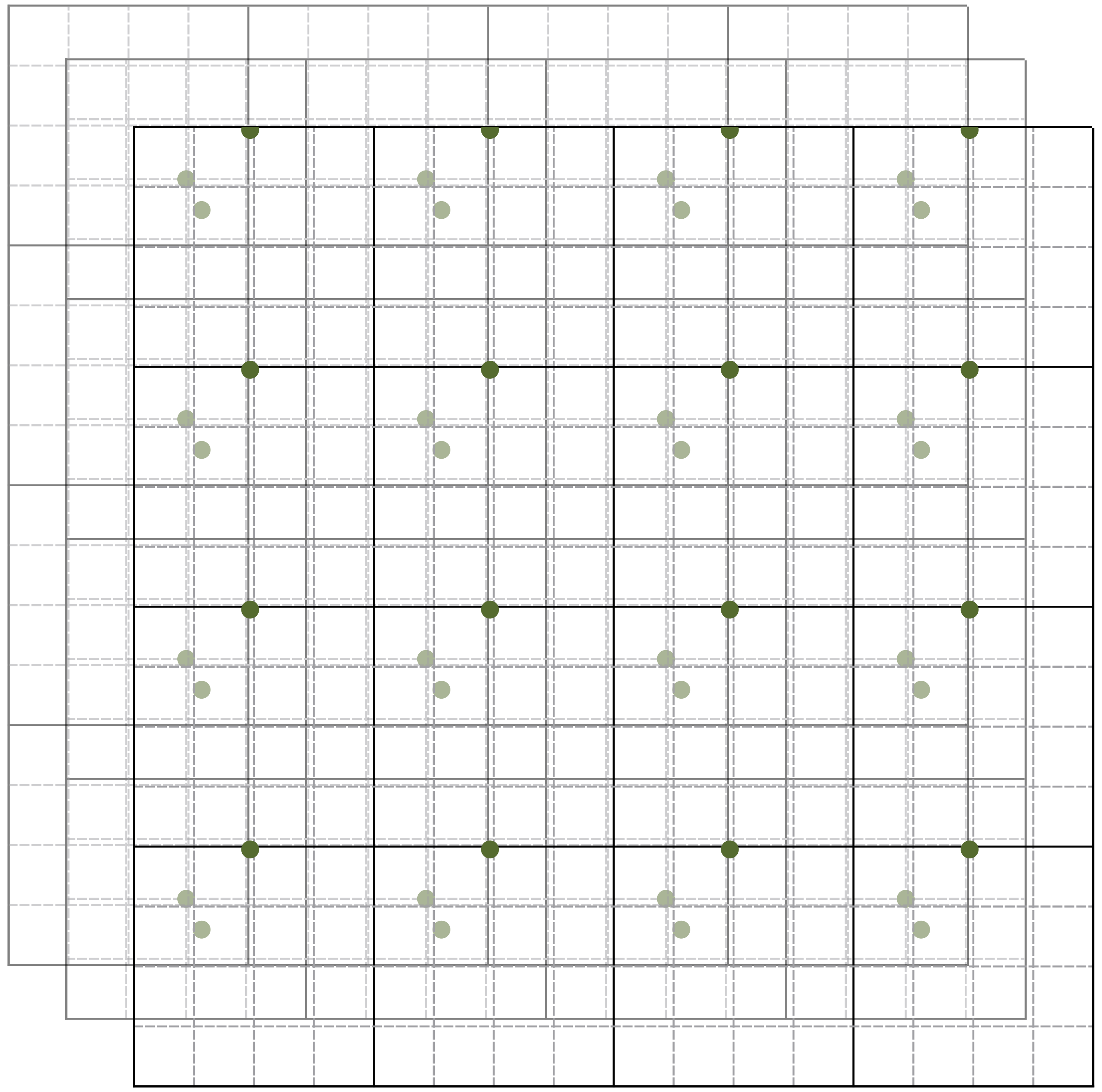

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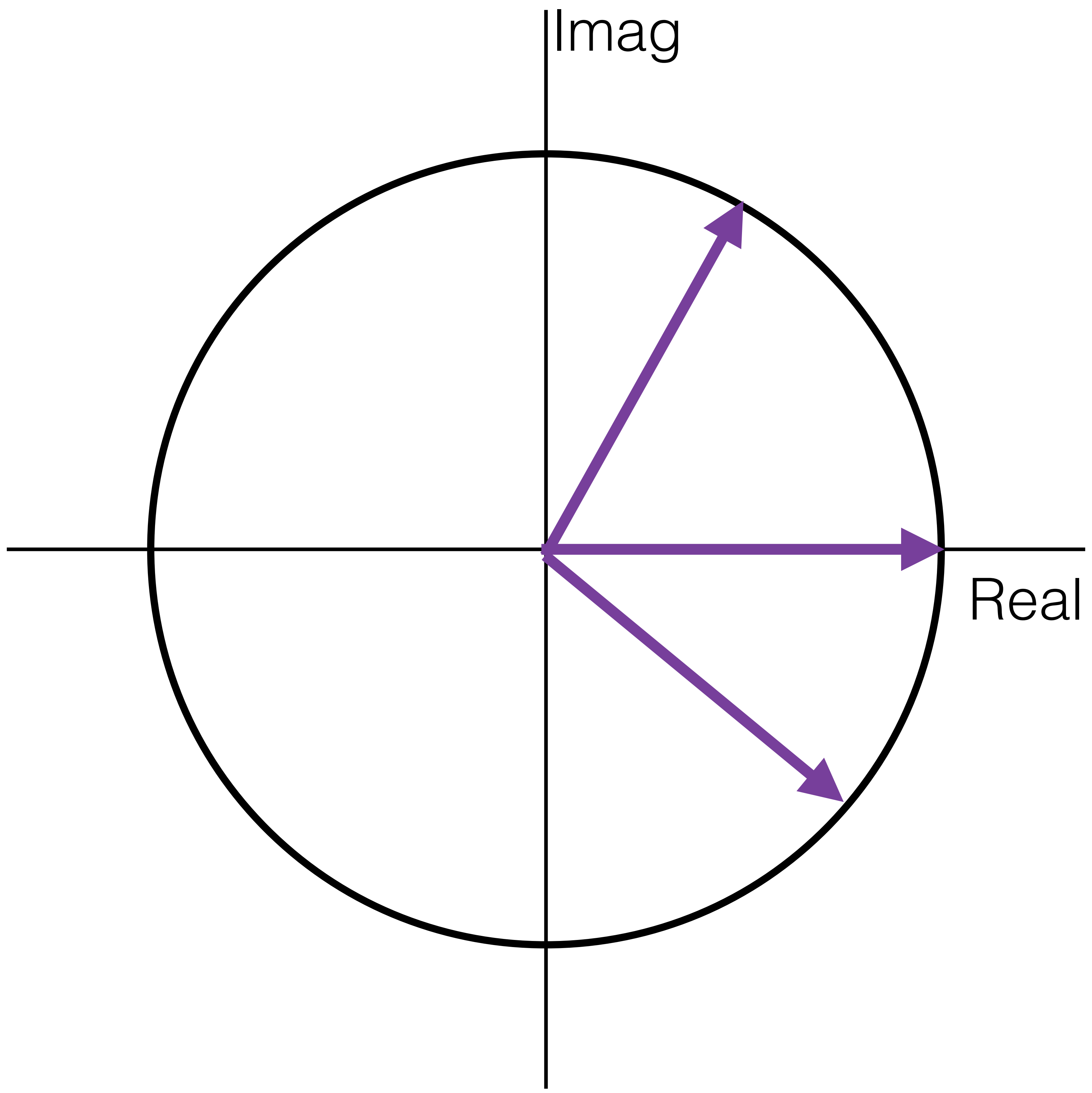

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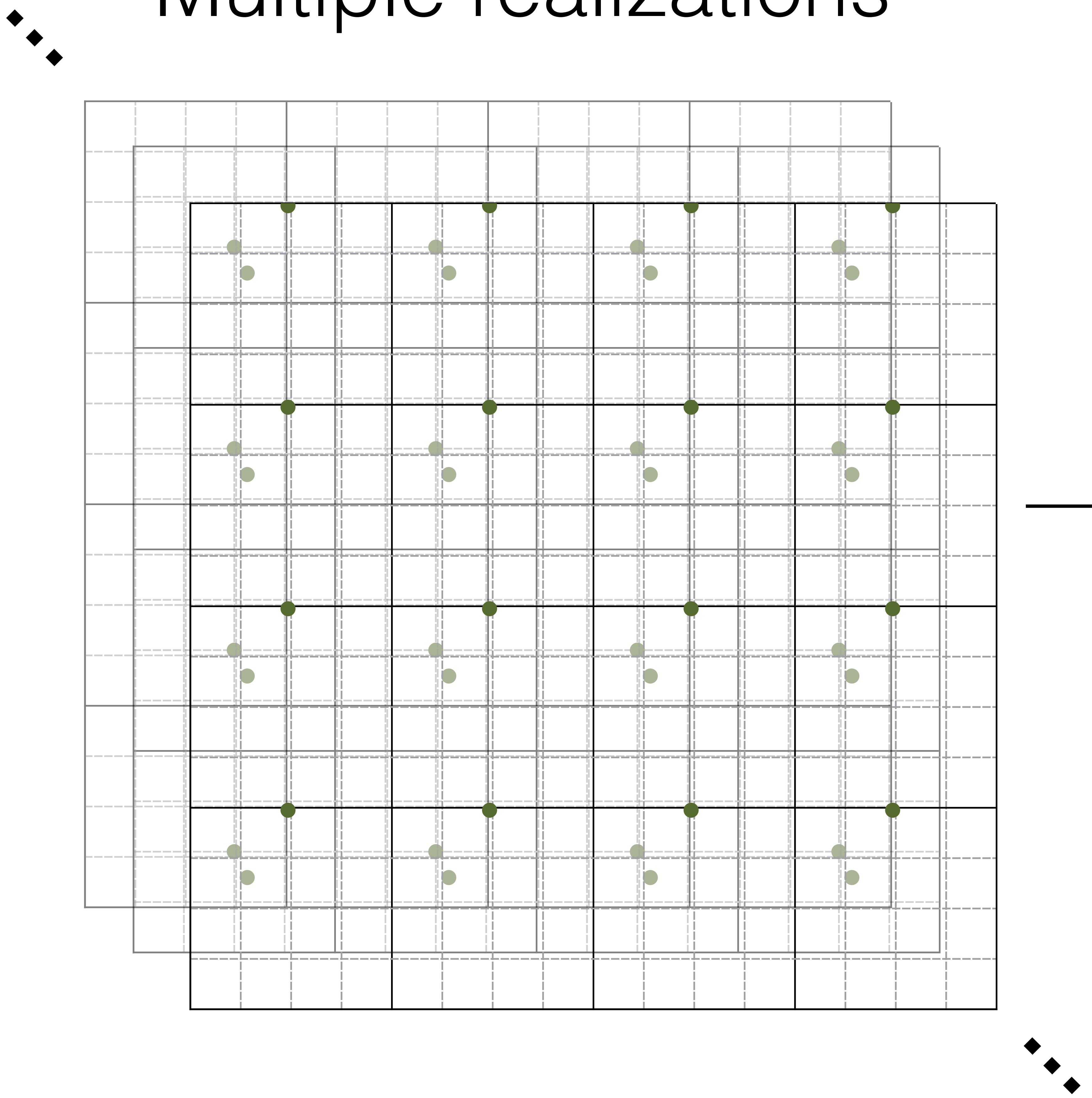

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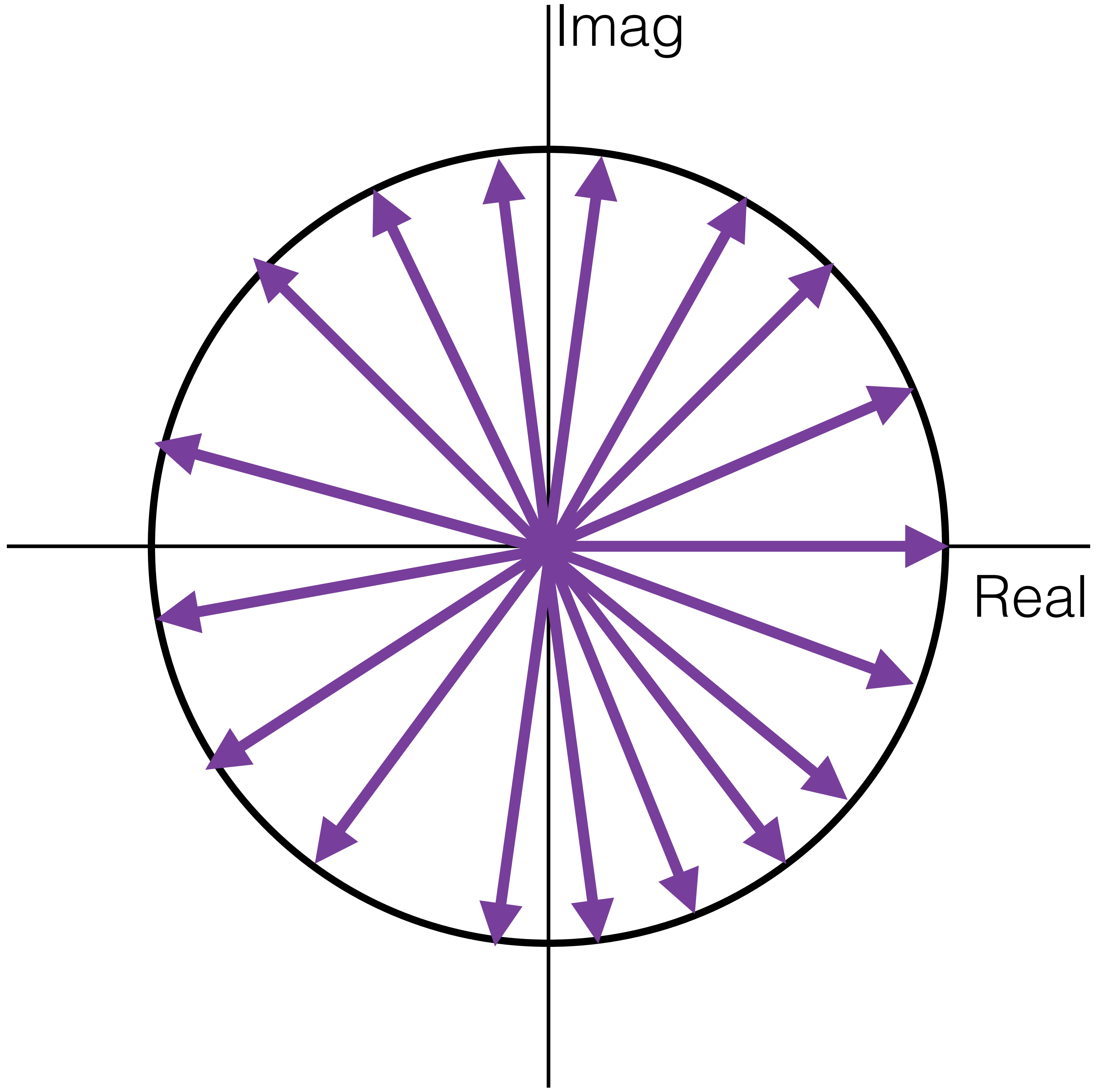

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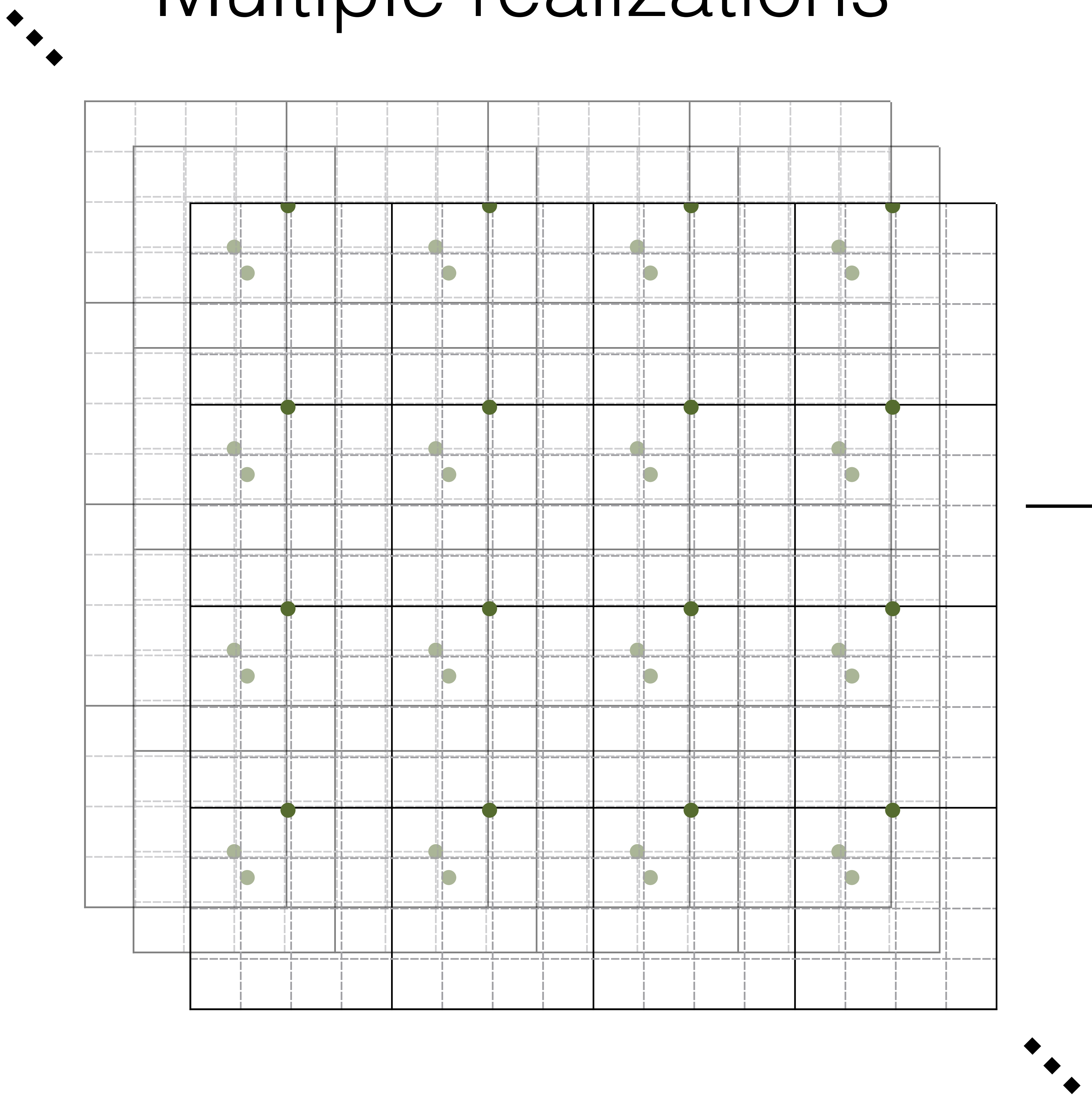

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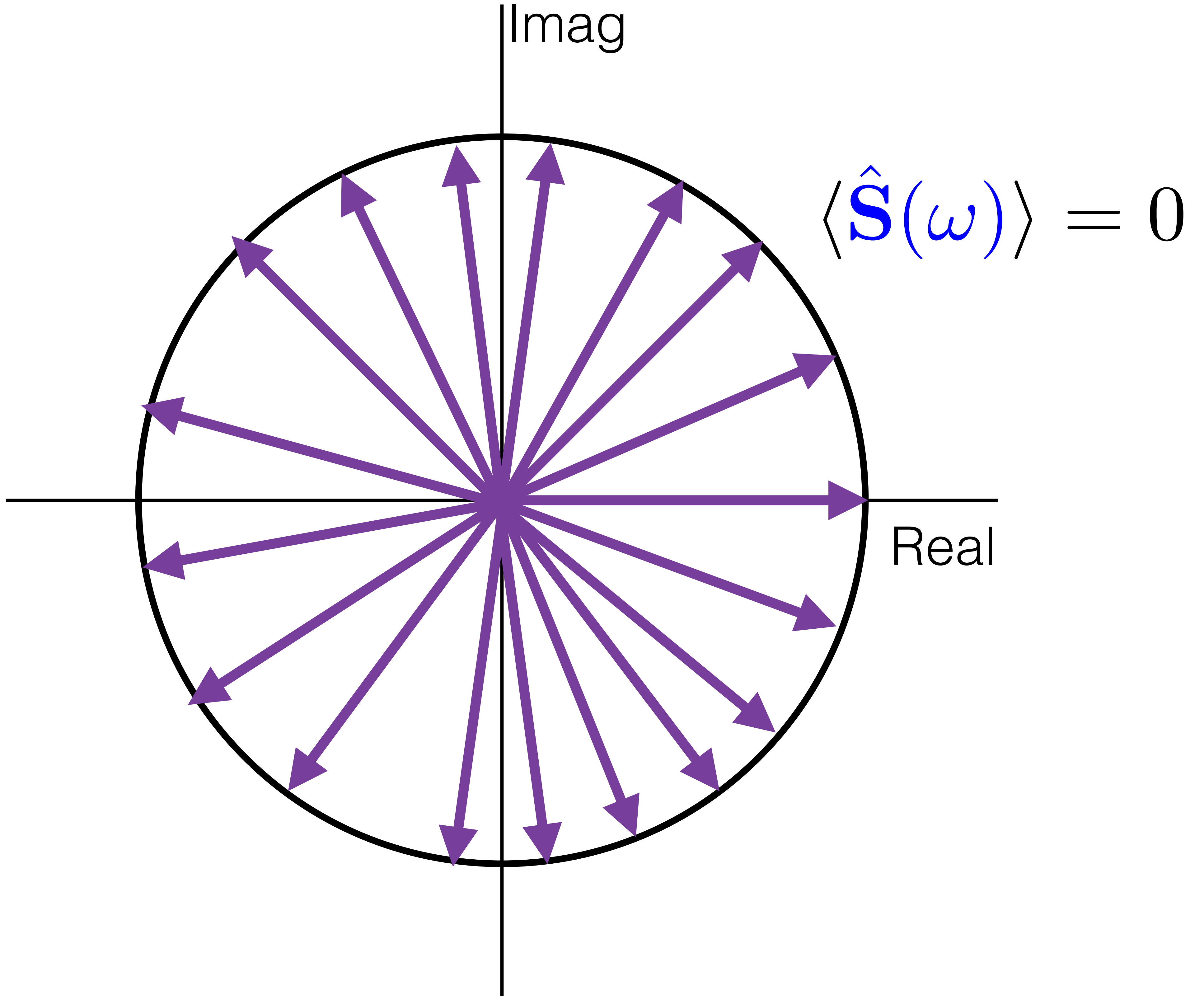

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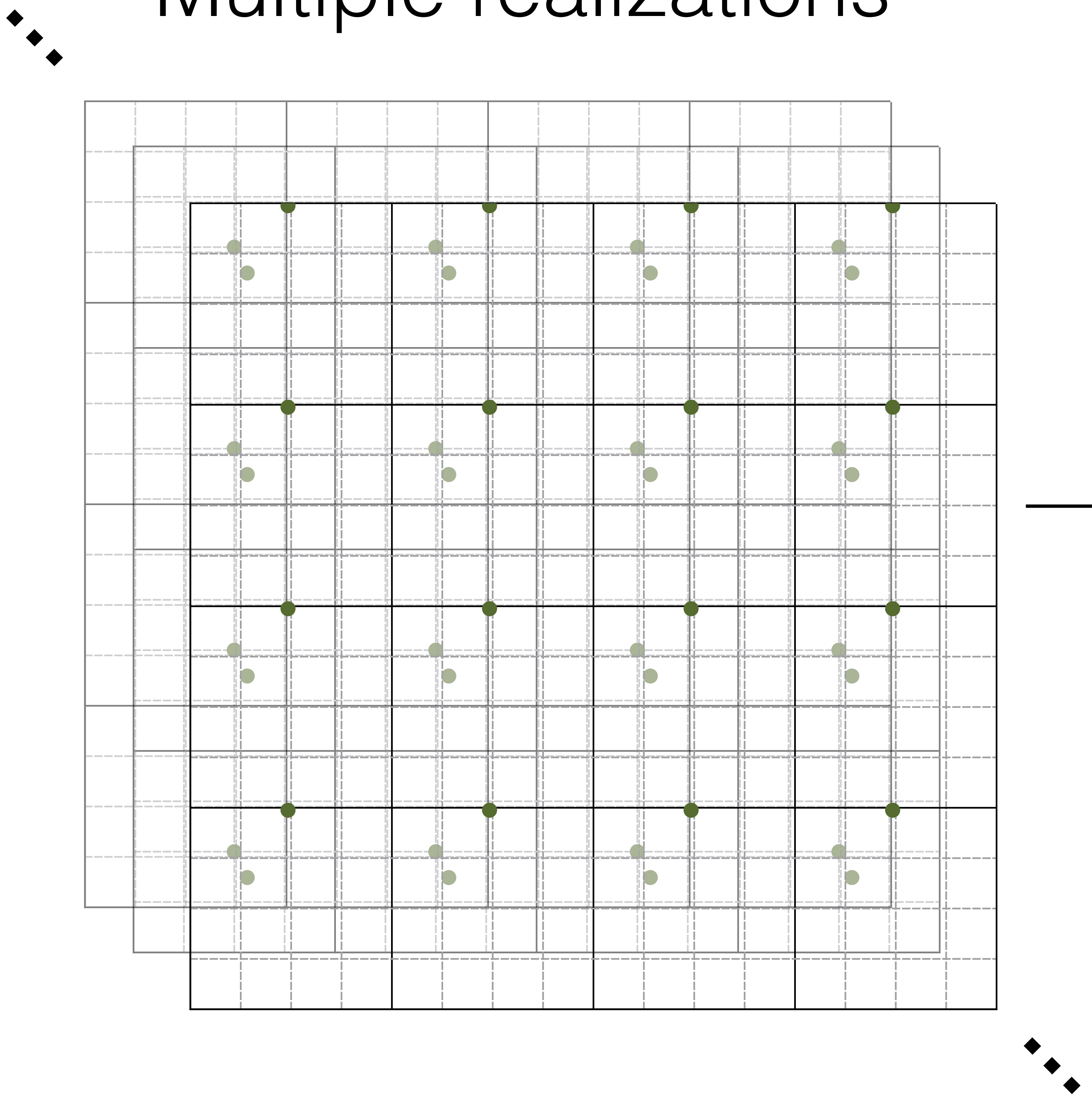

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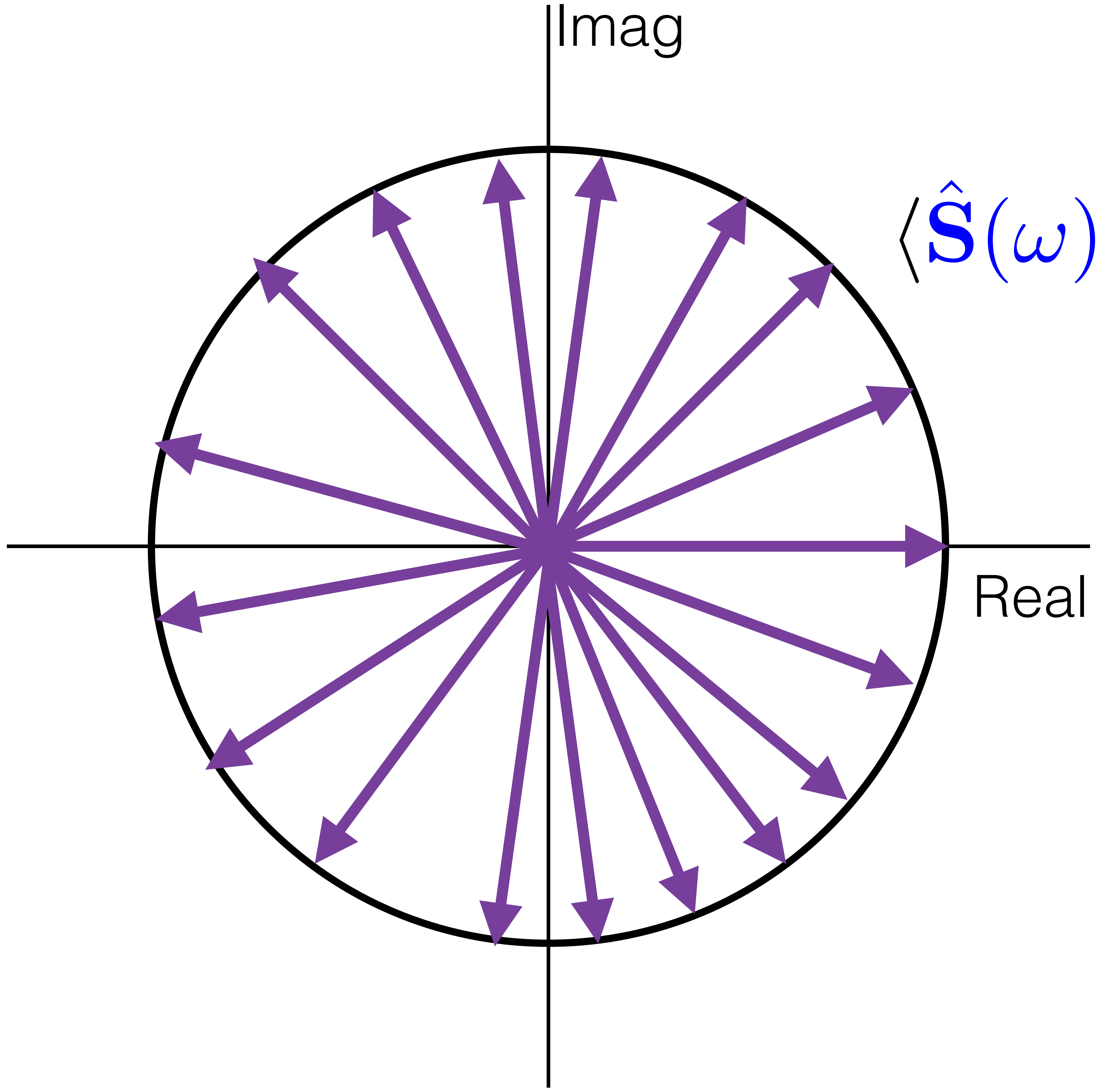

$$\langle \hat{\mathbf{S}}(\omega) \rangle = 0 \;\; \forall \omega \neq 0$$

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$$\text{Error} = \text{Bias}^2 + \text{Variance}$$

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- Homogenization allows representation of error only in terms of variance

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$$\text{Error} = \cancel{\text{Bias}^2} + \text{Variance}$$

- Homogenization allows representation of error only in terms of variance

- We can take any sampling pattern and homogenize it to make the Monte Carlo estimator unbiased.

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Variance in the Fourier domain

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Variance in the Fourier domain

Error: $$I - \tilde{\mu}_N = \hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

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Variance in the Fourier domain

Error:

$$I - \tilde{\mu}_N = \hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\hat{\mathbf{S}}(\omega)d\omega$$

$$\mathrm{Var}(I - \tilde{\mu}_N)$$

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$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left(\hat{f}(0) - \int_{\Omega} \hat{f}^*(\omega)\,\hat{\mathbf{S}}(\omega)\,d\omega\right)$$

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$$\mathrm{Var}(I - \tilde{\mu}_N) = \mathrm{Var}\left(\hat{f}(0) - \int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega)\, d\omega \right)$$

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Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \mathrm{Var}\left(\int_\Omega \hat{f}^*(\omega)\, \hat{\mathbf{S}}(\omega) d\omega\right)$$

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

where,

$P_f(\omega) = |\hat{f}^*(\omega)|^2$ Power Spectrum

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Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

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$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

Subr and Kautz [2013]

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Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\,\mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right)\,d\omega$$

Subr and Kautz [2013]

This is a general form, both for homogenised as well as non-homogenised sampling patterns

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For purely random samples:

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Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

For purely random samples:

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\, \langle P_S(\omega) \rangle\, d\omega$$

Fredo Durand [2011]

where,

$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

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Variance in the Fourier domain

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \mathrm{Var}\left(\hat{\mathbf{S}}(\omega)\right) d\omega$$

For purely random samples: $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\, \langle P_S(\omega) \rangle\, d\omega$$

Fredo Durand [2011]

where,

$$P_S(\omega) = |\hat{\mathbf{S}}(\omega)|^2$$

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Variance using Homogenized Samples

Homogenizing any sampling pattern makes $\langle \hat{\mathbf{S}}(\omega) \rangle = 0$

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Pilleboue et al. [2015]

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$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega) \, \langle P_S(\omega) \rangle \, d\omega$$

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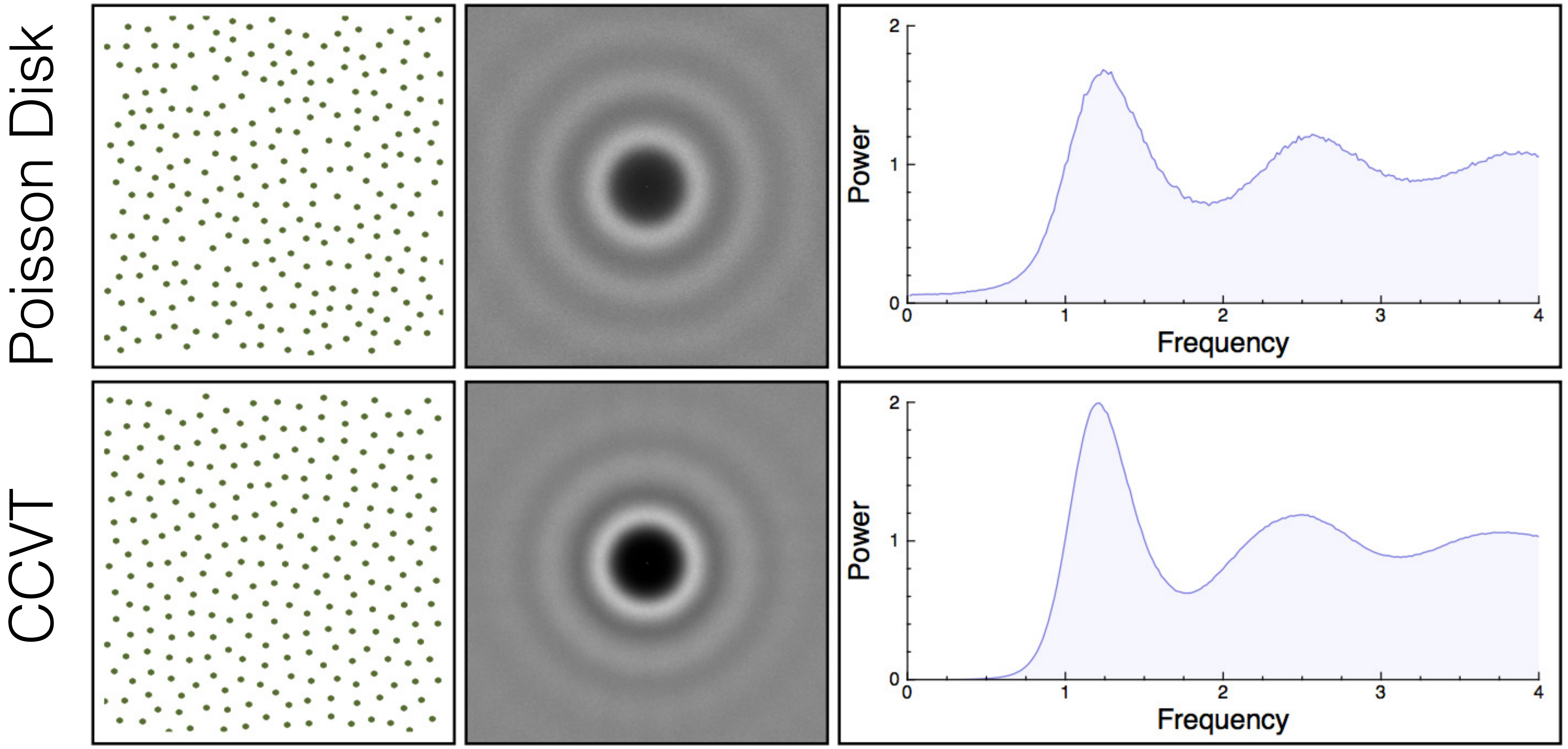
Variance in terms of n-dimensional Power Spectra

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega)\ \langle P_S(\omega) \rangle\ d\omega$$



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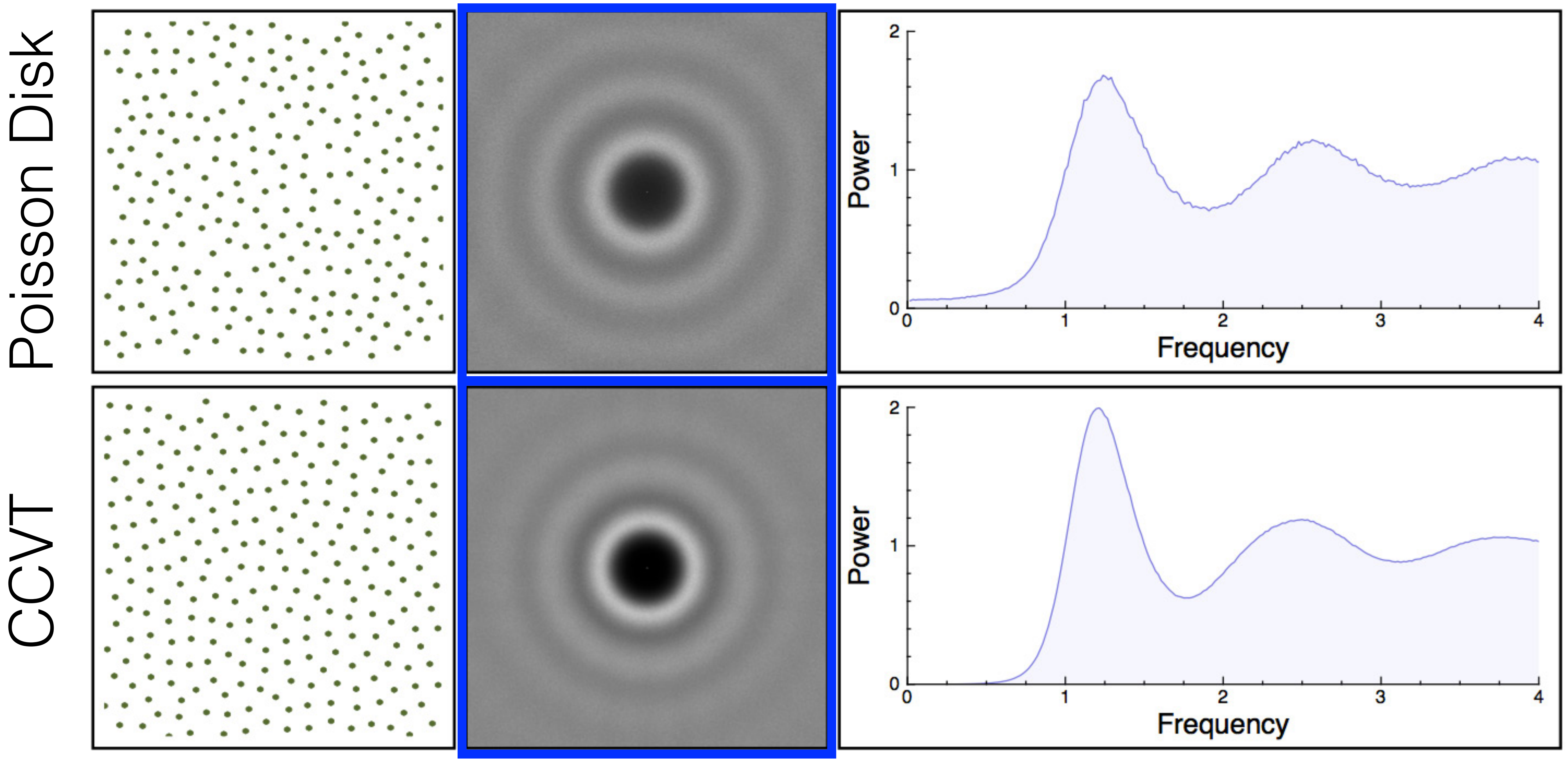

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Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\ \langle P_S(\omega)\rangle\, d\omega$$

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Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_\Omega P_f(\omega)\ \langle P_S(\omega)\rangle\, d\omega$$

In polar coordinates:

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Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega) \, \langle P_S(\omega) \rangle \, d\omega$$

In polar coordinates:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho \mathbf{n}) \, \langle P_{\mathbf{S}}(\rho \mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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Variance in the Polar Coordinates

$$\mathrm{Var}(\tilde{\mu}_N) = \int_{\Omega} P_f(\omega) \, \langle P_S(\omega) \rangle \, d\omega$$

In polar coordinates:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho \mathbf{n}) \, \langle P_{\mathbf{S}}(\rho \mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

Render the Possibilities
SIGGRAPH2016

Variance in the Polar Coordinates

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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Variance in the Polar Coordinates

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \, \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

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Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \, \langle P_\mathbf{S}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:

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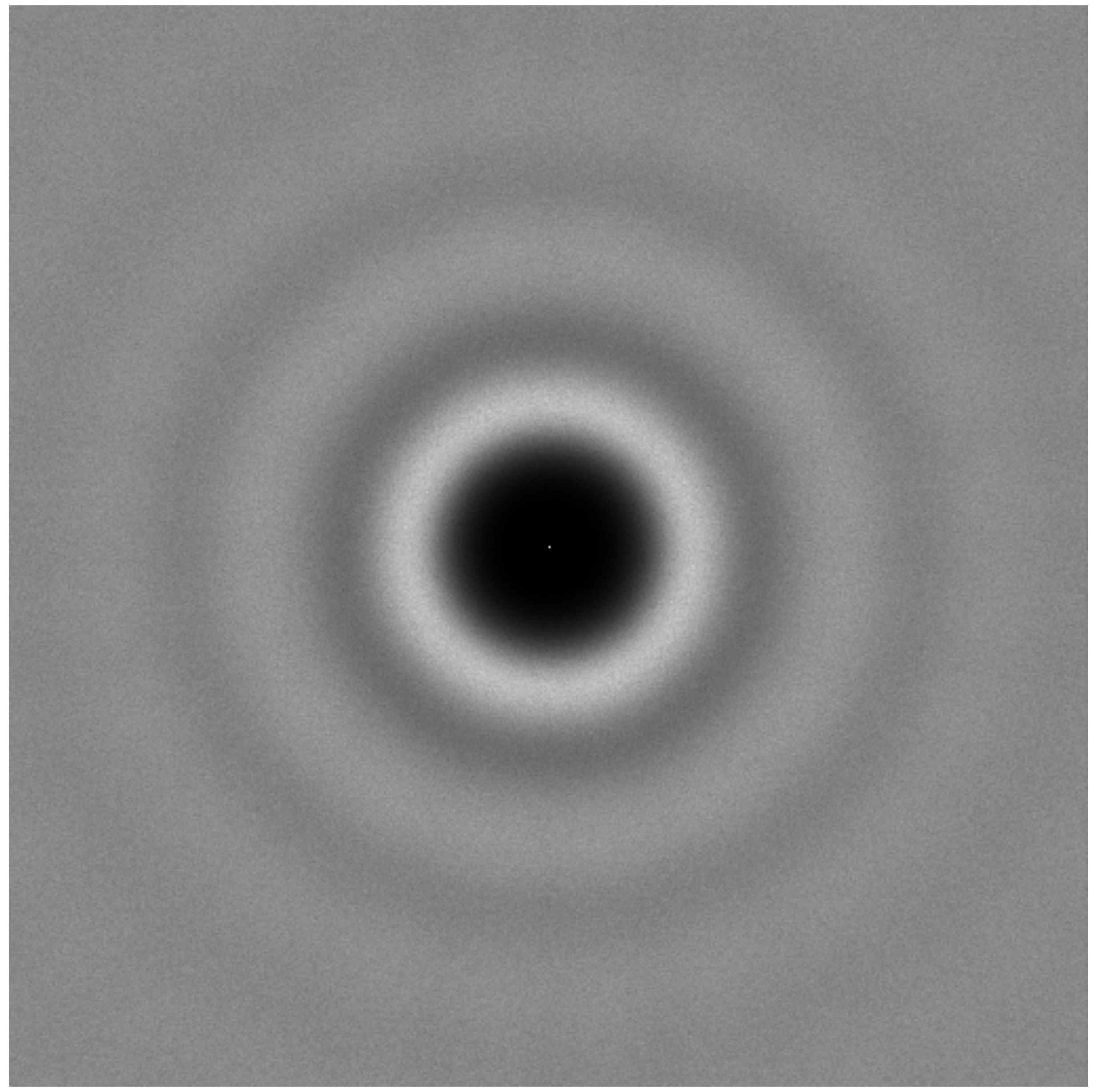
Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:



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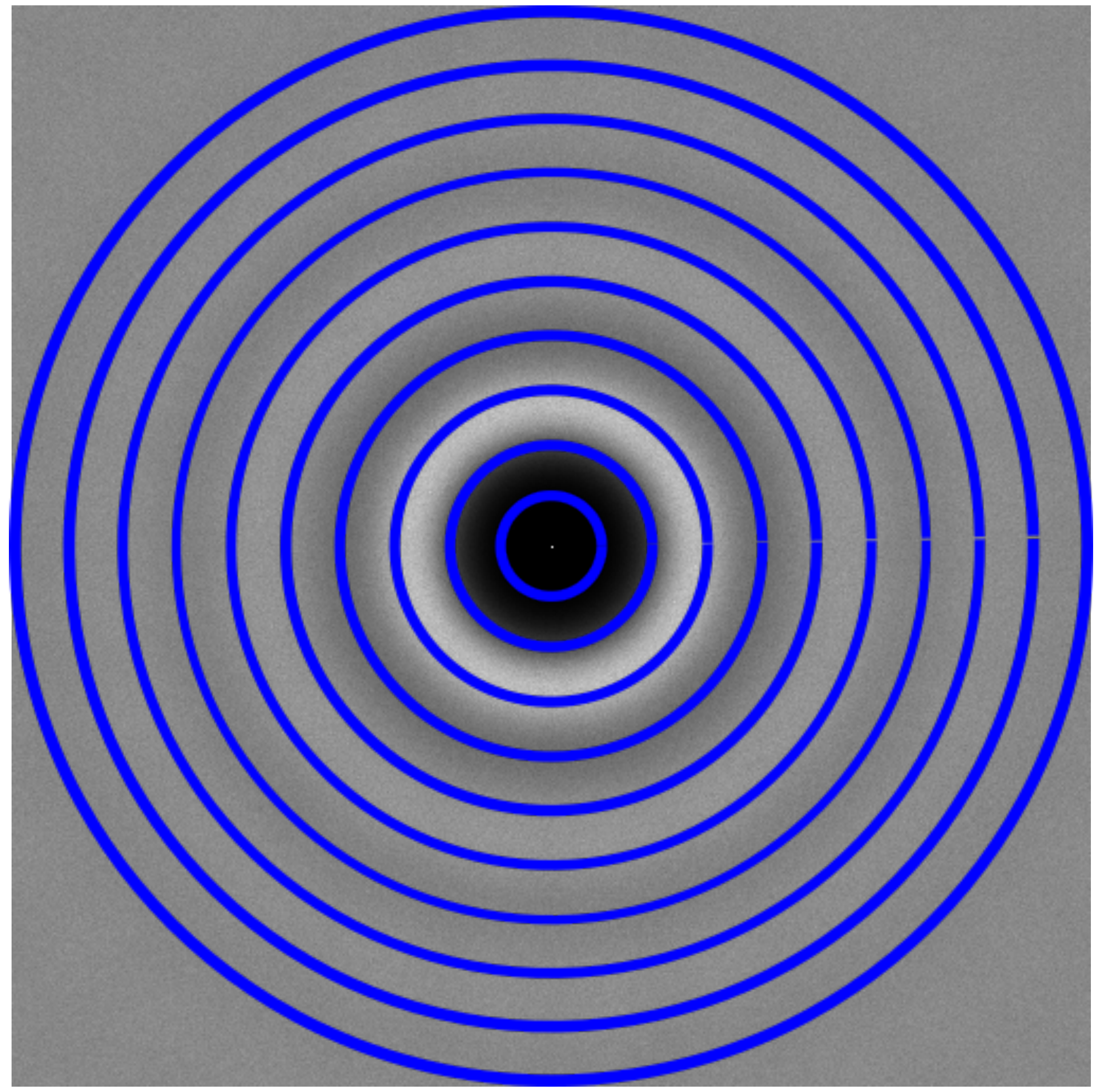
Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:



Render the Possibilities
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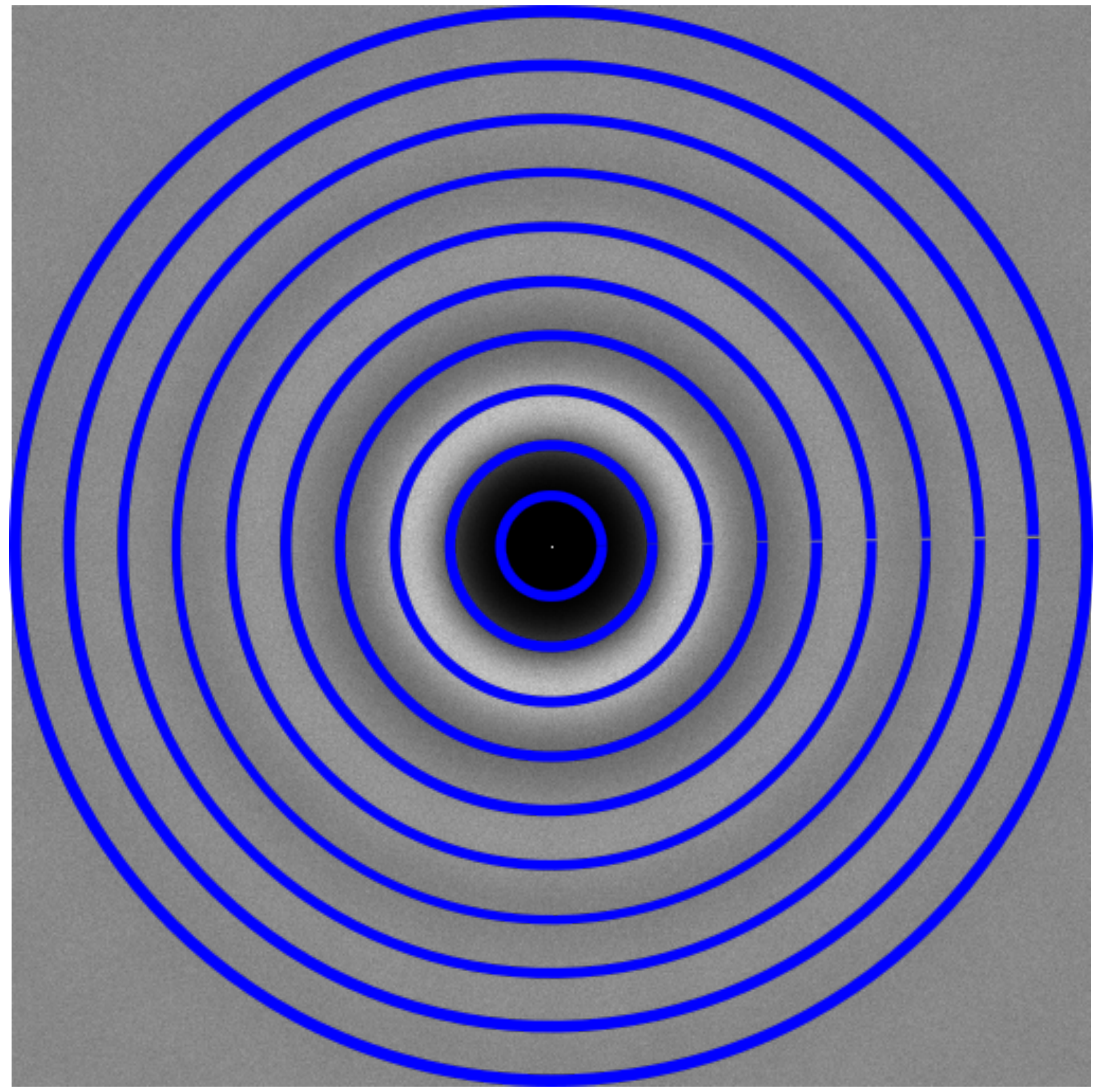
Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



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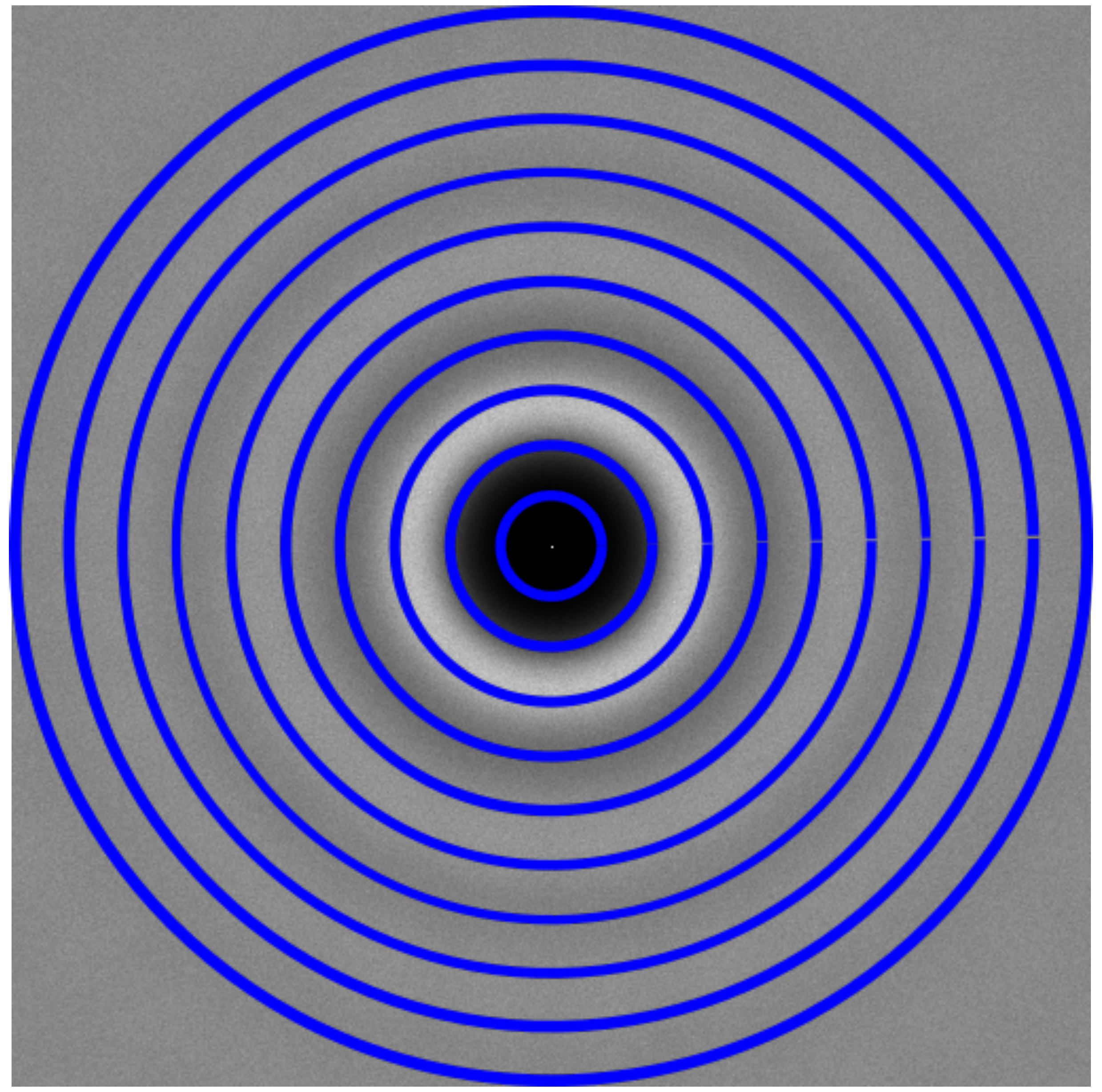
Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



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Variance for Isotropic Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \int_{\mathcal{S}^{d-1}} P_f(\rho\mathbf{n}) \langle P_{\mathbf{S}}(\rho\mathbf{n}) \rangle \, d\mathbf{n} \, d\rho$$

For isotropic power spectra:

$$\boxed{Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho}$$

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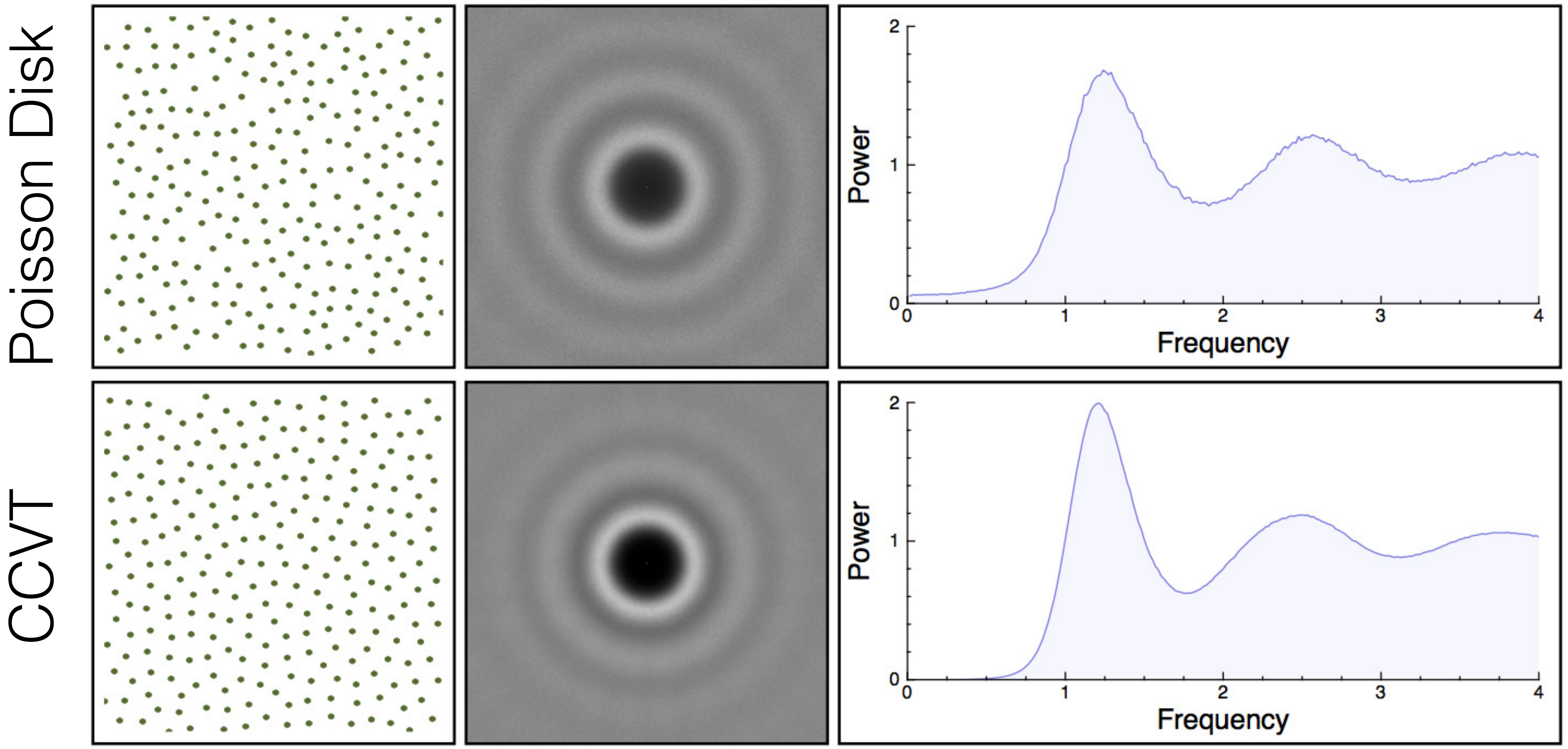
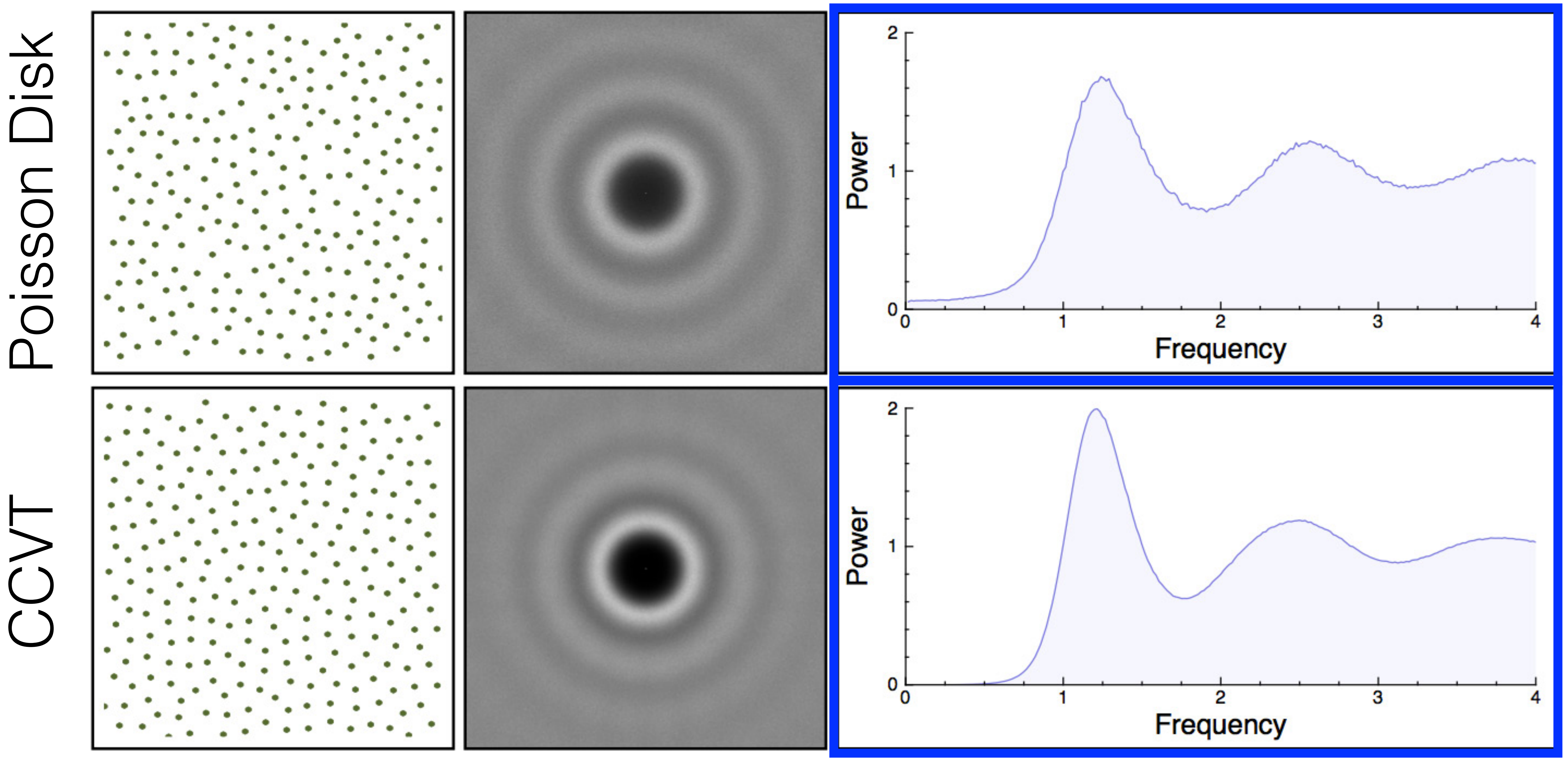
Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



Variance in terms of 1-dimensional Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



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Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho$$

Render the Possibilities
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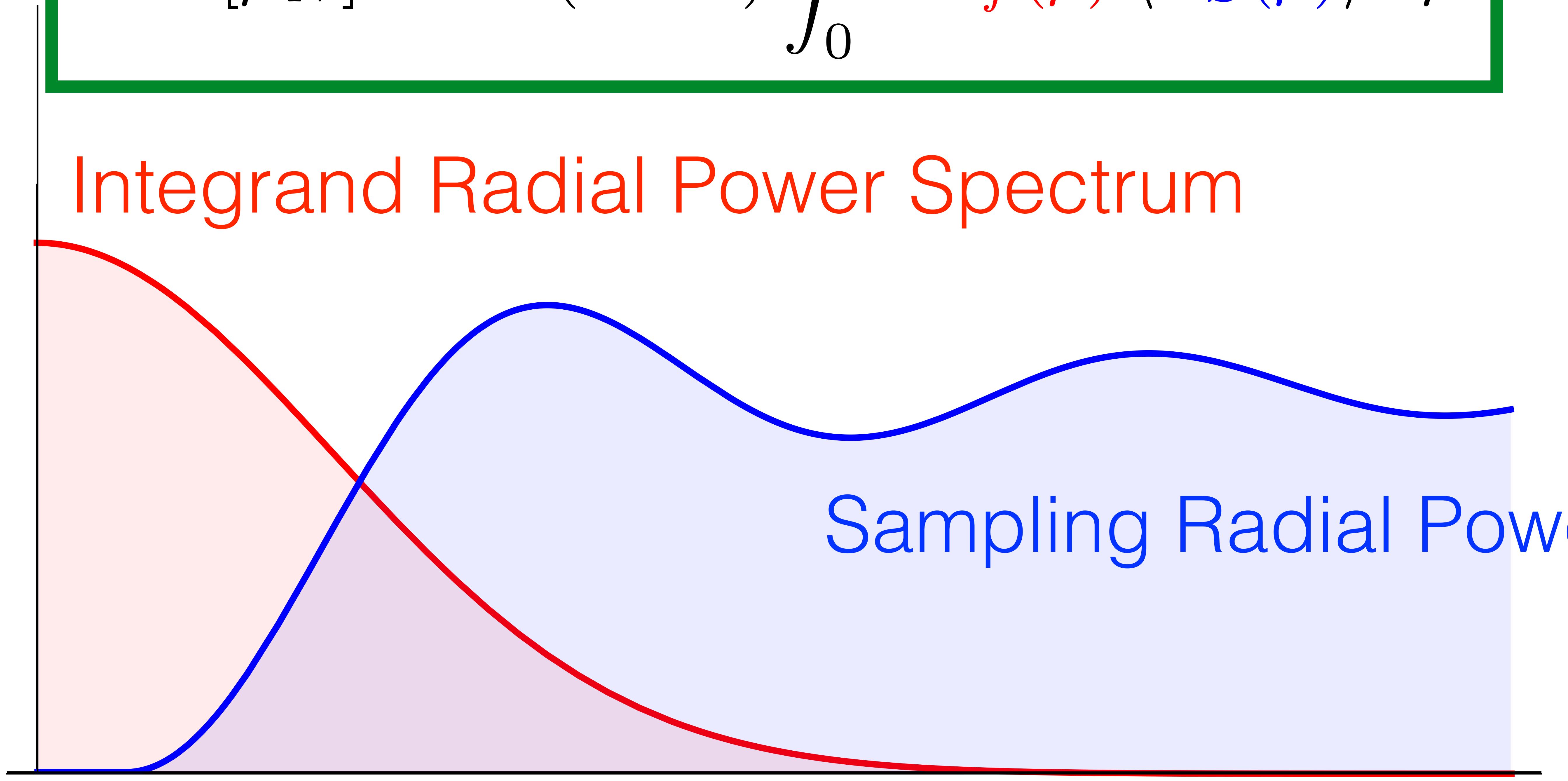
Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho$$



For given number of Samples

Render the Possibilities
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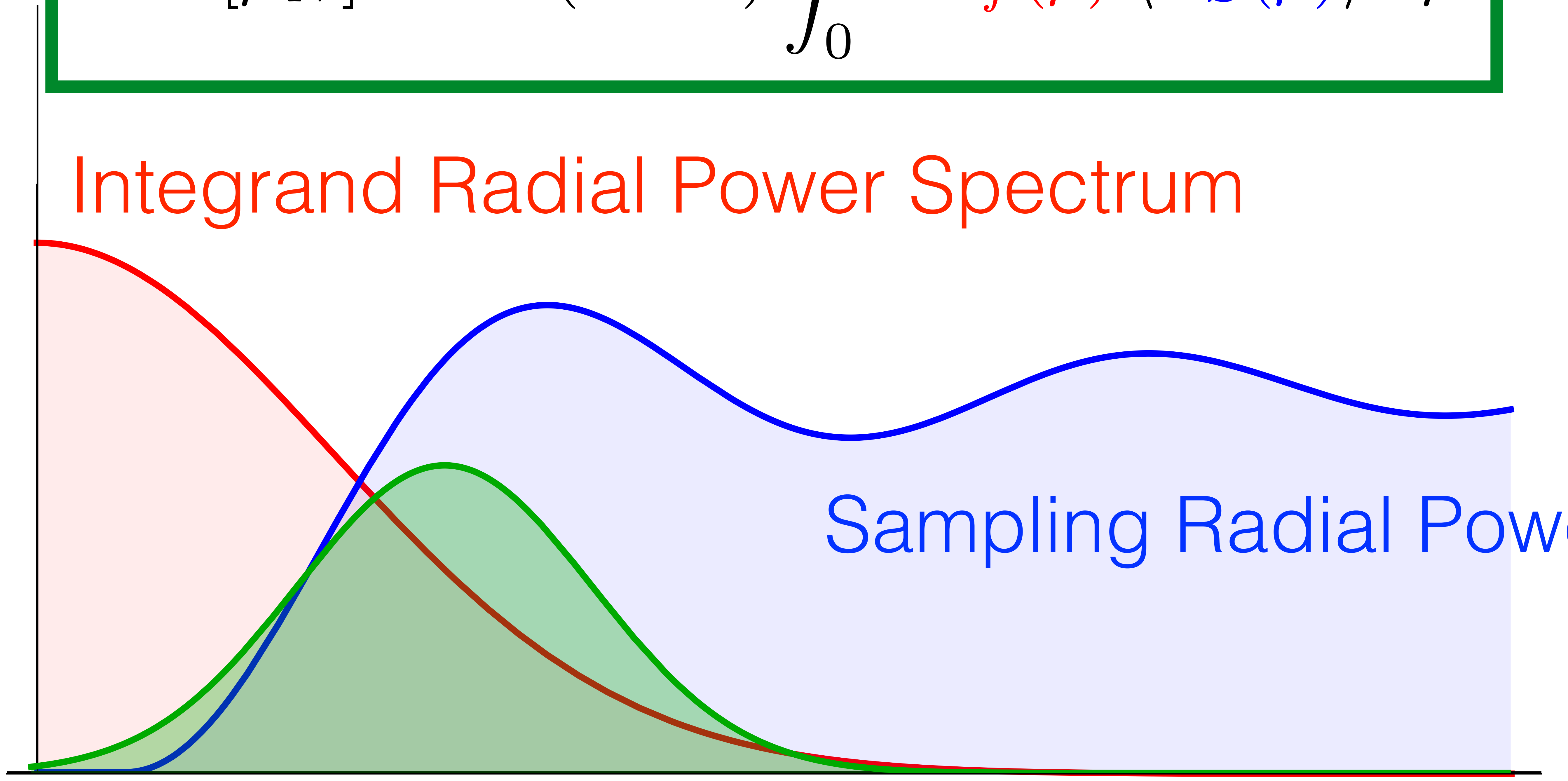
Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \left\langle \tilde{P}_{\mathbf{S}}(\rho) \right\rangle d\rho$$



For given number of Samples

Render the Possibilities
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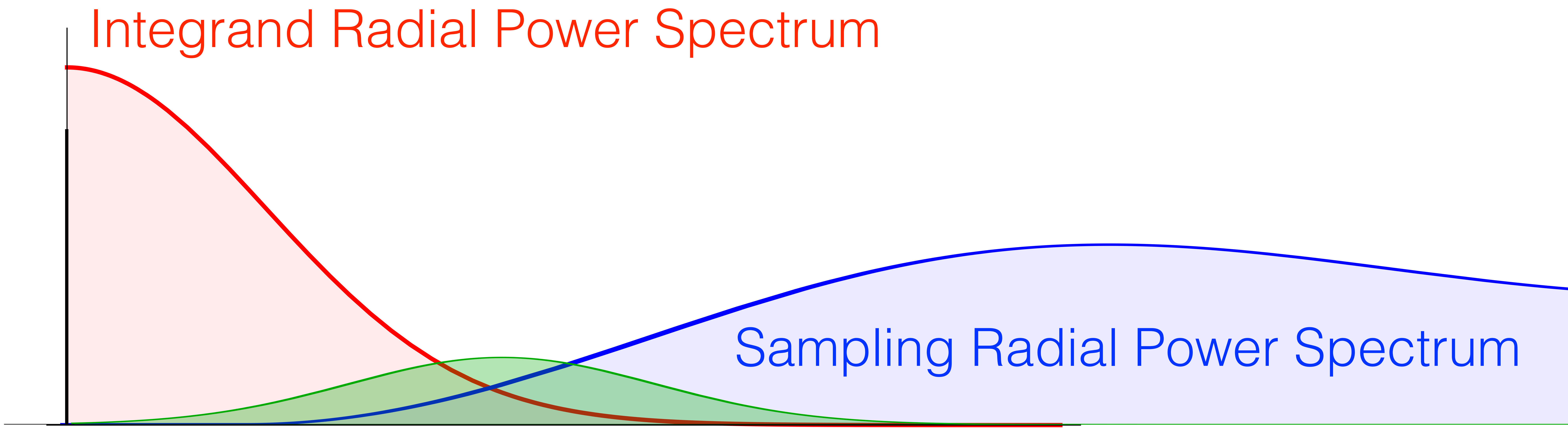
Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



For given number of Samples

Render the Possibilities
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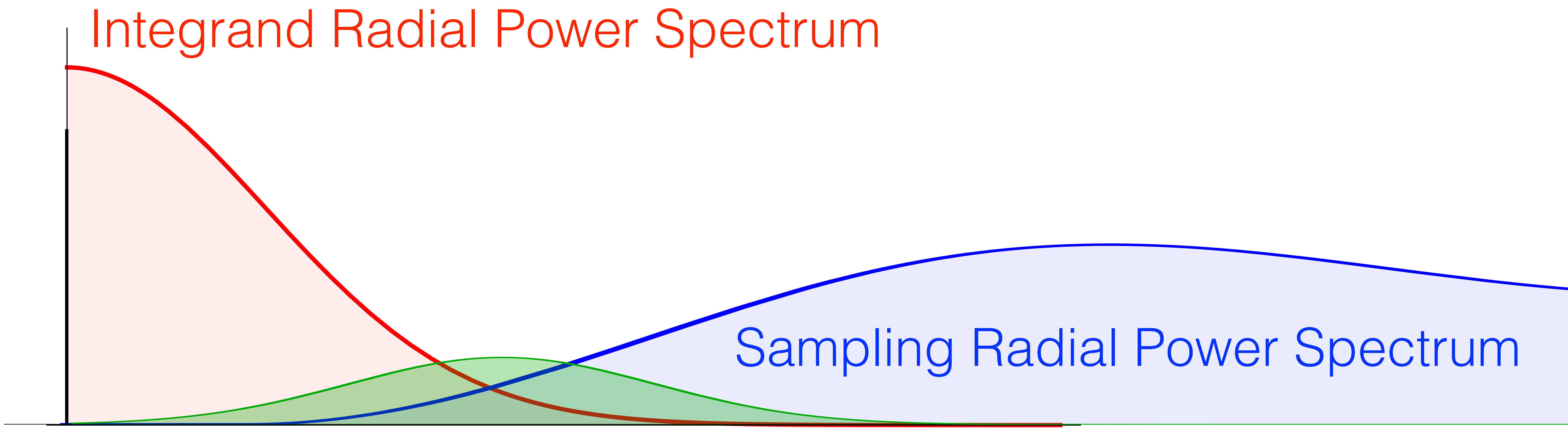
Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^\infty \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



For given number of Samples

Render the Possibilities
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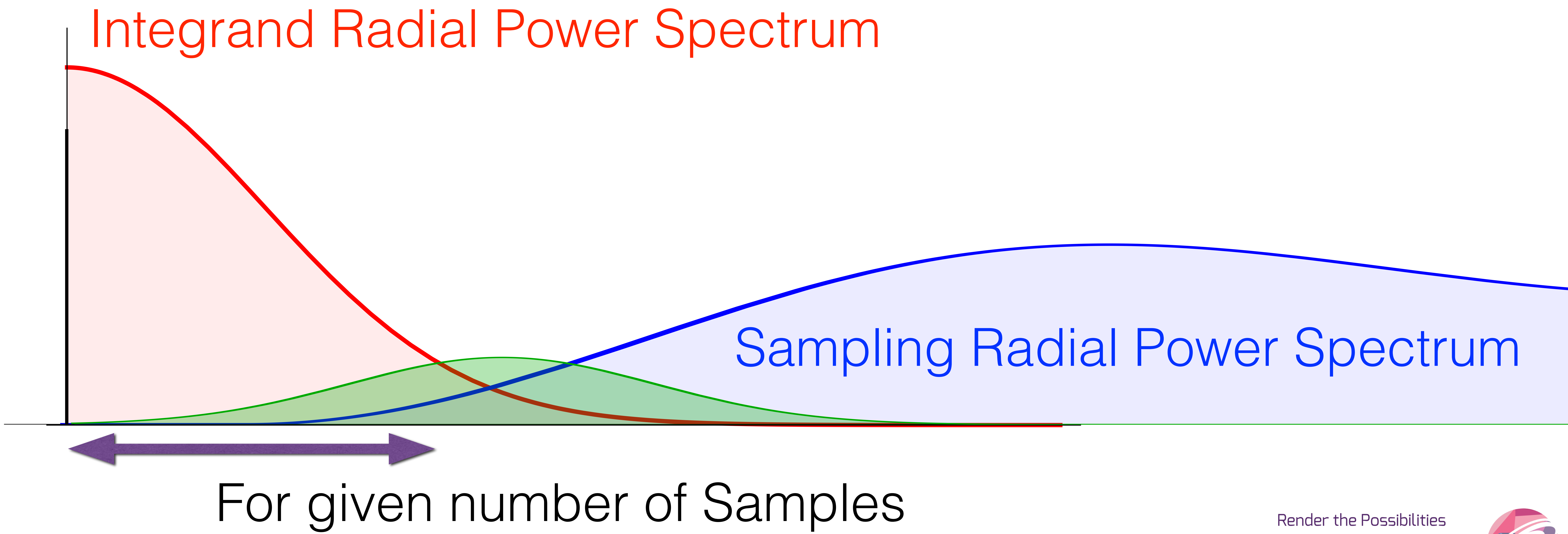
Variance: Integral over Product of Power Spectra

$$Var[\tilde{\mu}_N] = \mathcal{M}(\mathcal{S}^{d-1}) \int_0^{\infty} \tilde{P}_f(\rho) \, \langle \tilde{P}_{\mathbf{S}}(\rho) \rangle \, d\rho$$



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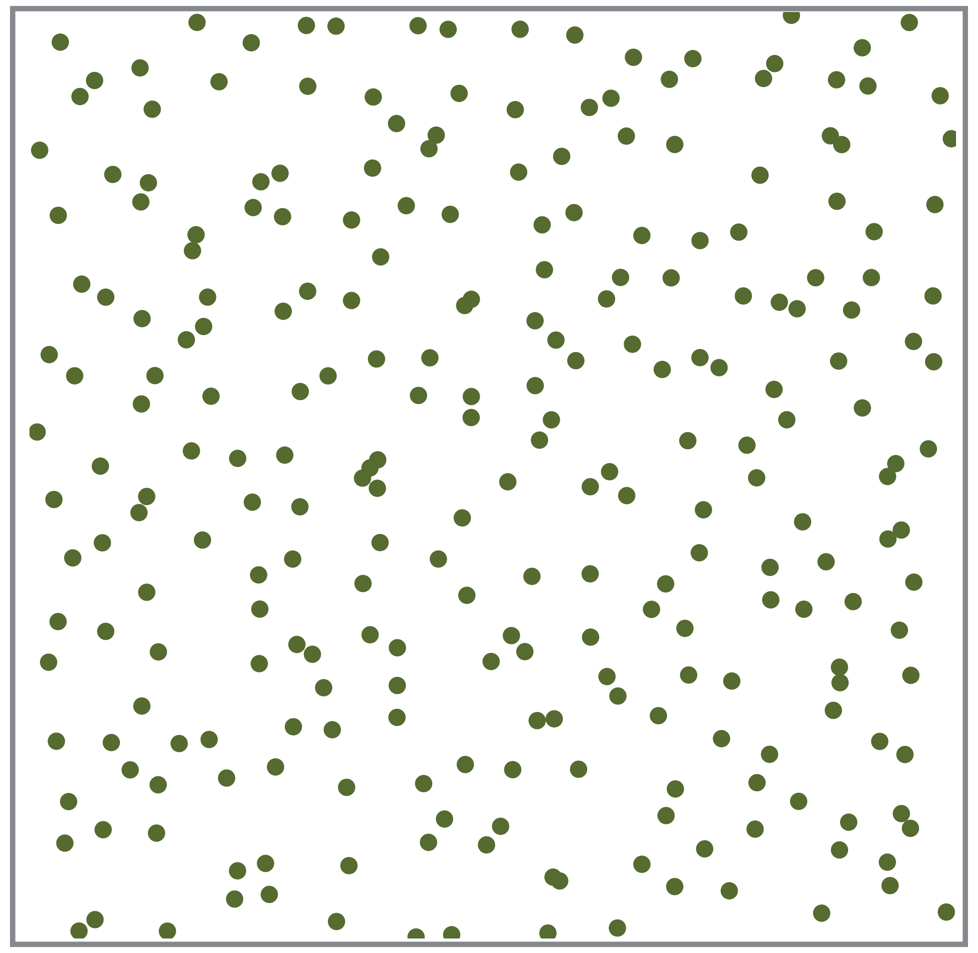
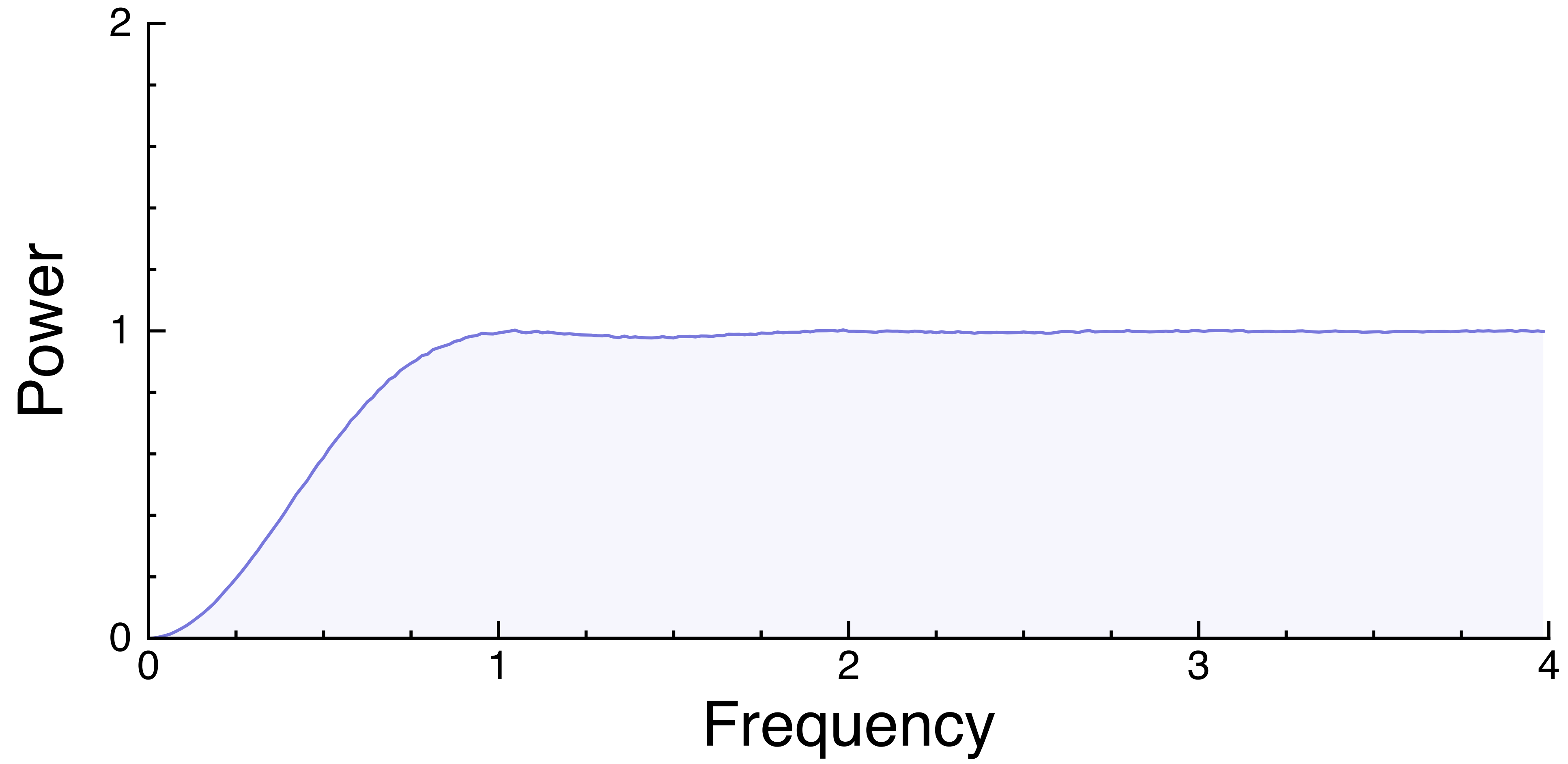
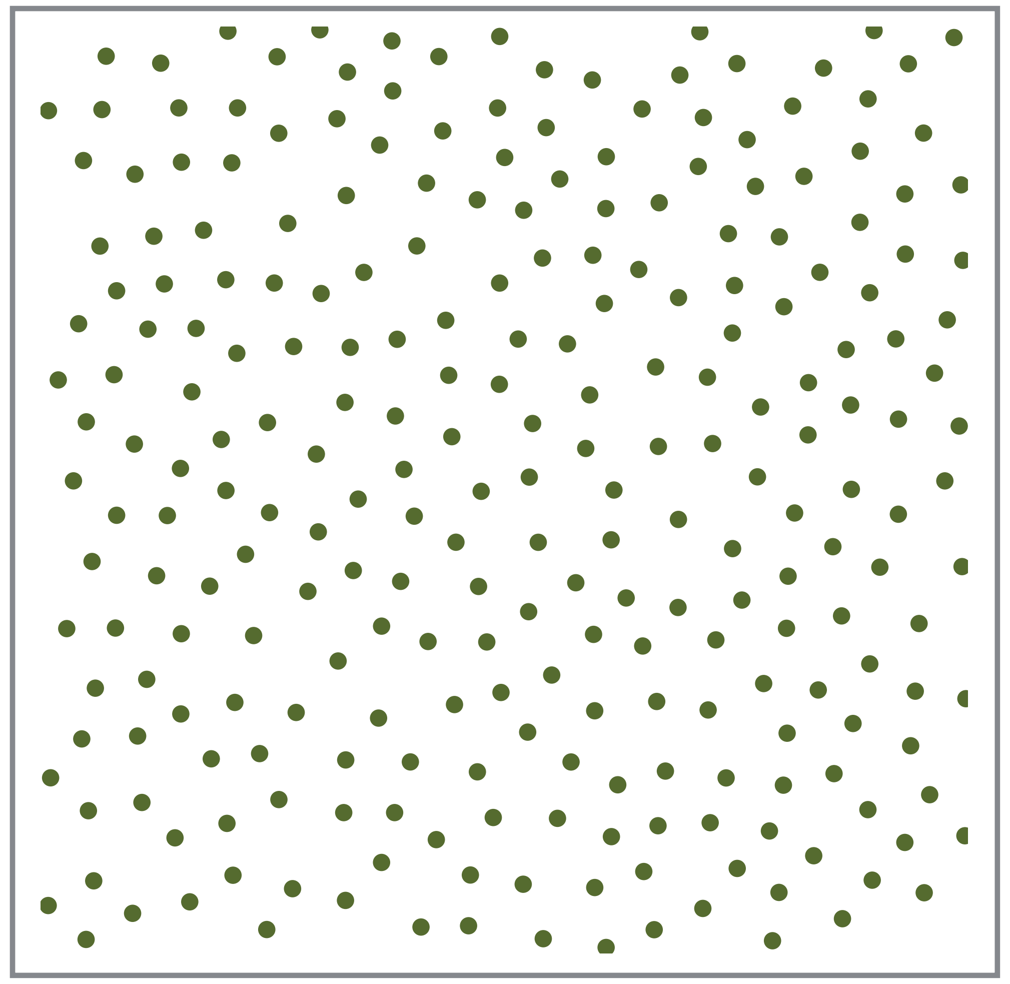
Spatial Distribution vs Radial Mean Power Spectra

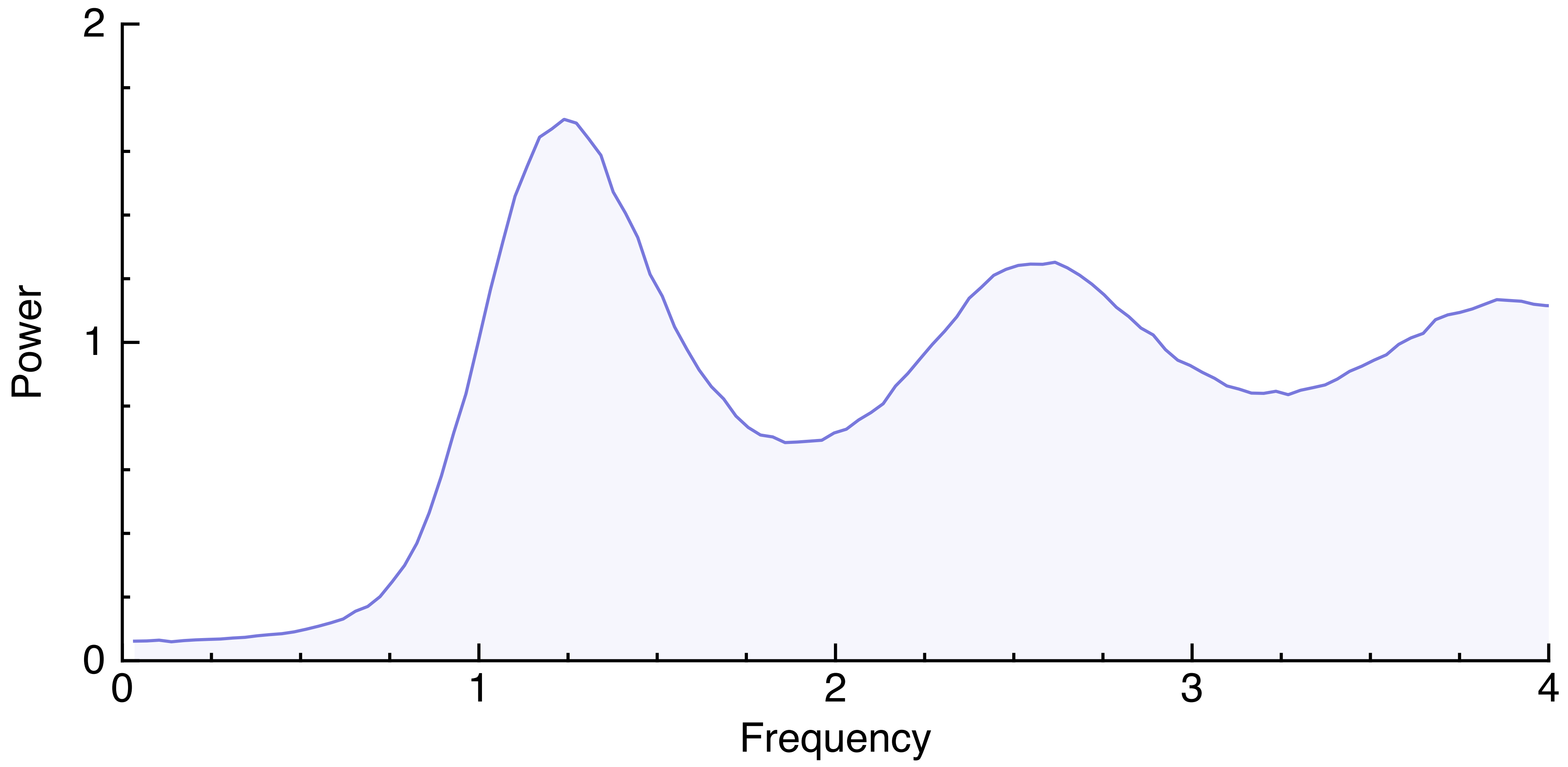
Jitter





Poisson Disk





Render the Possibilities
SIGGRAPH2016

For 2-dimensions

Samplers	Worst Case	Best Case
Random		
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
SIGGRAPH2016

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
SIGGRAPH2016

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter		
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
SIGGRAPH2016

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
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For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk		
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
SIGGRAPH2016

For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
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For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT		

Pilleboue et al. [2015]

Render the Possibilities
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For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	

Pilleboue et al. [2015]

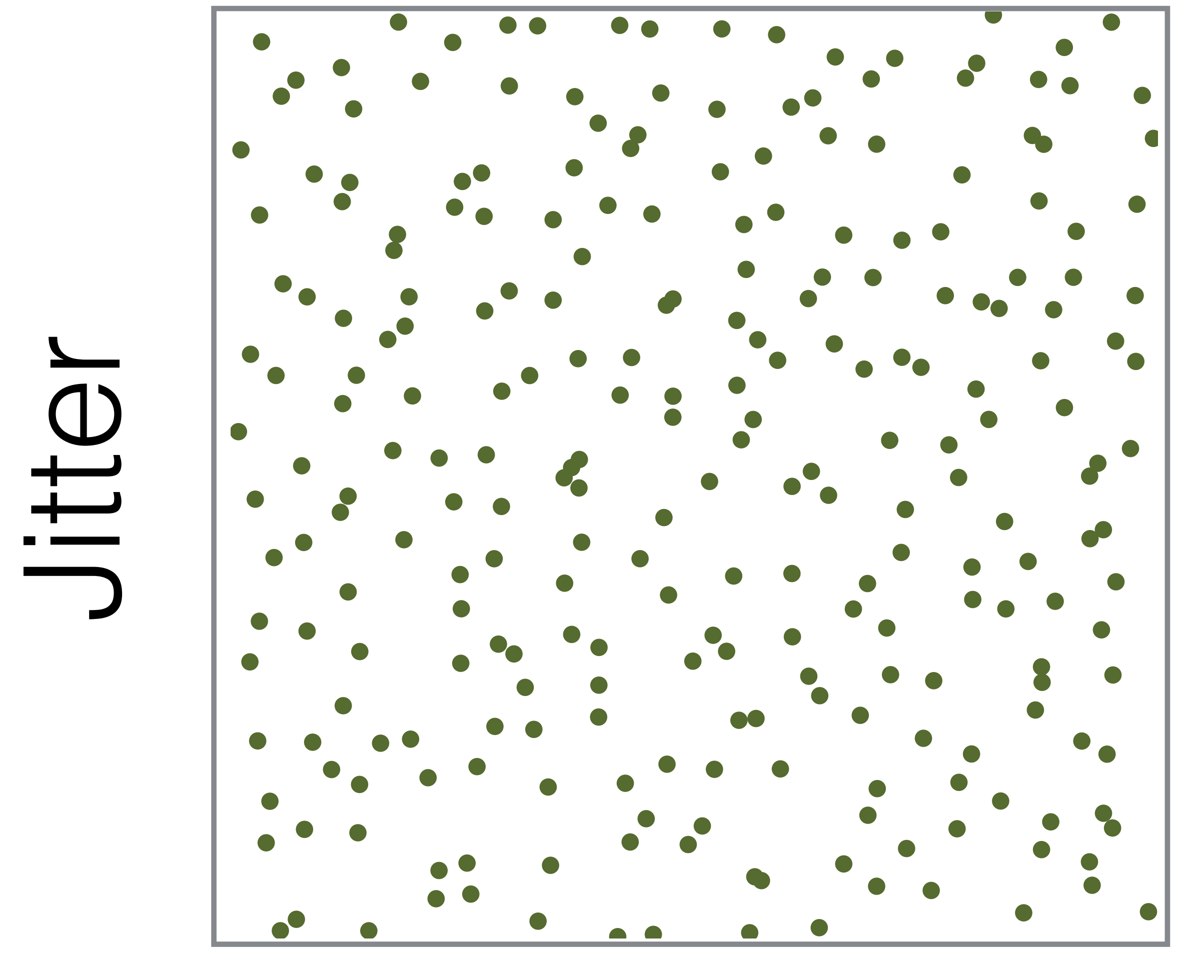
Render the Possibilities
SIGGRAPH2016

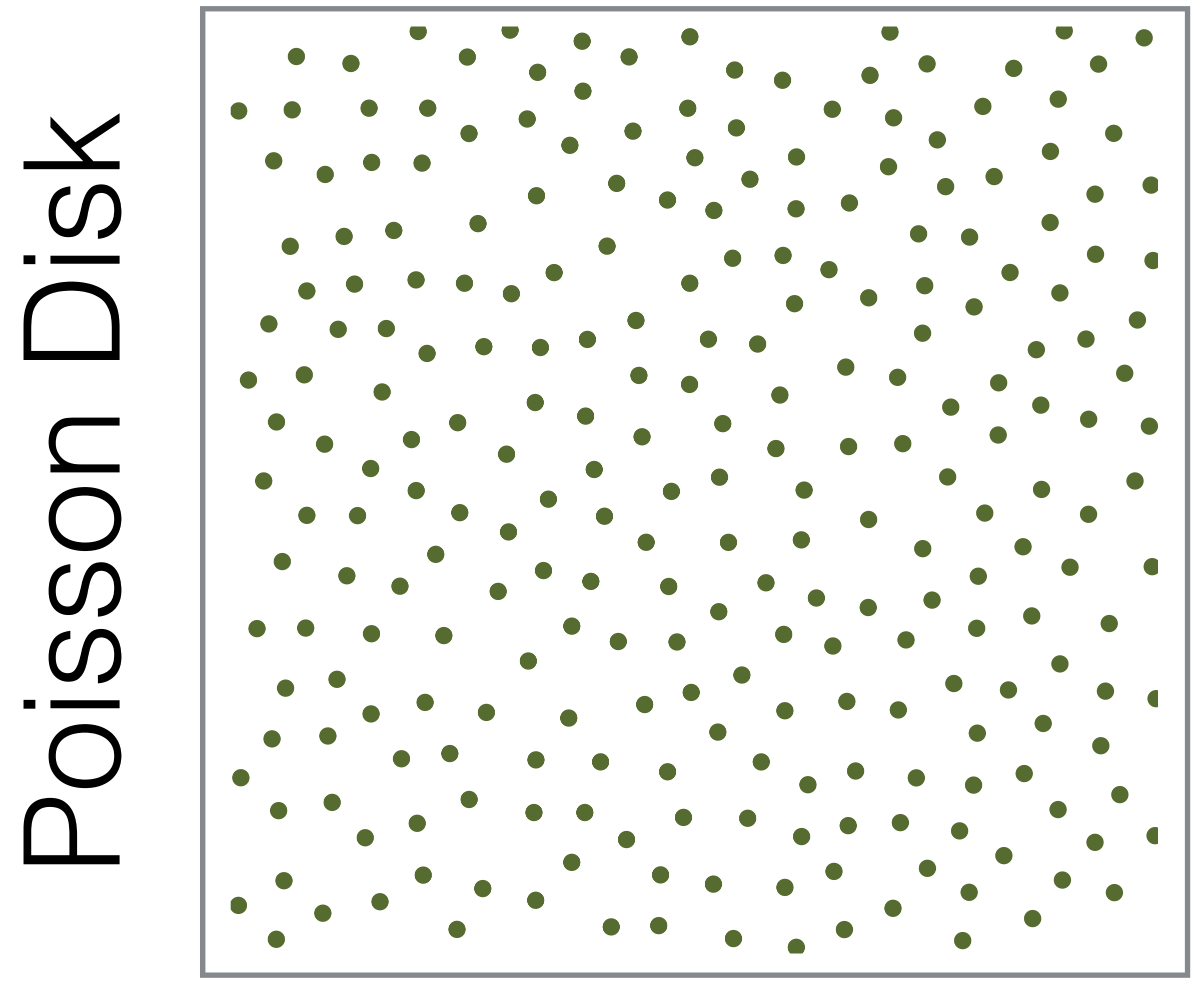
For 2-dimensions

Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Pilleboue et al. [2015]

For 2-dimensions





Samplers	Worst Case	Best Case
Random	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
Jitter	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-2})$
Poisson Disk	$\mathcal{O}(N^{-1})$	$\mathcal{O}(N^{-1})$
CCVT	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-3})$

Pilleboue et al. [2015]

Render the Possibilities
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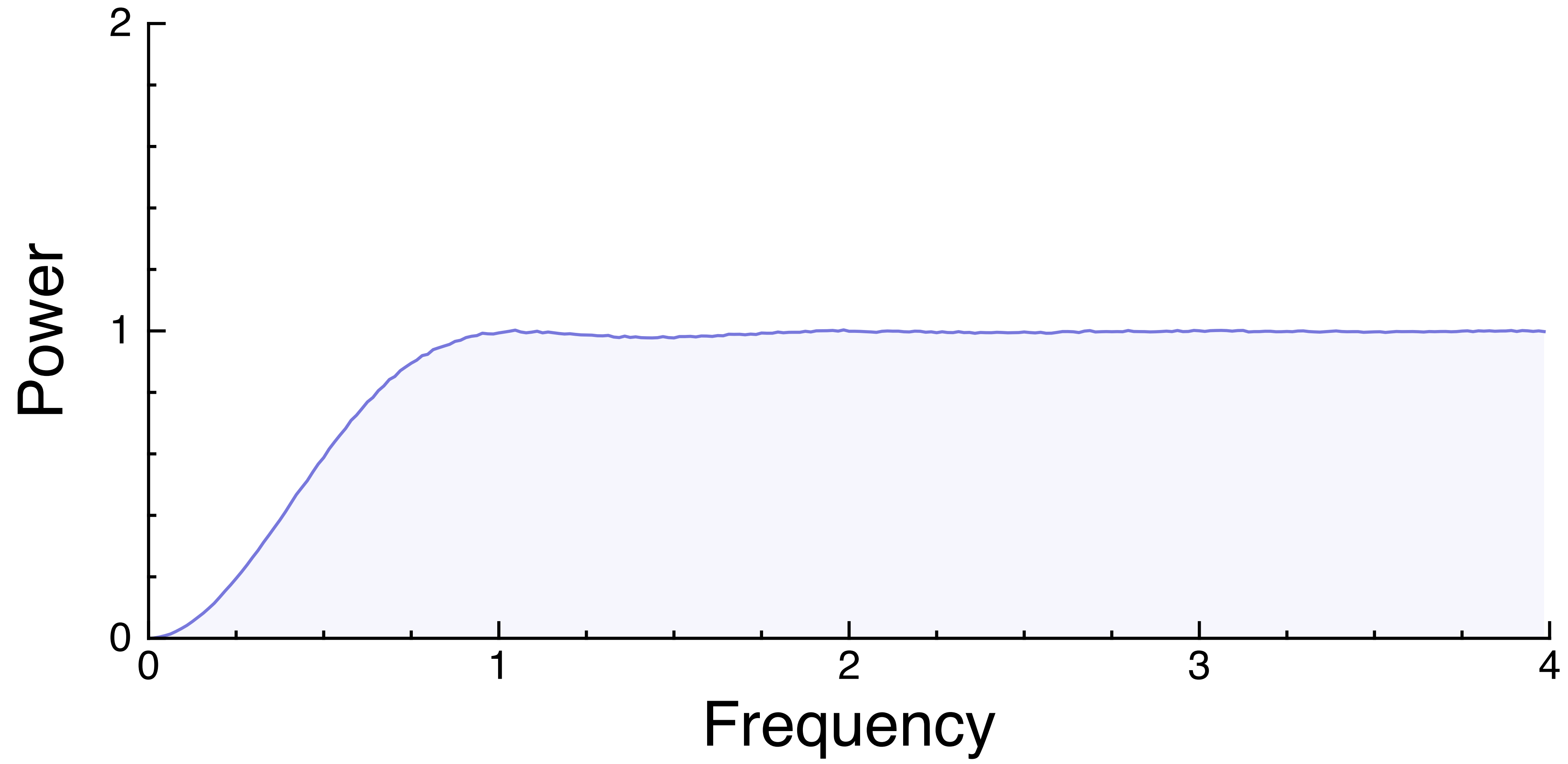
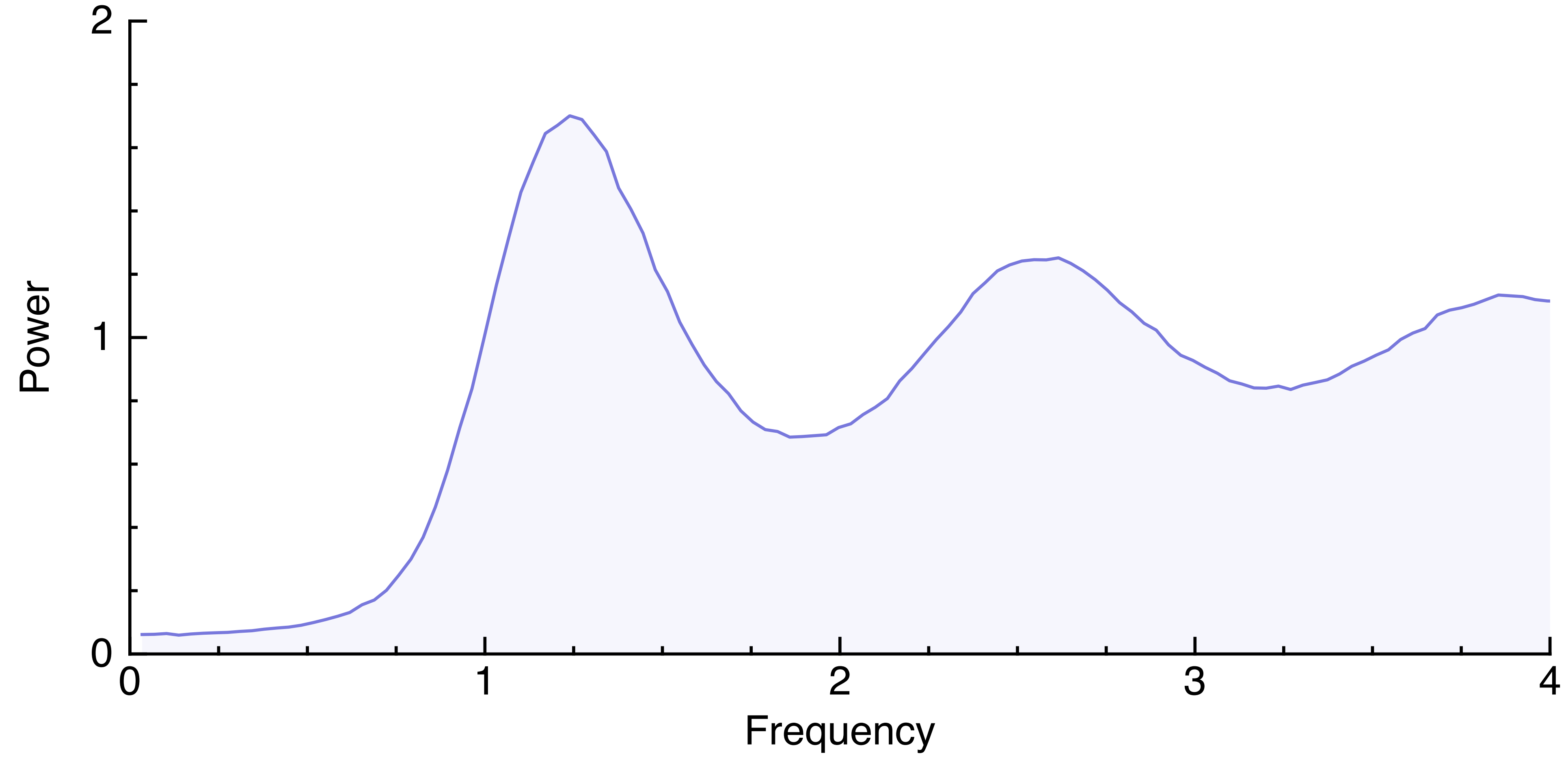
Low Frequency Region

Jitter



Poisson Disk



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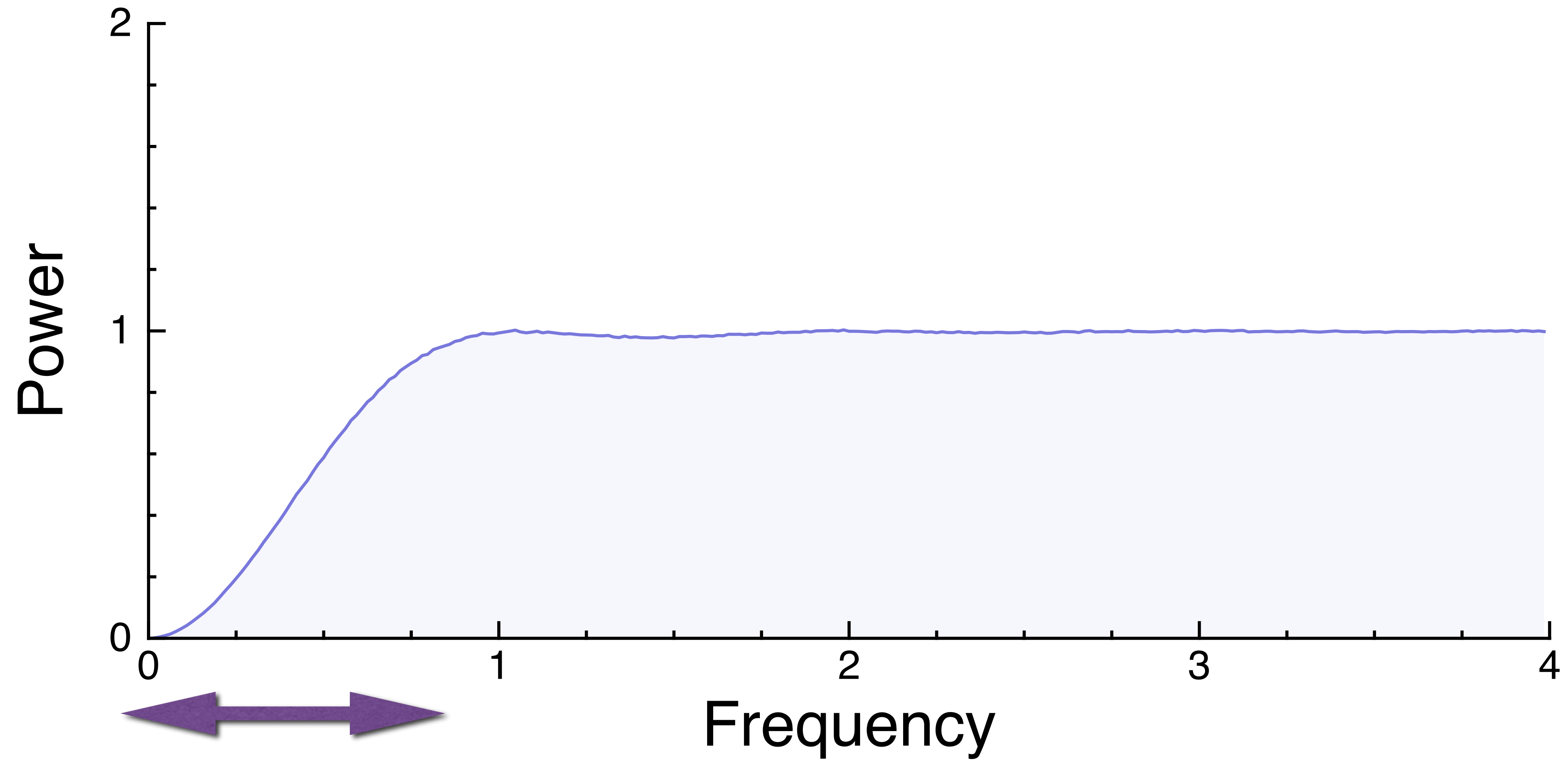
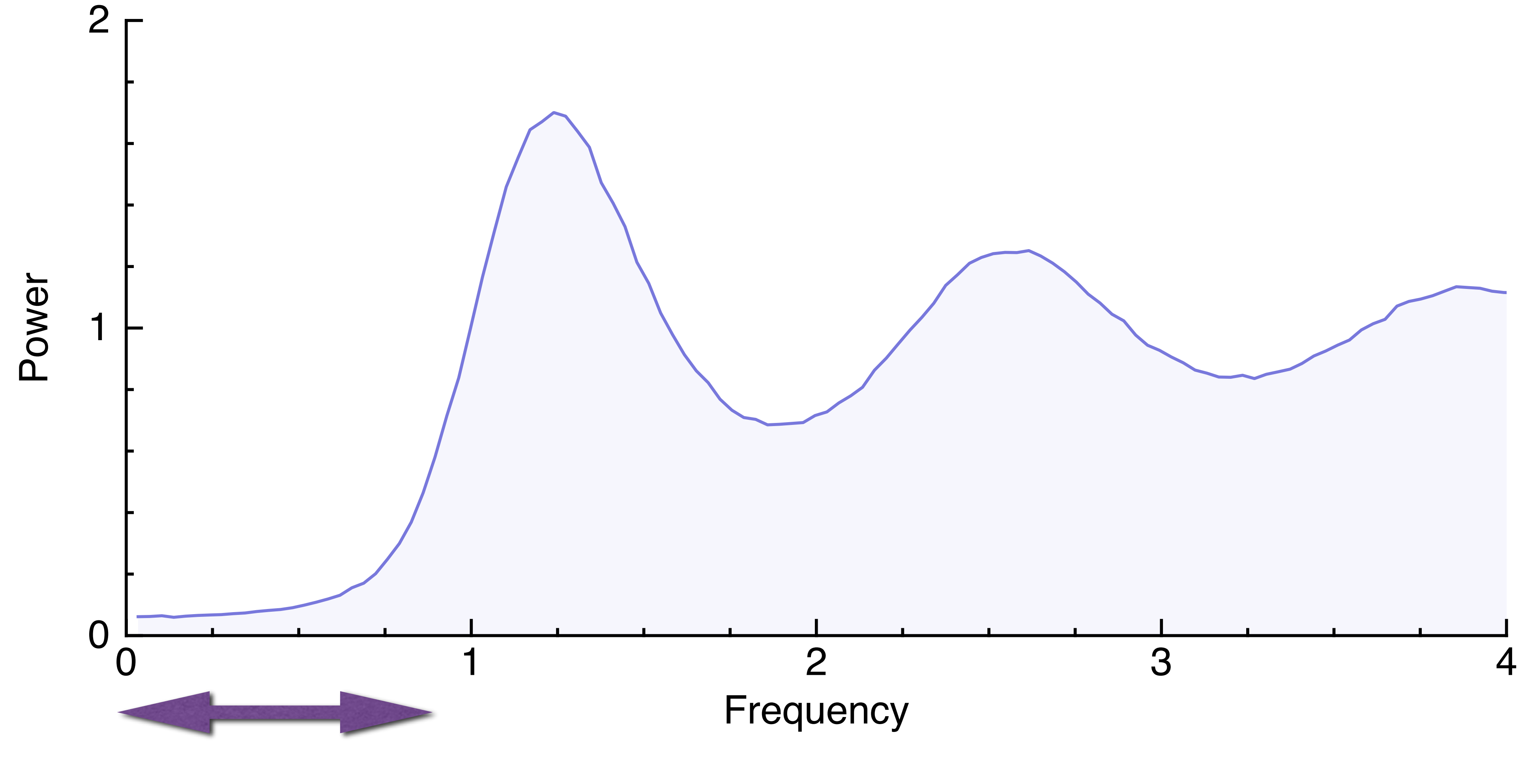
Low Frequency Region

Jitter



Poisson Disk



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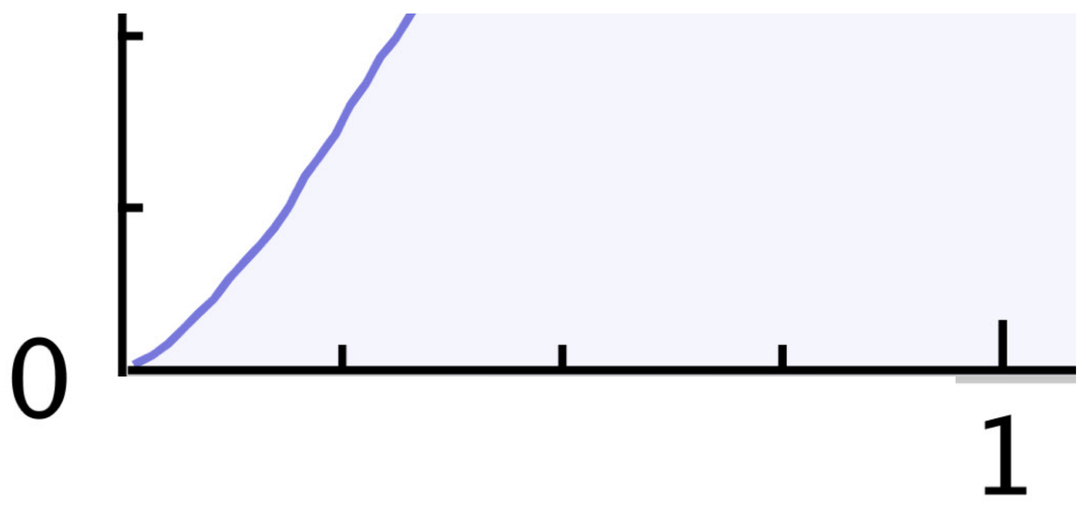
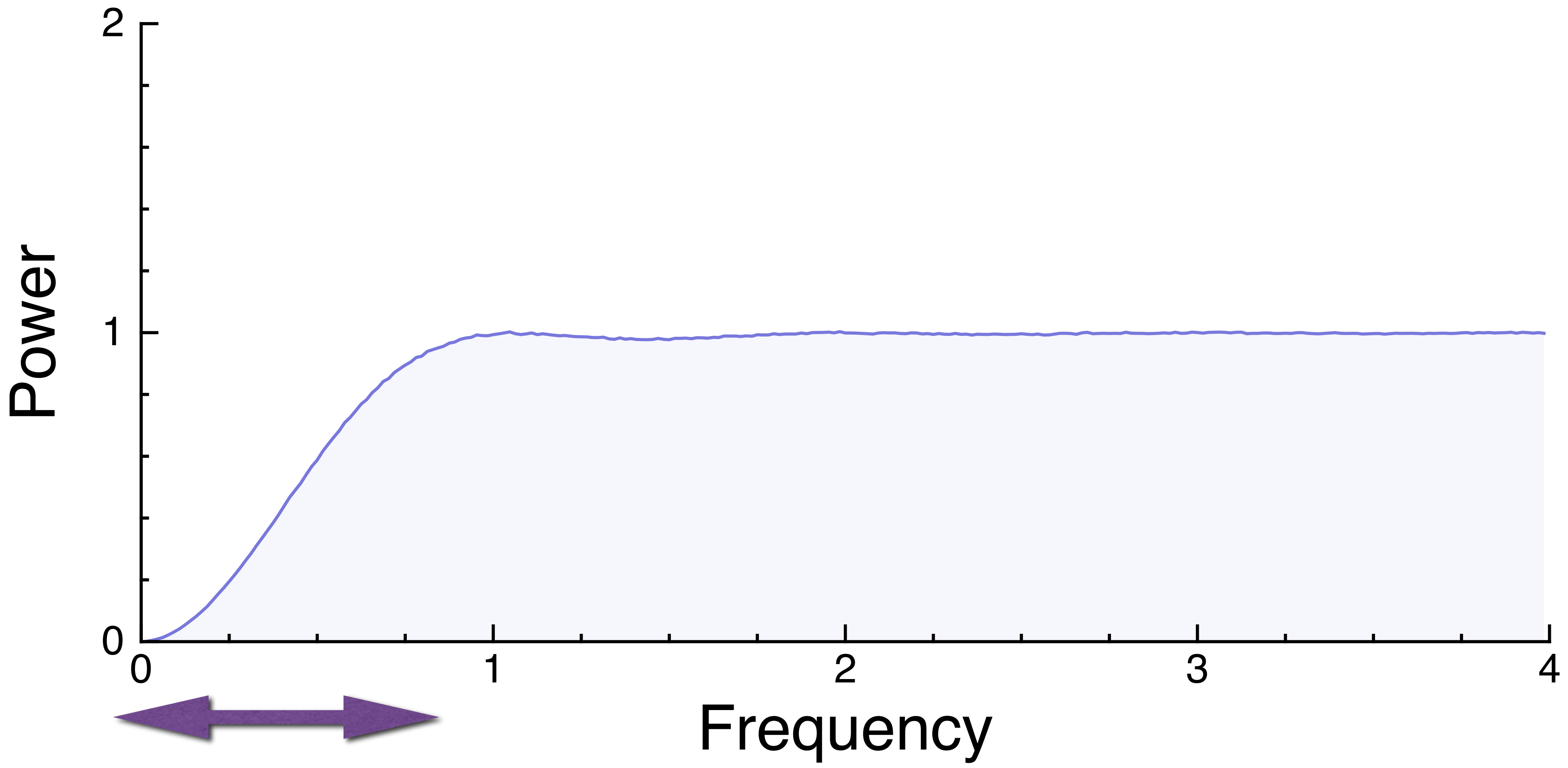
Low Frequency Region

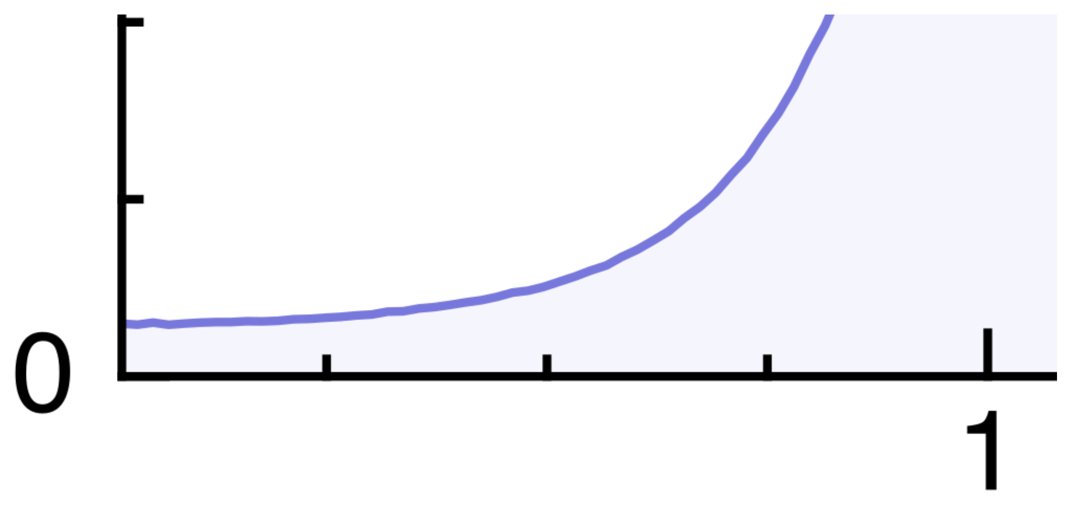
Zoom-in

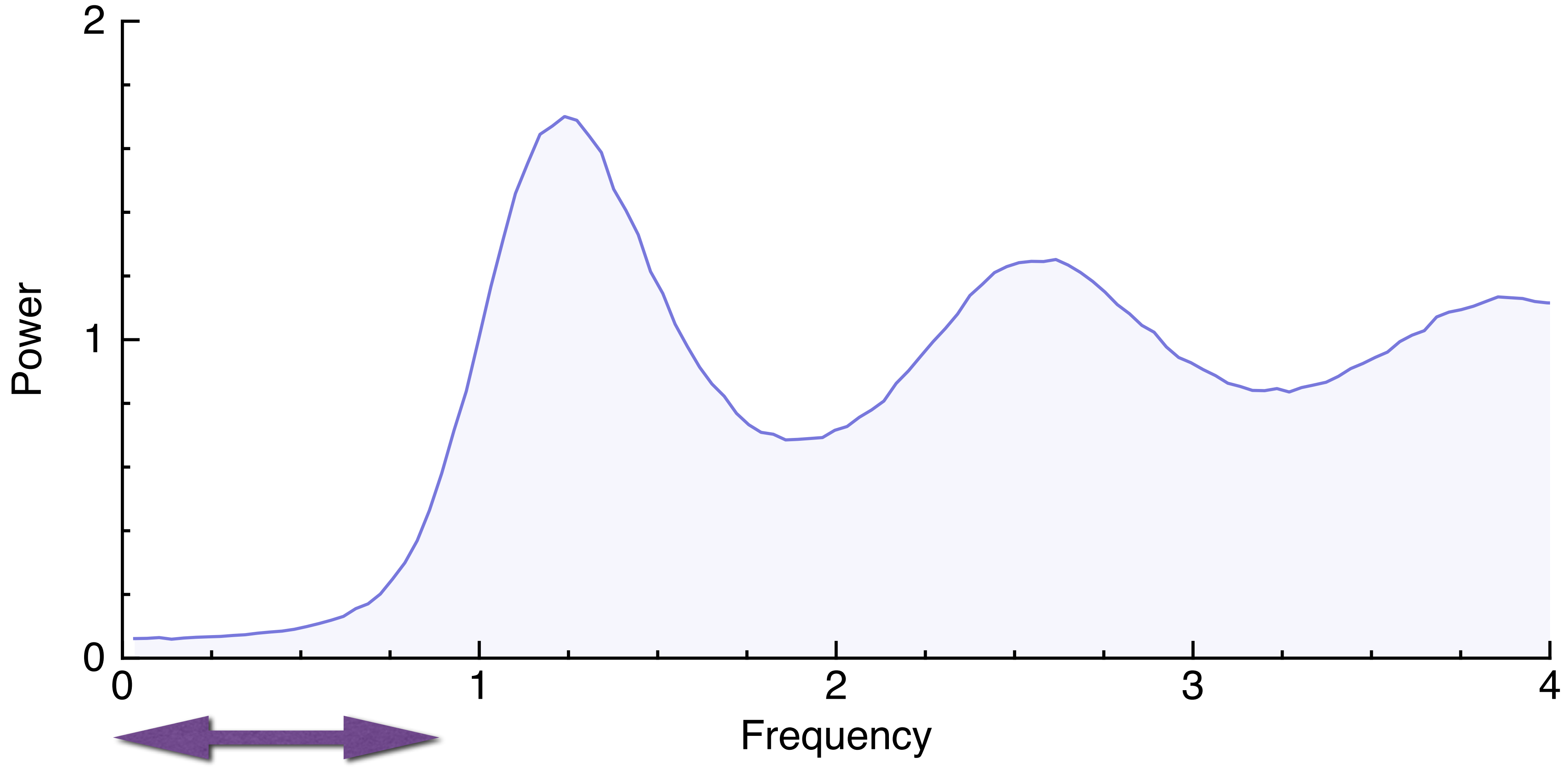
Jitter





Poisson Disk





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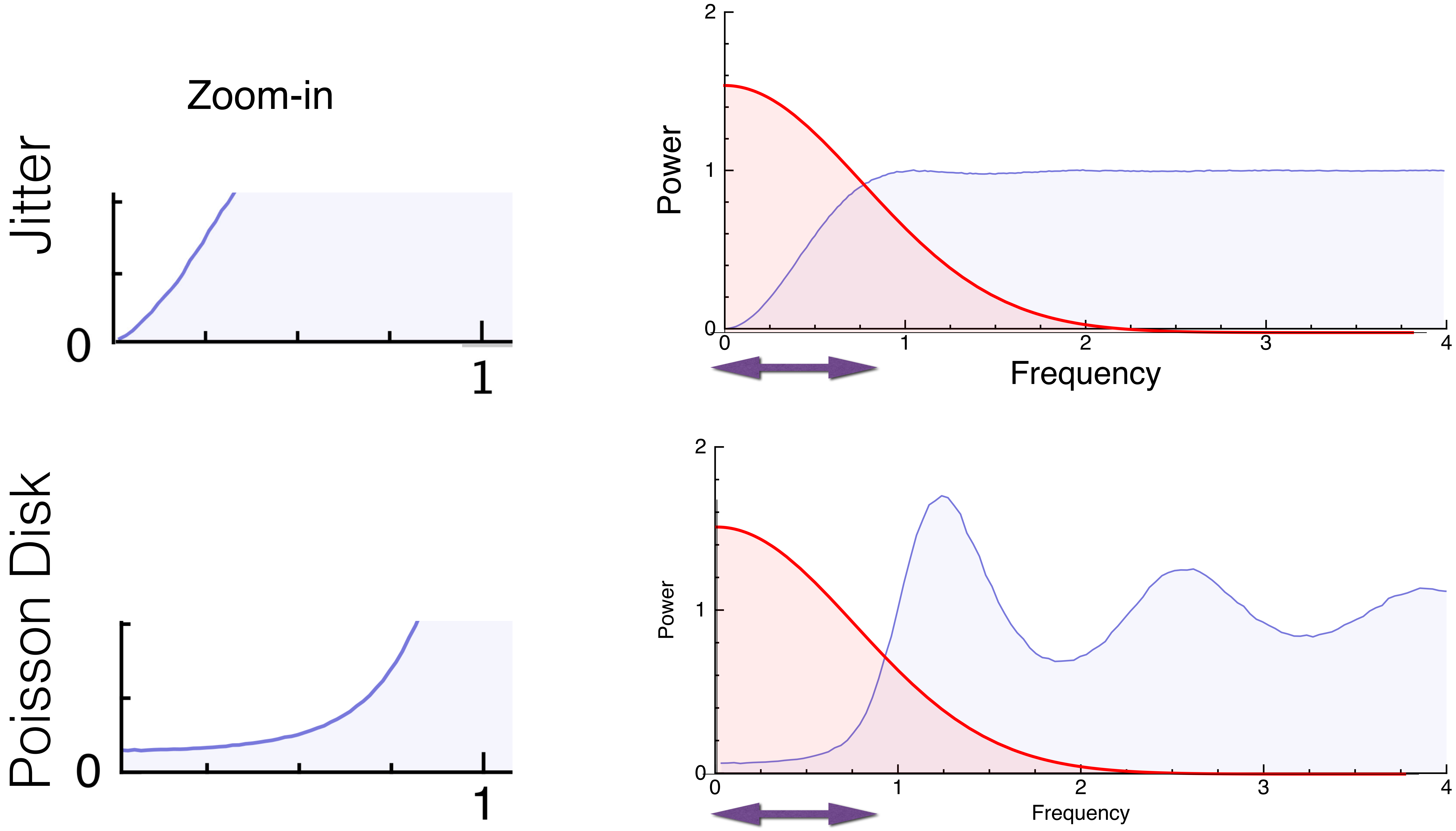
Variance for Low Sample Count



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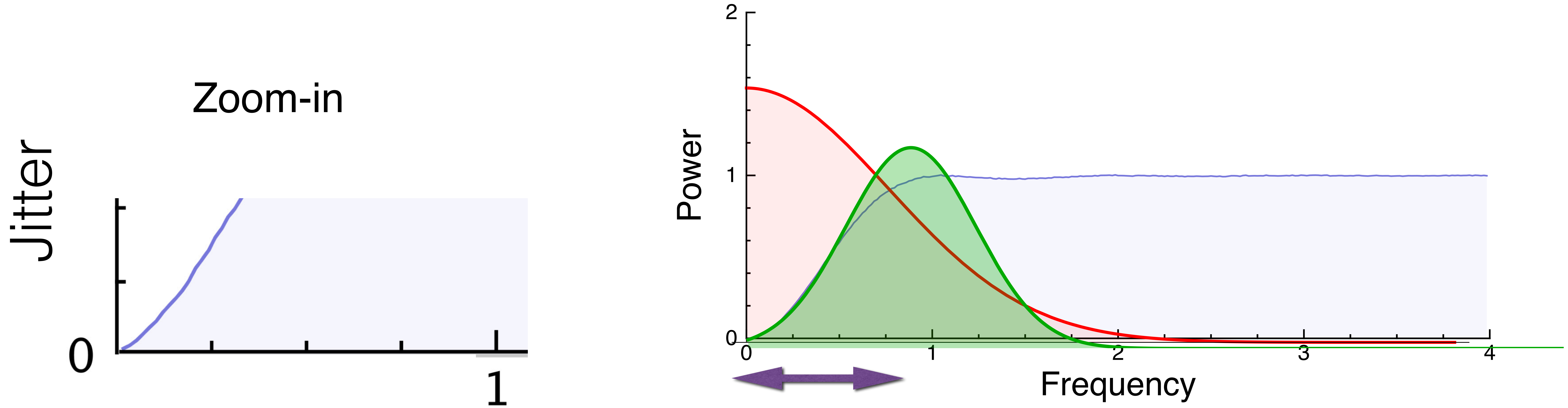
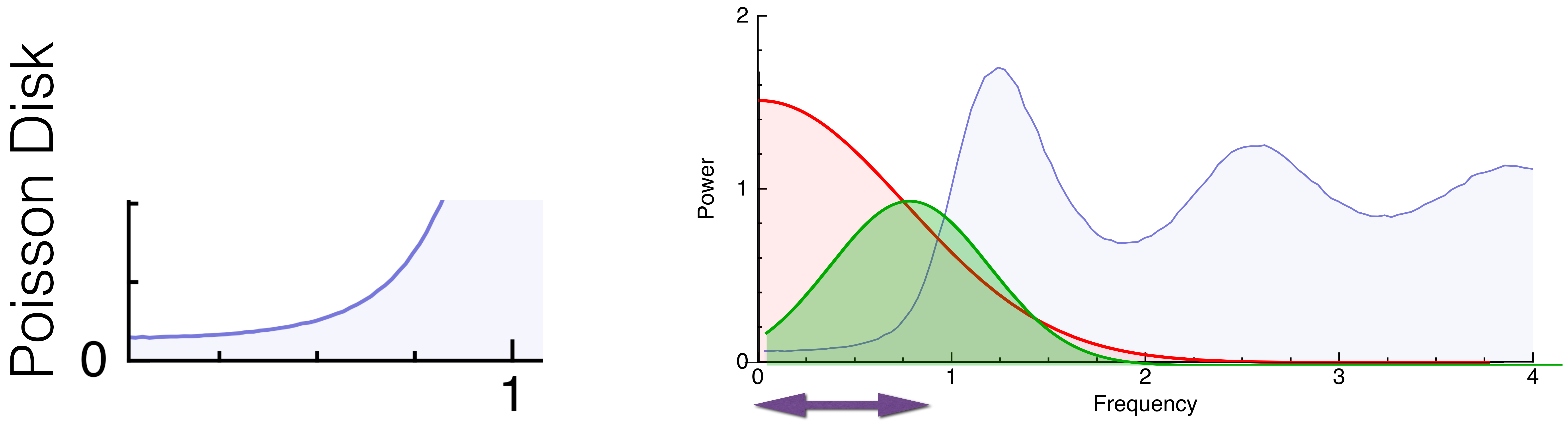
Variance for Low Sample Count

Zoom-in





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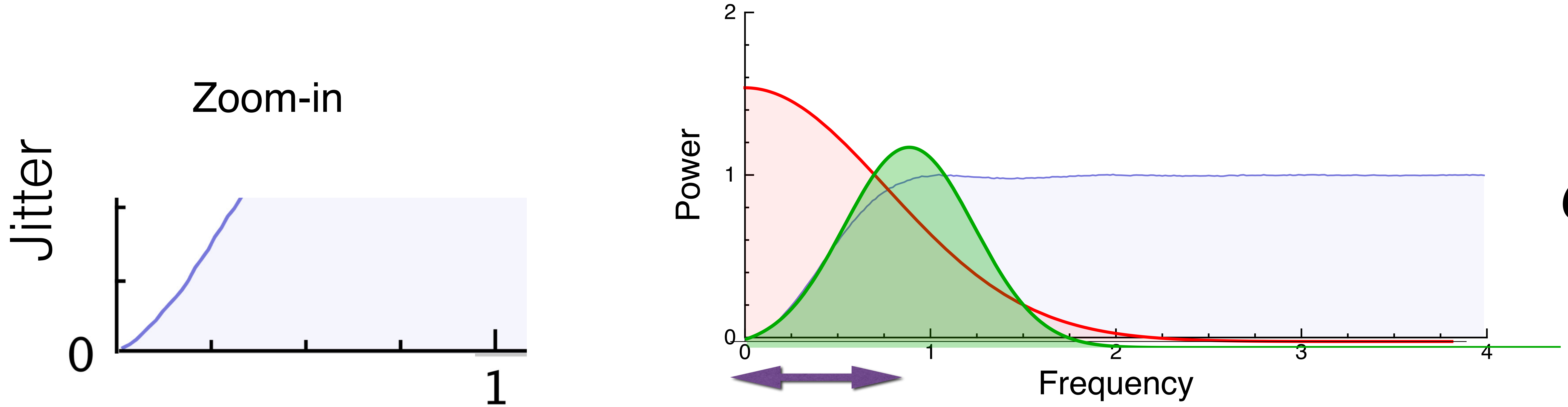
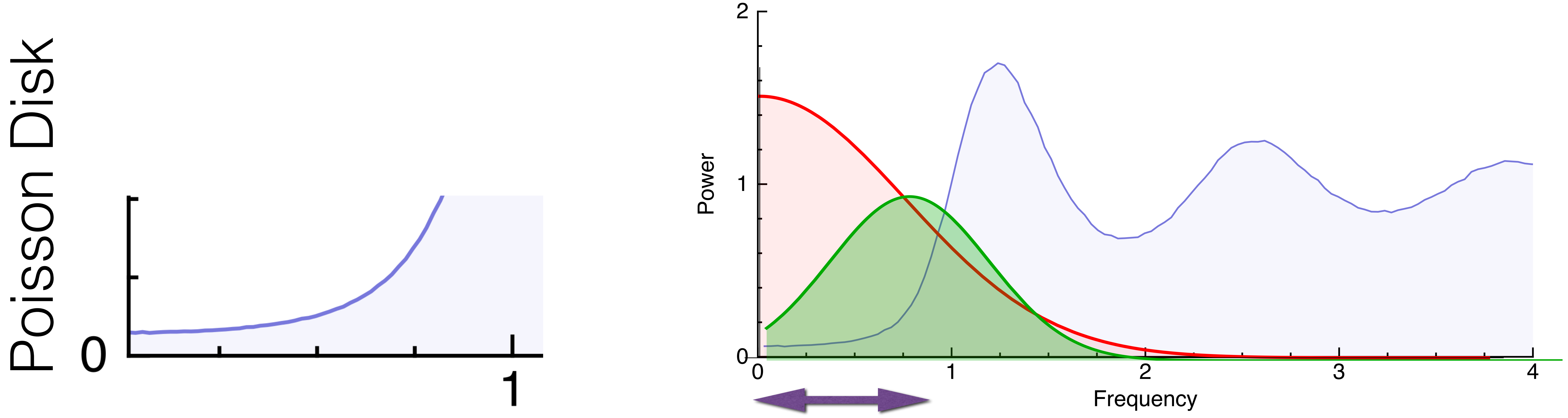
Variance for Increasing Sample Count



$\mathcal{O}(N^{-2})$



$\mathcal{O}(N^{-1})$

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Experimental Verification

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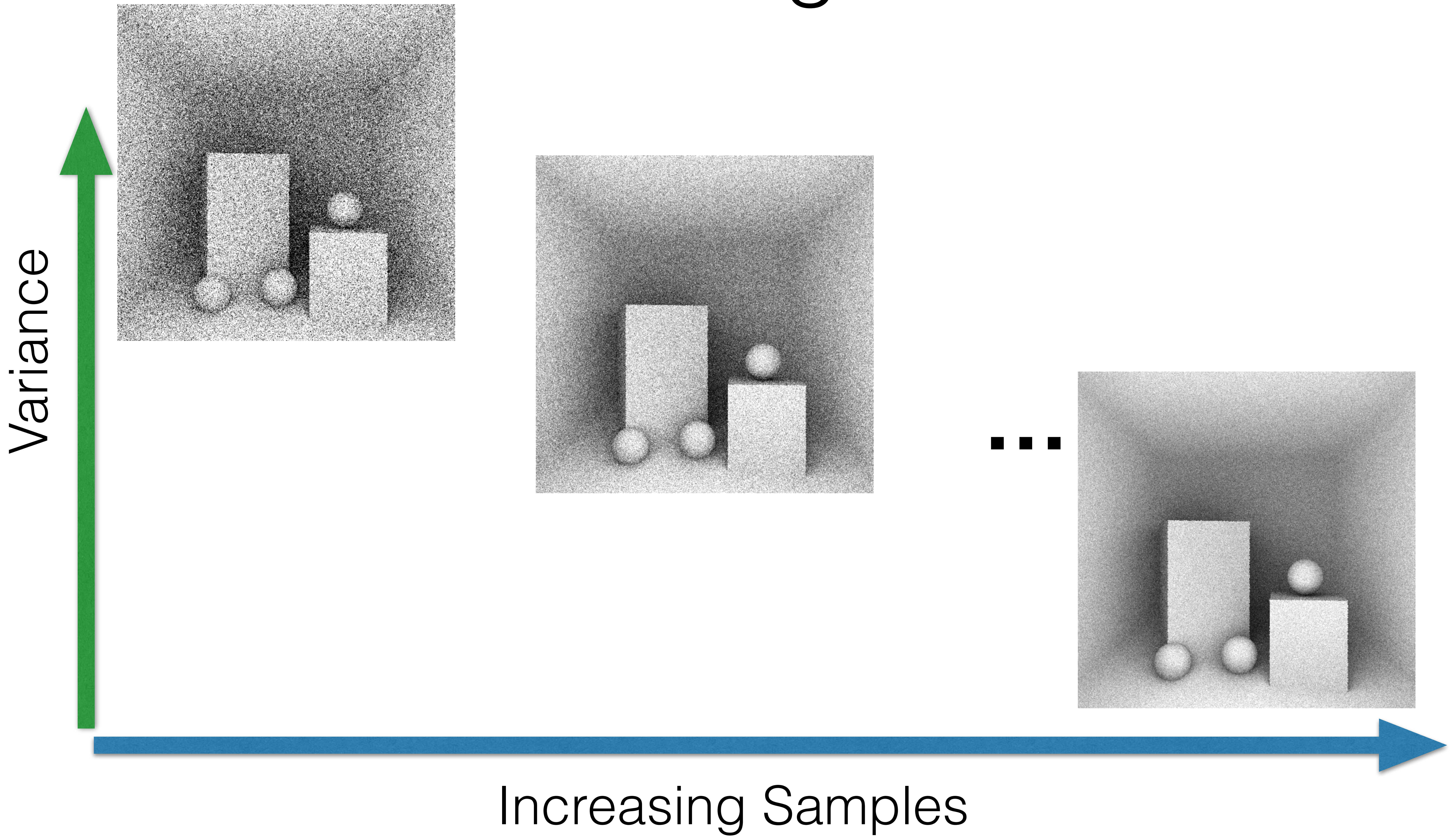
Convergence rate



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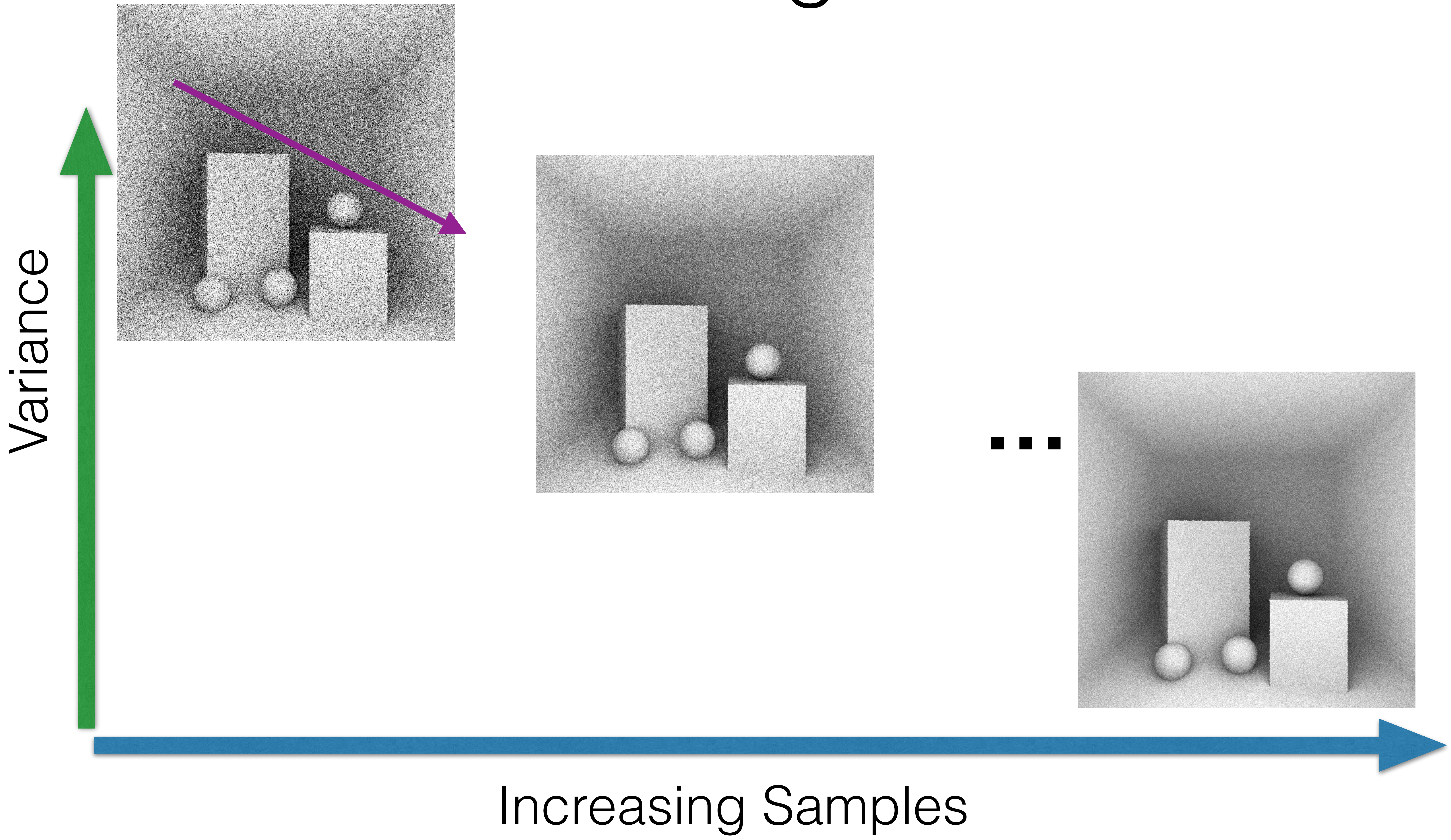
Convergence rate



Render the Possibilities
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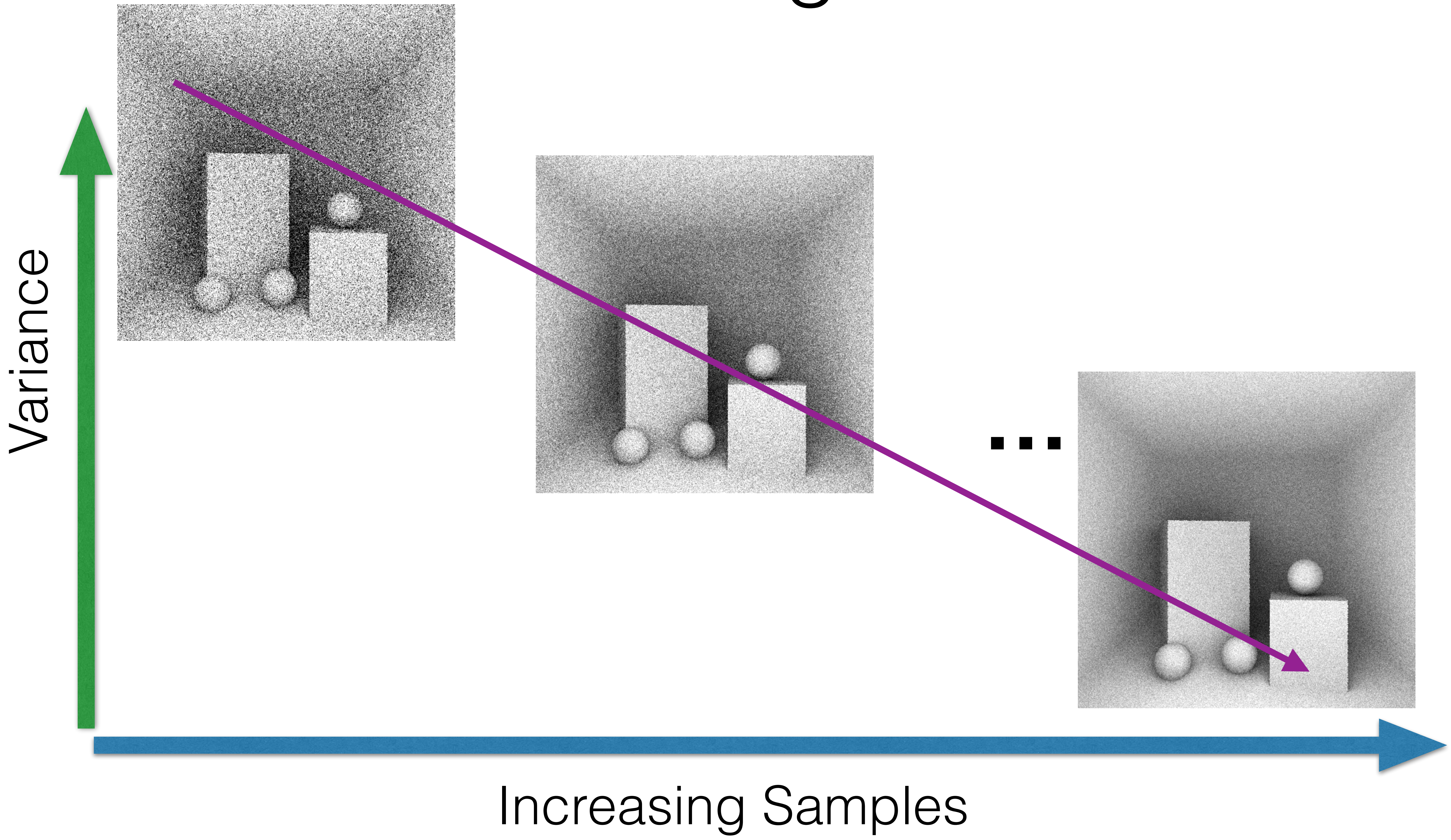
Convergence rate



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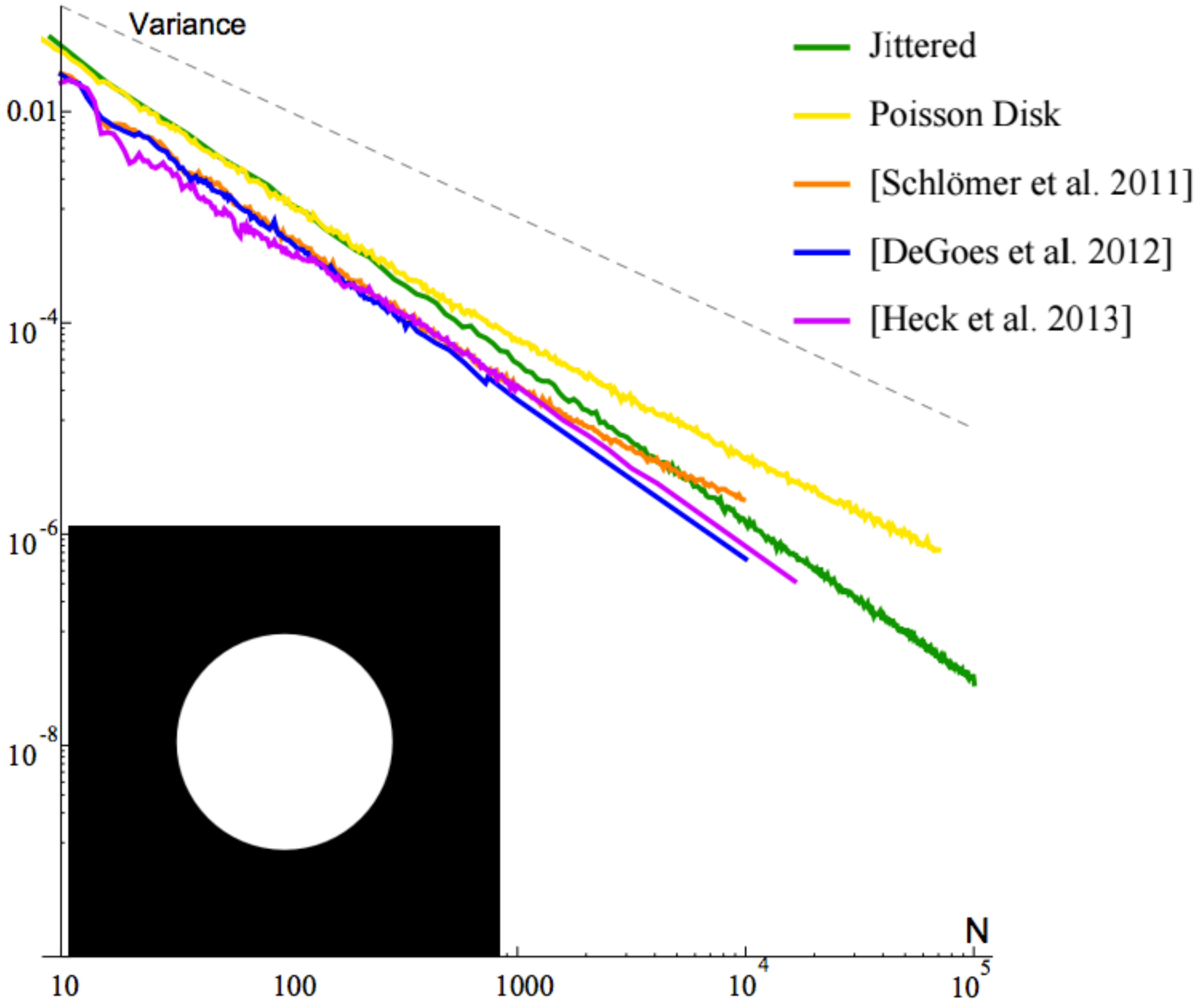
Disk Function as Worst Case



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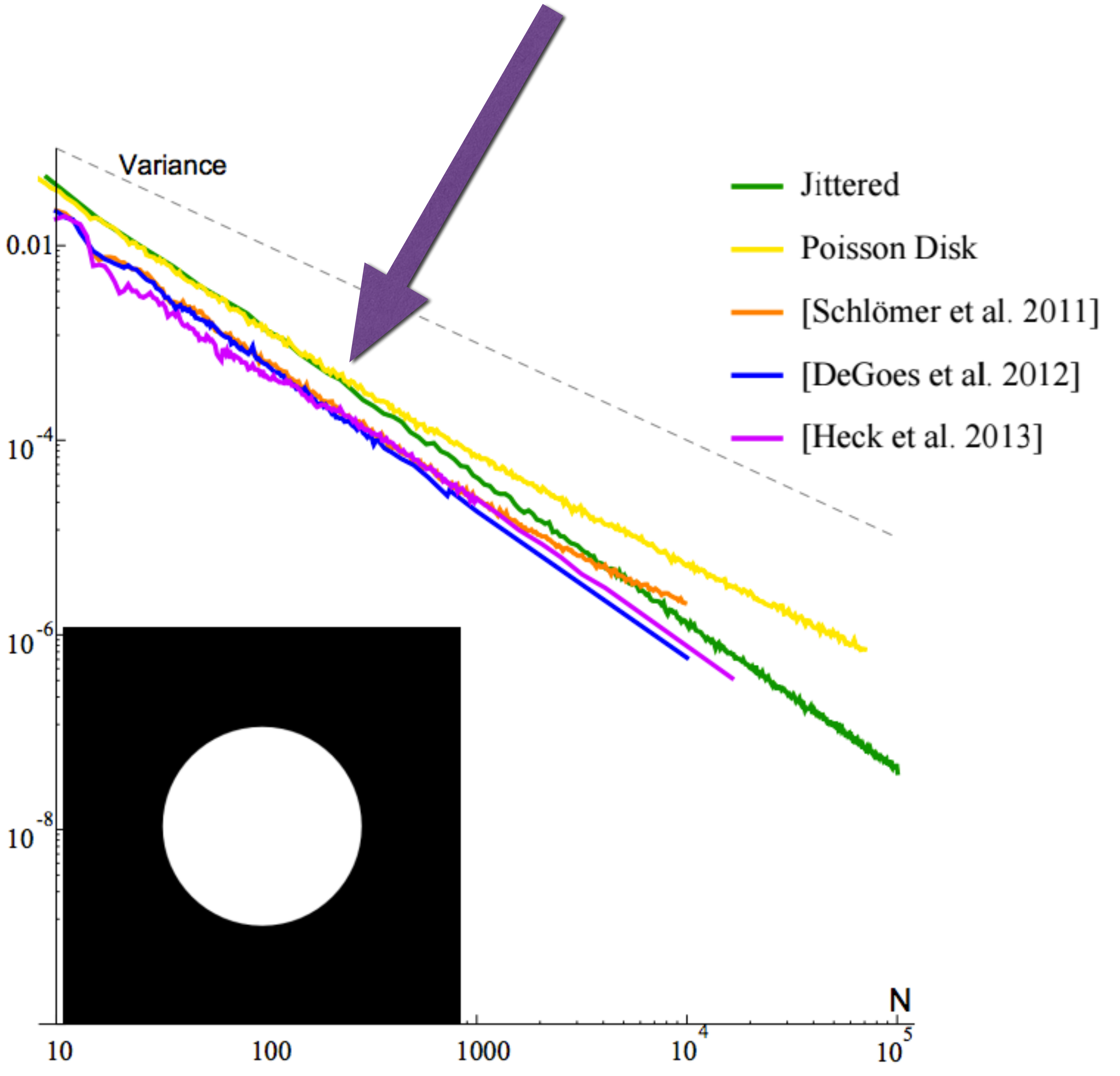
Disk Function as Worst Case



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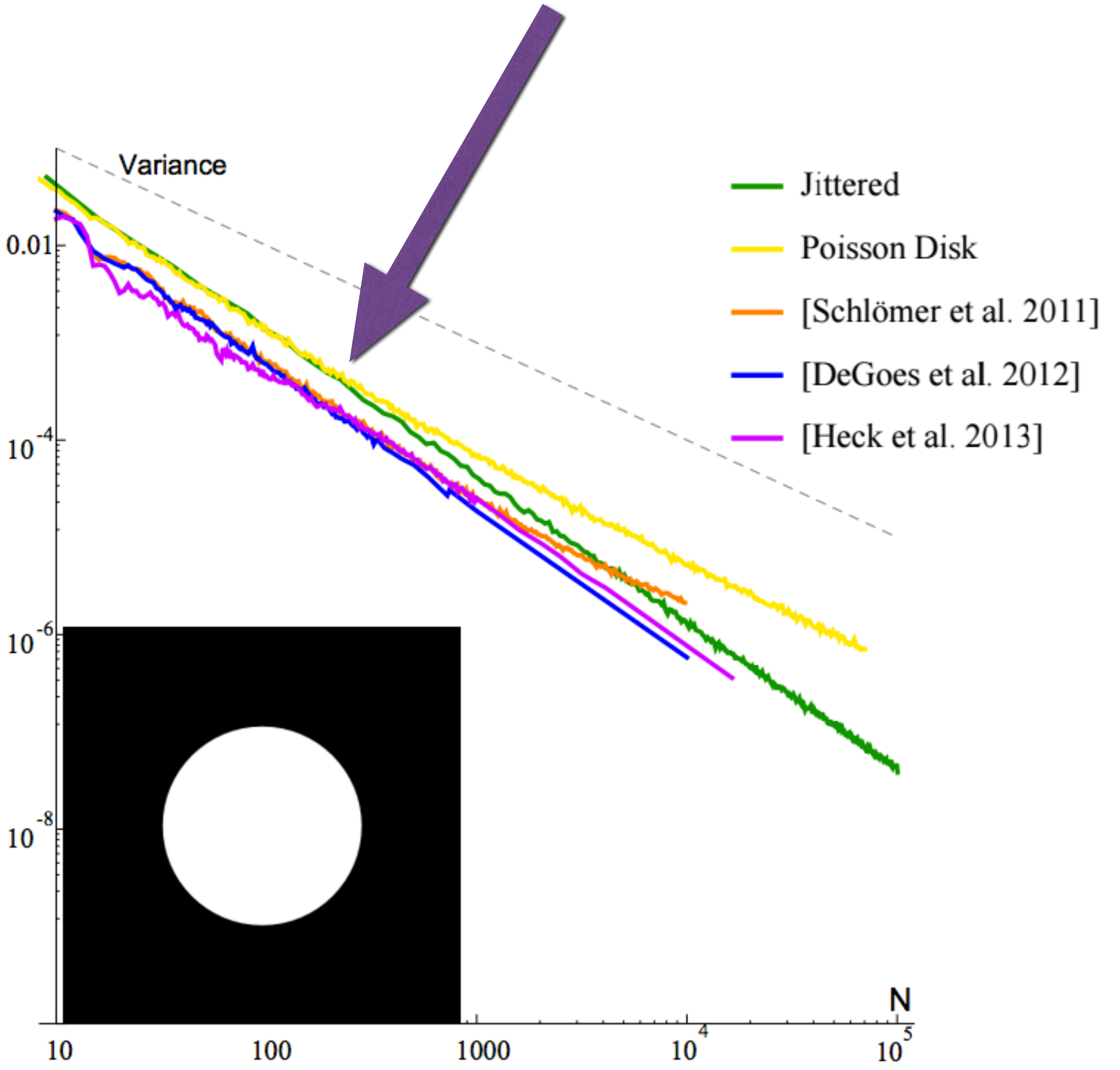
Disk Function as Worst Case



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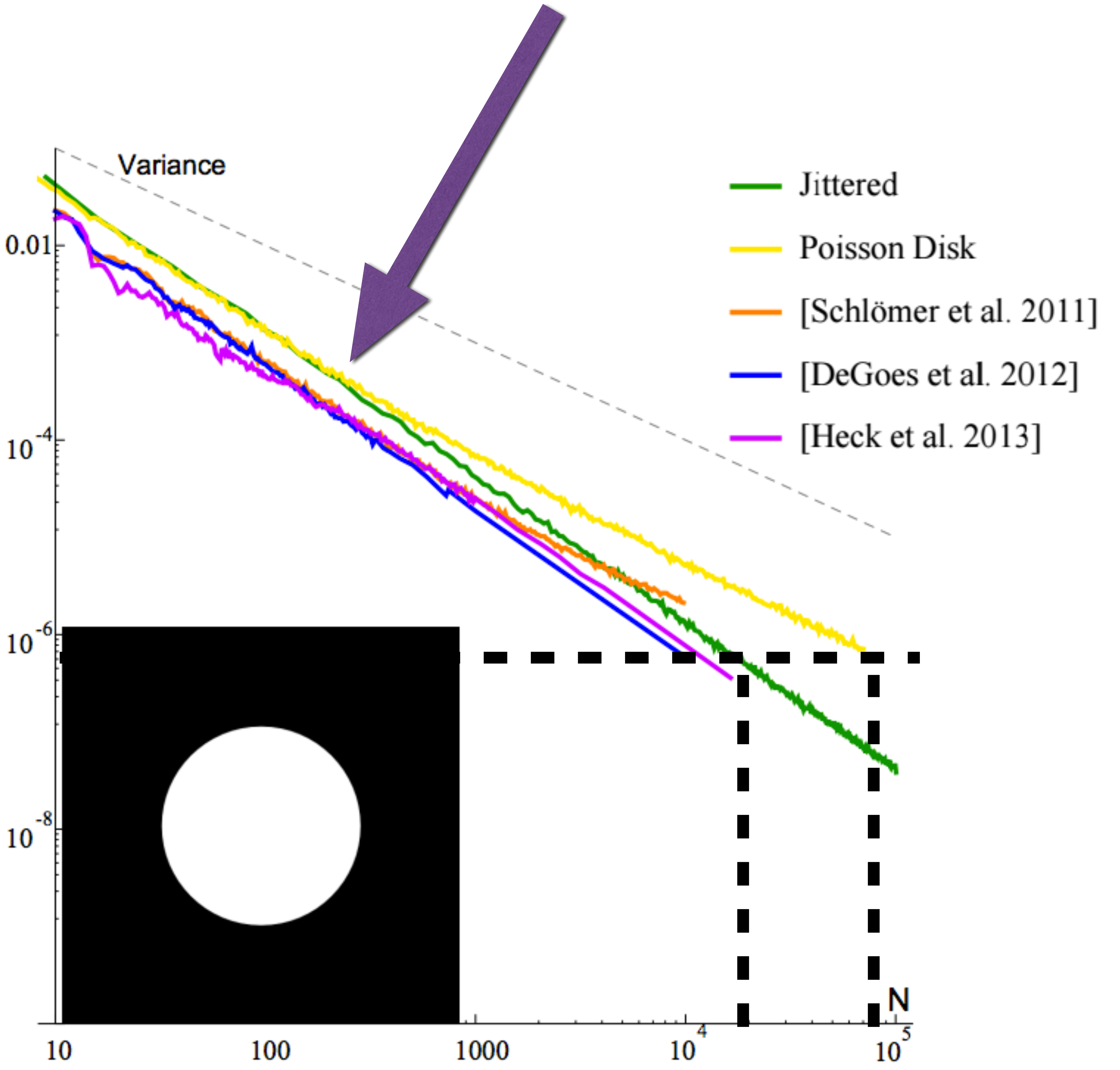
Disk Function as Worst Case



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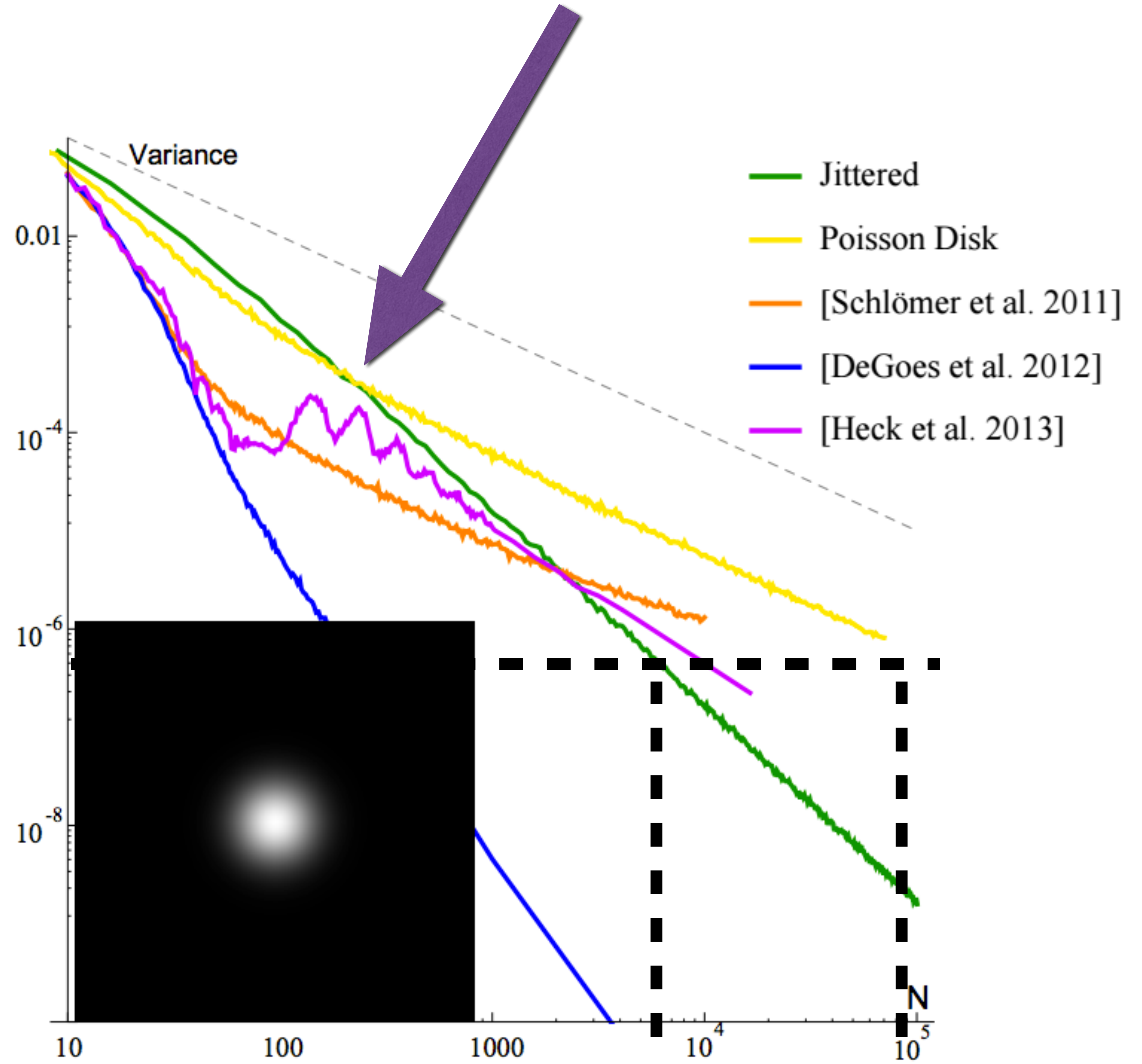
Gaussian as Best Case



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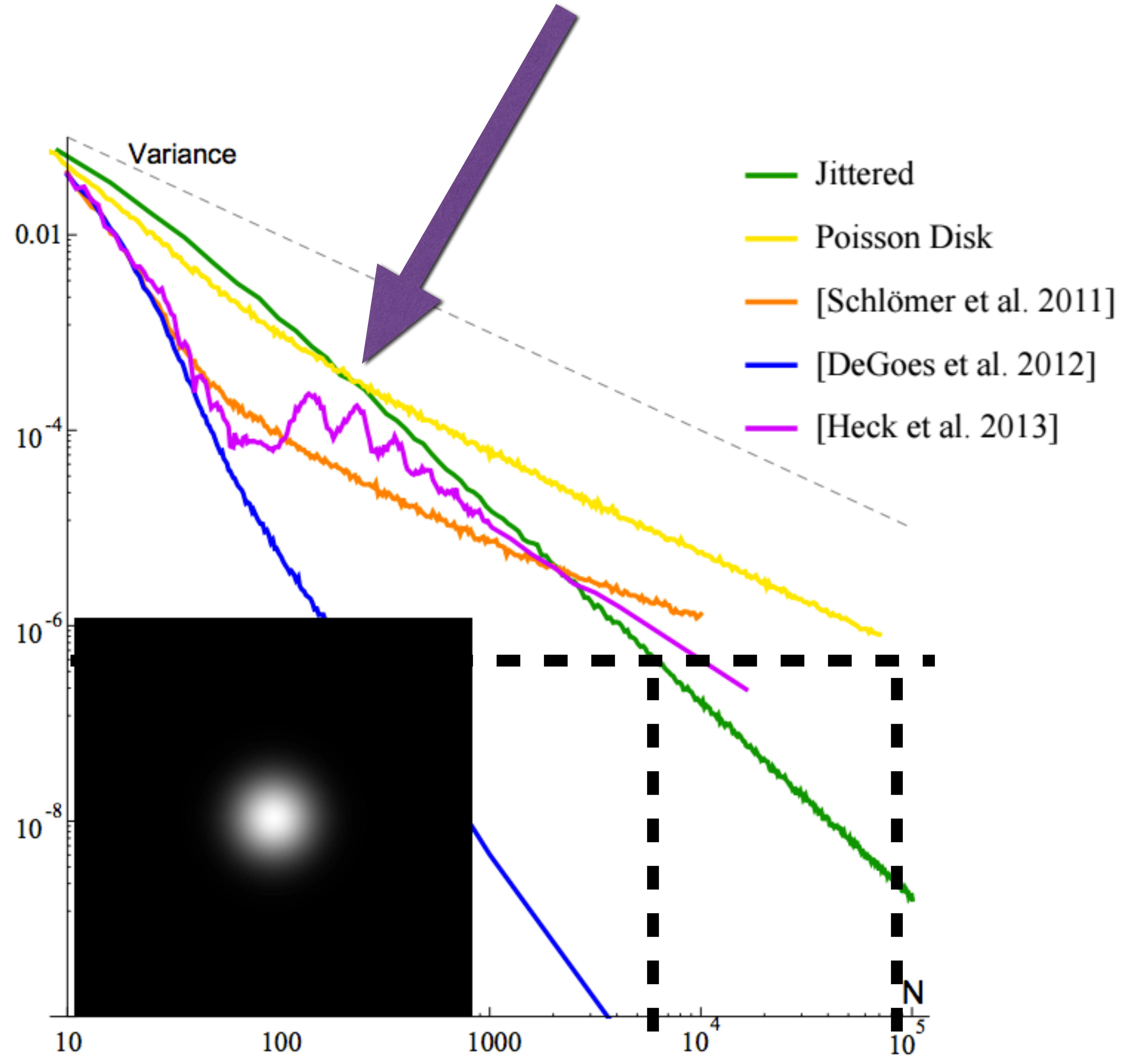
Gaussian as Best Case



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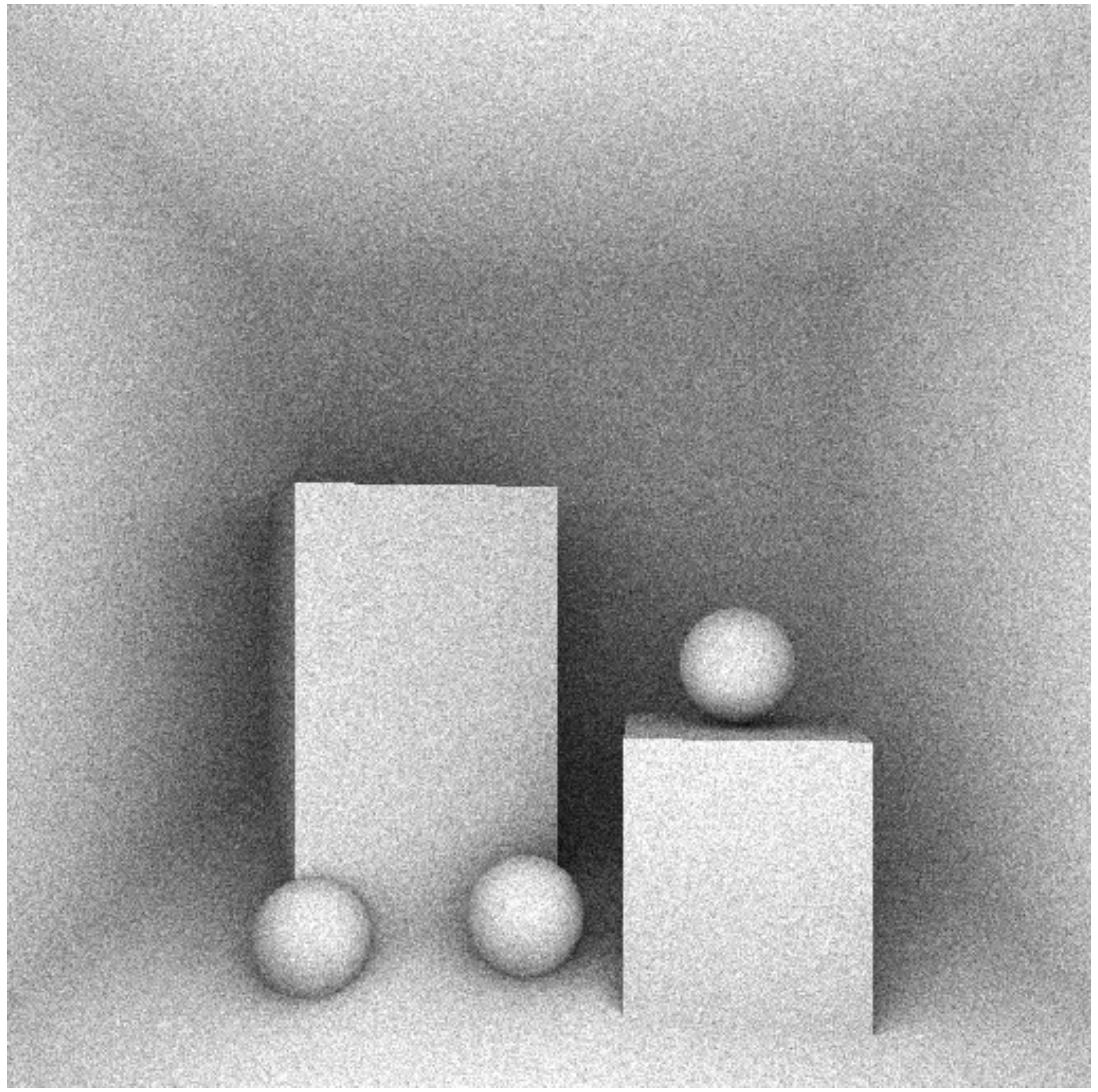
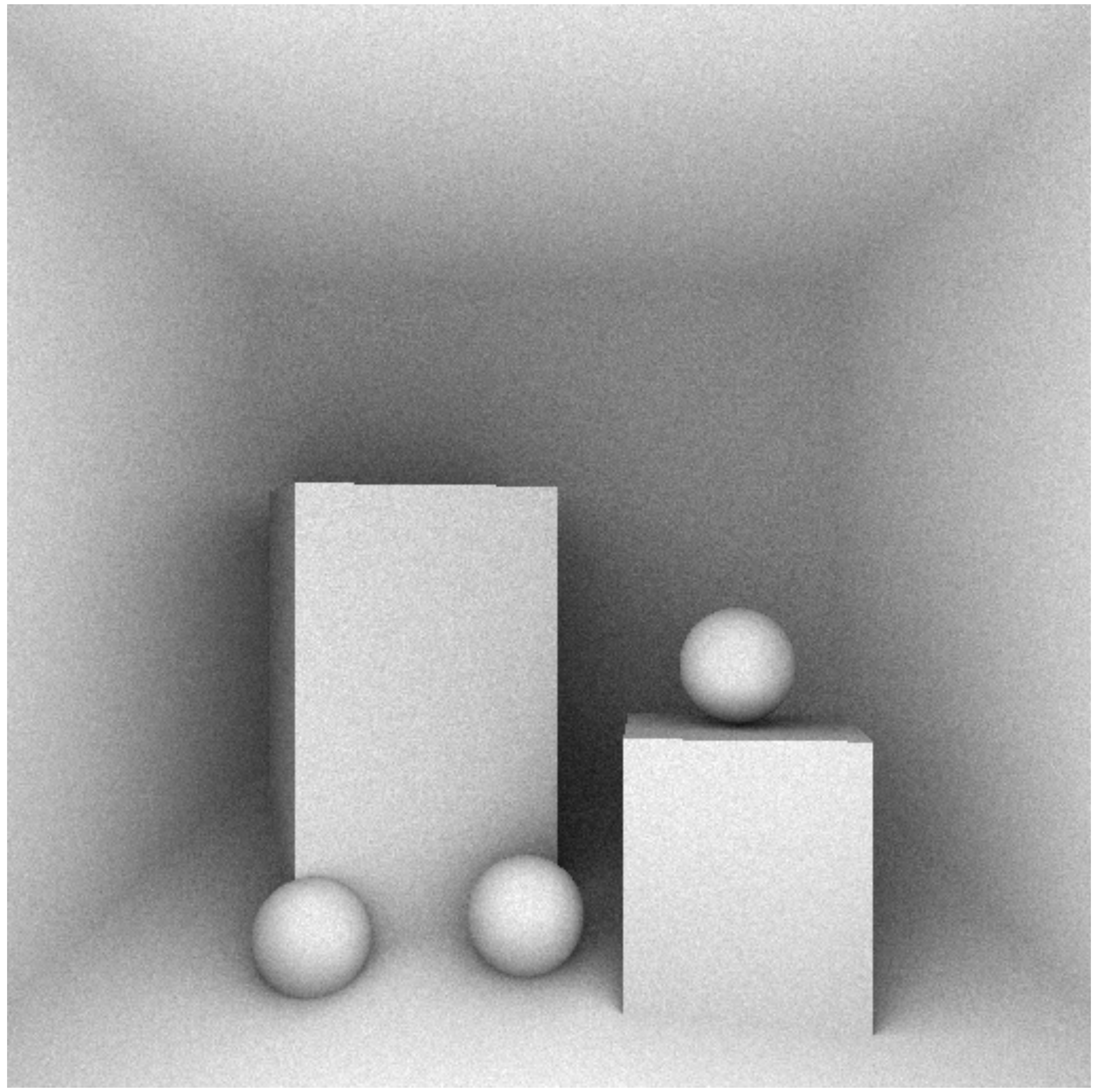
Ambient Occlusion Examples

Random vs Jittered

96 Secondary Rays



MSE: 4.74 x 10e-3



MSE: 8.56 x 10e-4

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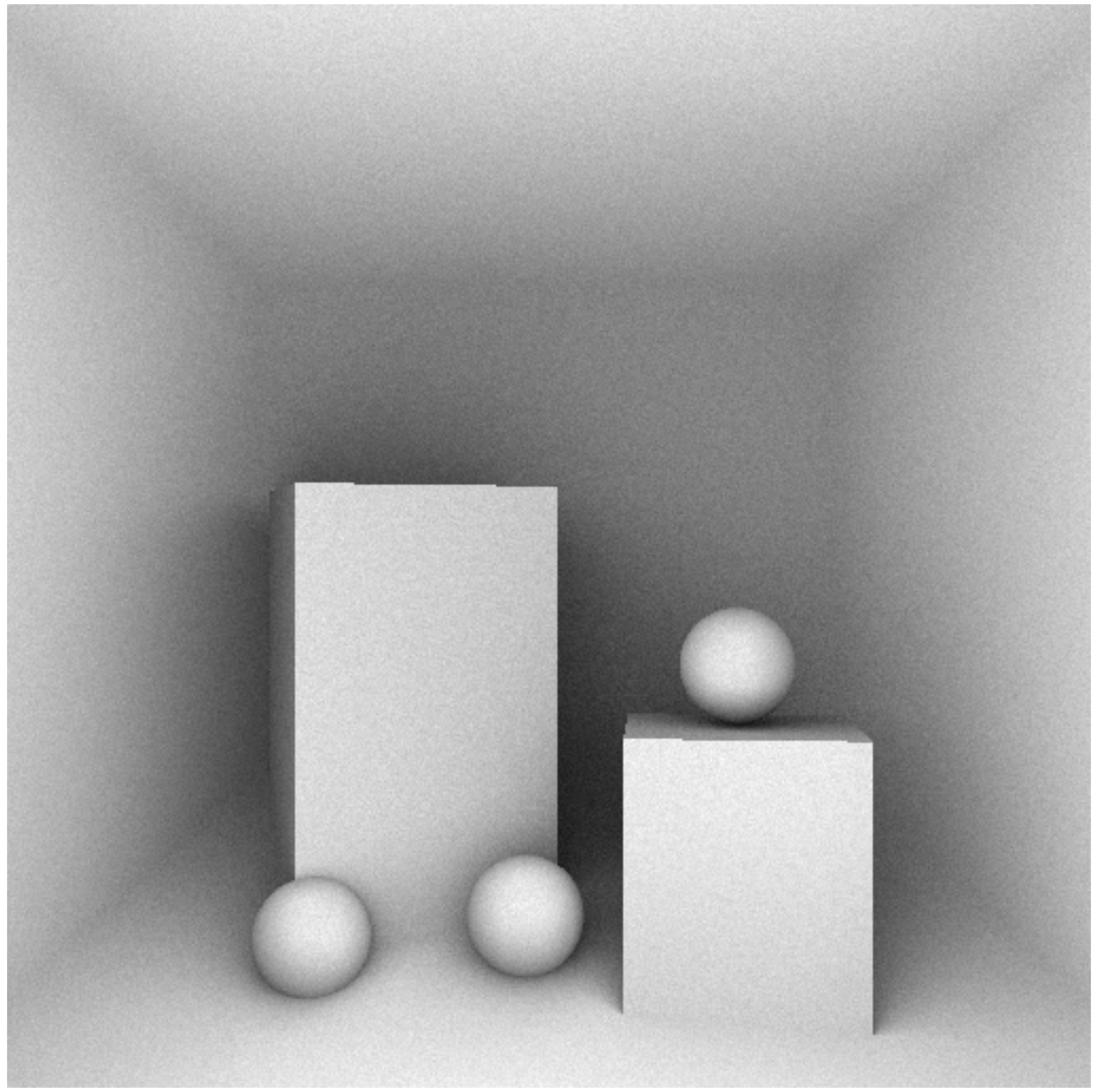
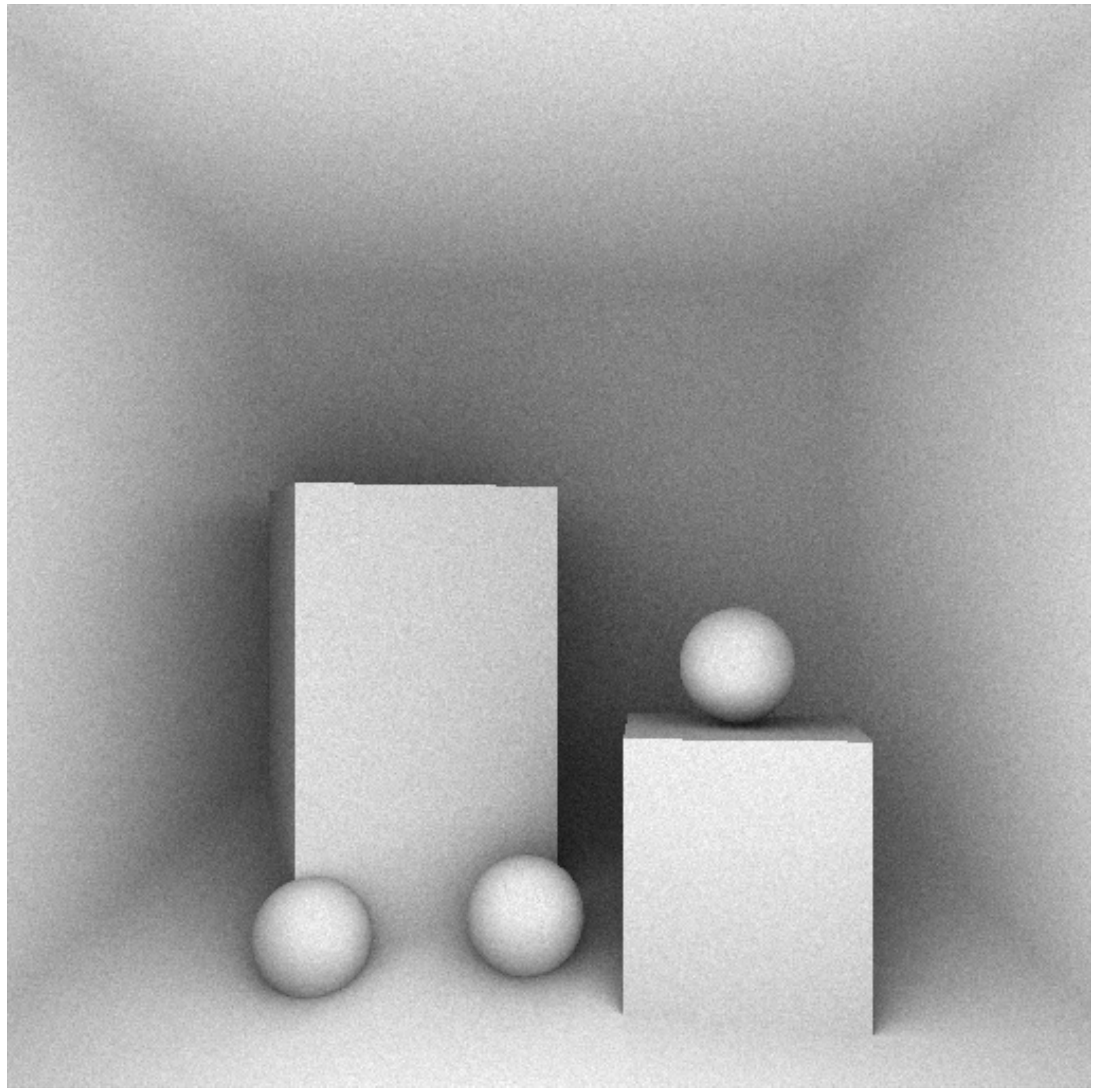
CCVT vs. Poisson Disk

96 Secondary Rays



MSE: 4.24 x 10e-4



MSE: 6.95 x 10e-4

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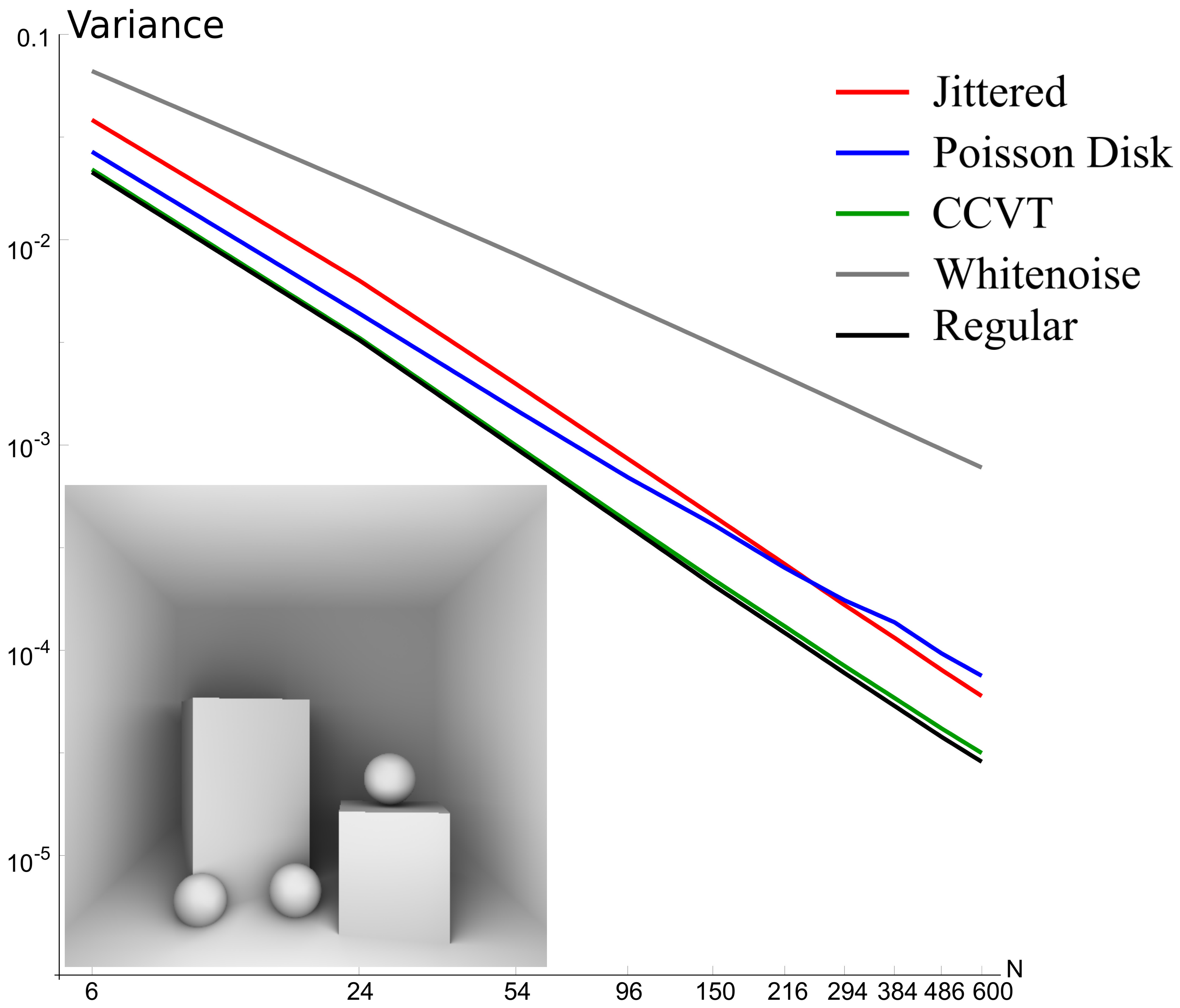
Convergence rates



Render the Possibilities
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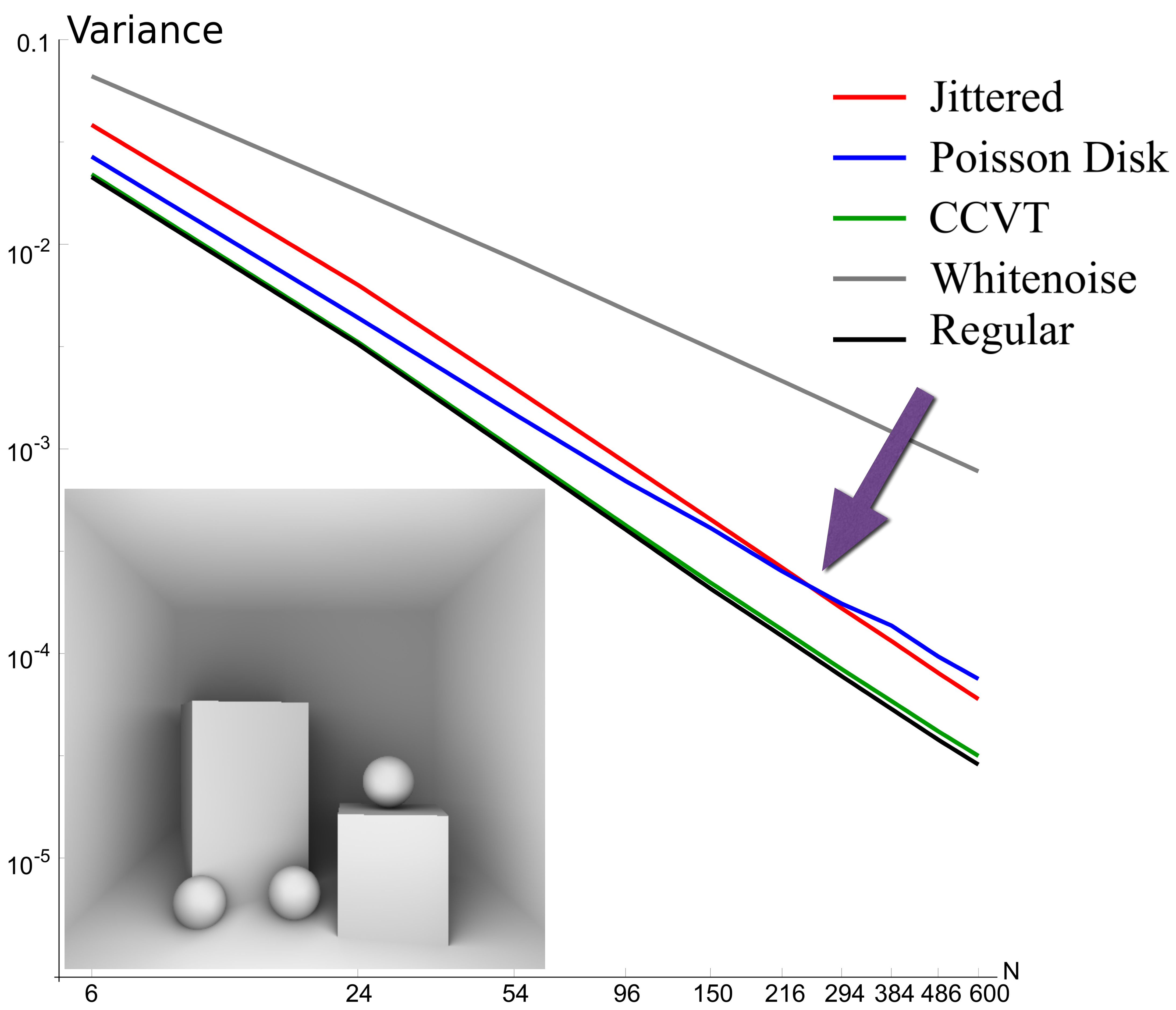
Convergence rates



Render the Possibilities
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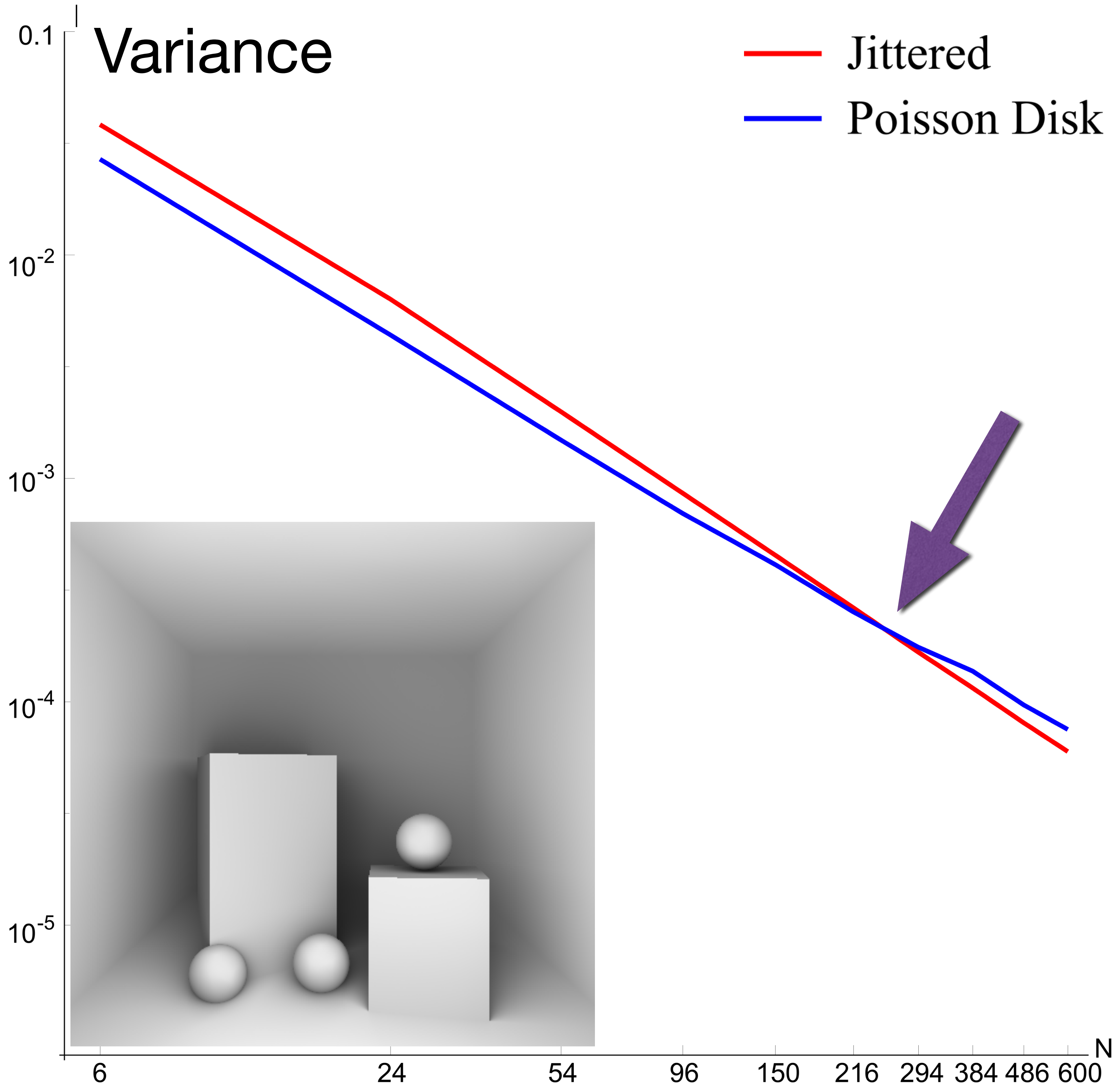
Jittered vs Poisson Disk



Render the Possibilities
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What are the benefits of this analysis ?

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- For offline rendering, analysis tells which samplers would converge faster.
- For real time rendering, blue noise samples are more effective in reducing variance for a given number of samples