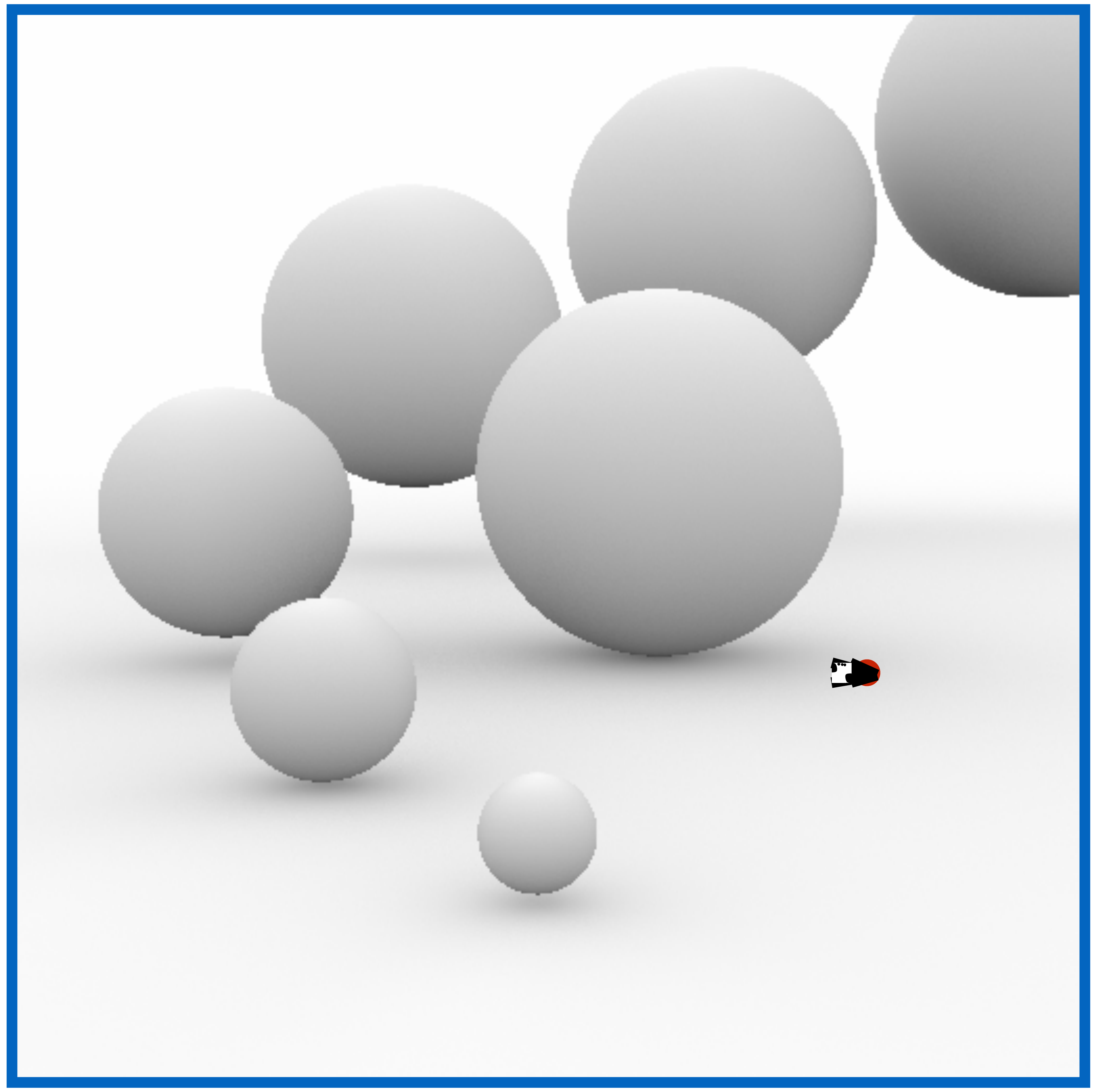
Variance and Convergence Analysis of Monte Carlo Line and Segment Sampling

Gurprit Singh **Bailey Miller** **Wojciech Jarosz**

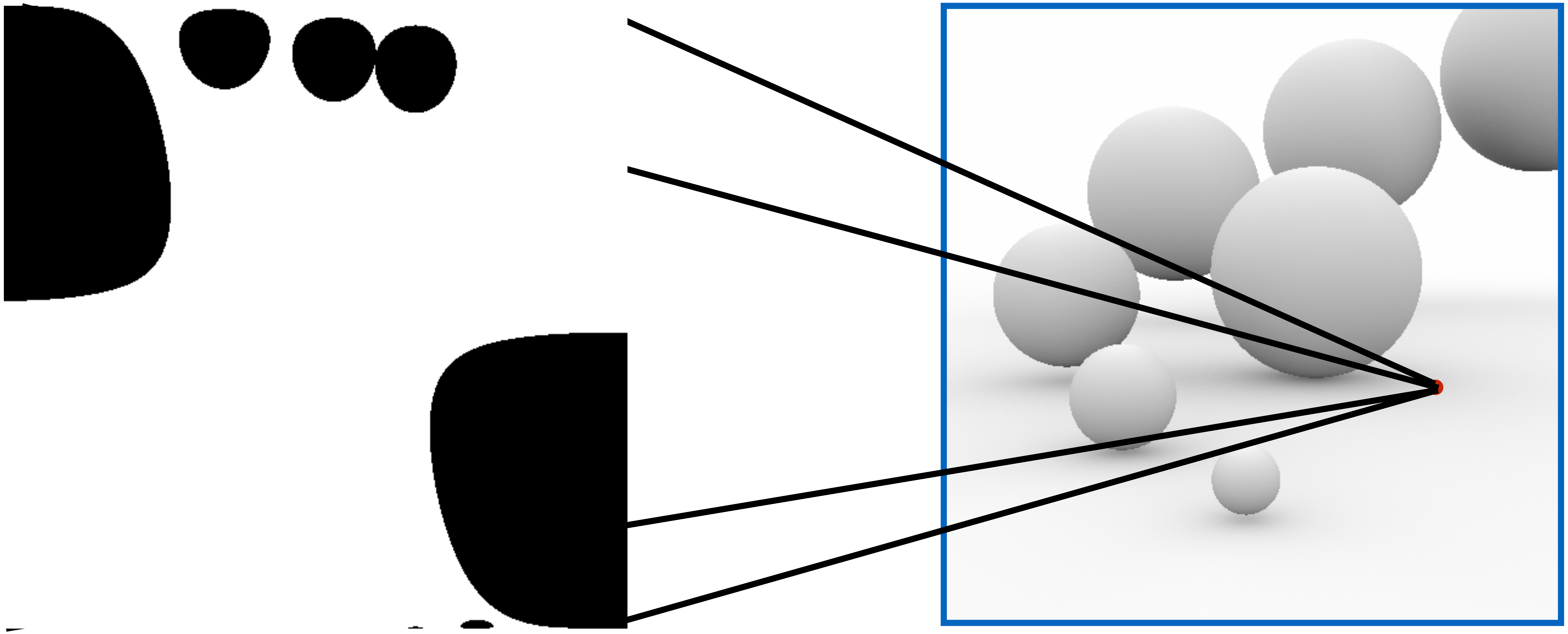
DARTMOUTH
VISUAL COMPUTING LAB

EGSR 2017
Helsinki, Finland, June 19–21

Monte Carlo Integration



Monte Carlo Integration



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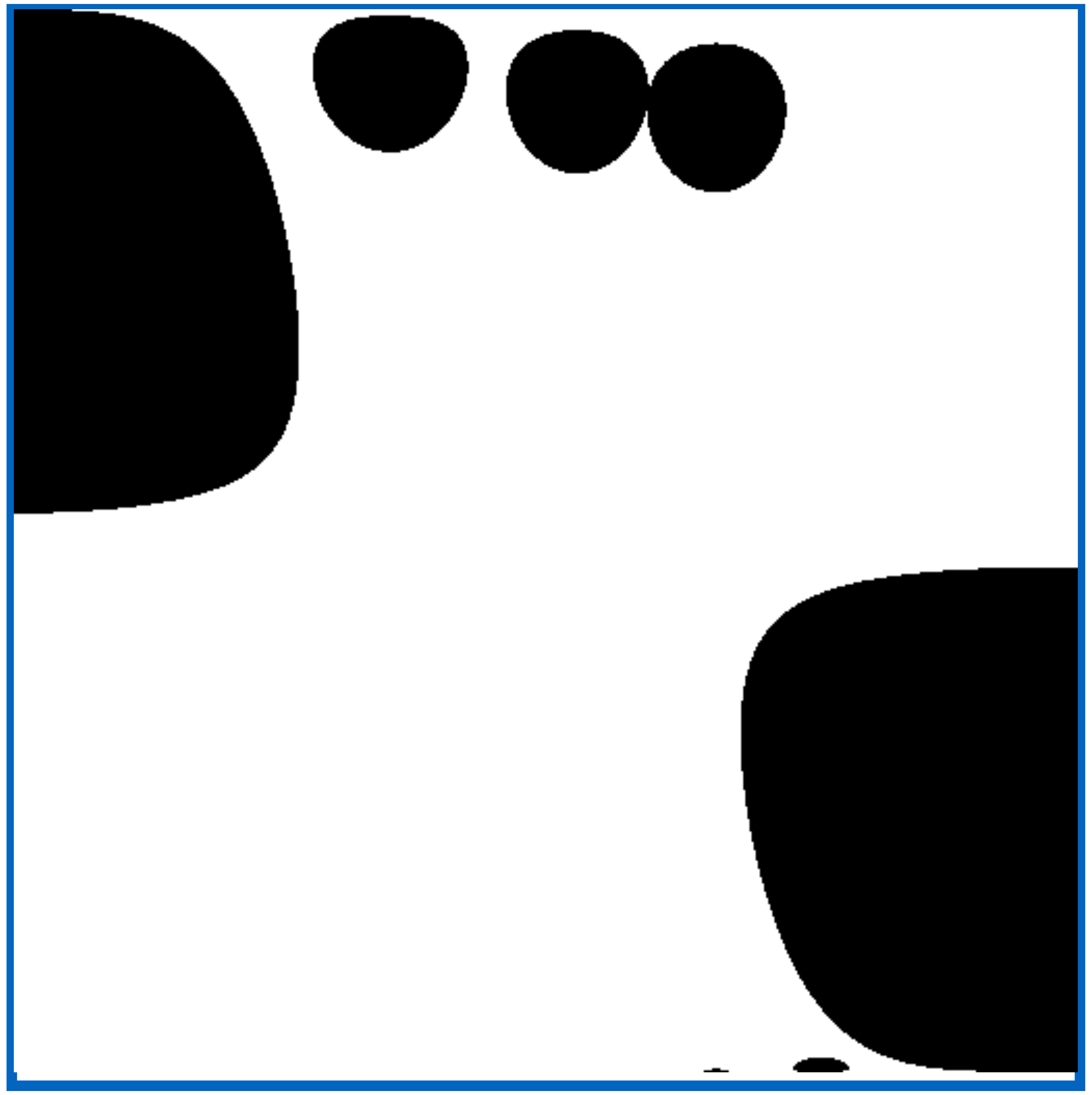
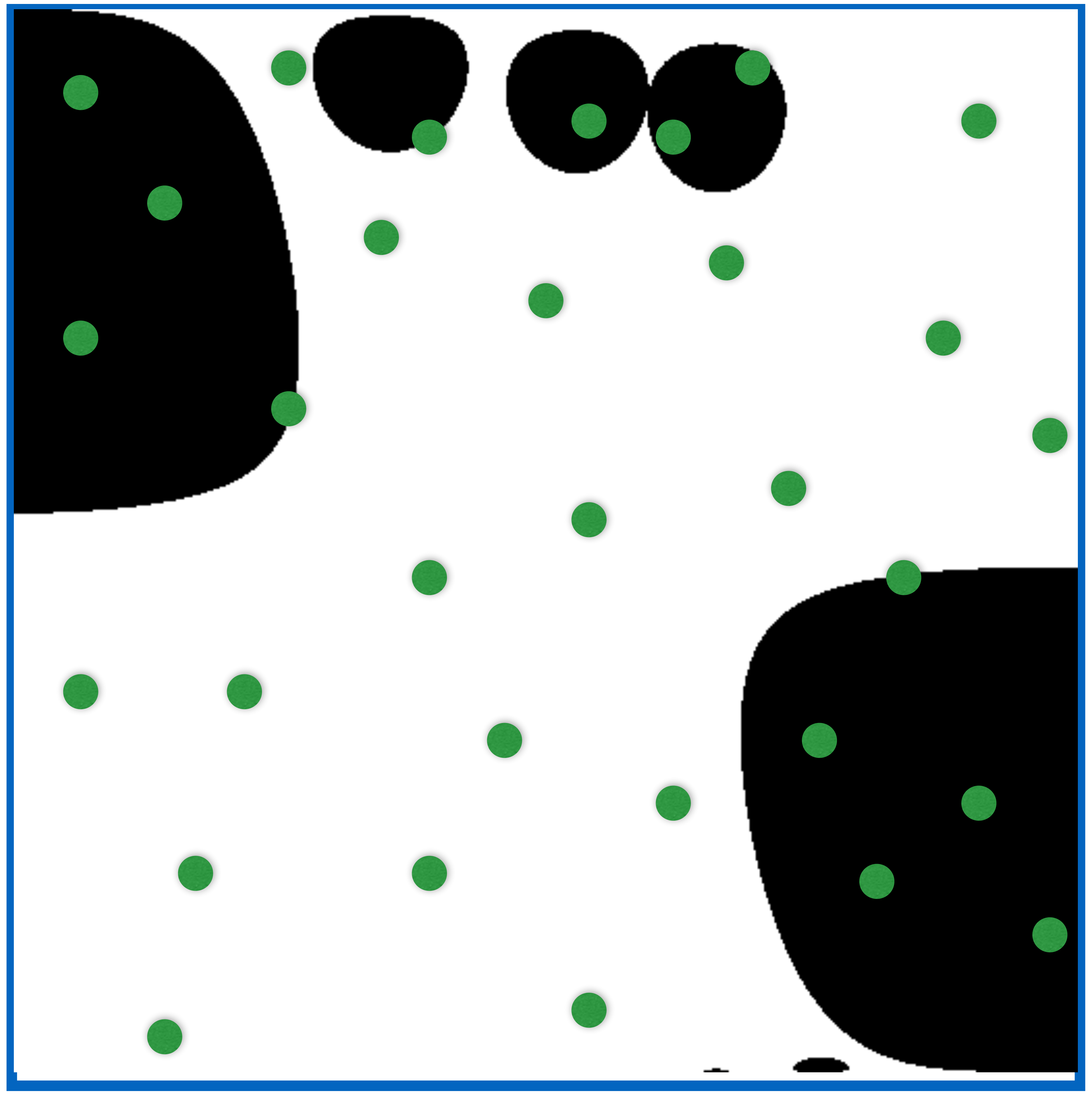


$f(u, v)$

$$I = \int_u \int_v f(u, v) du dv$$

Monte Carlo Integration

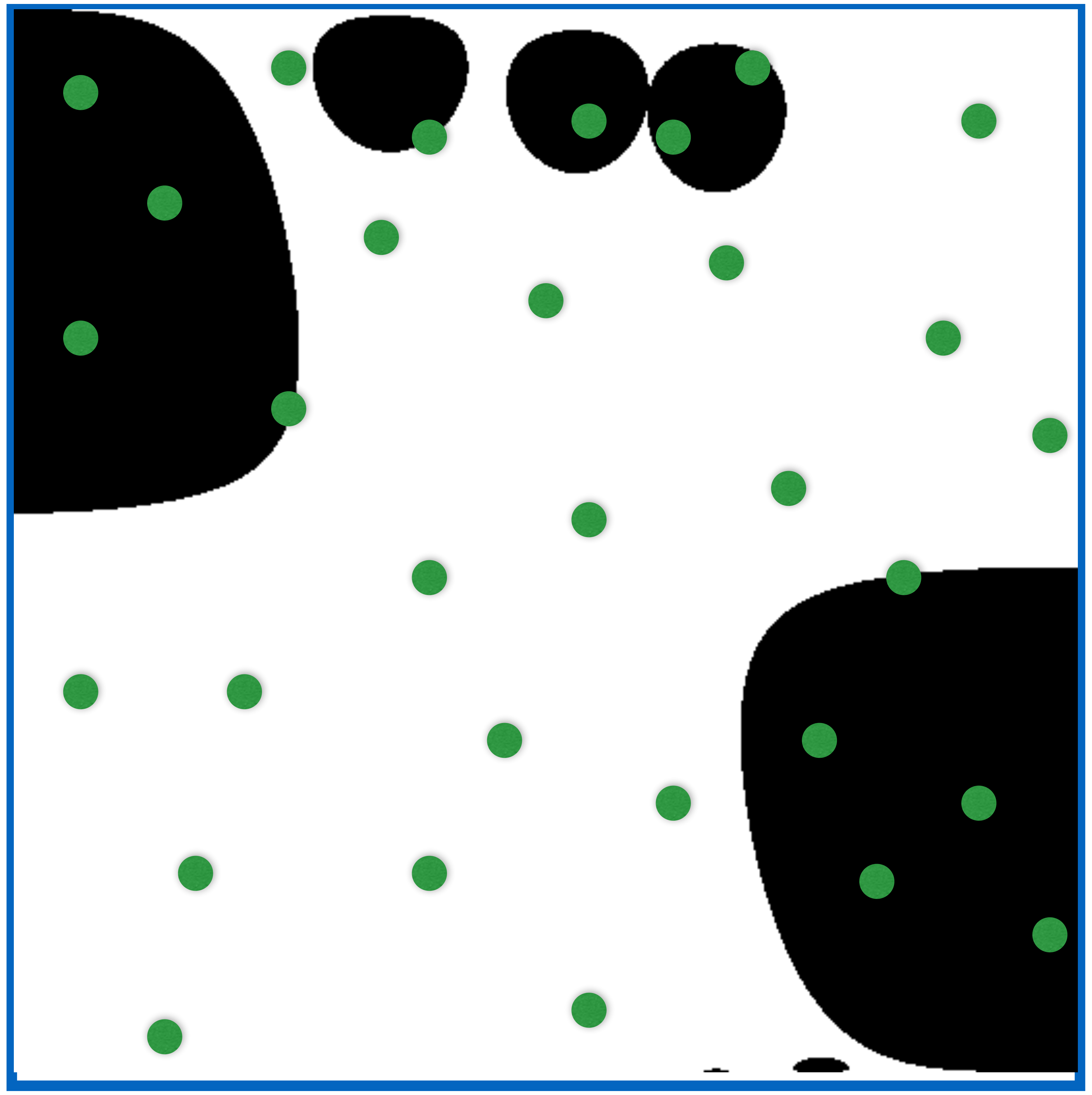


$f(u, v)$

$$I = \int_u \int_v f(u, v) du dv$$

$$I \approx \frac{1}{N} \sum_{k=1}^{N} \frac{f(u_k, v_k)}{p(u_k, v_k)}$$

Monte Carlo Estimator

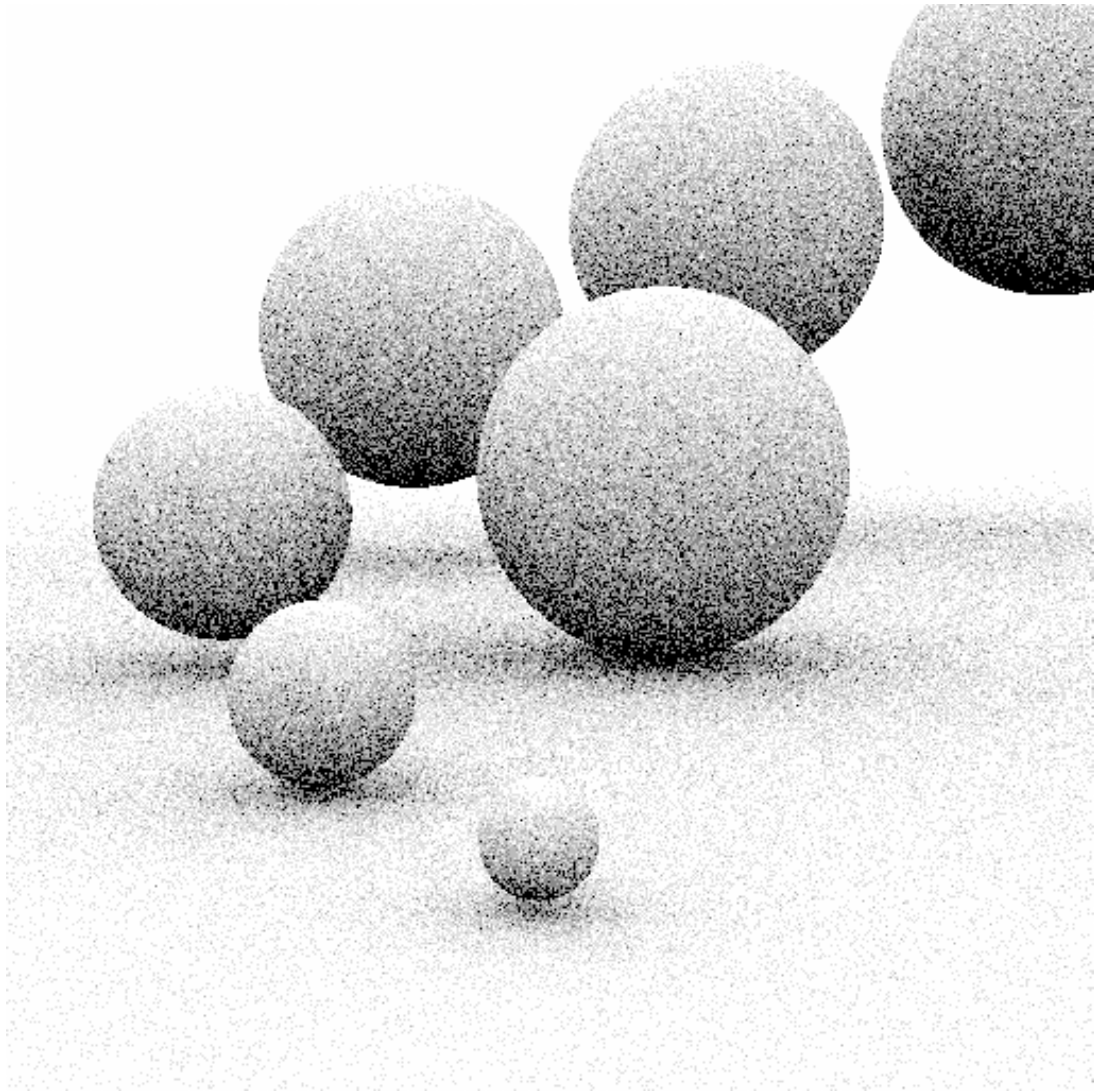


$f(u,v)$

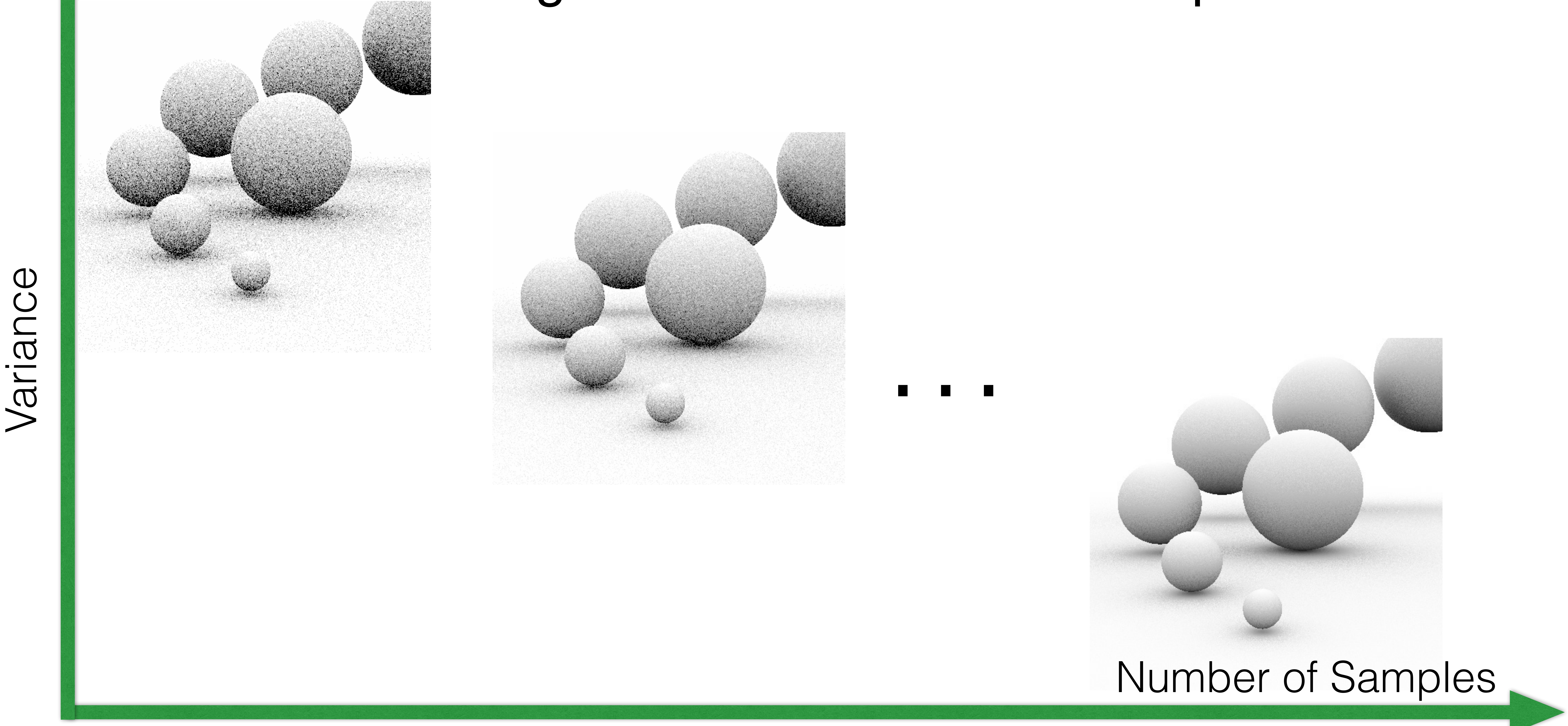
$$I = \int_u \int_v f(u,v)\,du\,dv$$

$$I_N = \frac{1}{N}\sum_{k=1}^{N} \frac{f(u_k, v_k)}{p(u_k, v_k)}$$

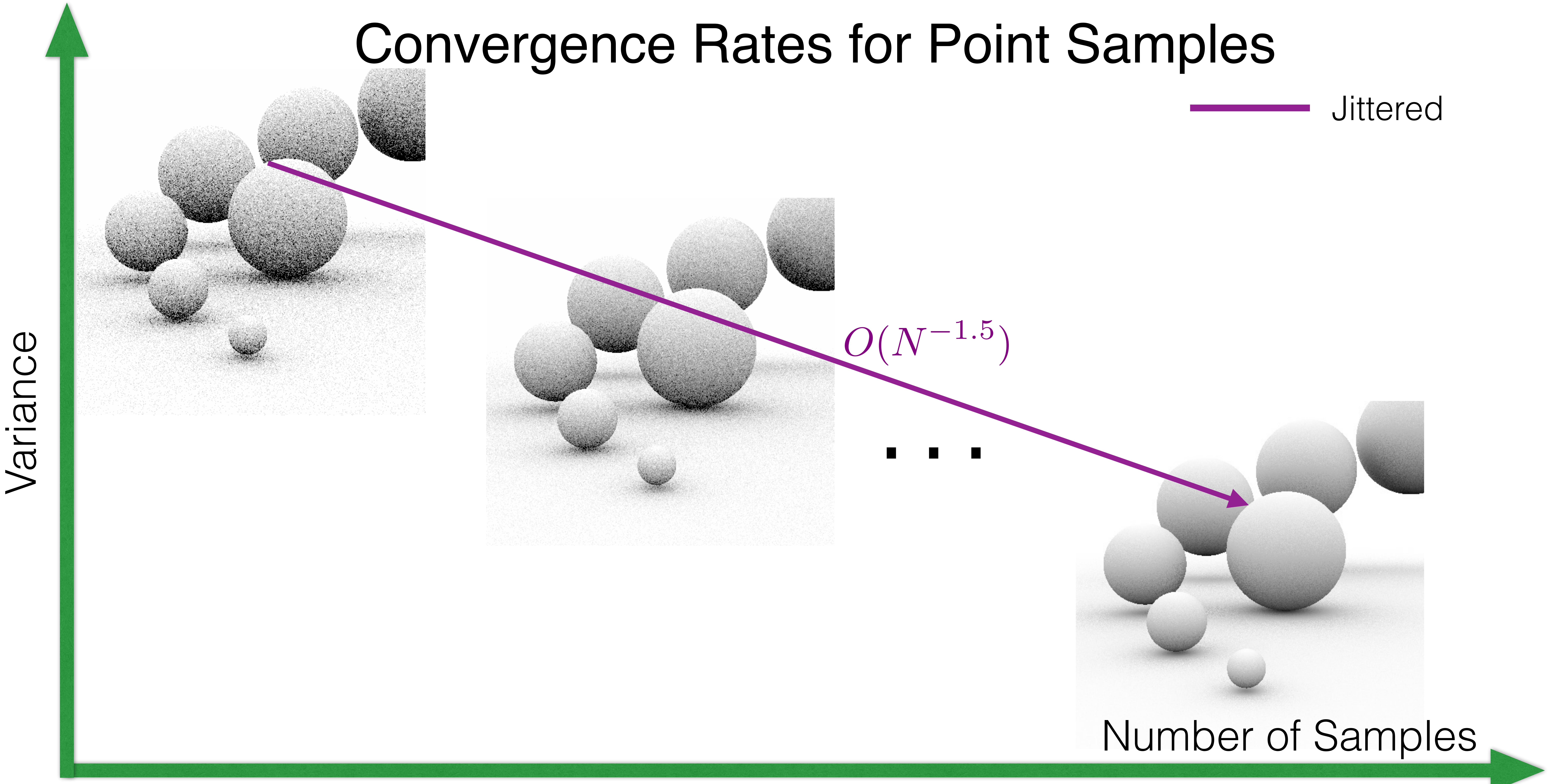
Variance



Convergence Rates for Point Samples



Convergence Rates for Point Samples



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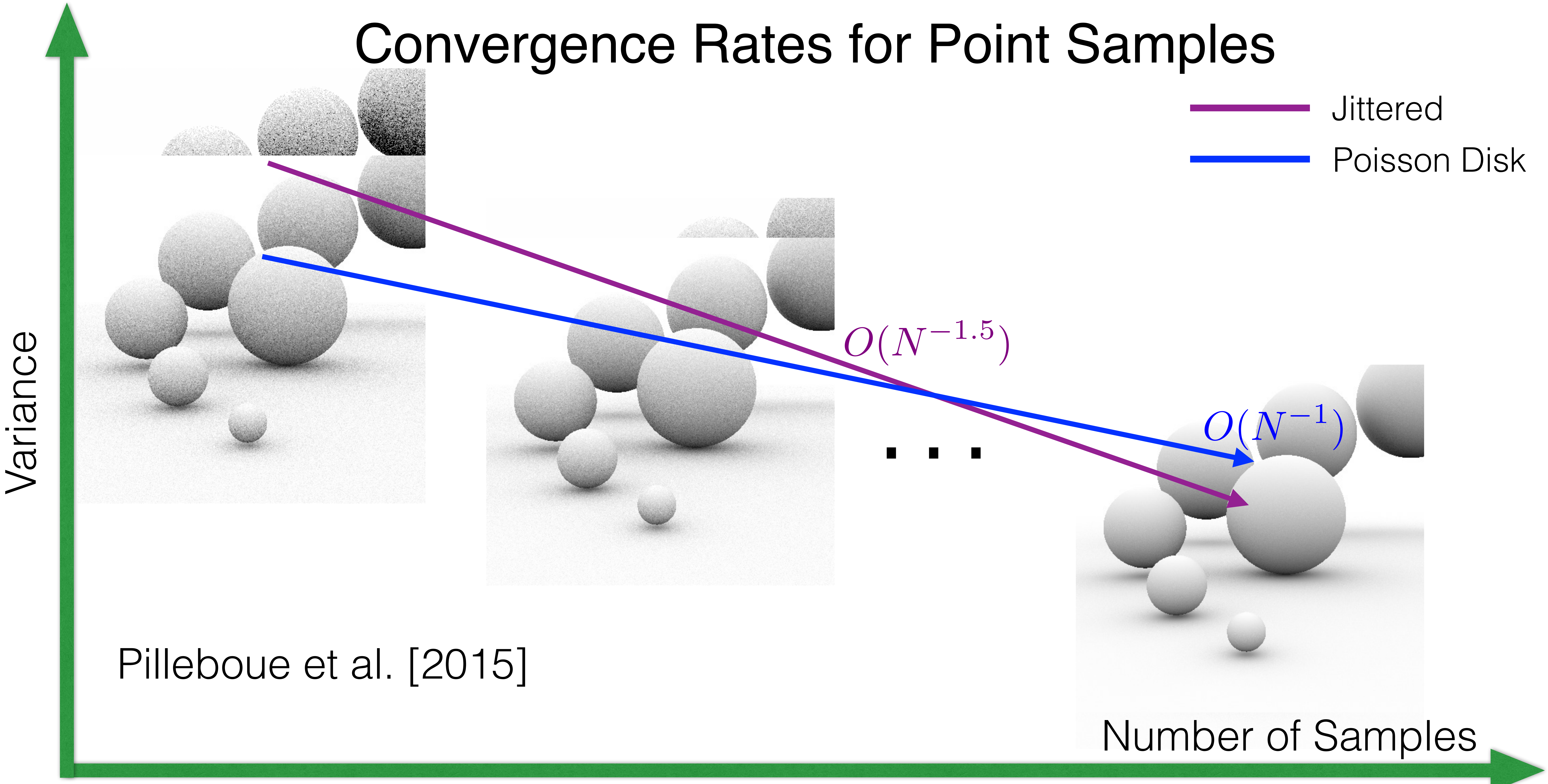
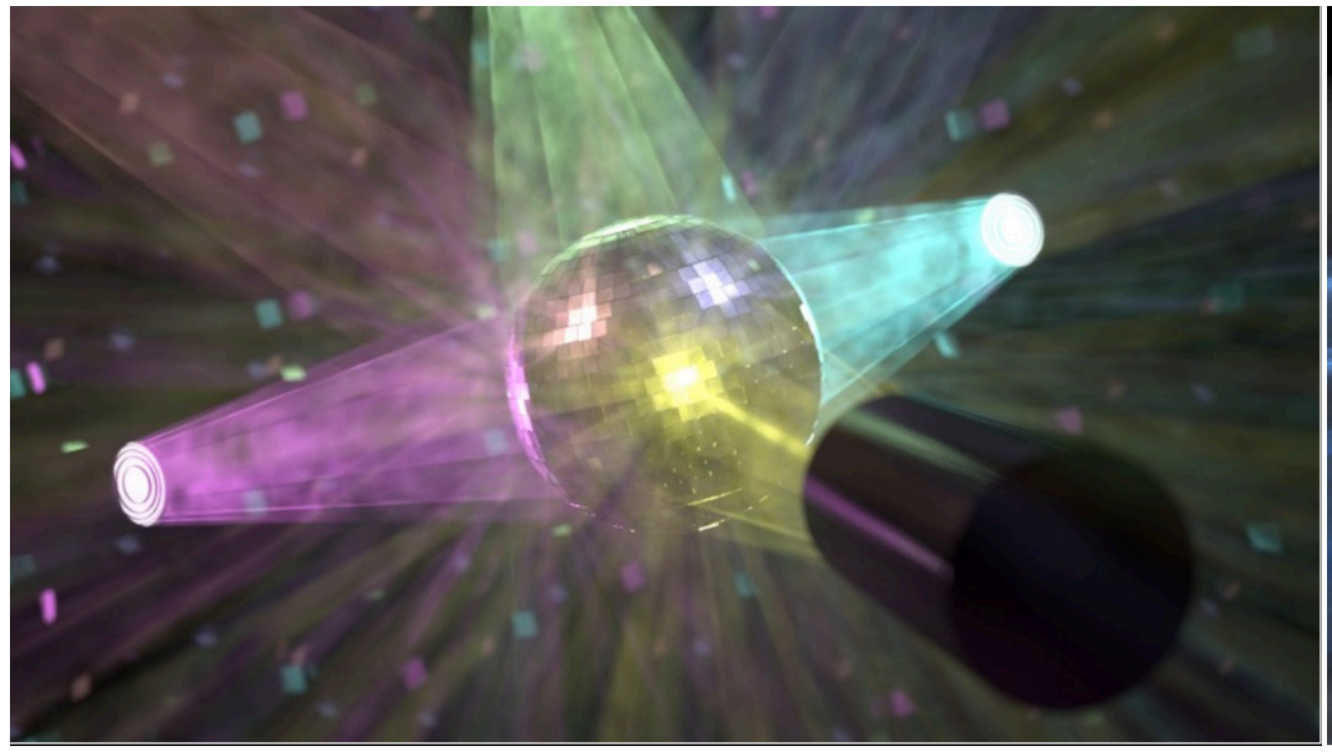
Convergence Rates for Point Samples



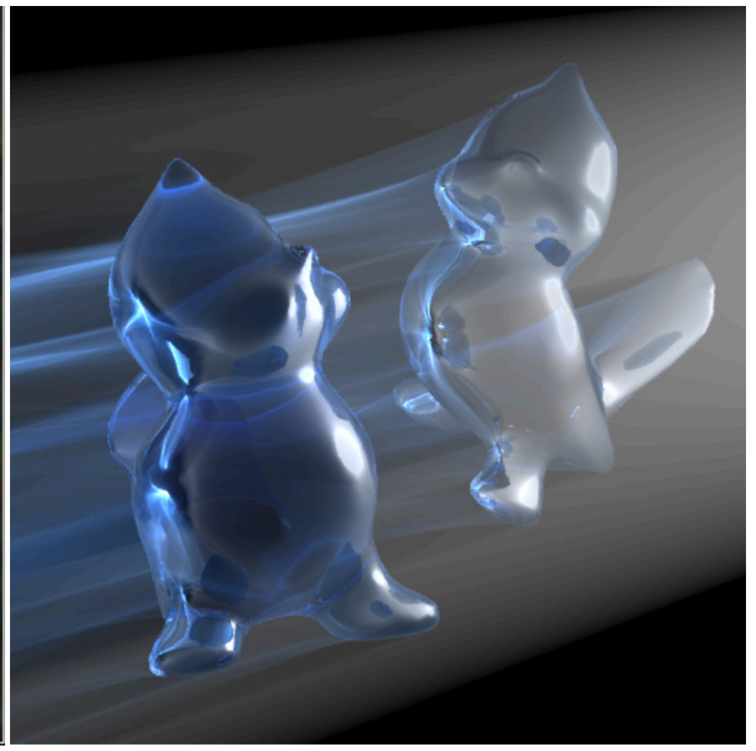
Previous Work

Krivanek et al. 2014

Habel et al. 2013



Jarosz et al. 2011

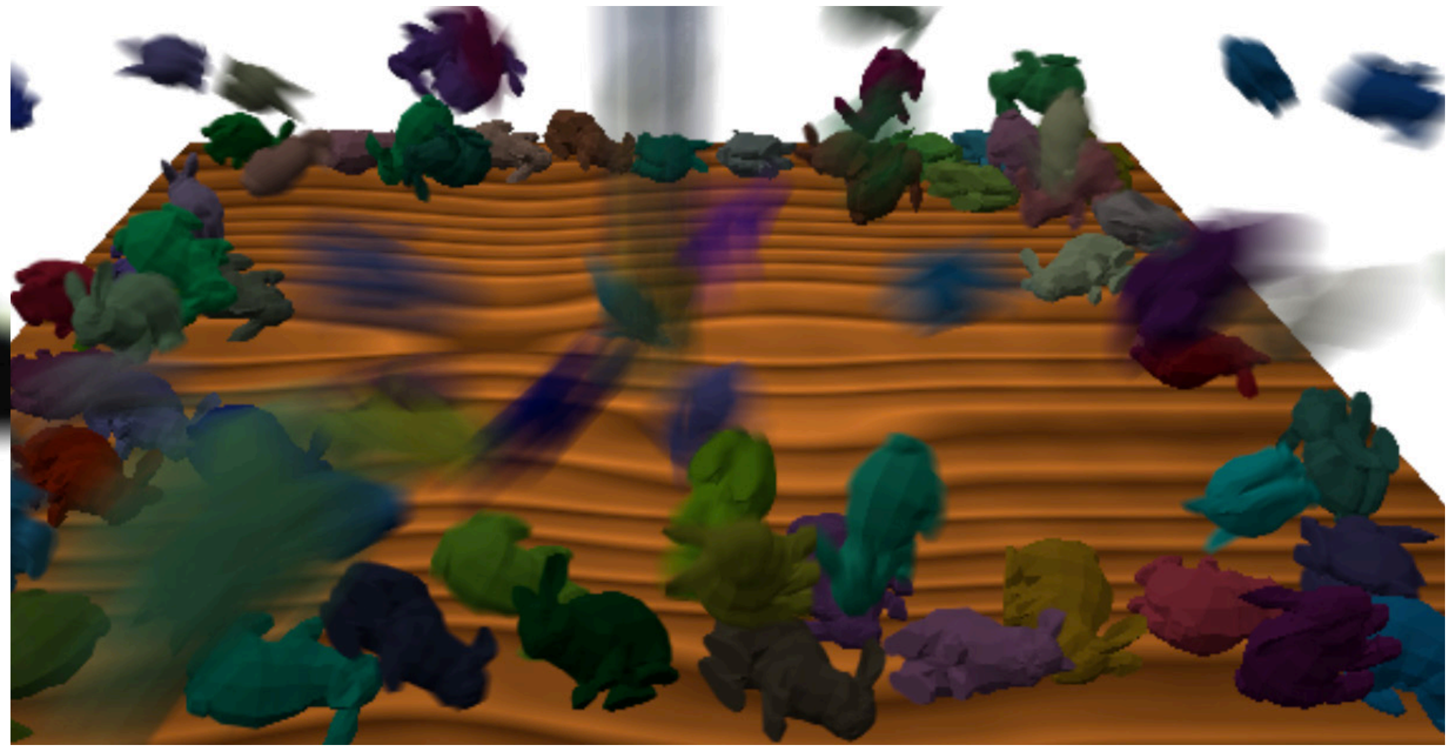


Sun et al. 2010

Gribel et al. 2012

Barringer et al. 2012

Tzeng et al. 2012



Gribel et al. 2010

- Monte Carlo Estimators for Point, Segment & Line Samples
- Variance Formulation for Point Samples
- Variance Formulation for Segment and Line Samples
- Experimental Verification

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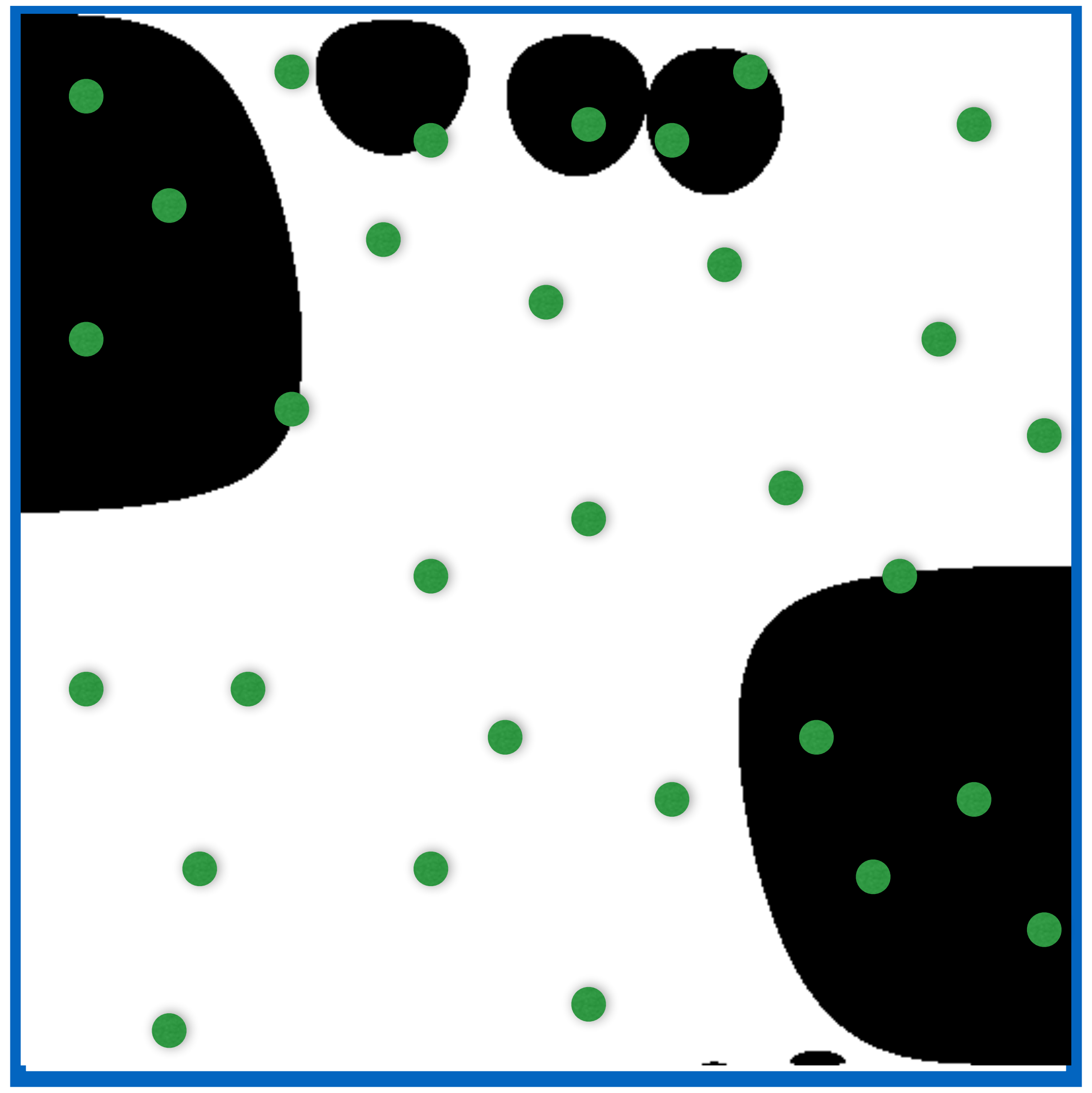
Monte Carlo Estimator

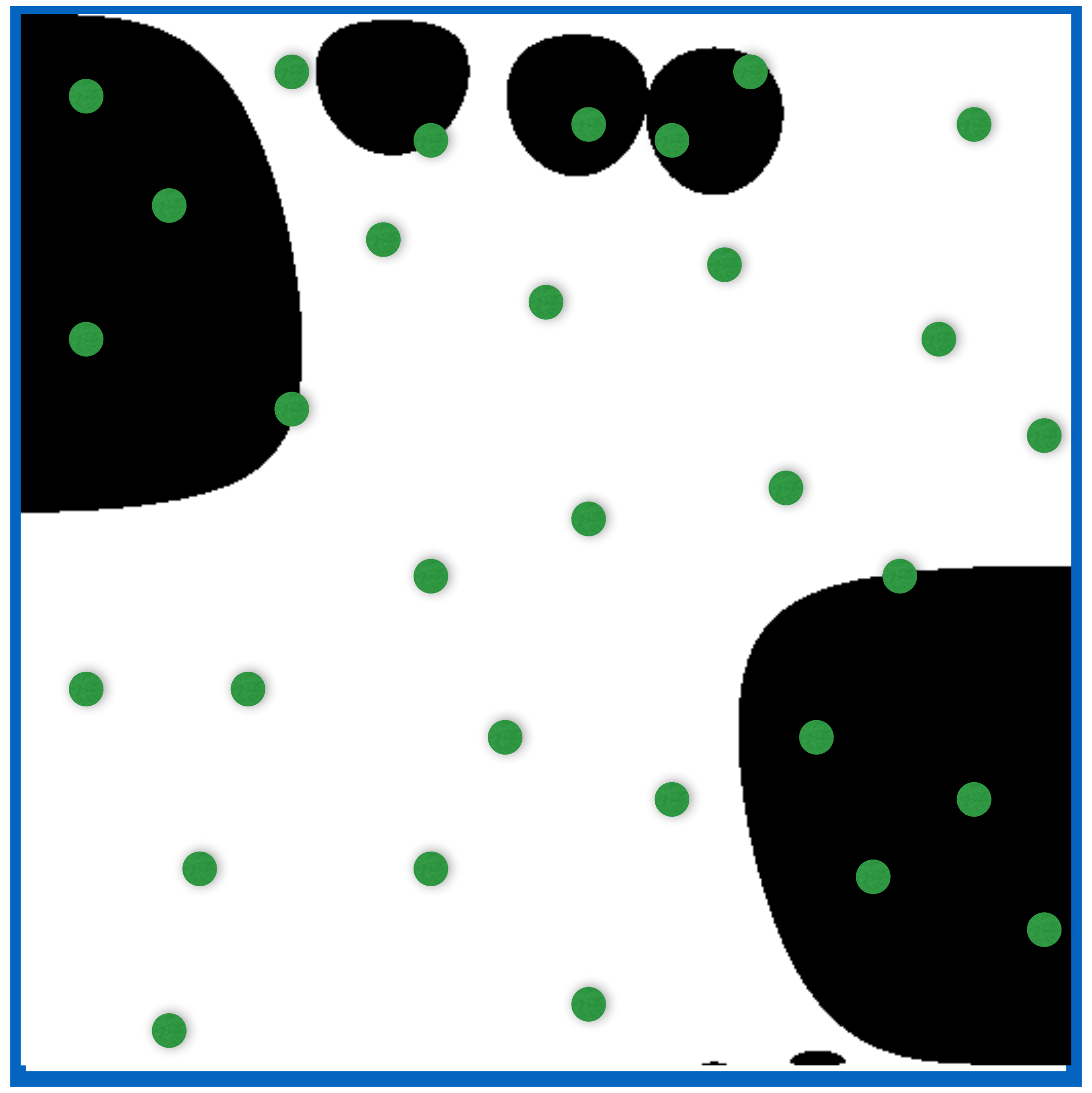
Points



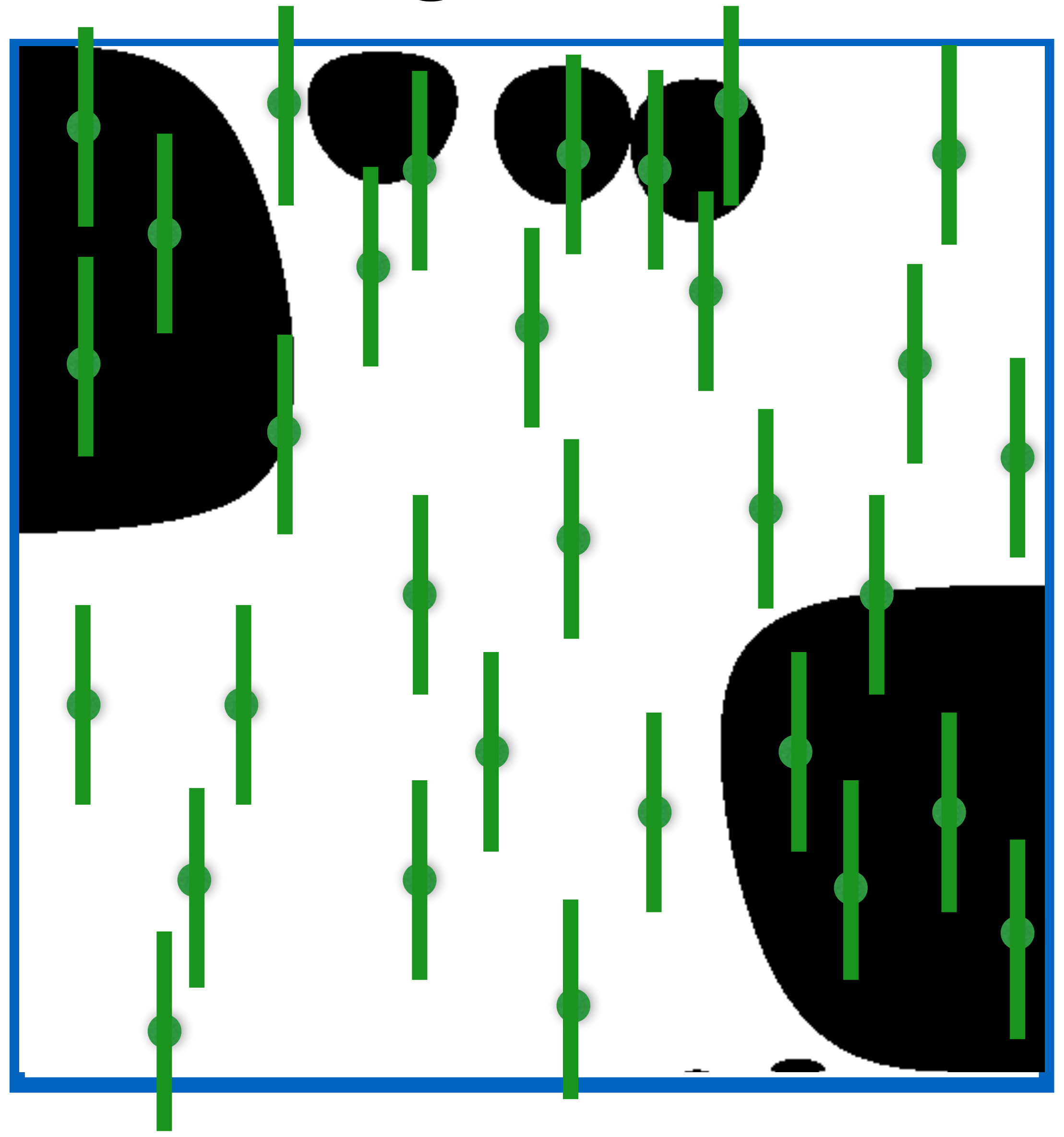
$$I_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f(x_k, y_k)}{p(x_k, y_k)}$$

Monte Carlo Estimator

Points



Segments



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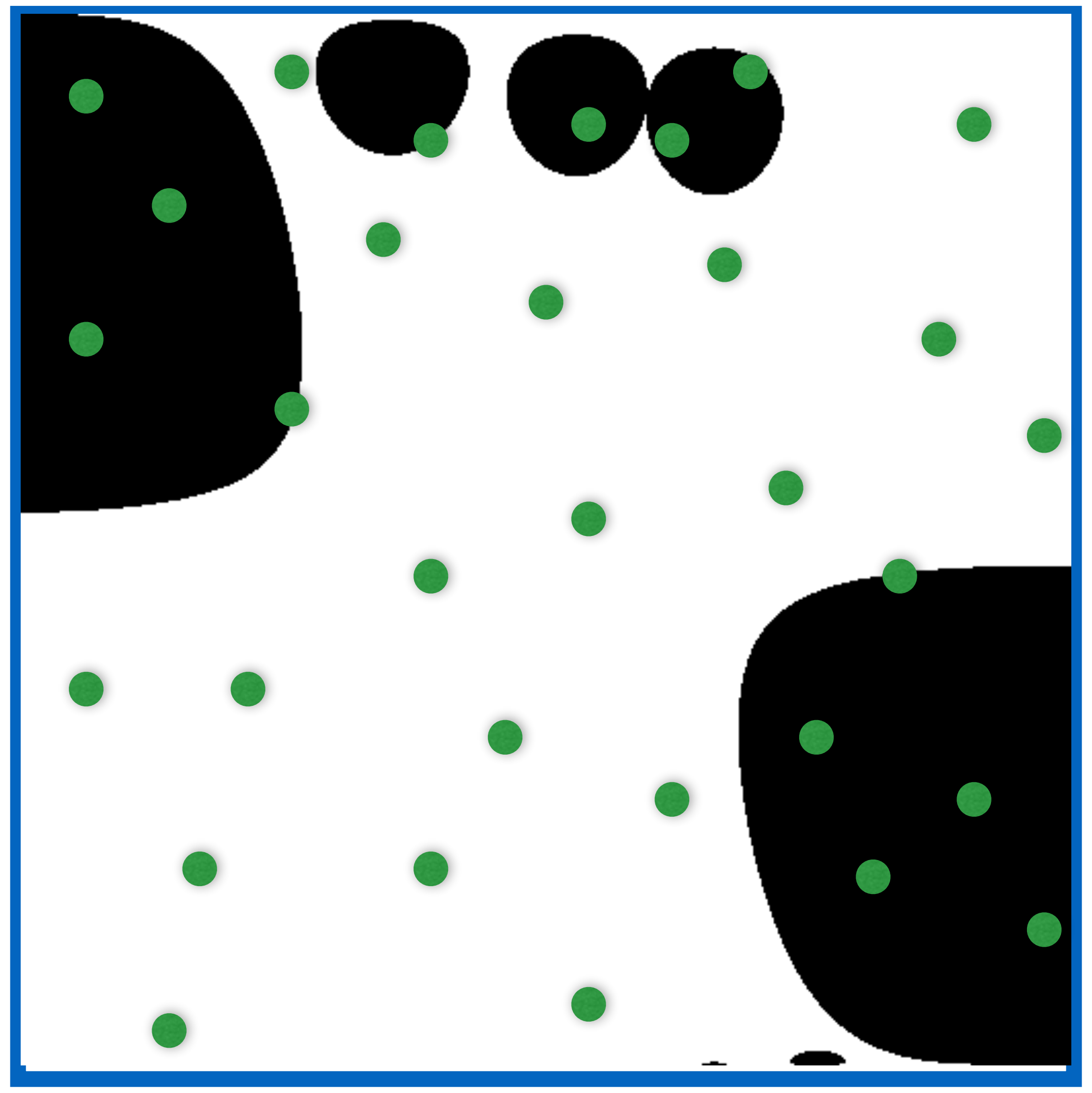
Monte Carlo Estimator

Points



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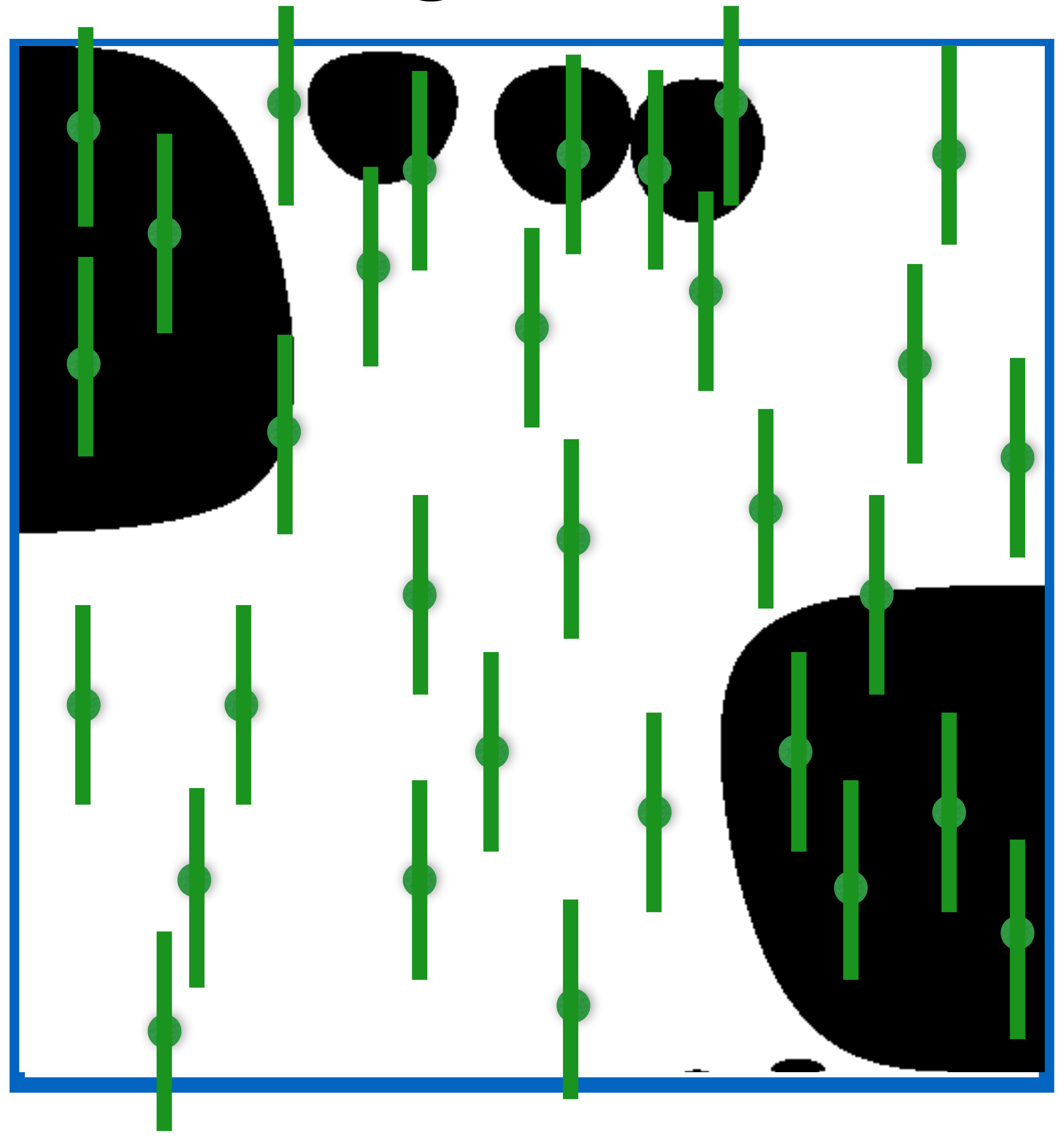

$$I_N = \frac{1}{N}\sum_{k=1}^{N}\frac{\frac{1}{2a}\int_{y_k-a}^{y_k+a} f(x_k, y)dy}{p(x_k, y_k)}$$

Monte Carlo Estimator

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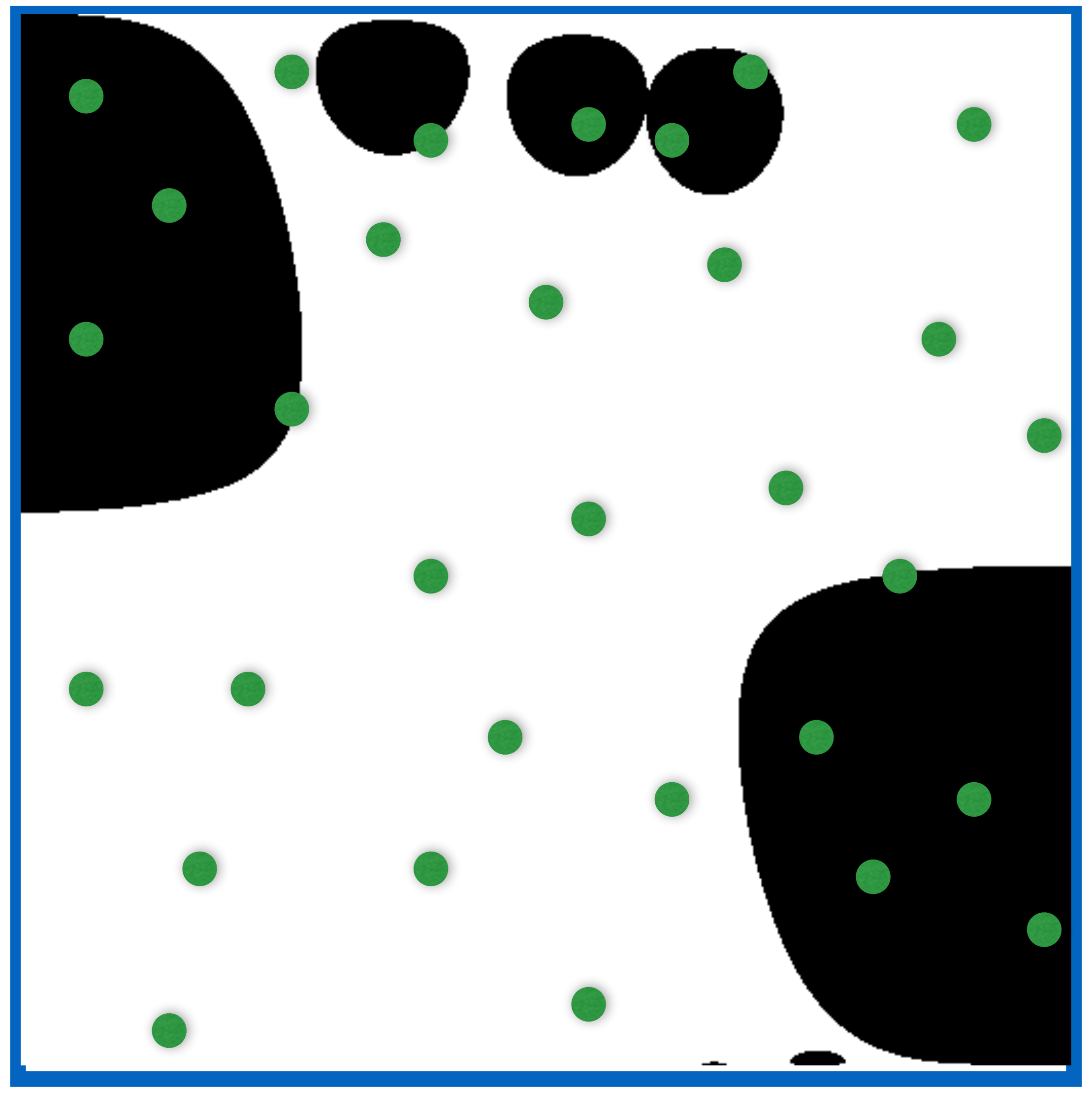

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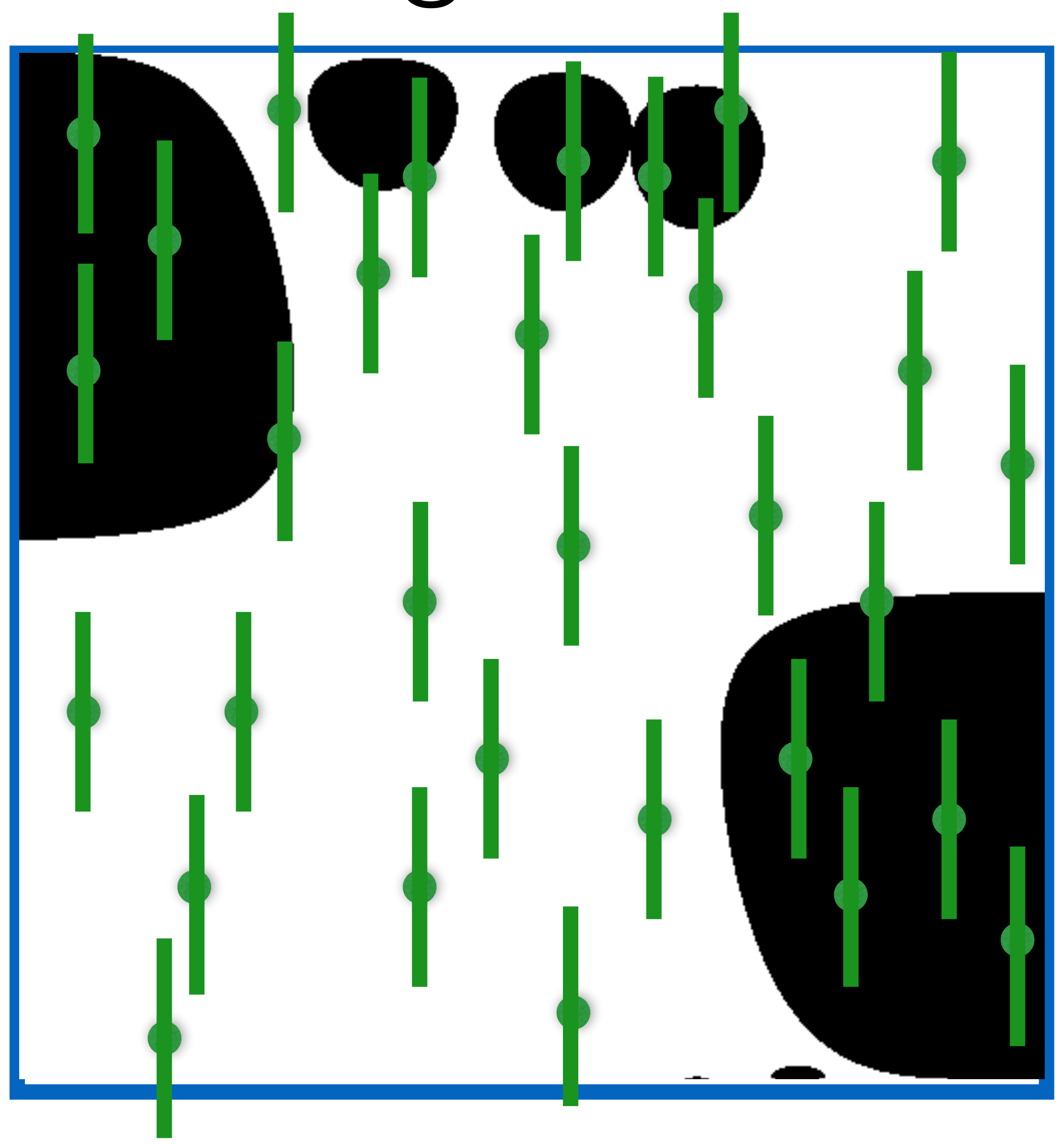

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Monte Carlo Estimator

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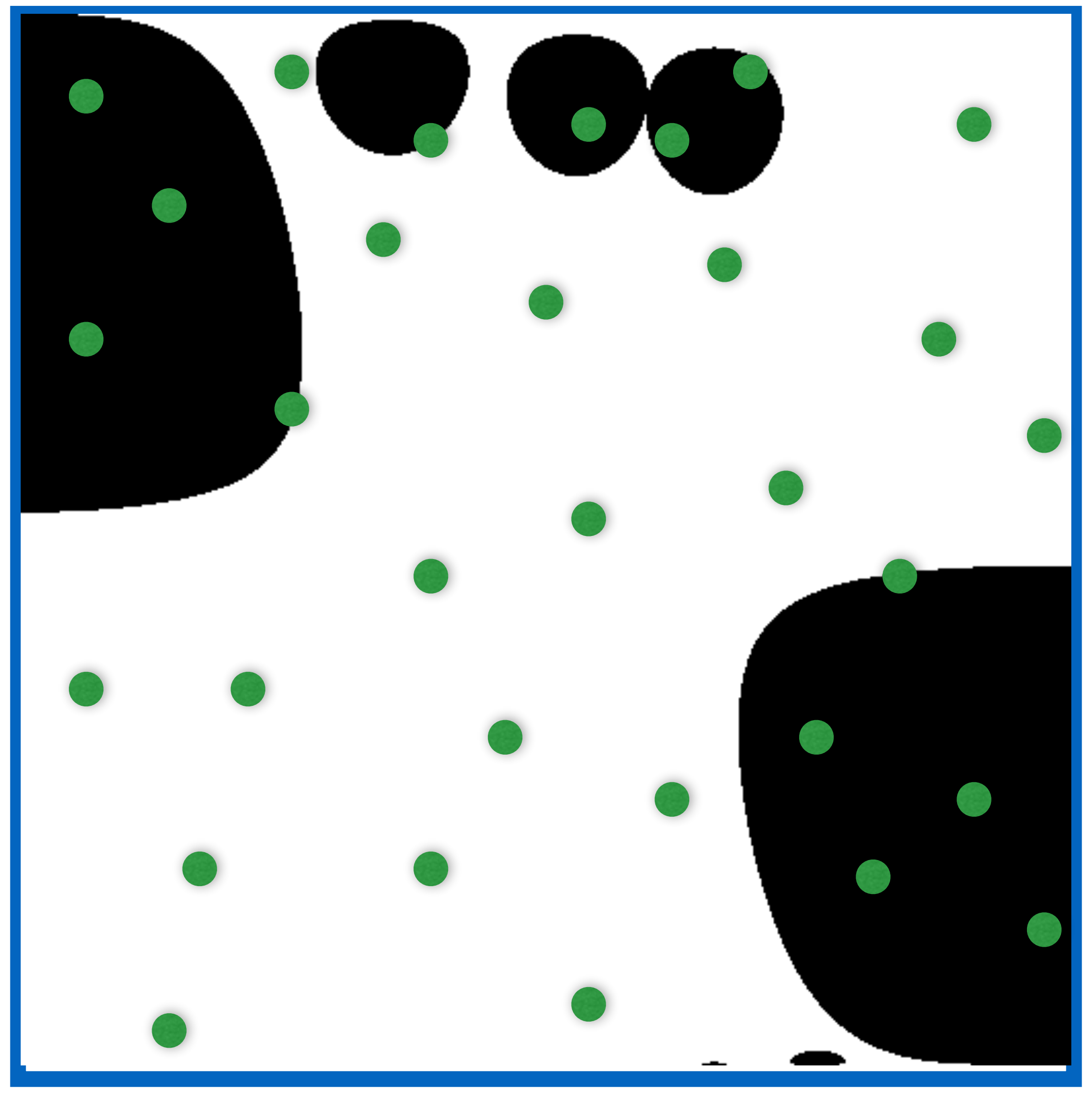

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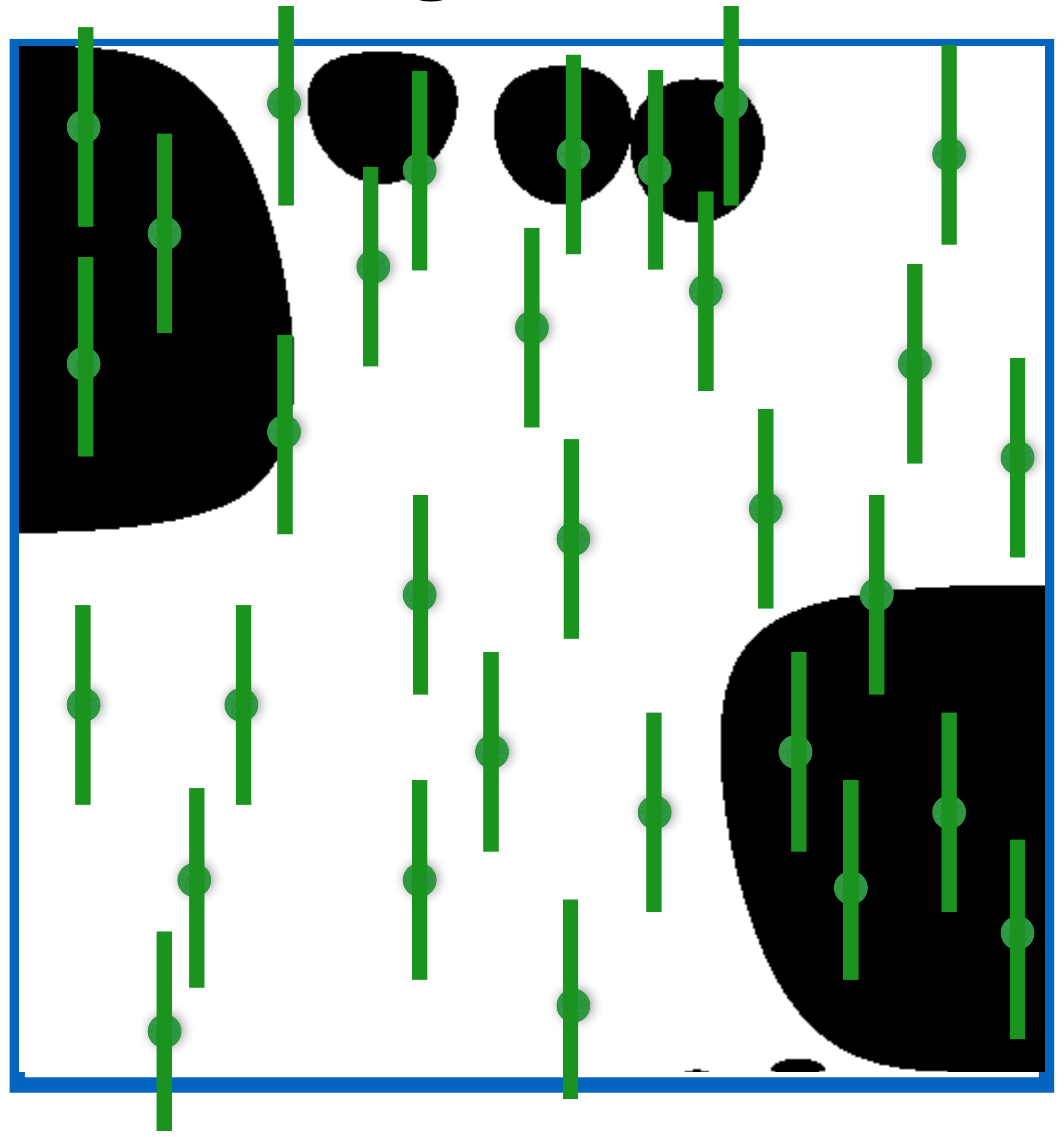

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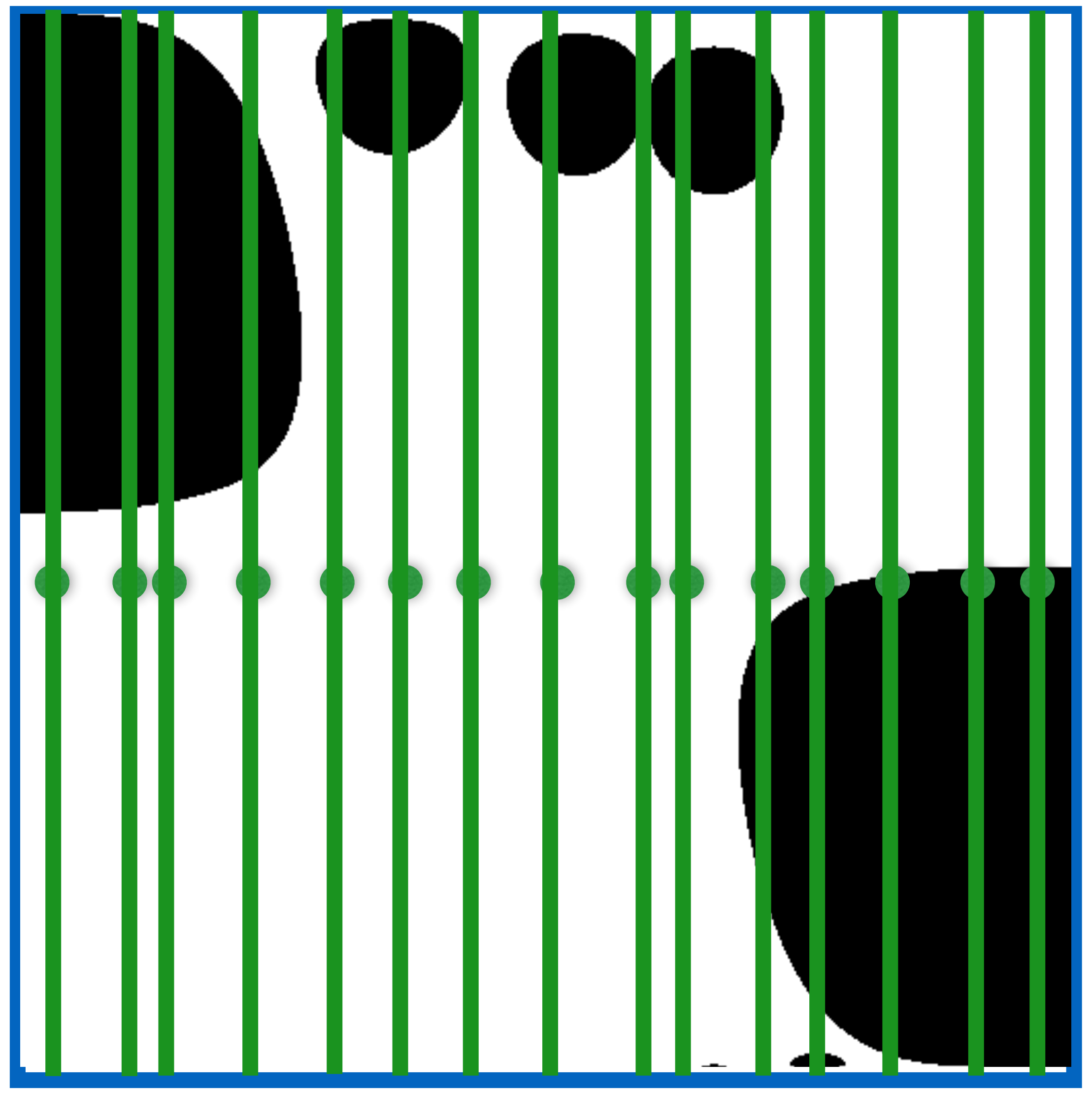
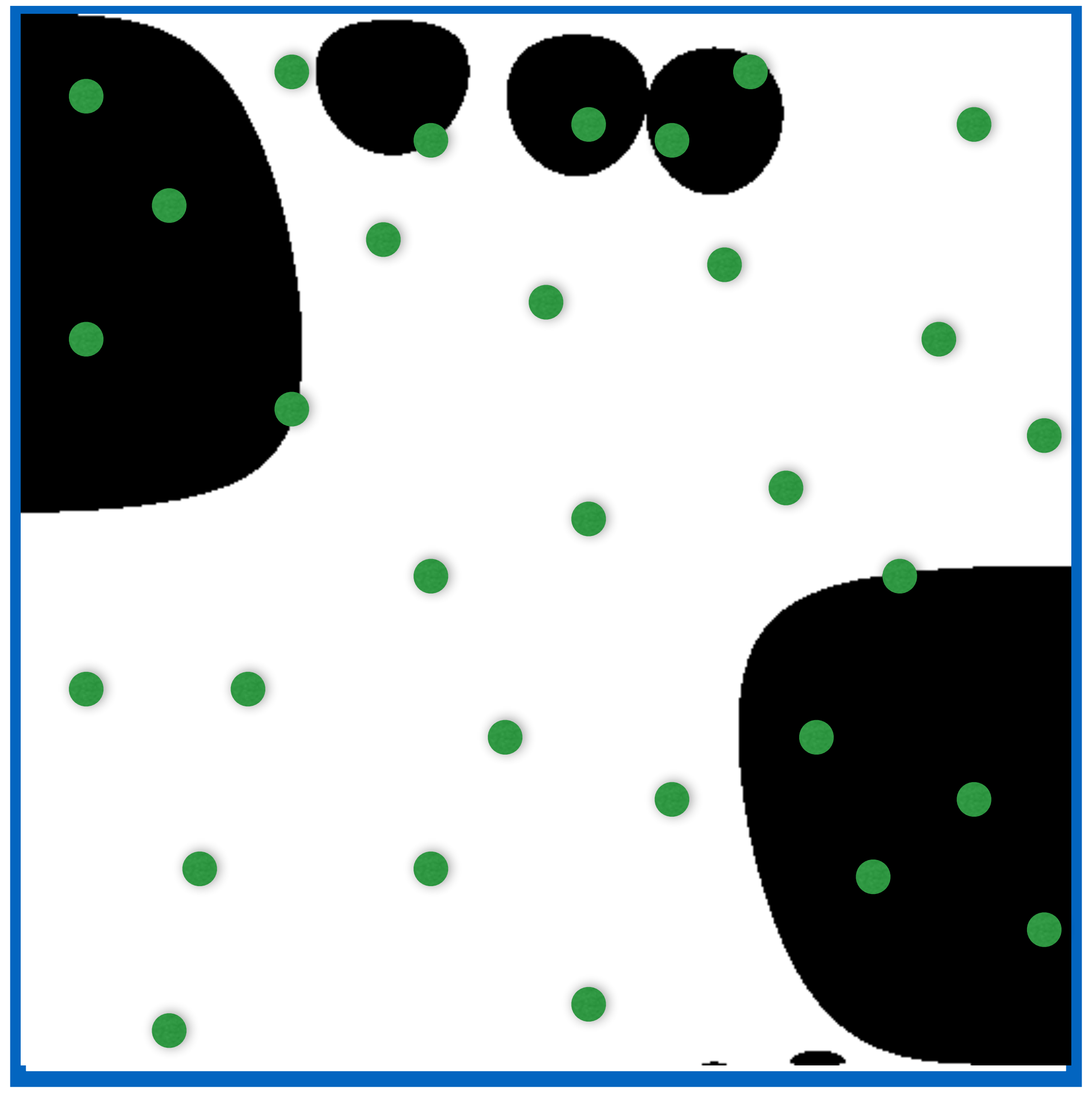
Lines



Monte Carlo Estimator

Points



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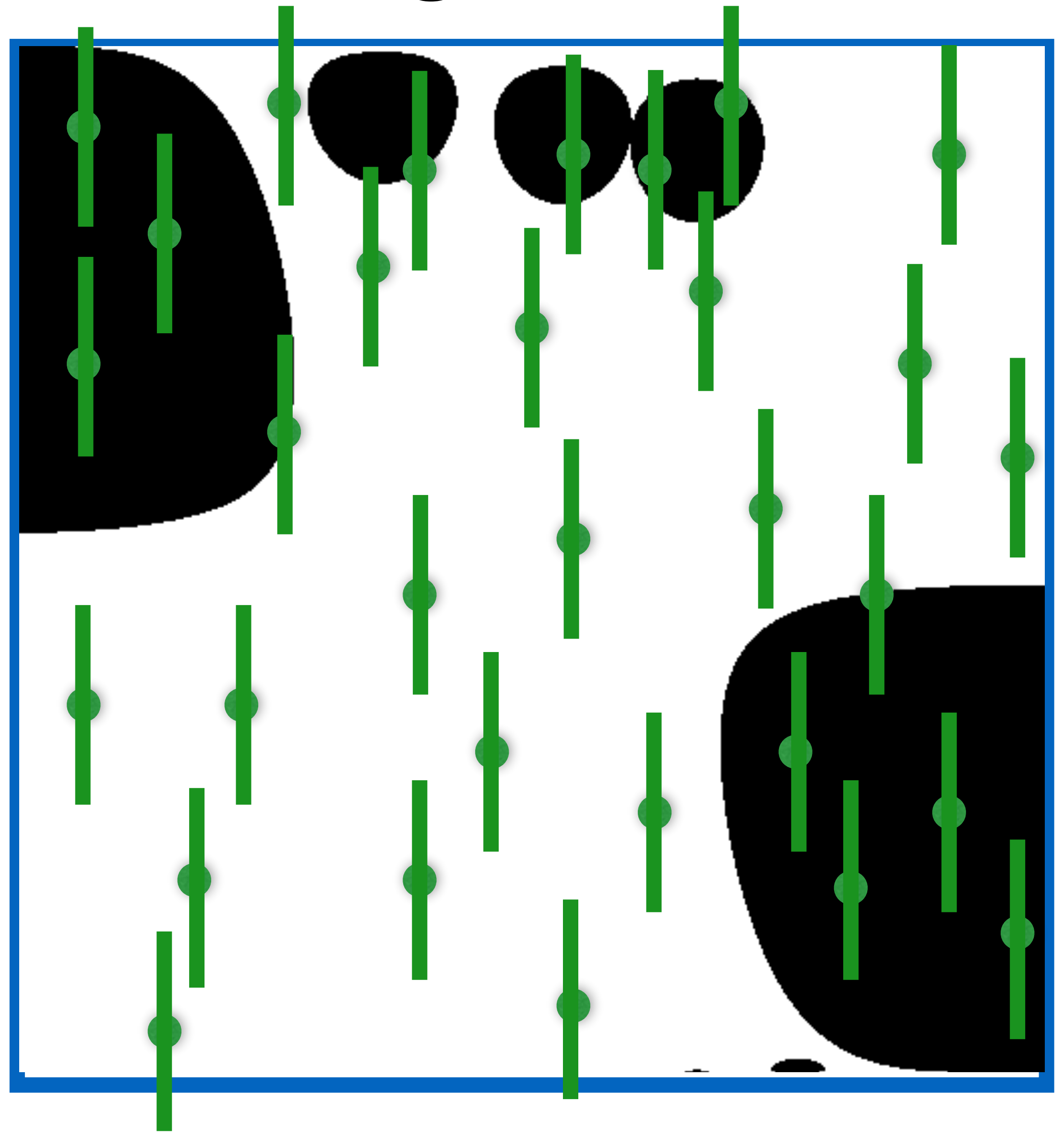

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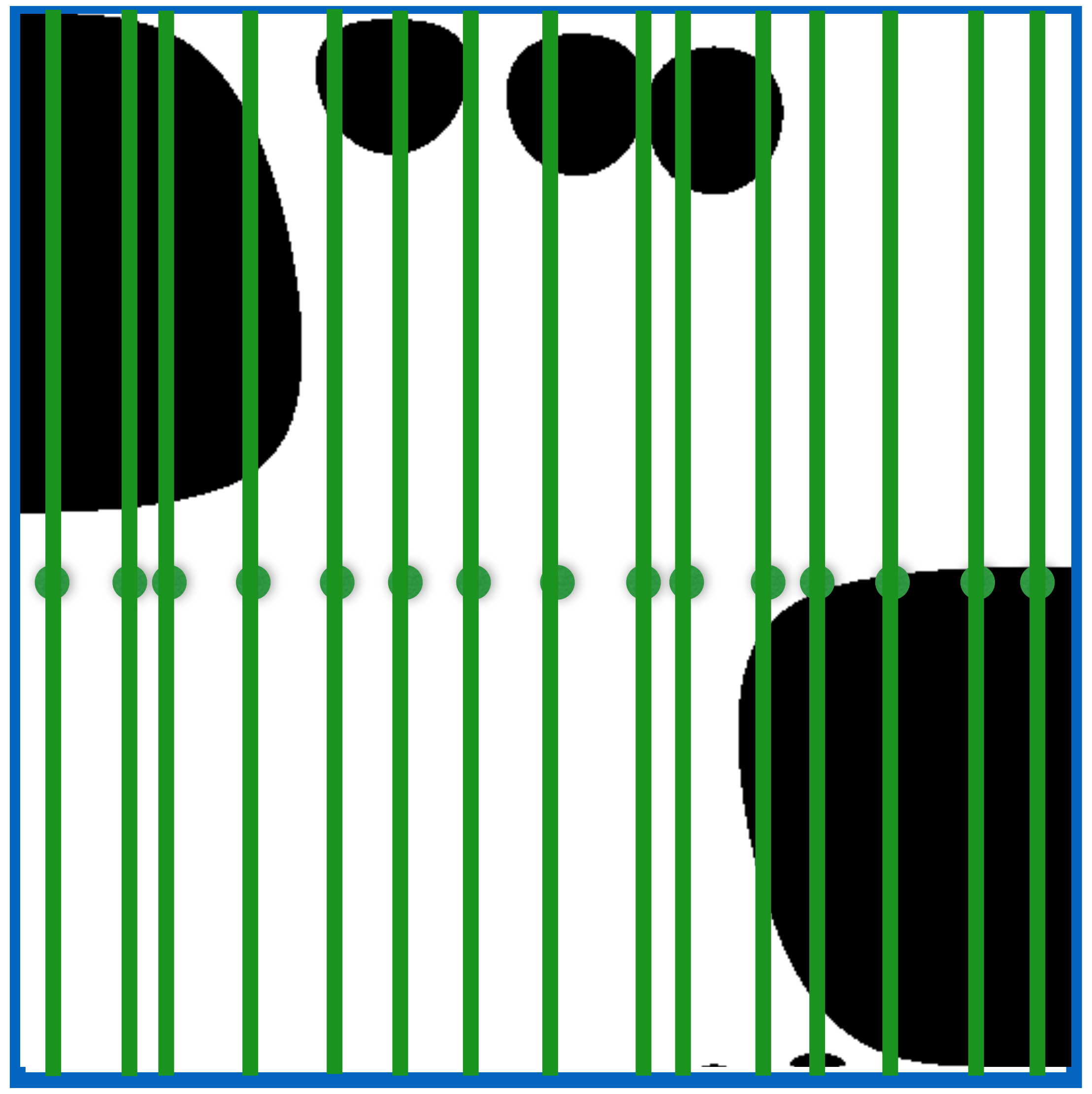
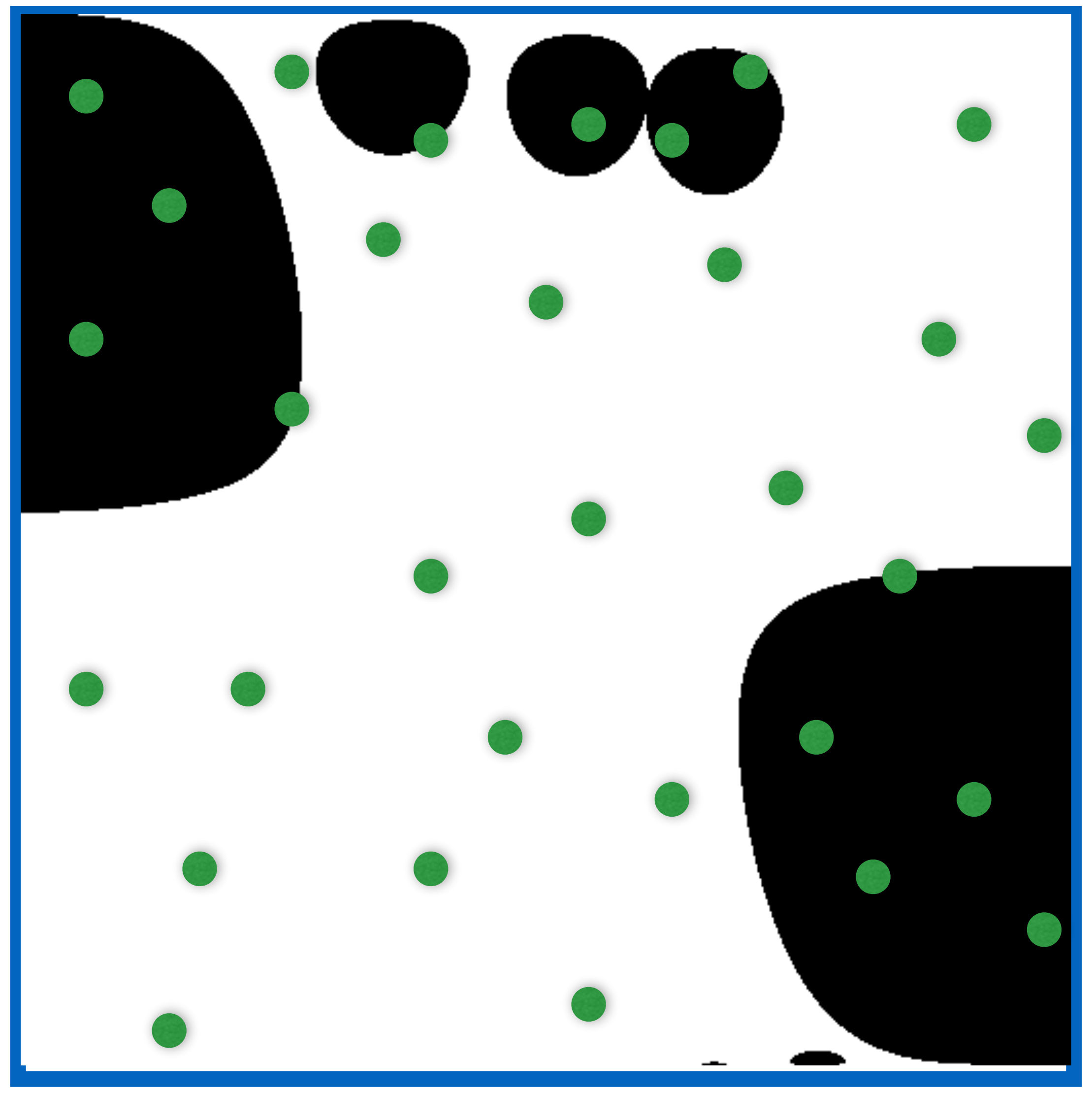


$$I_N = \sum_{k=1}^{N}\frac{\frac{1}{l}\int_y f(x_k, y)dy}{p(x_k)}$$

Monte Carlo Estimator

Points



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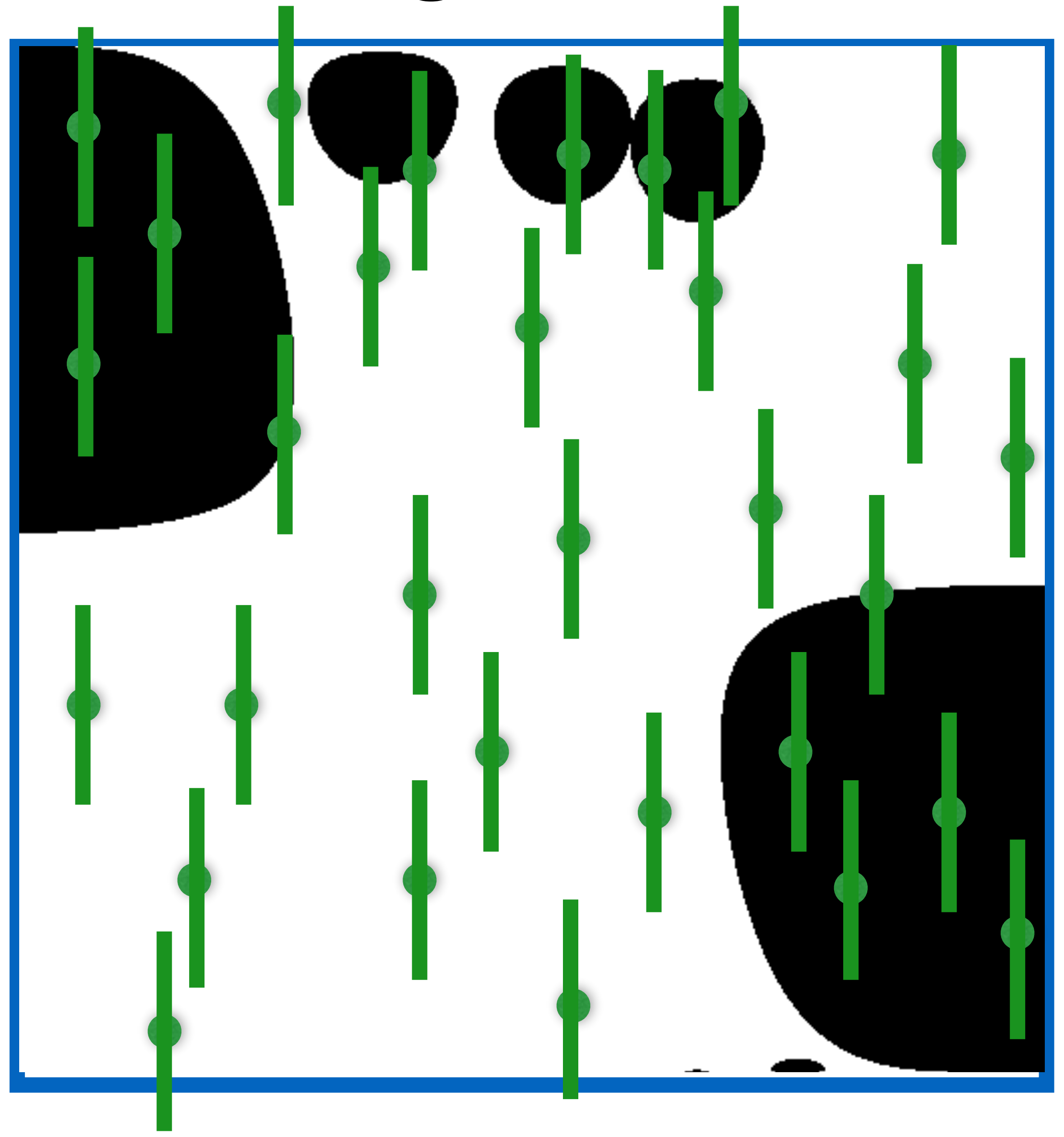

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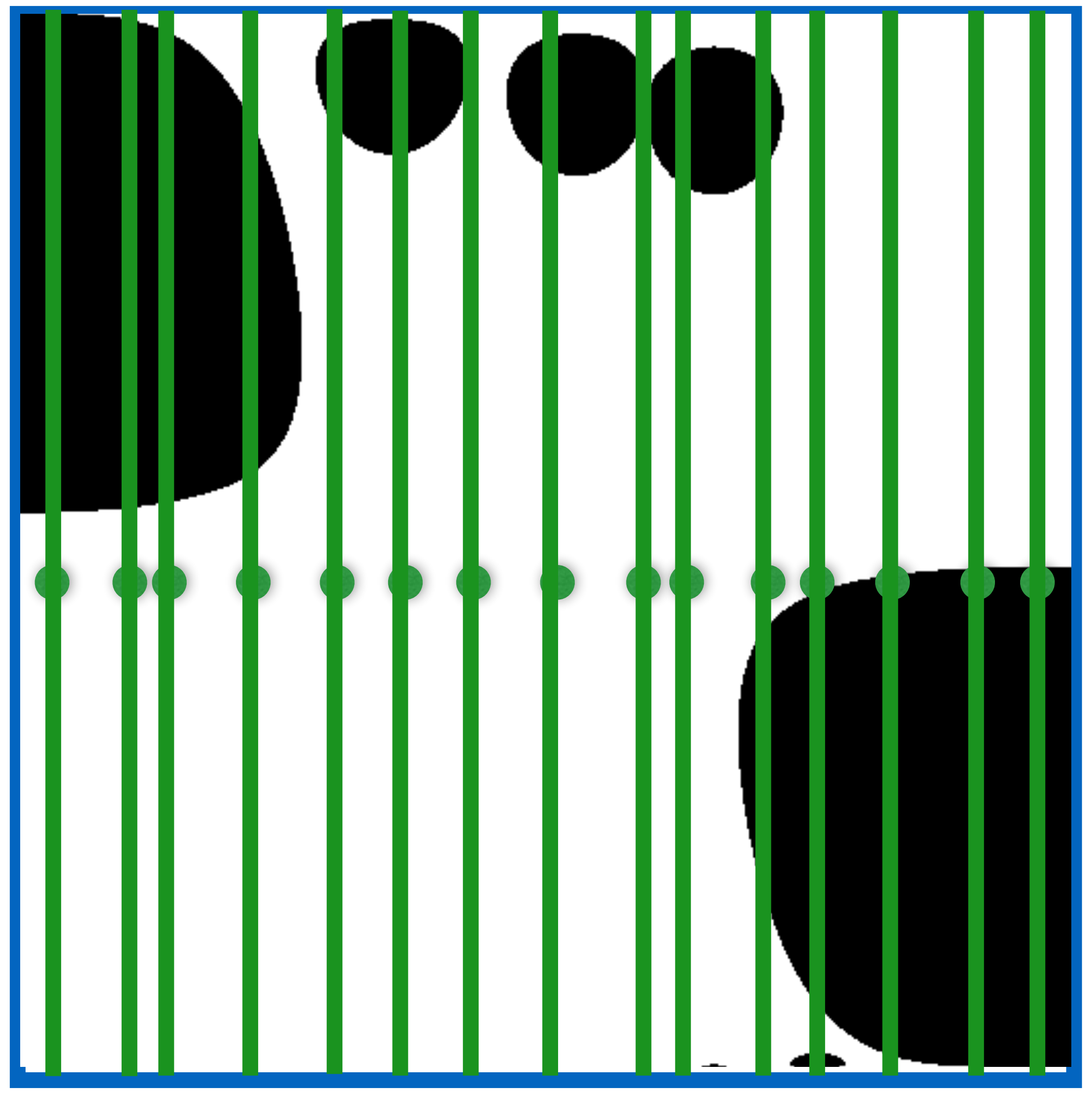

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Monte Carlo Estimator

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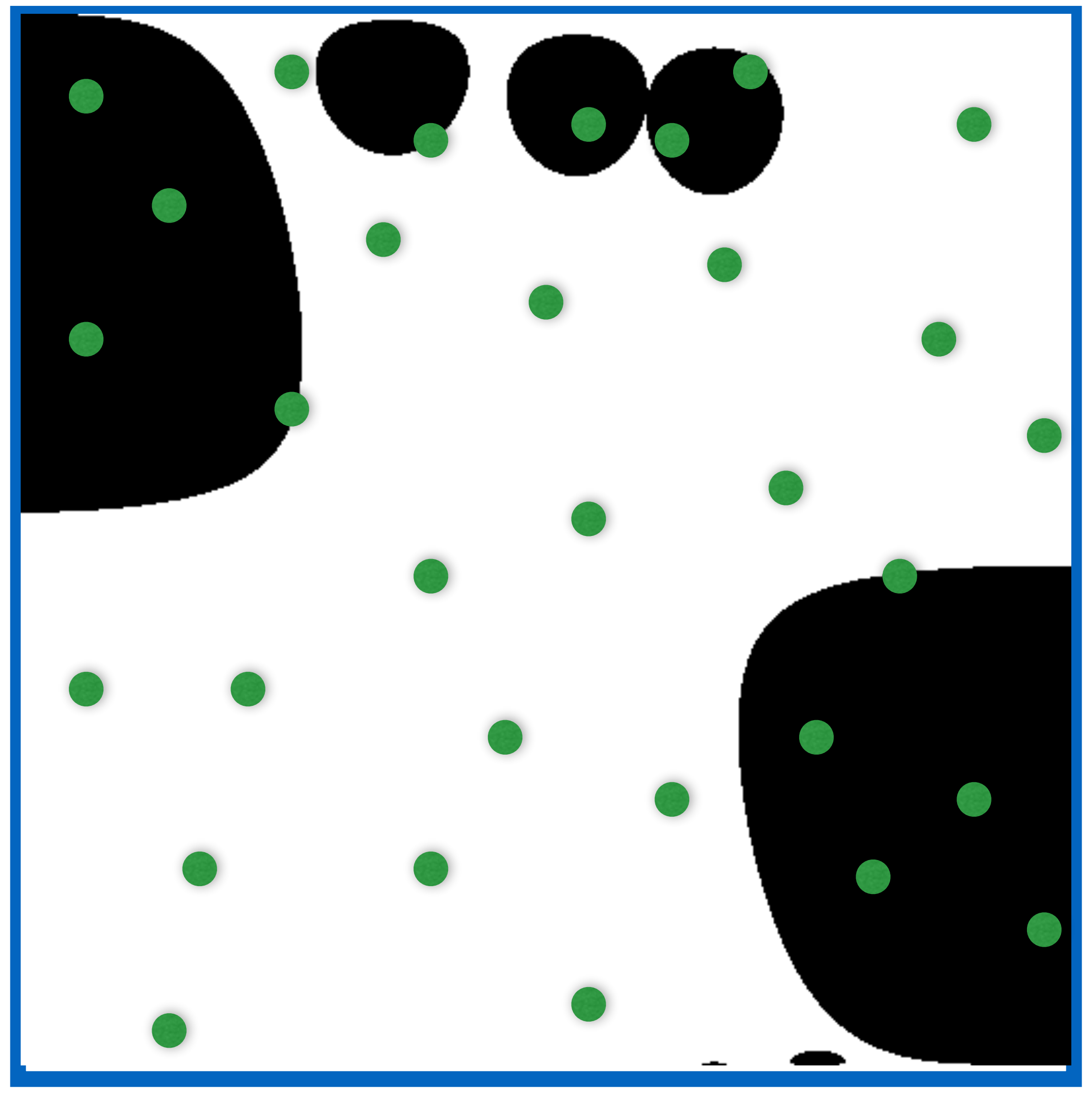

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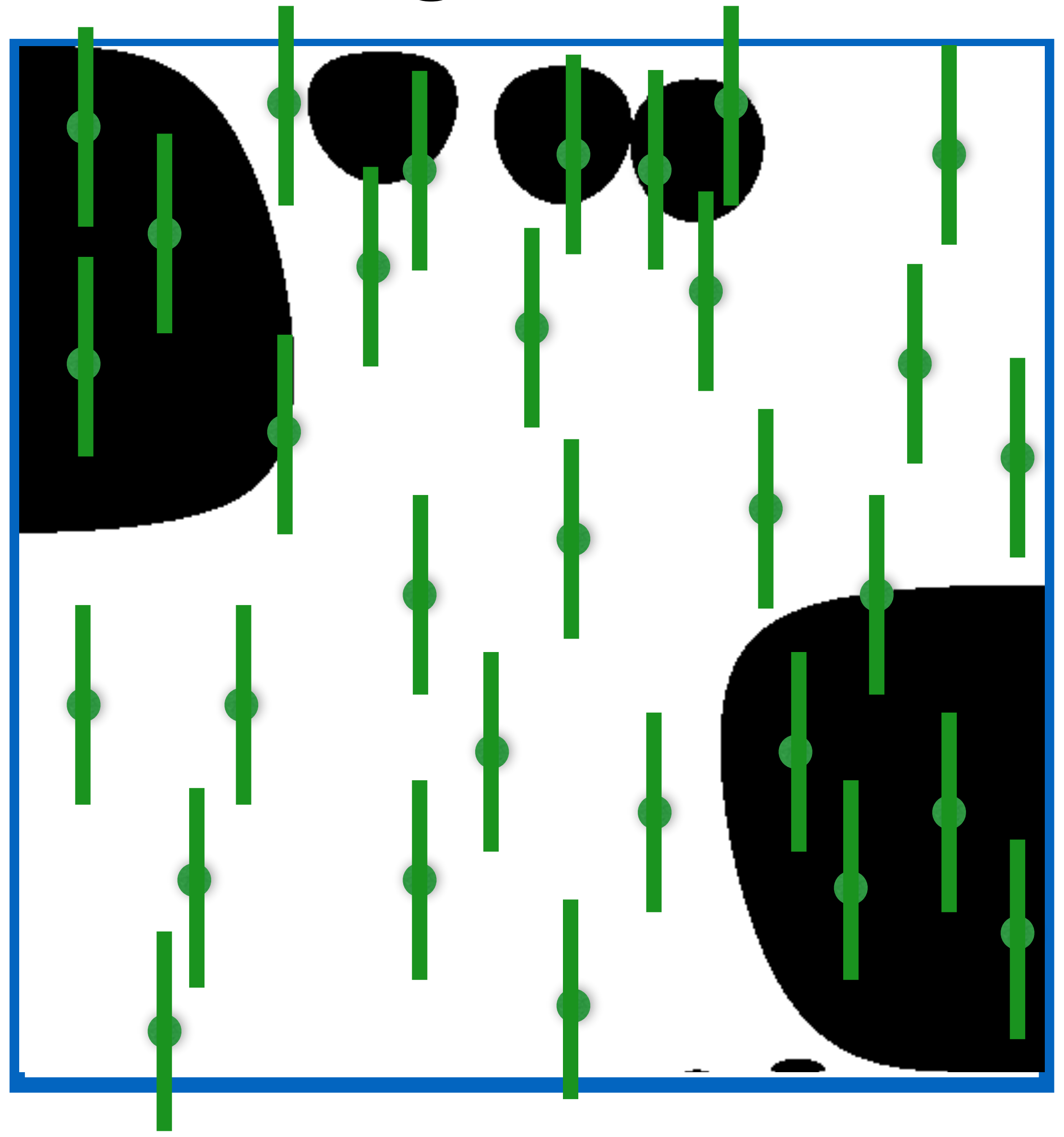

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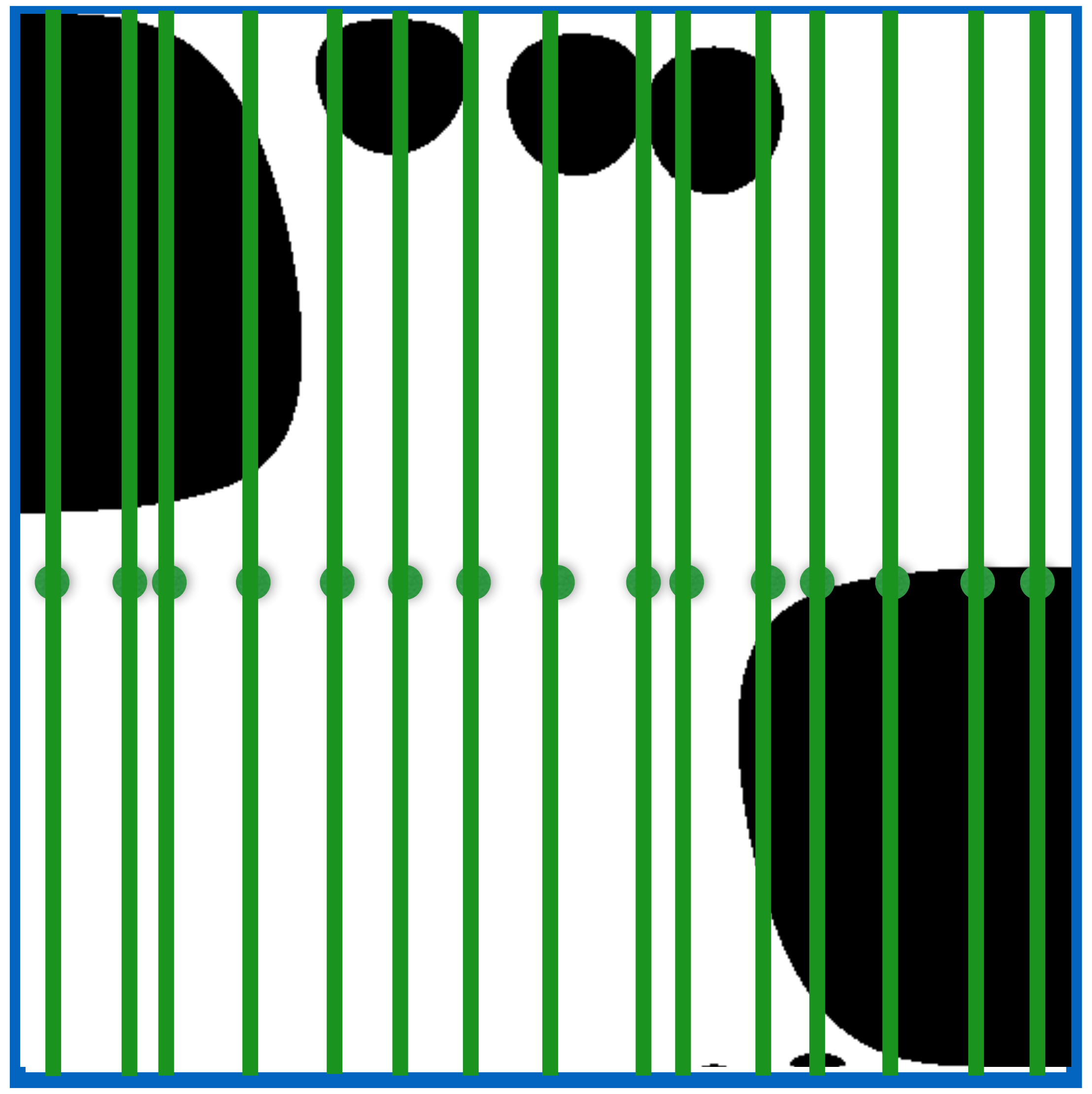

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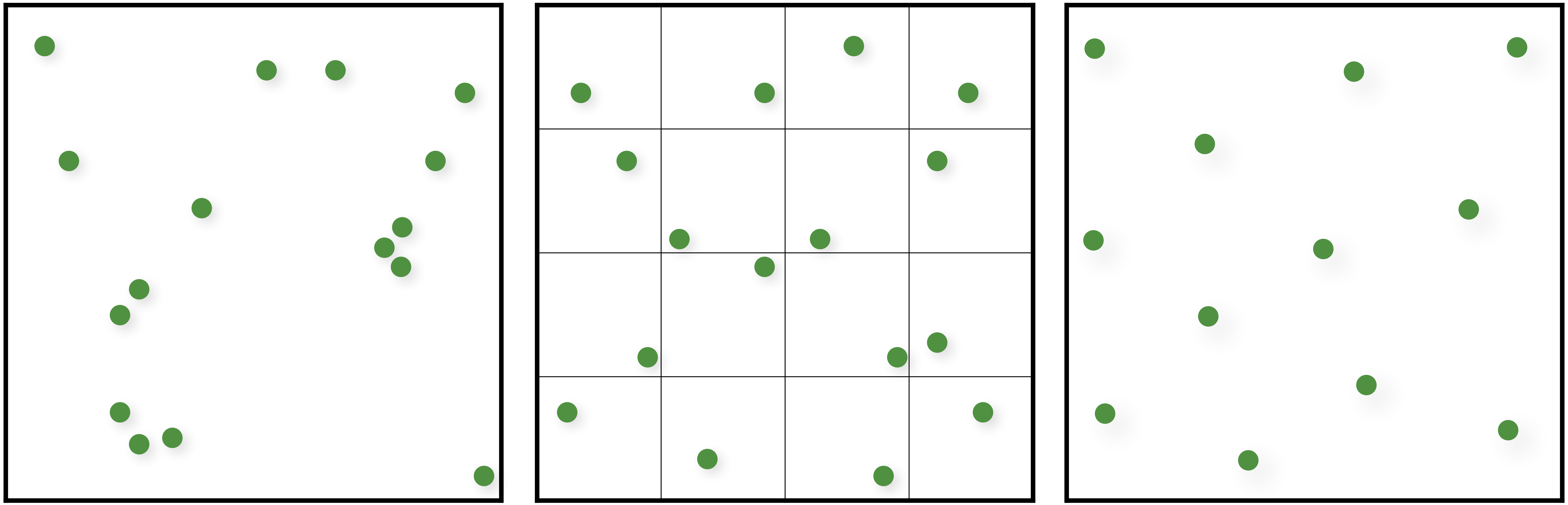
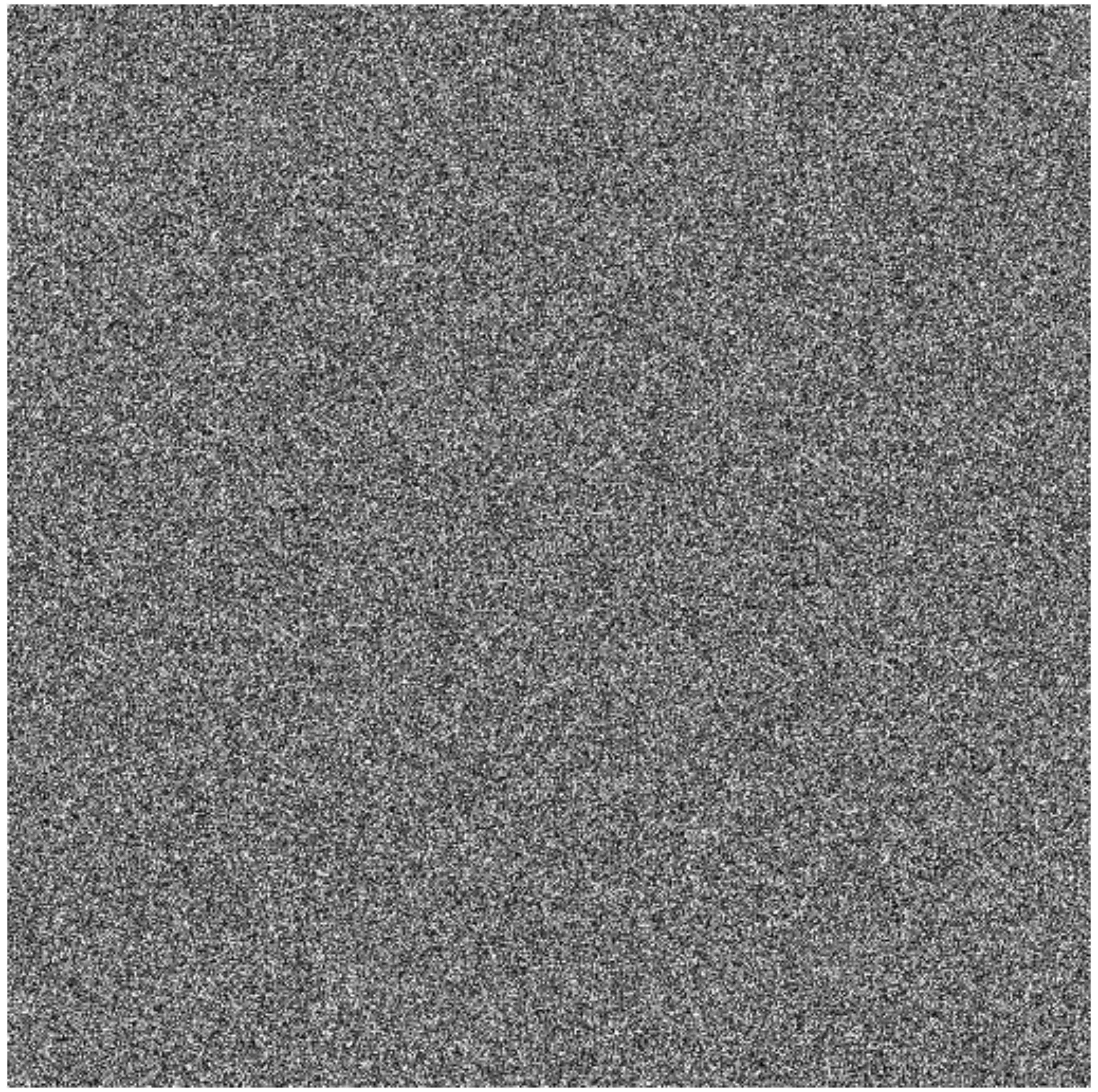
Point Sampling Patterns



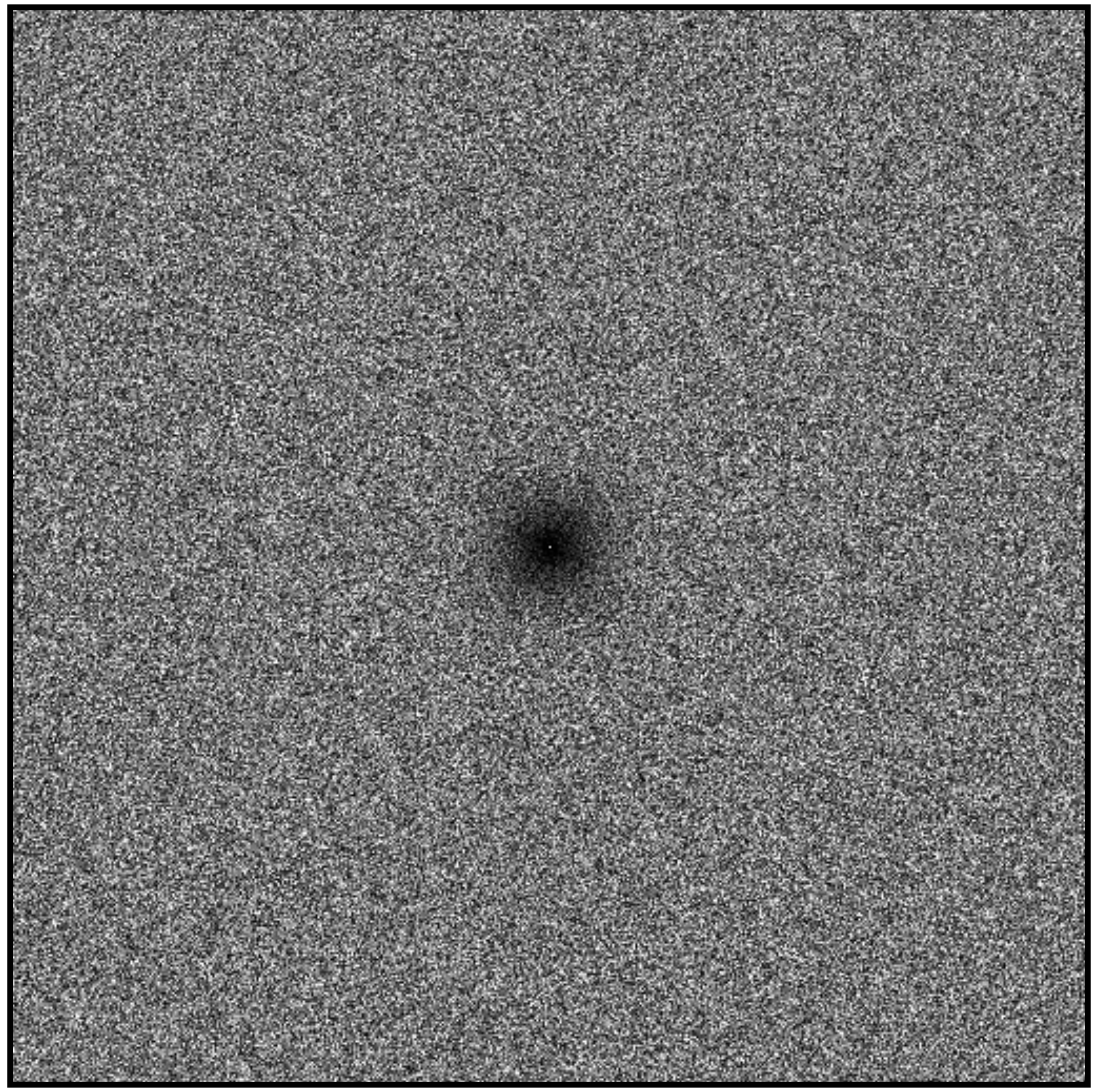
Random | Jitter | Poisson Disk

$$P_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N}\delta(\vec{x} - \vec{x}_k)$$

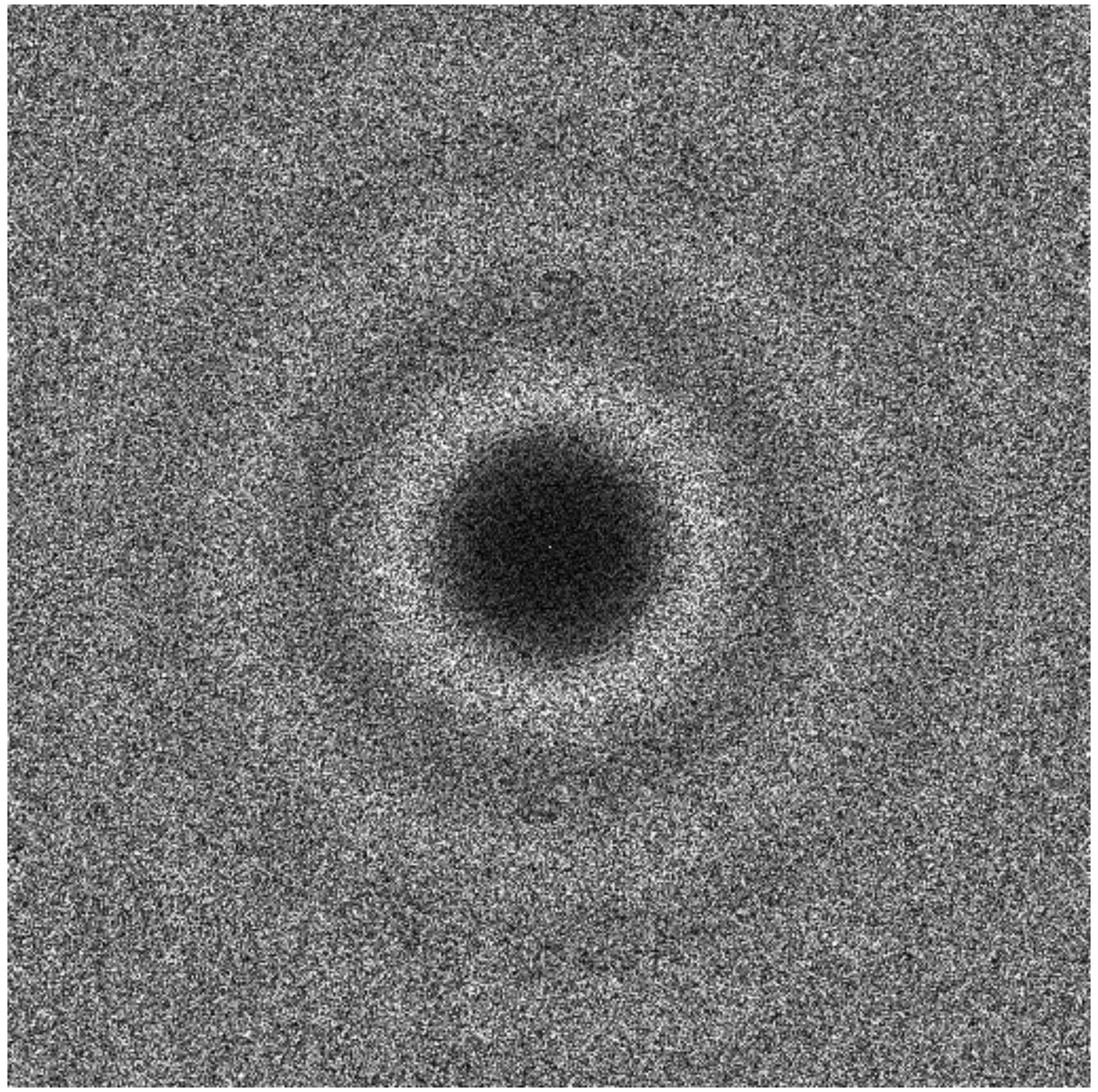
Point Samples' Expected Power Spectra



Random



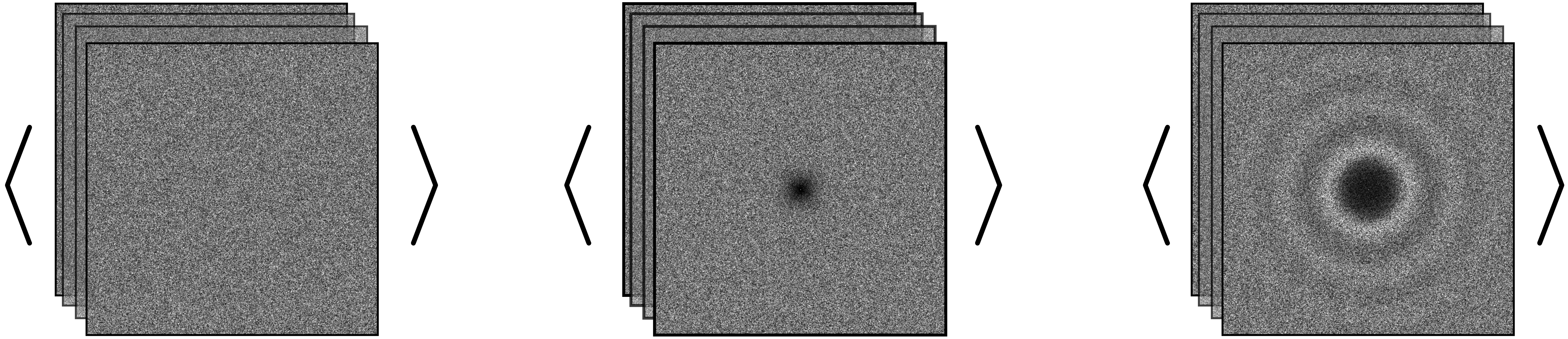
Jitter



Poisson Disk

$$\mathcal{P}_{P_N}(\nu) = \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2$$

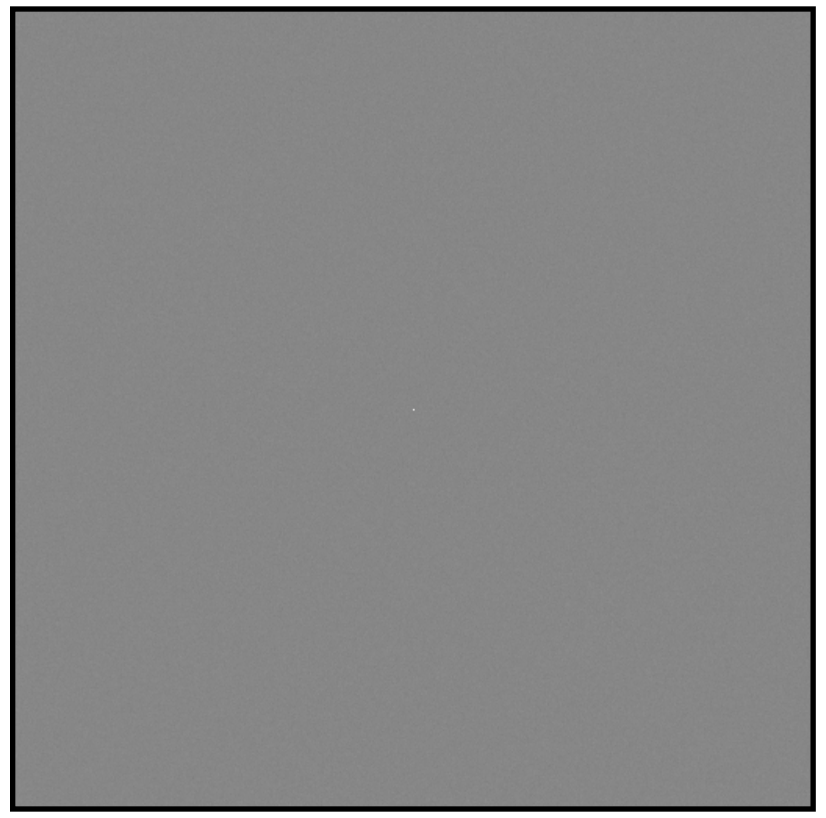
Point Samples' Expected Power Spectra



Random   Jitter   Poisson Disk

$$\left\langle \mathcal{P}_{P_N}(\nu) \right\rangle = \left\langle \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2 \right\rangle$$

Point Samples' Expected Power Spectra



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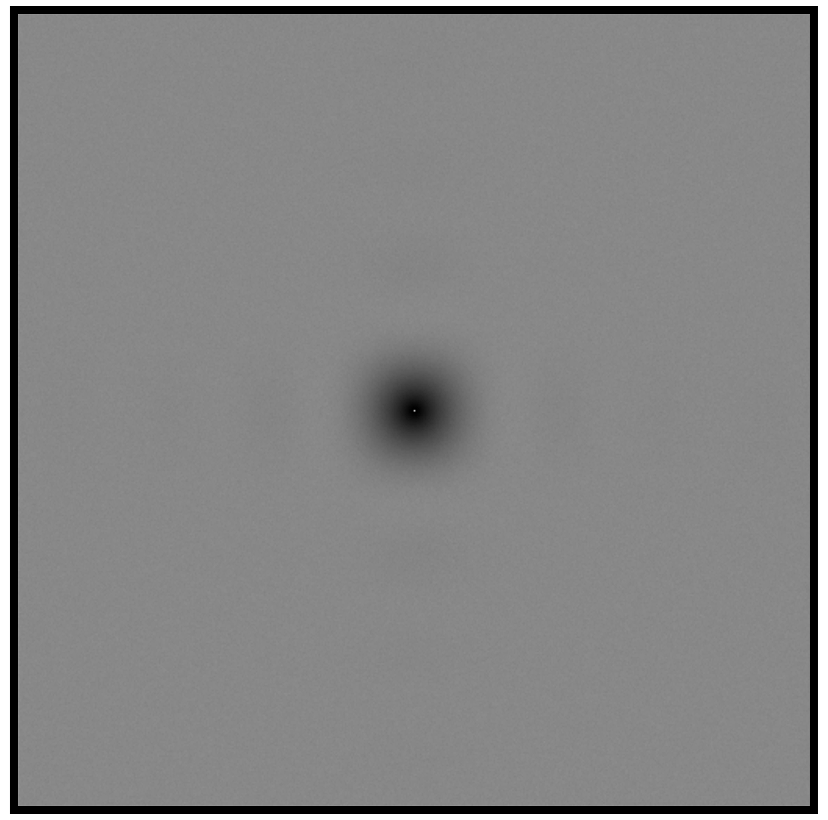

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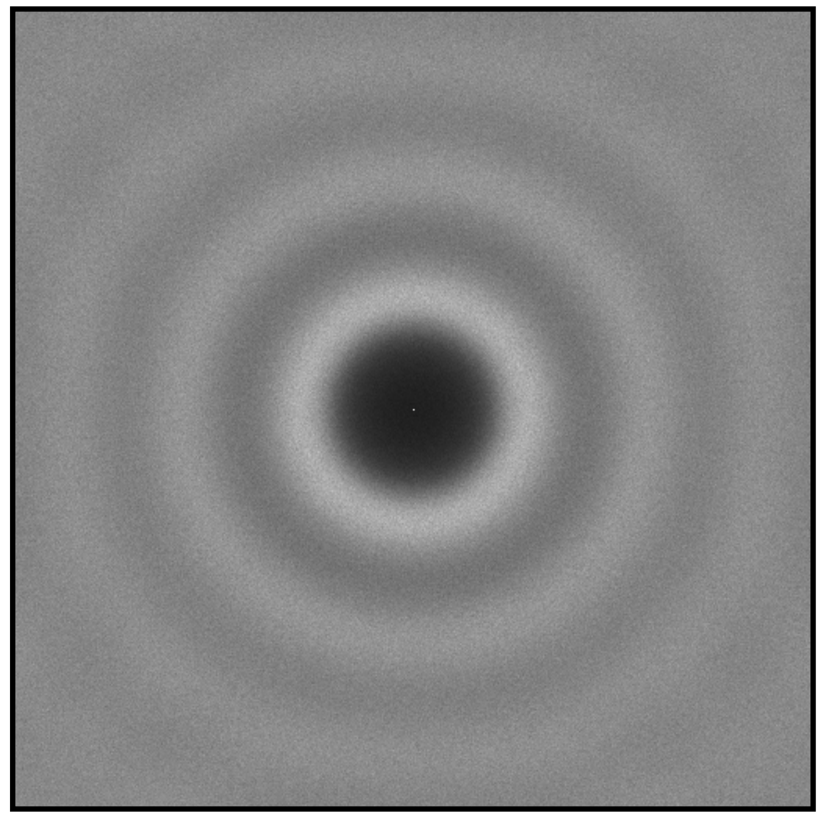

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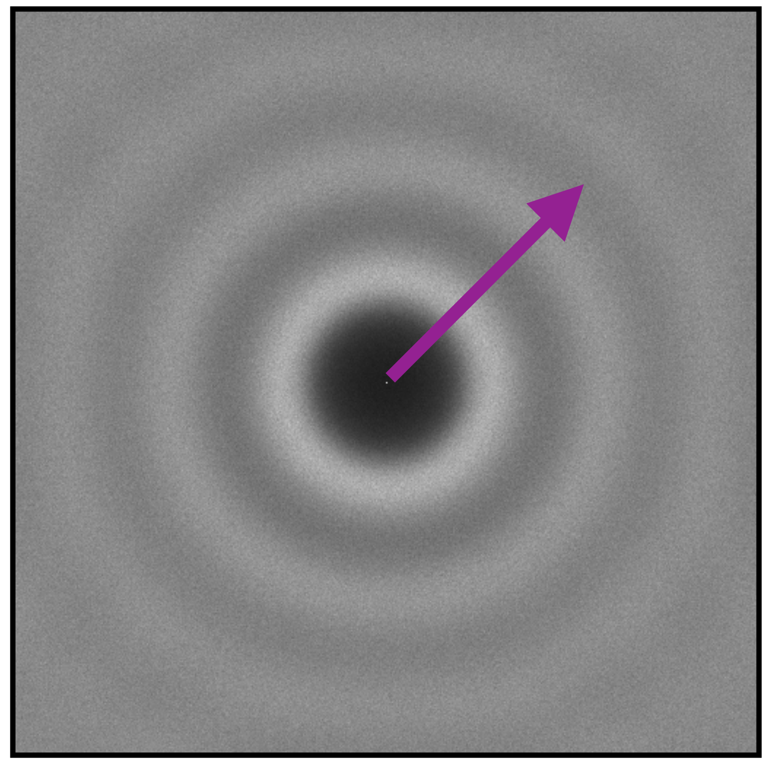
$$\left\langle \mathcal{P}_{P_N}(\nu) \right\rangle = \left\langle \left| \frac{1}{N} \sum_{k=1}^{N} e^{-i2\pi\nu\cdot\vec{x}_k} \right|^2 \right\rangle$$

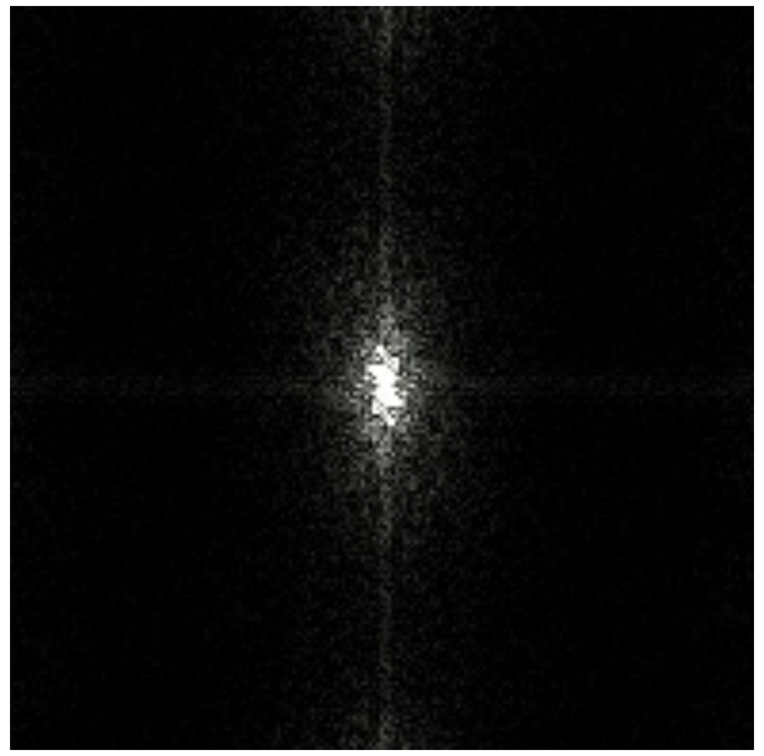
Variance Formulation in d-Dimensions

$\langle \mathcal{P}_{P_N}(\nu) \rangle$ $\qquad$ $\mathcal{P}_f(\nu)$

$\mathrm{Var}(I_N) = \int_\Omega$

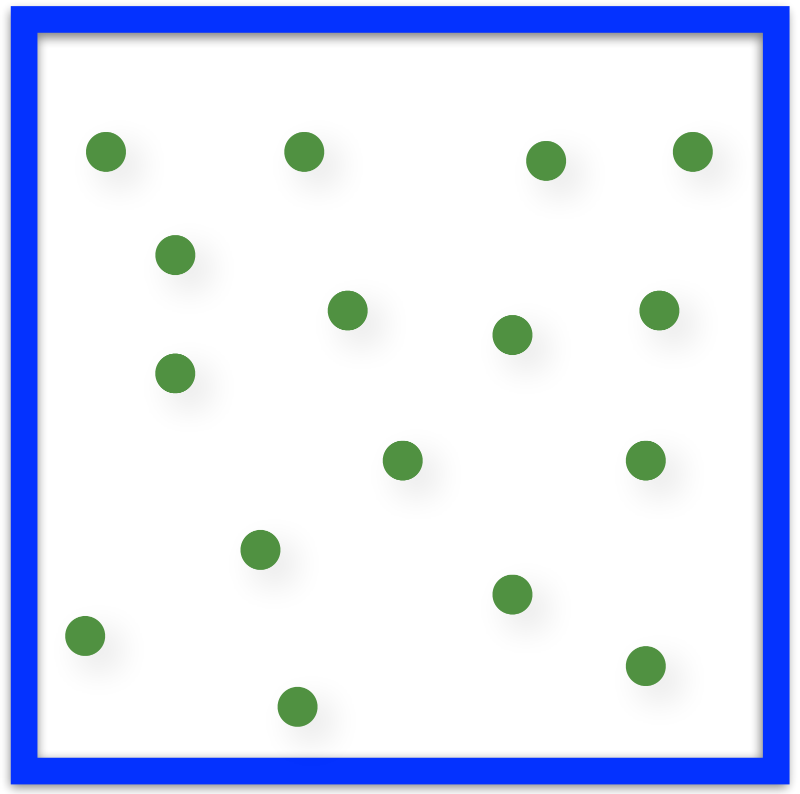


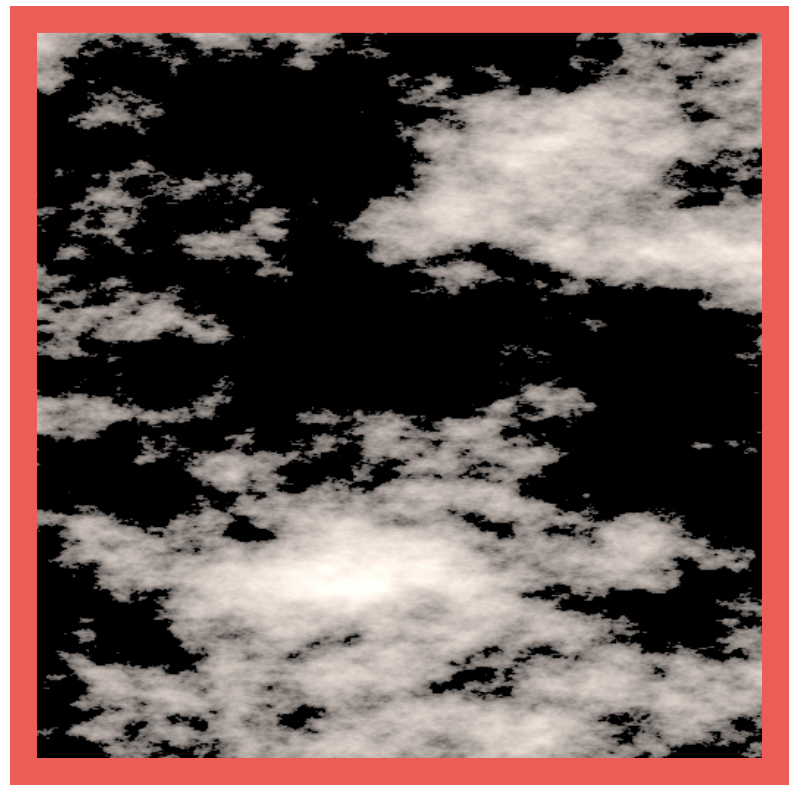
$\times$



$d\nu$

$P_N(\vec{x})$ $\qquad$ $f(\vec{x})$





Poisson Disk Samples

Fredo Durand [2011]

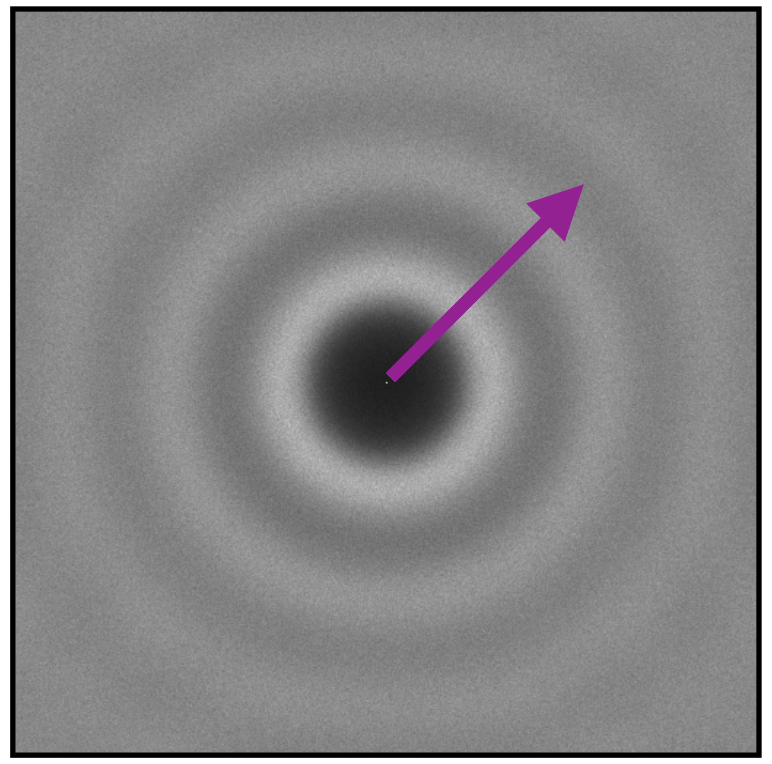
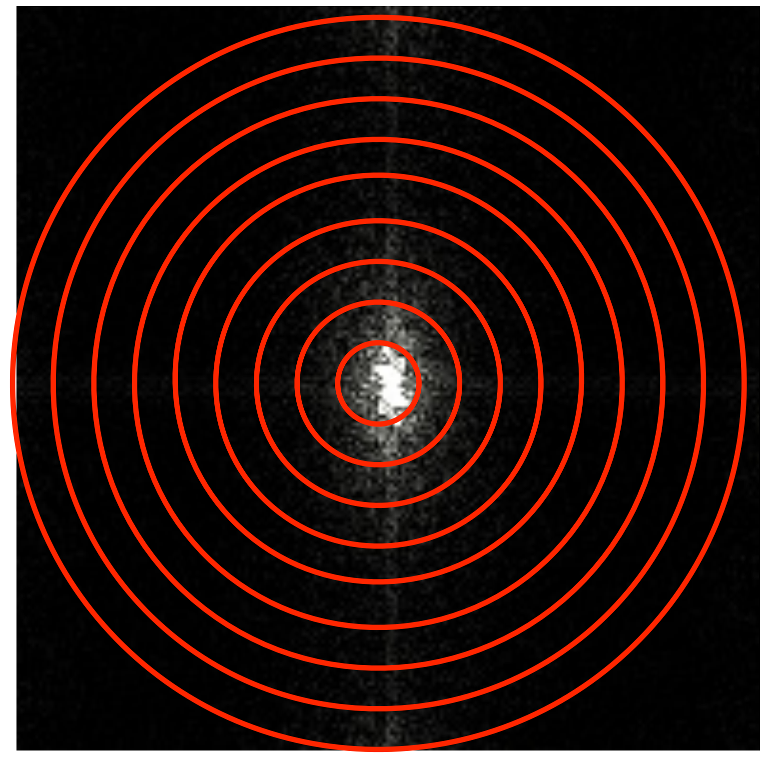
Subr & Kautz [2013]

Pilleboue et al. [2015]

Helsinki, Finland, June 19–21

Variance Formulation in d-Dimensions

$\langle \mathcal{P}_{P_N}(\nu) \rangle$ $\quad$ $\mathcal{P}_f(\nu)$

$\mathrm{Var}(I_N) = \int_\Omega$  $\times$  $d\nu$

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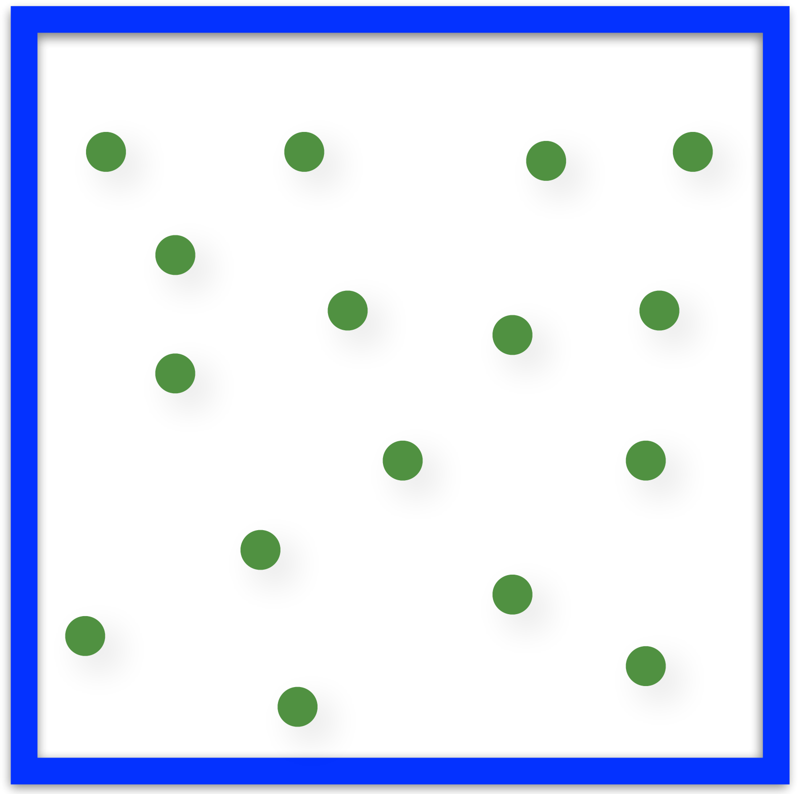

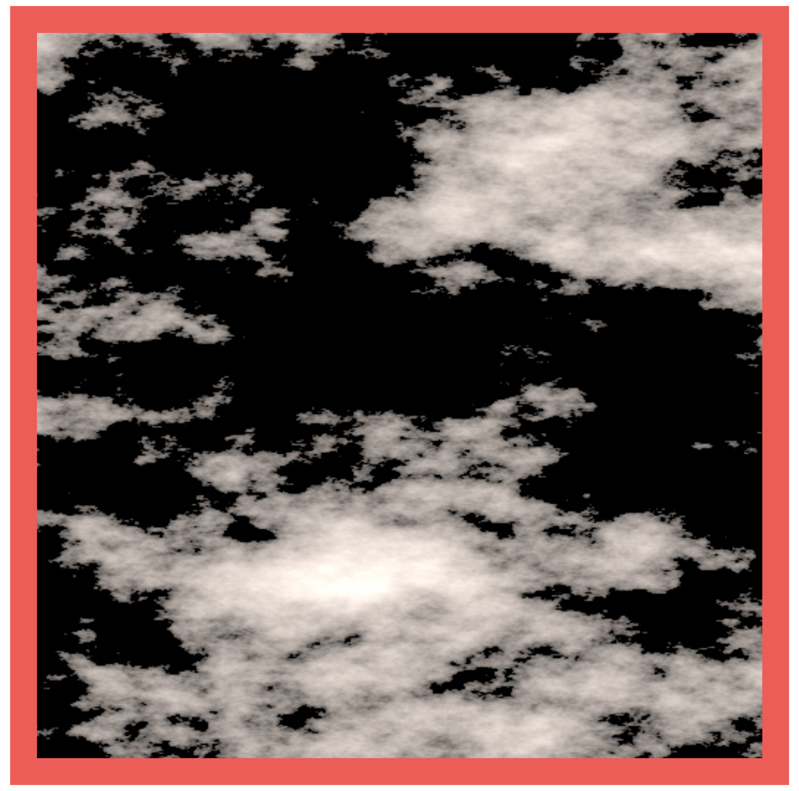

Poisson Disk Samples

Pilleboue et al. [2015]

Variance Formulation in d-Dimensions

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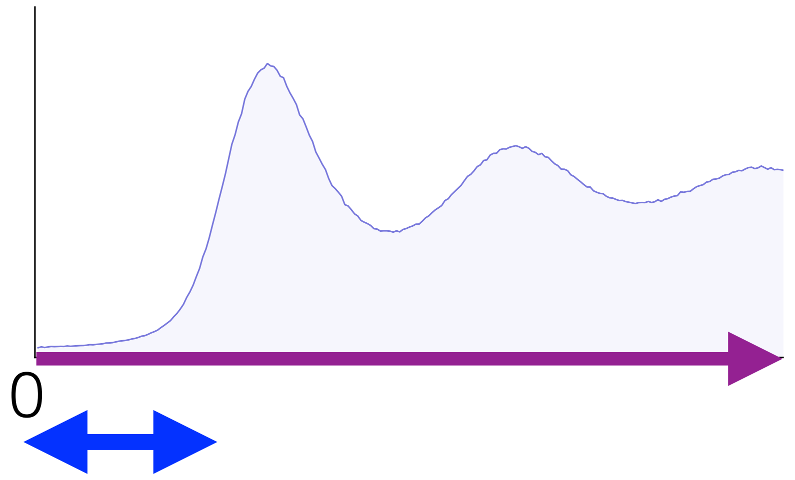
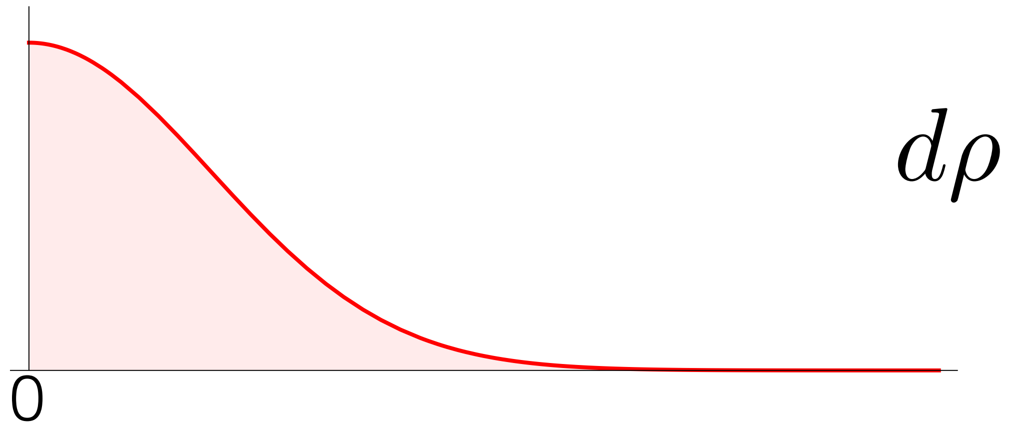
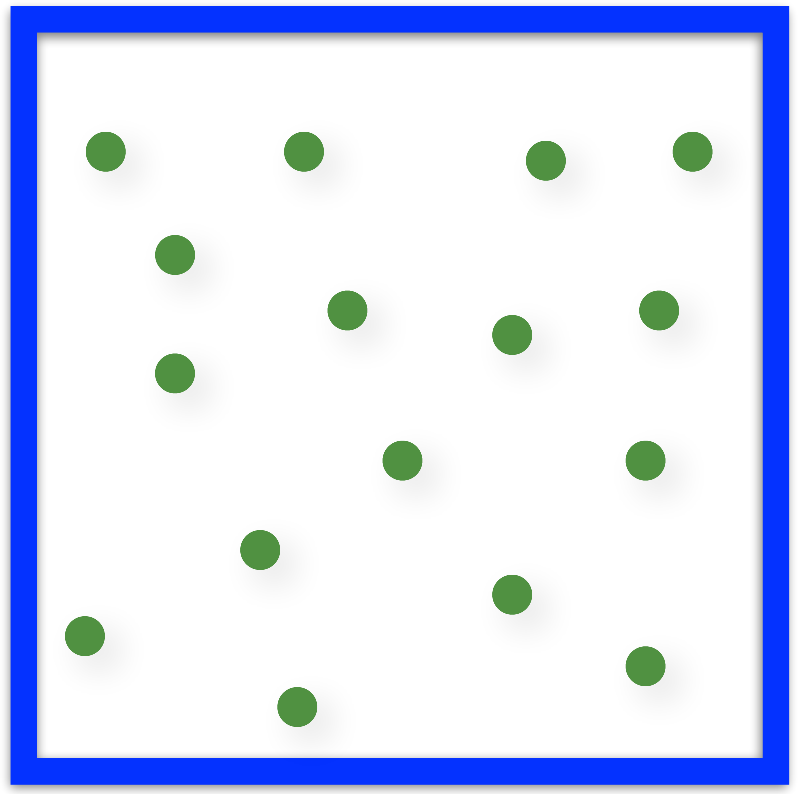
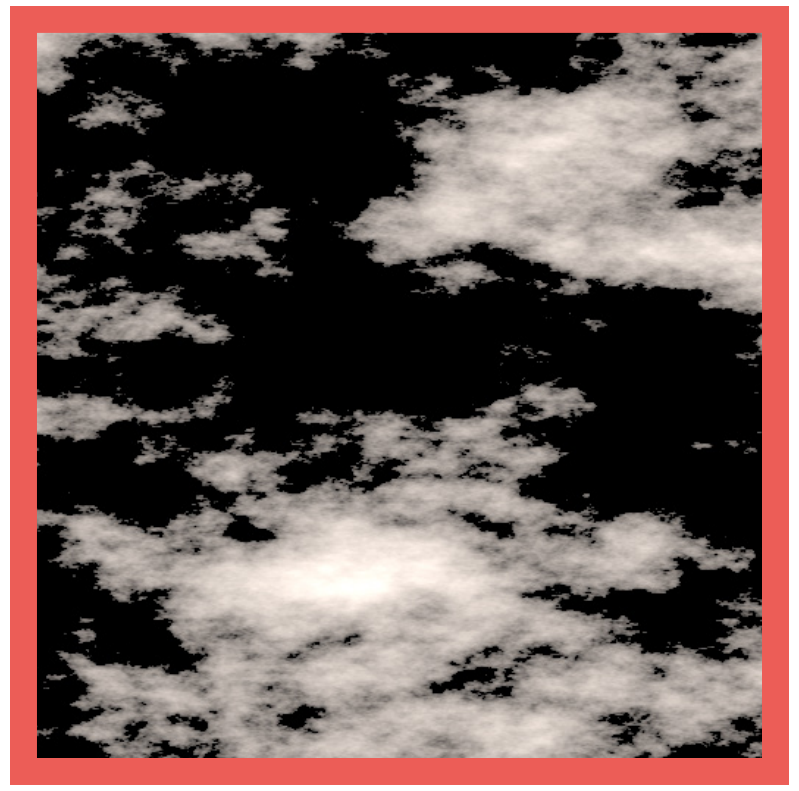
$\mathrm{Var}(I_N) = \int_0^\infty$  $\times$  $d\rho$

$f(\vec{x})$



Poisson Disk Samples



Pilleboue et al. [2015]

Expected Power Spectra:
Line and Segment Samples

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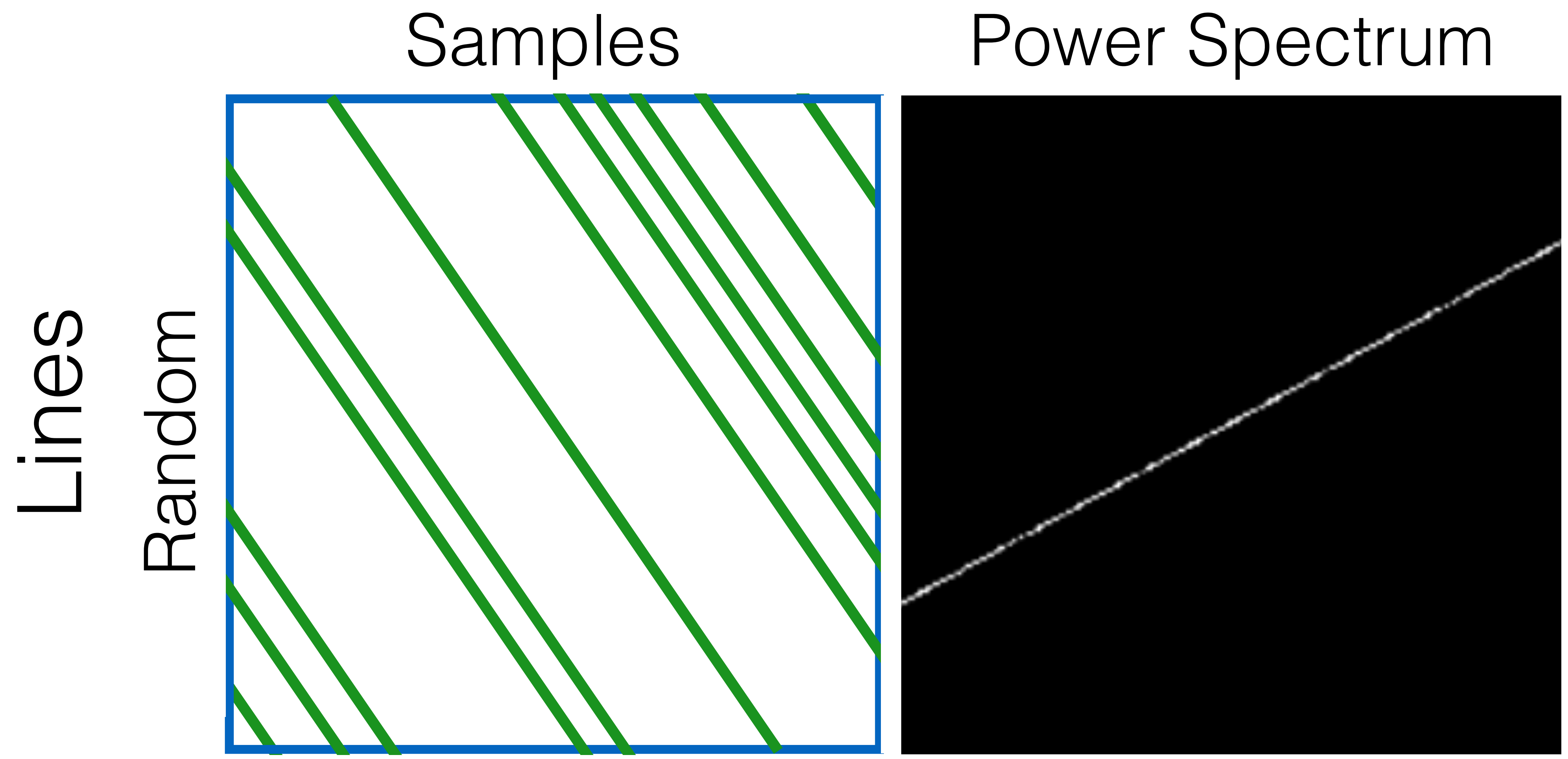
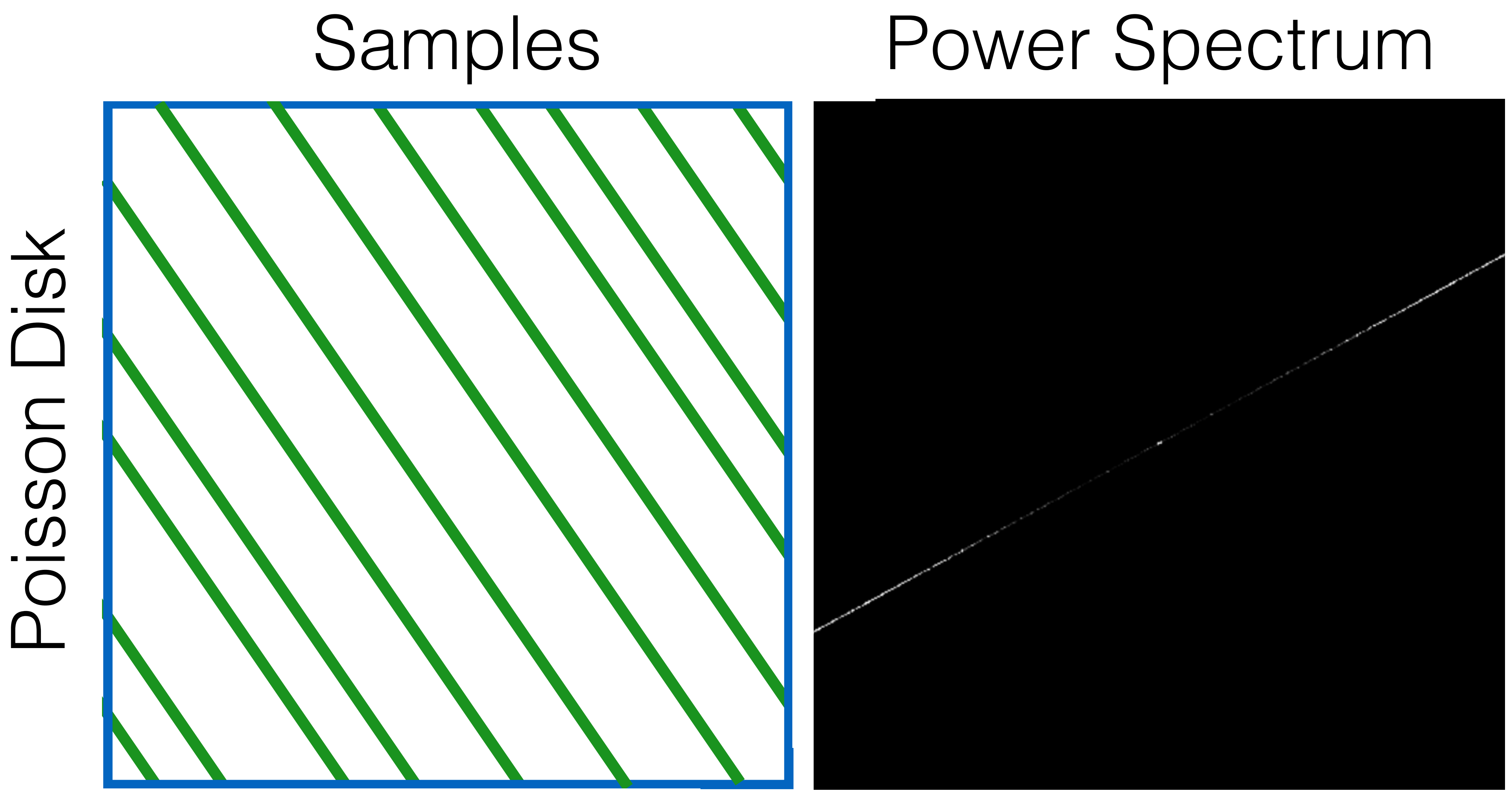
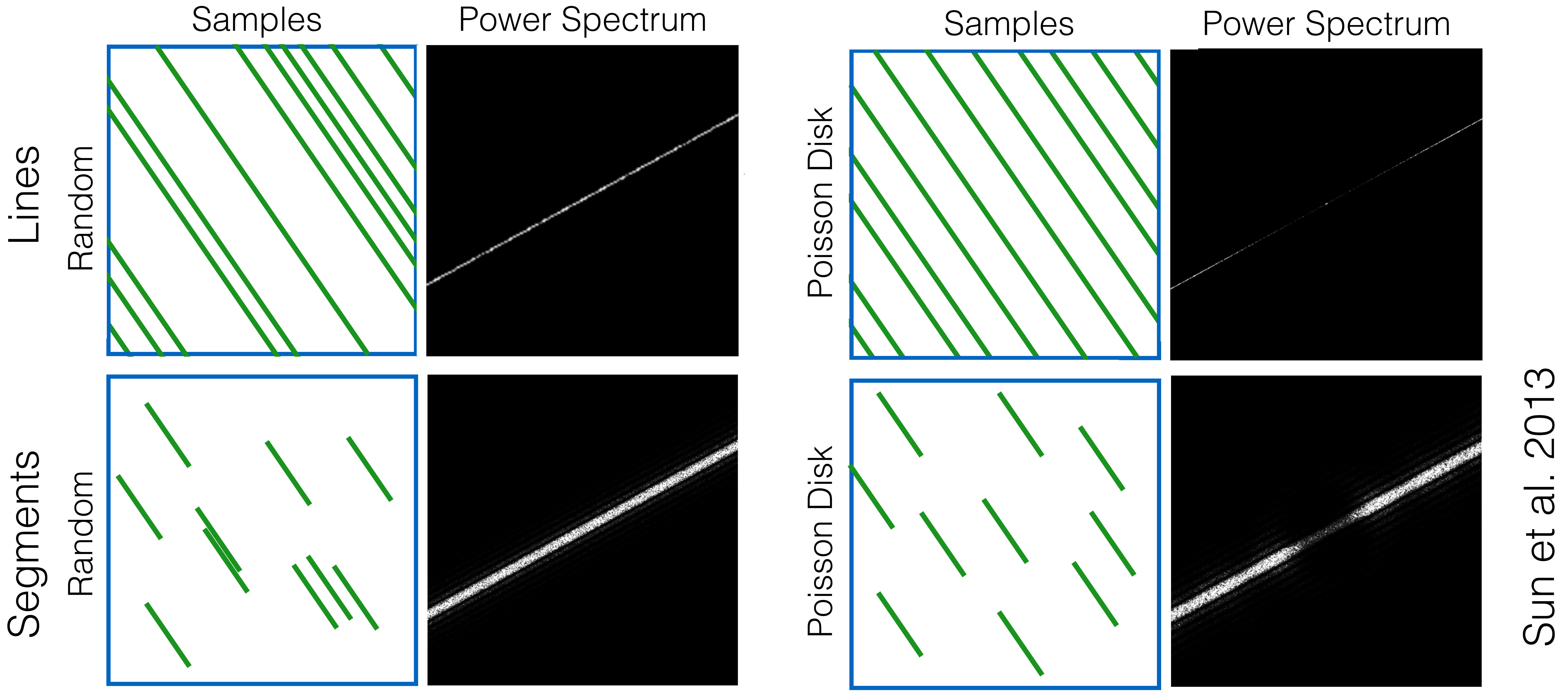




Sun et al. 2013

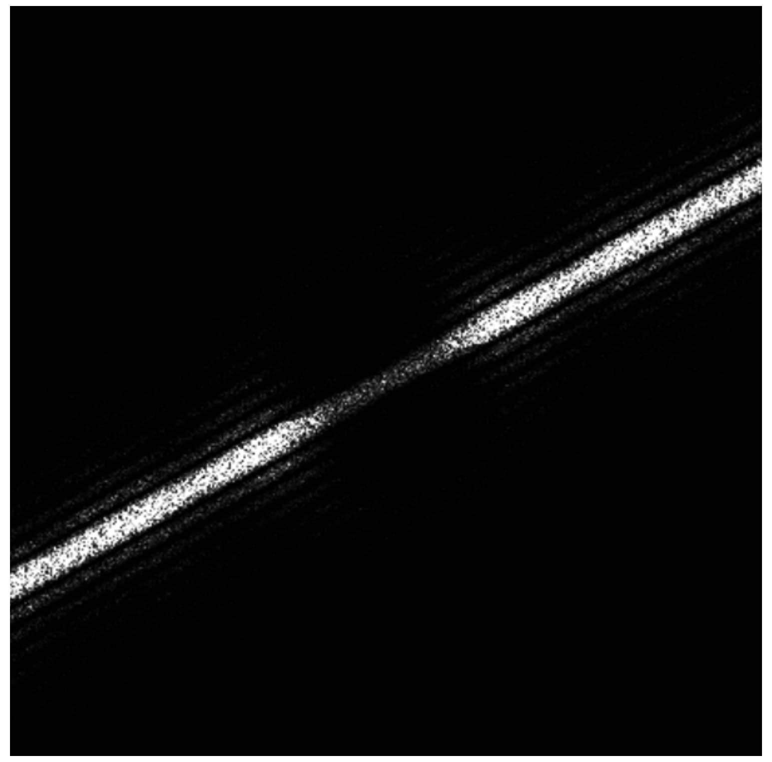
Expected Power Spectra: Line and Segment Samples

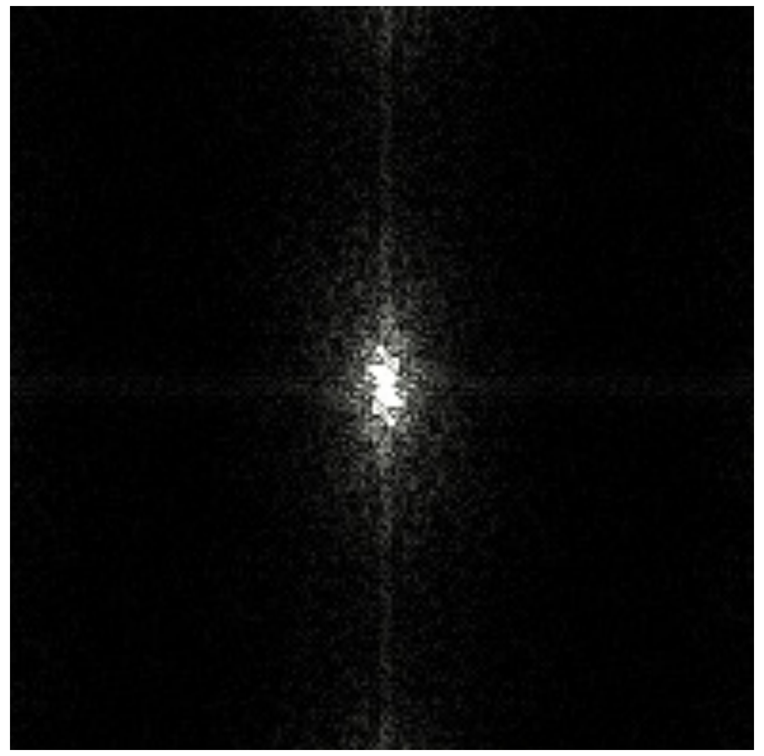


Variance formulation for Line Samples

Segment Sampling

$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{L_N}(\nu) \rangle \times \mathcal{P}_f(\nu) \, d\nu$$

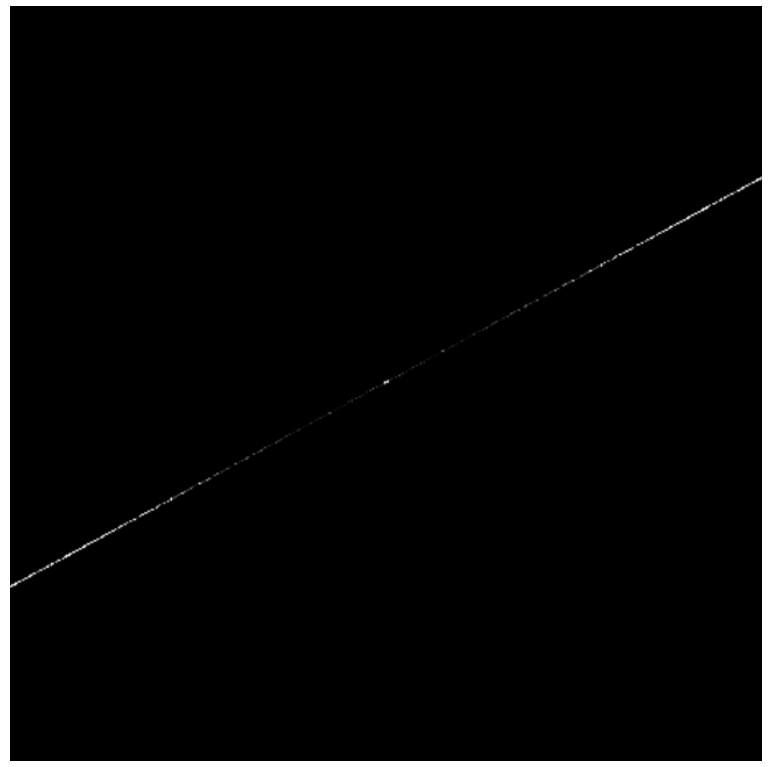


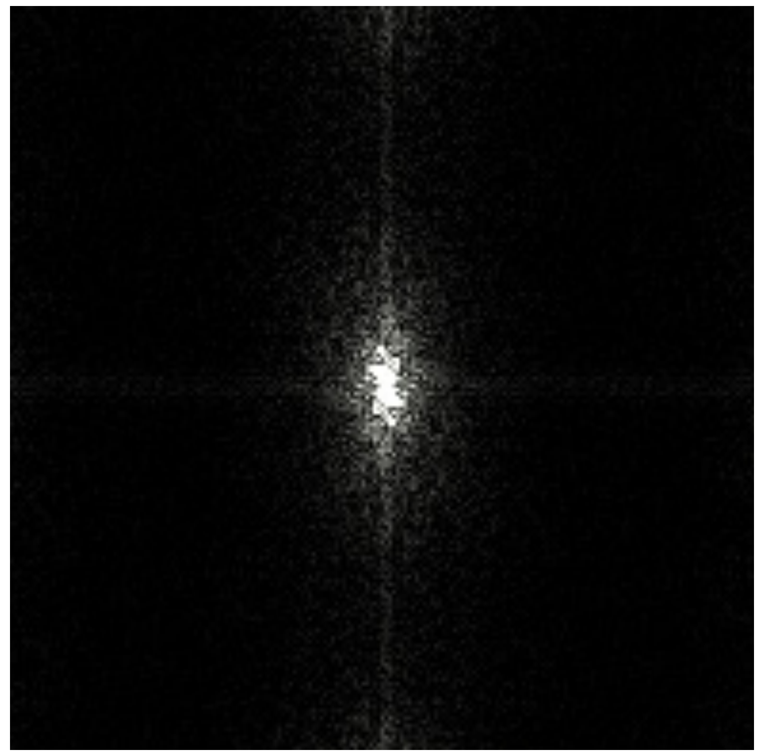


Poisson Disk offsets

Line Sampling

$$\mathrm{Var}(I_N) = \int_\Omega \langle \mathcal{P}_{S_N}(\nu) \rangle \times \mathcal{P}_f(\nu) \, d\nu$$





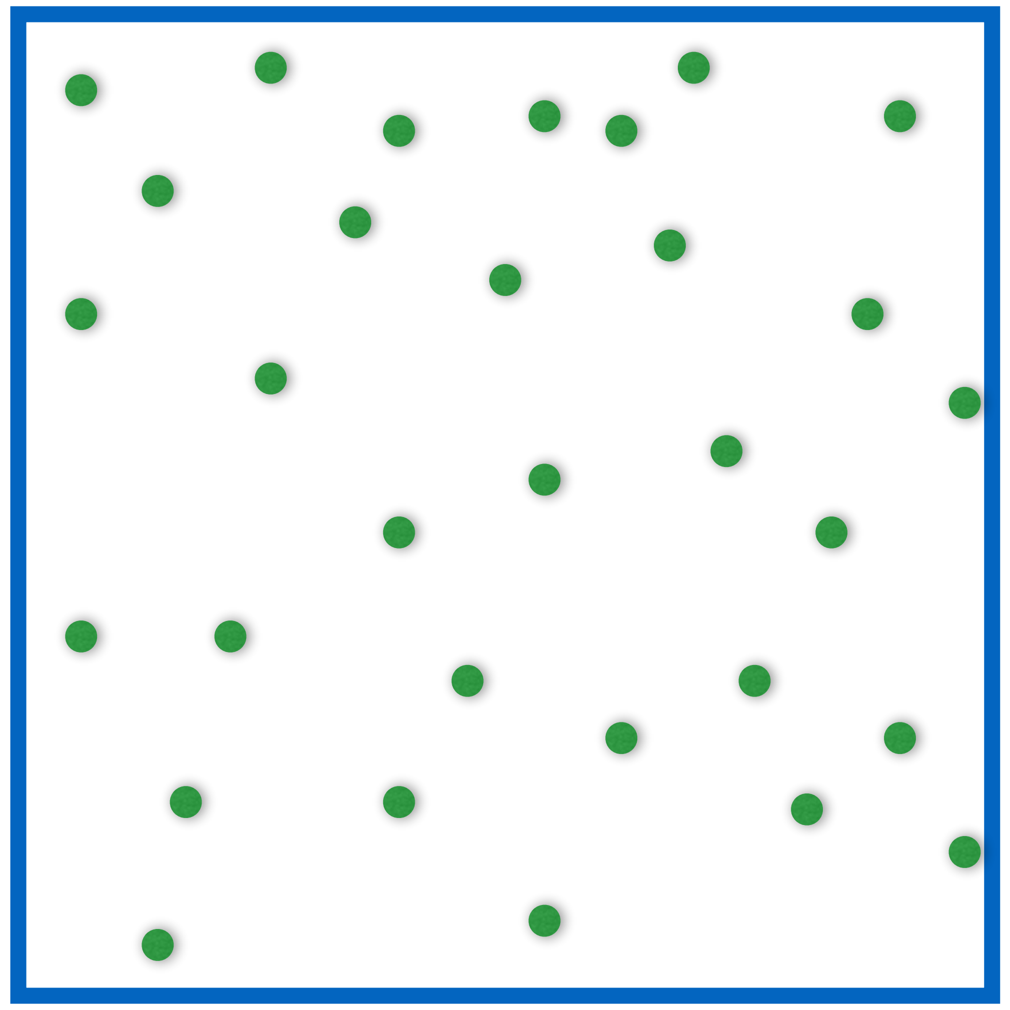
How can we perform convergence analysis for Segment and Line Sampling ?

- Monte Carlo Estimators for Point, Segment & Line Samples
- Variance Formulation for Point Samples
- Variance Formulation for Segment and Line Samples
- Experimental Verification

Monte Carlo Estimator

$$I_N = \frac{1}{N}\sum_{k=1}^{N} f(\vec{x}_k) \;=\int_0^1 \frac{1}{N}\sum_{k=1}^{N} f(\vec{x})\delta(\vec{x}-\vec{x}_k)d\vec{x} \;=\int_0^1 P_N(\vec{x})f(\vec{x})d\vec{x}$$

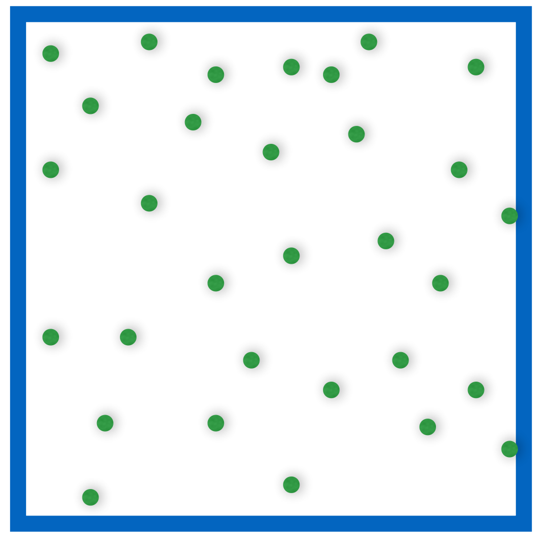
$$P_N(\vec{x}) = \frac{1}{N}\sum_{k=1}^{N}\delta(\vec{x}-\vec{x}_k)$$

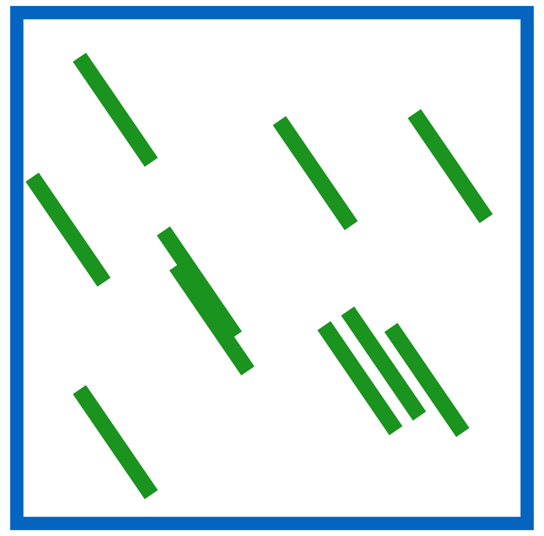


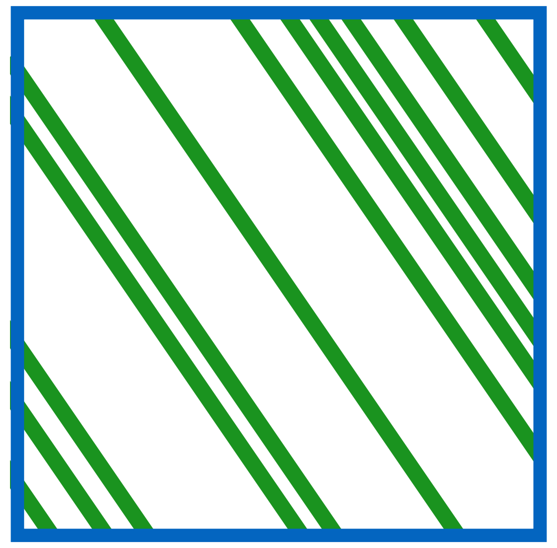
Fredo Durand [2011]

Monte Carlo Estimator in Frequency Domain

Spatial Domain

Points $I_N = \int_0^1 P_N(\vec{x}) f(\vec{x}) d\vec{x}$ $P_N(\vec{x}) =$ 

Segments $I_N = \int_0^1 S_N(\vec{x}) f(\vec{x}) d\vec{x}$ $S_N(\vec{x}) =$ 

Lines $I_N = \int_0^1 L_N(\vec{x}) f(\vec{x}) d\vec{x}$ $L_N(\vec{x}) =$ 

Monte Carlo Estimator in Frequency Domain

	Spatial Domain	Fourier Domain
Points	$I_N = \int_0^1 P_N(\vec{x}) f(\vec{x}) d\vec{x}$	$I_N = \int_\Omega \mathcal{F}_{P_N}(\nu)\mathcal{F}_f(\nu) d\nu$
Segments	$I_N = \int_0^1 S_N(\vec{x}) f(\vec{x}) d\vec{x}$	$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu) d\nu$
Lines	$I_N = \int_0^1 L_N(\vec{x}) f(\vec{x}) d\vec{x}$	$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu) d\nu$

Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Lines

$$I_N = \int_{\Omega} \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$$\mathcal{K}_S(\lambda, \nu_k^{||}) = \lambda sinc(\lambda\nu_k^{||})$$

Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Monte Carlo Estimator in Frequency Domain

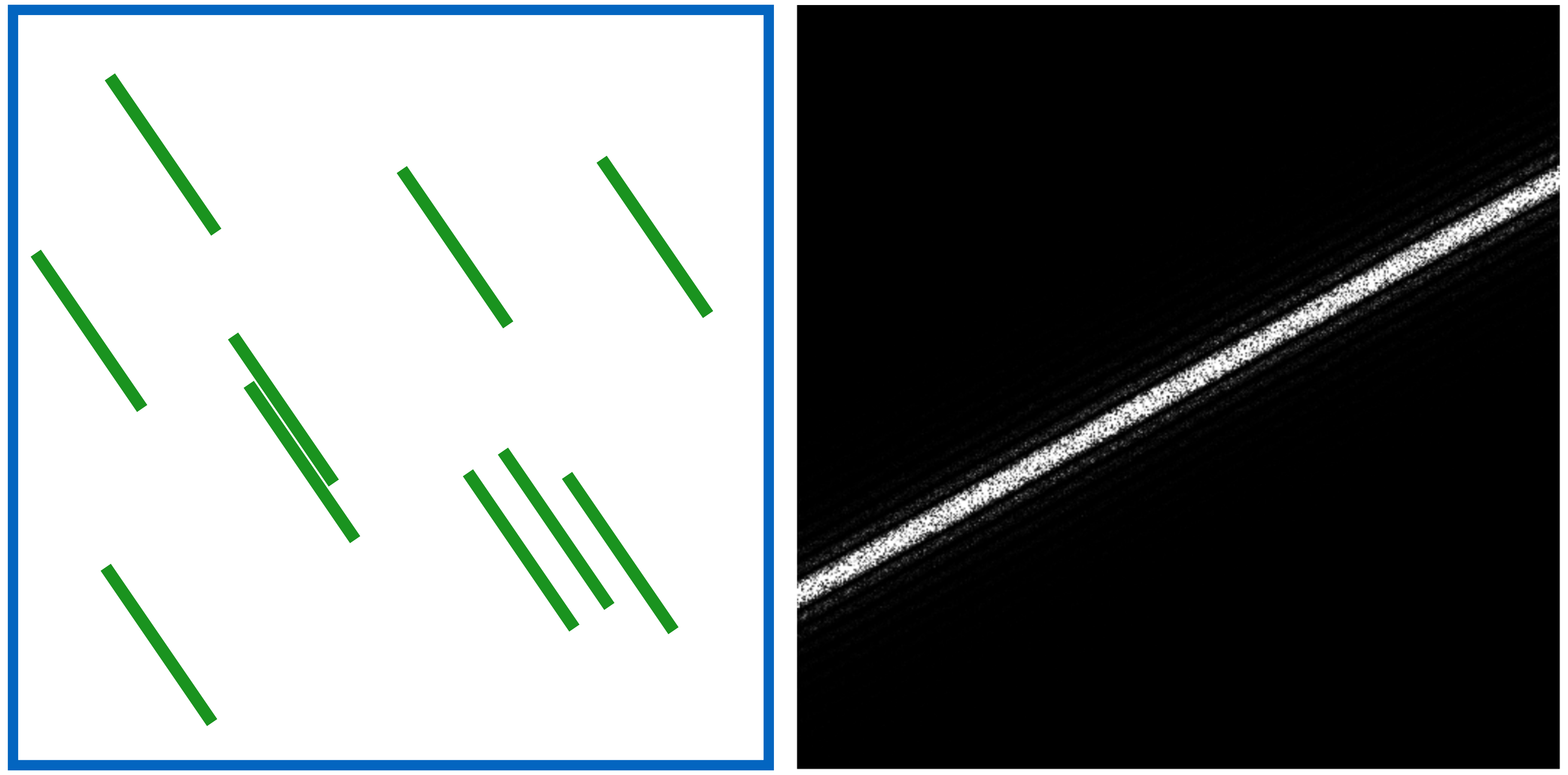
Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$|\mathcal{F}_{S_N}(\nu)|^2$

Random



Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Monte Carlo Estimator in Frequency Domain

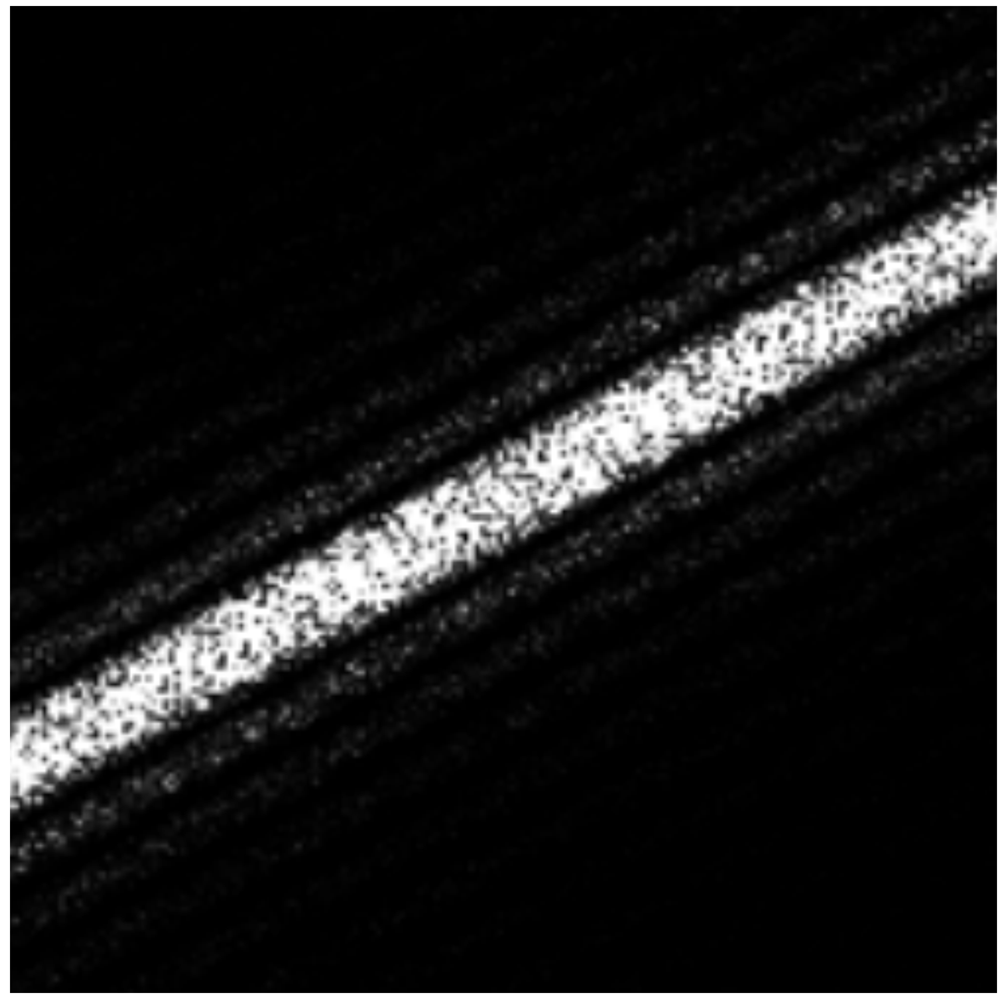
Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

Random

$|\mathcal{F}_{S_N}(\nu)|^2$



Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Sun et al. 2013

Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

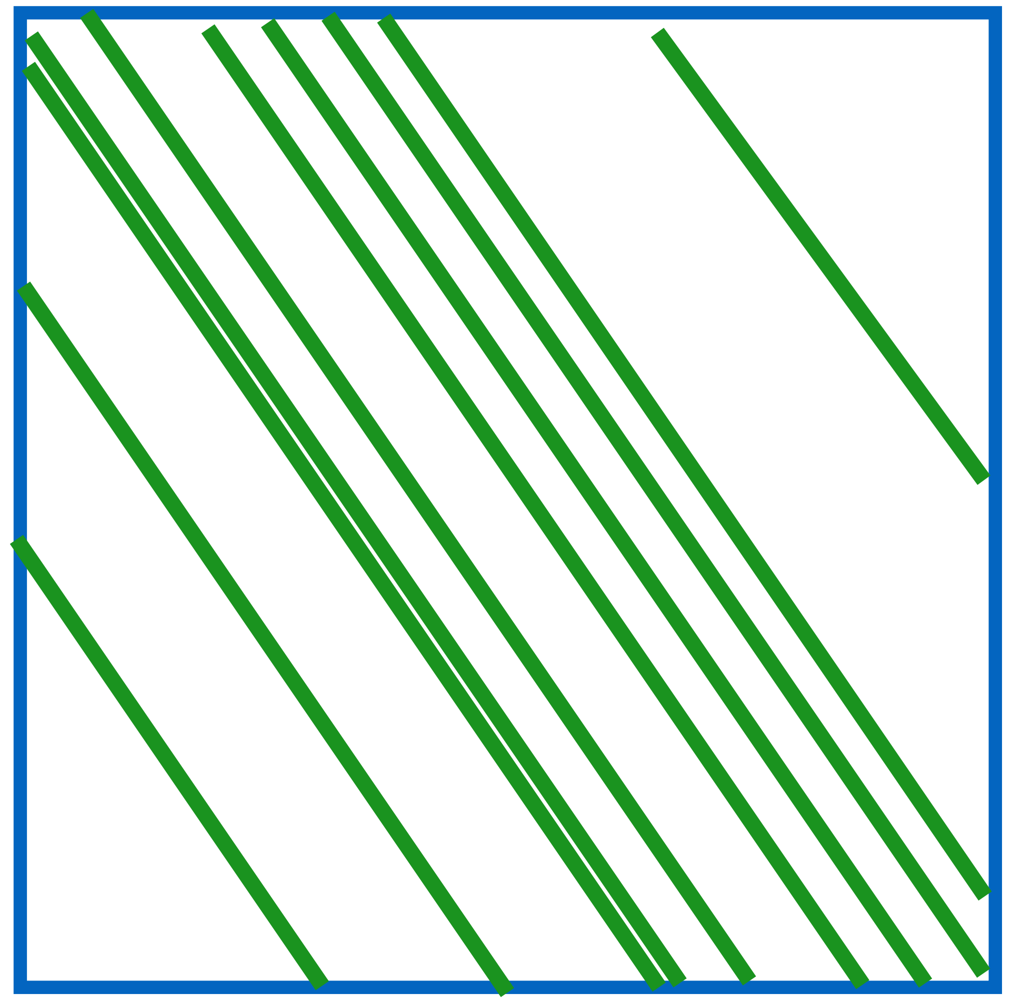
$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

Lines

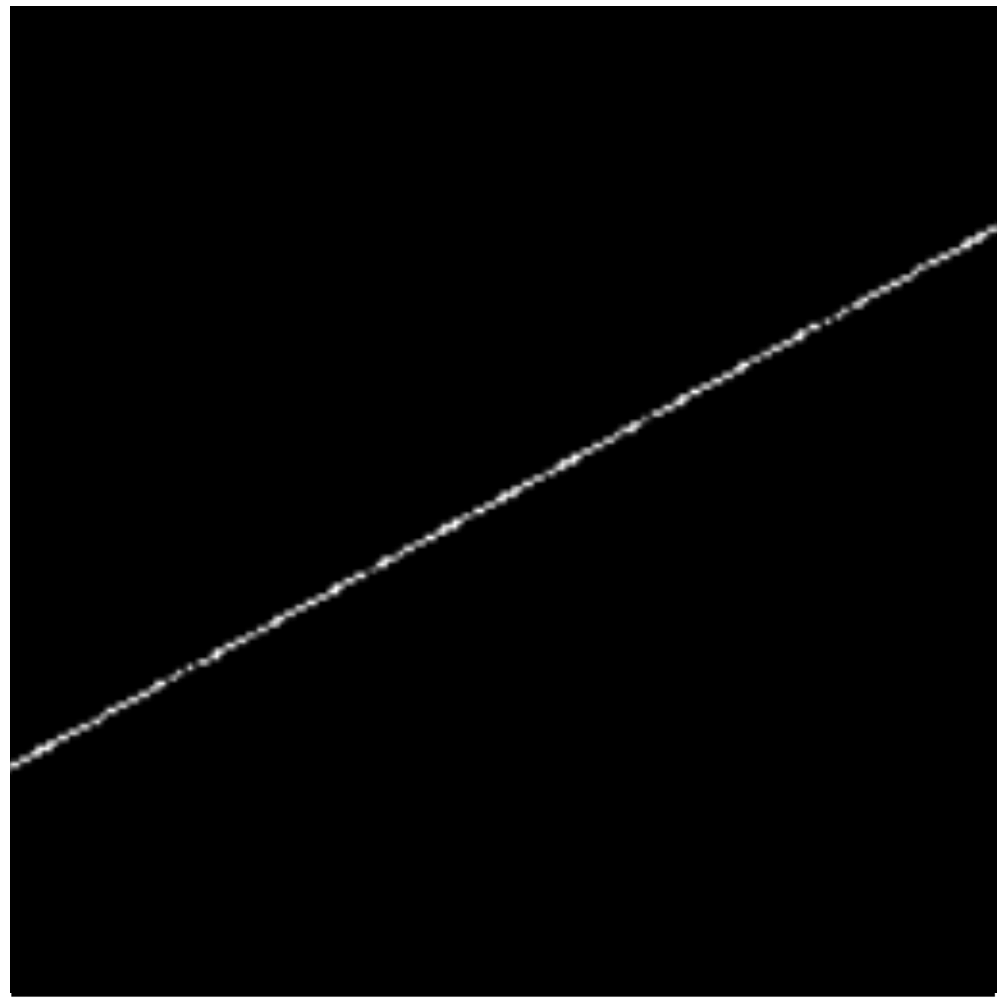
$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Segments

Random



$|\mathcal{F}_{S_N}(\nu)|^2$



Sun et al. 2013

Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

Lines

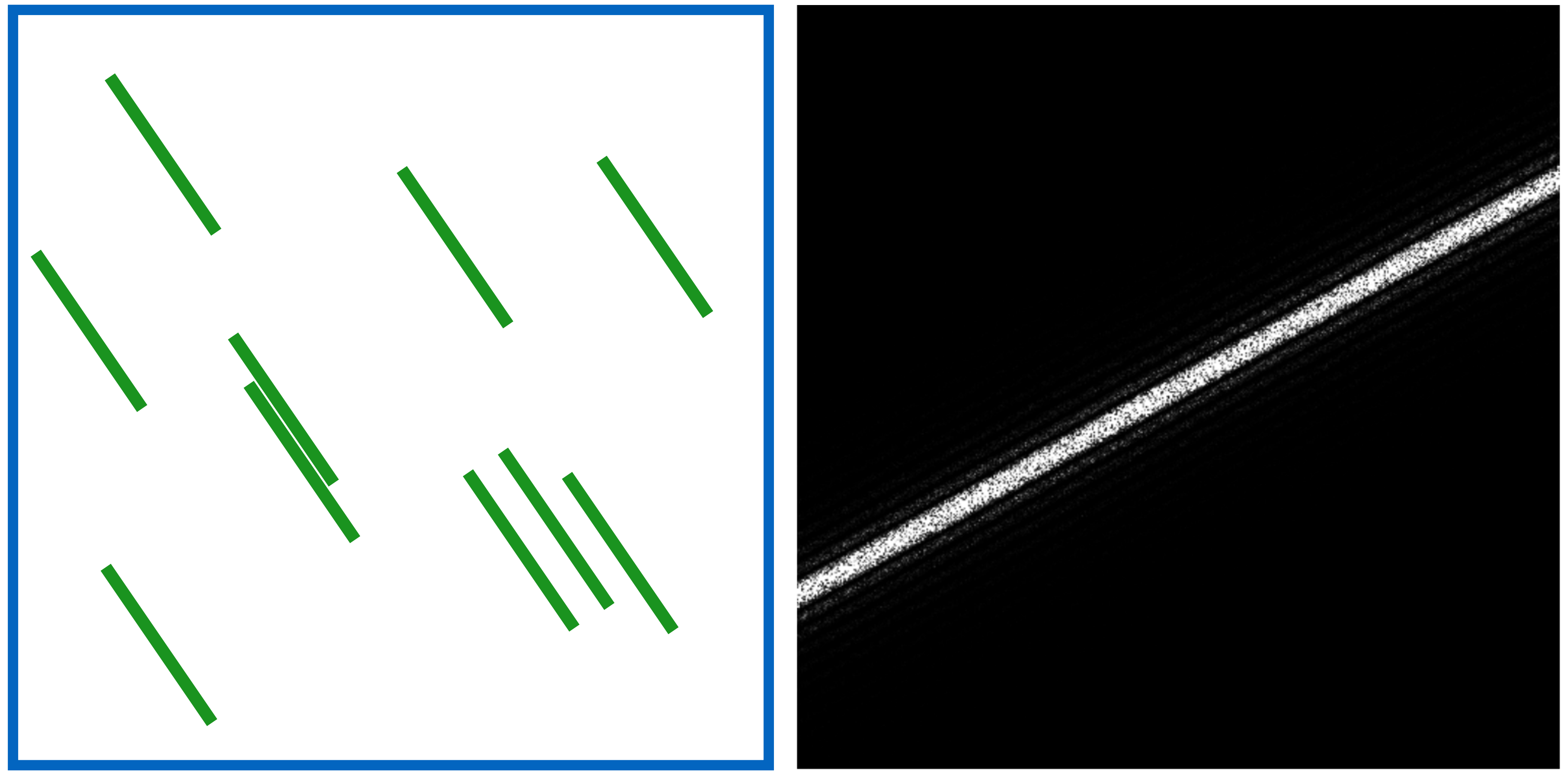
$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$$|\mathcal{F}_{S_N}(\nu)|^2$$

Segments

Random



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Monte Carlo Estimator in Frequency Domain

Segments

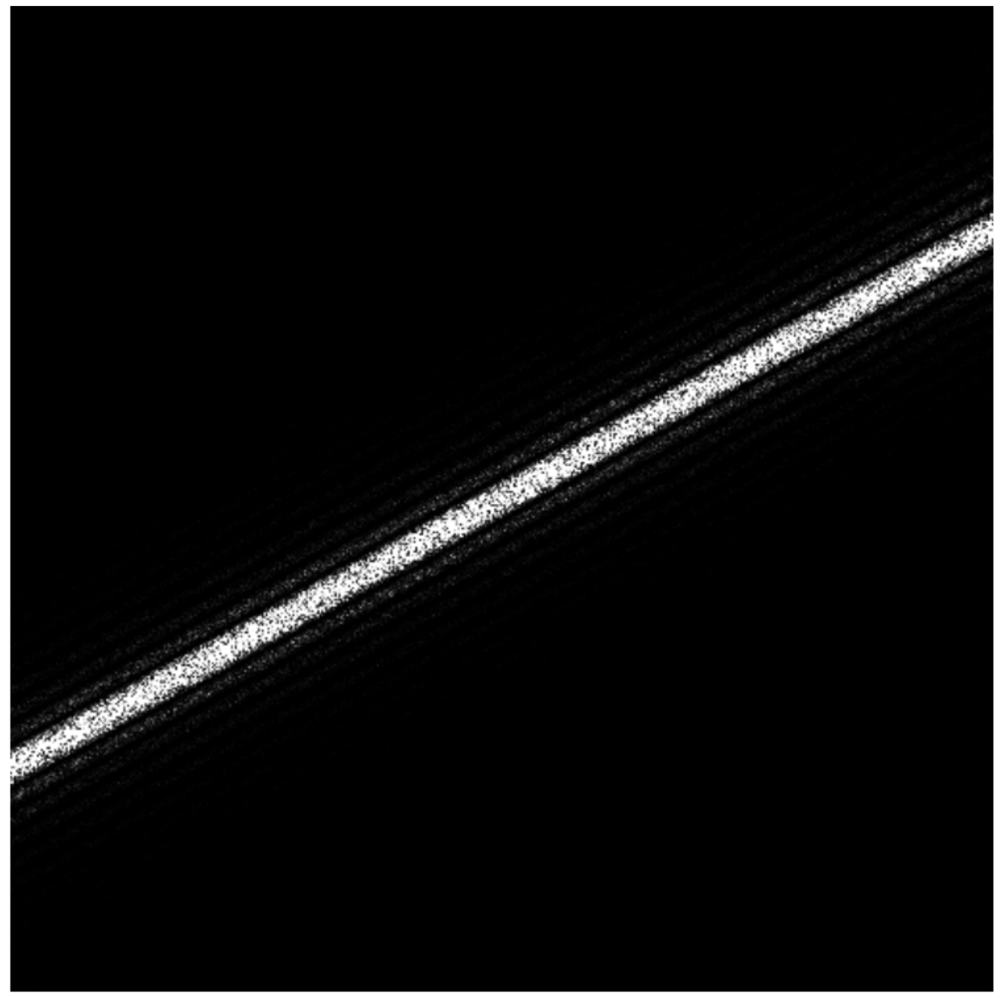
$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$$|\mathcal{F}_{S_N}(\nu)|^2$$

Segments

Random



Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{L_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp})\mathcal{K}_L(\nu_k^{||})$$

Helsinki, Finland, June 19–21

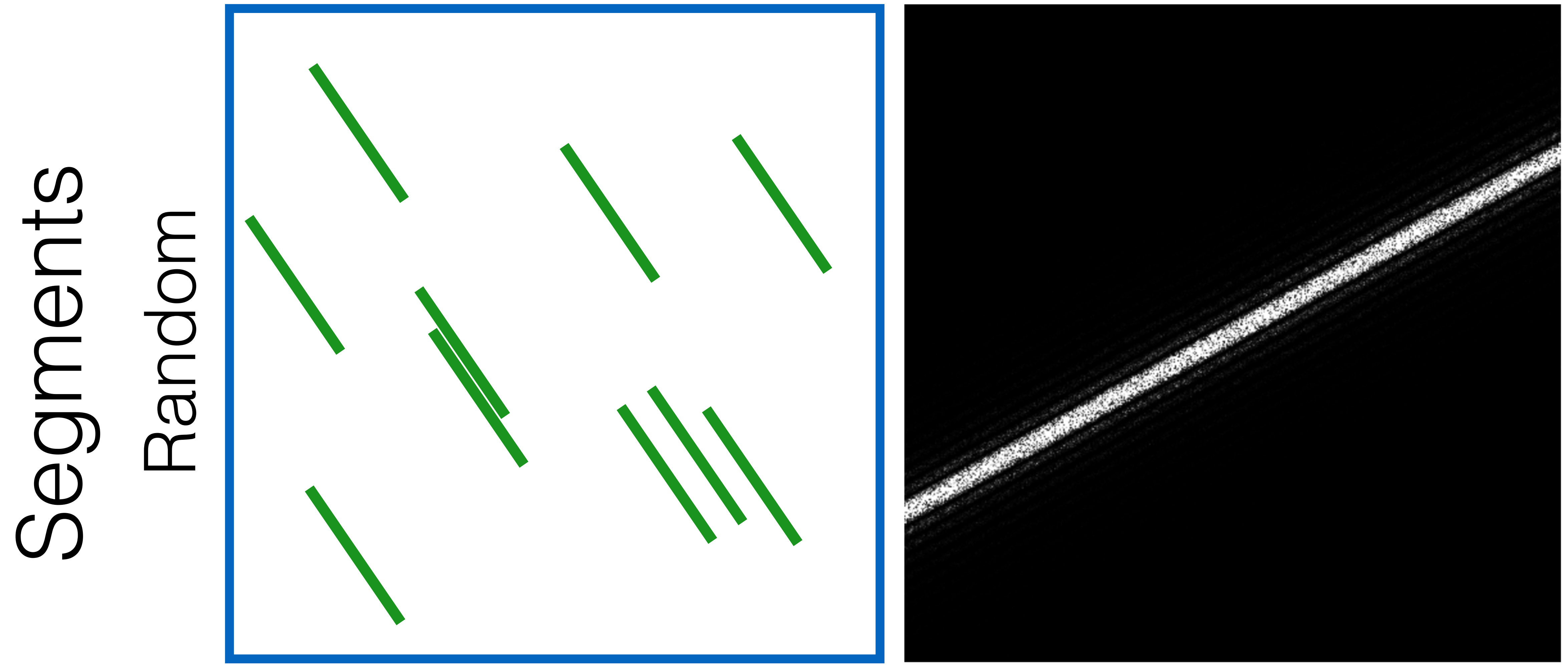
Monte Carlo Estimator in Frequency Domain

Segments

$$I_N = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$|\mathcal{F}_{S_N}(\nu)|^2$



Lines

$$I_N = \int_{\Omega} \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{L_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp})\mathcal{K}_L(\nu_k^{||})$$

$$\mathcal{K}_L(\nu_k^{||}) = \delta(\nu_k^{||})$$

Monte Carlo Estimator in Frequency Domain

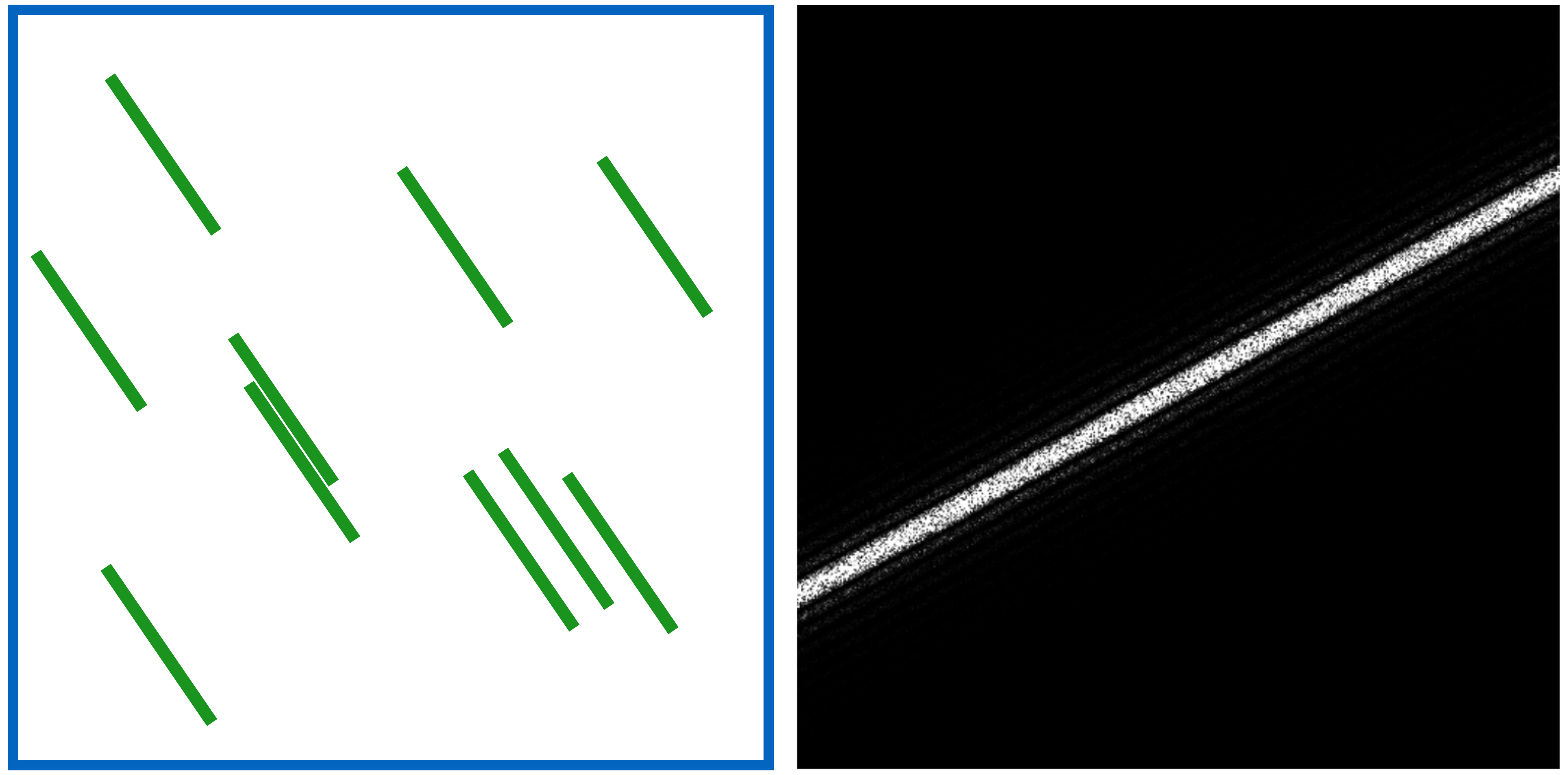
Segments

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

$|\mathcal{F}_{S_N}(\nu)|^2$

Segments
Random

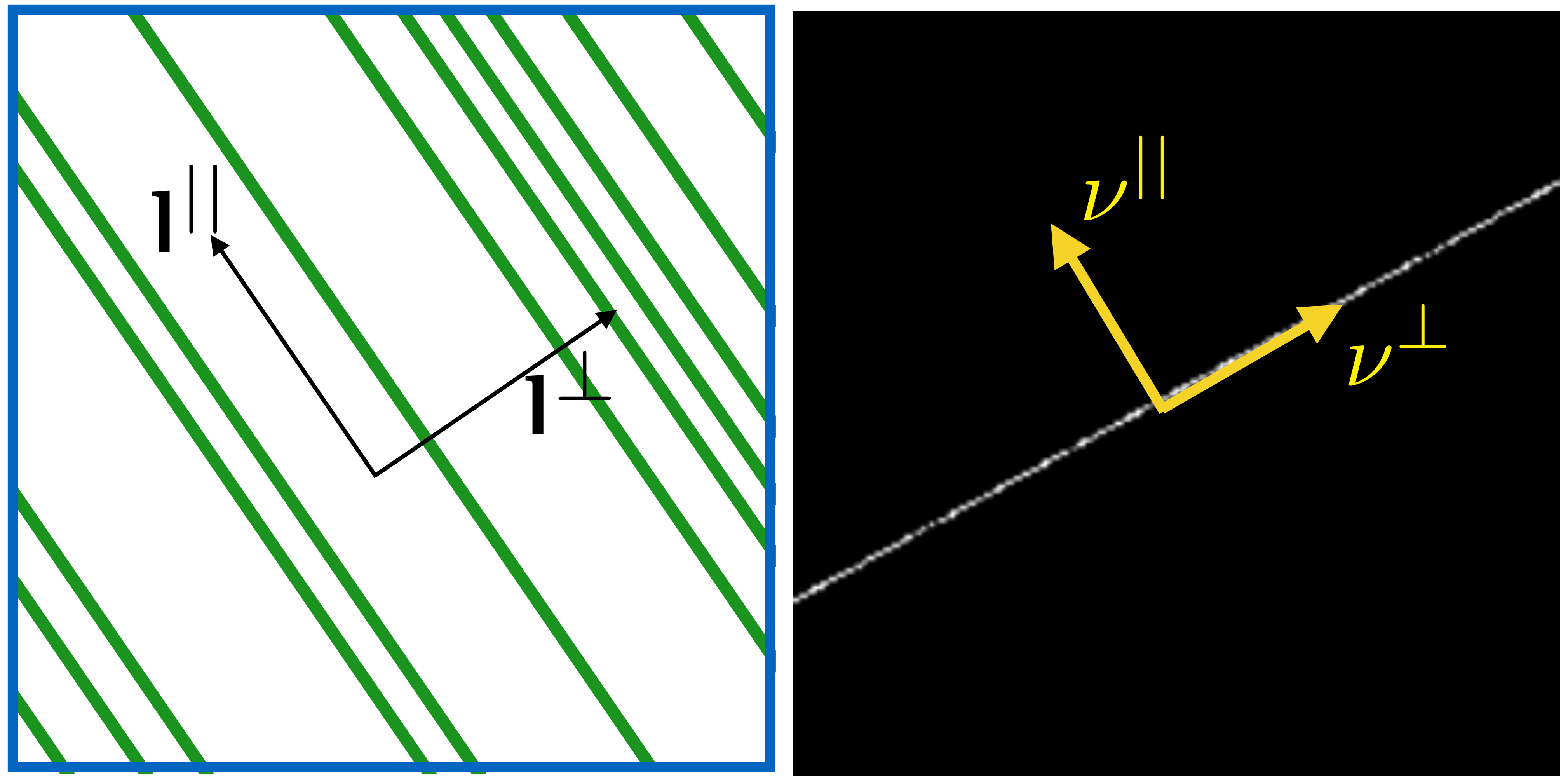


Lines

$$I_N = \int_\Omega \mathcal{F}_{L_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{L_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k^{\perp})\mathcal{K}_L(\nu_k^{||})$$

$|\mathcal{F}_{L_N}(\nu)|^2$



Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_\Omega \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_{\Omega} \mathcal{F}_{S_N}(\nu)\mathcal{F}_f(\nu)d\nu$$

$$\mathcal{F}_{S_N}(\nu) = \frac{1}{N}\sum_{k=1}^{N}\mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})$$

Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_\Omega \frac{1}{N} \sum_{k=1}^{N} \mathcal{F}(\nu_k) \mathcal{K}_S(\lambda, \nu_k^{||}) \mathcal{F}_f(\nu) d\nu$$

Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

First Interpretation

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Second Interpretation

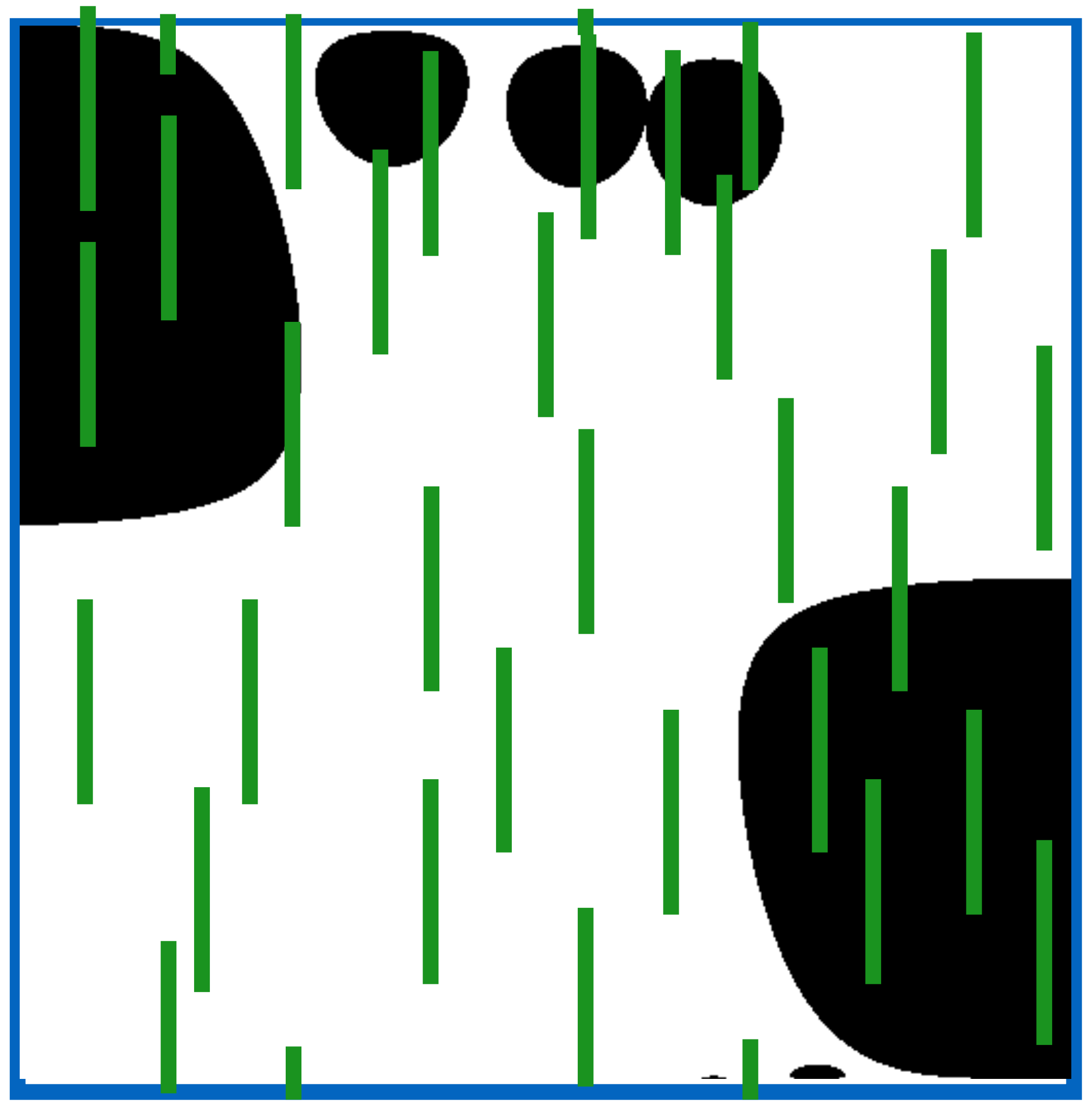
$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\,\mathcal{F}_f(\nu)d\nu$$

Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Original Integrand

Segments in d-dimensions



First Interpretation

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Second Interpretation

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Monte Carlo Estimator in Frequency Domain for Segment Samples

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

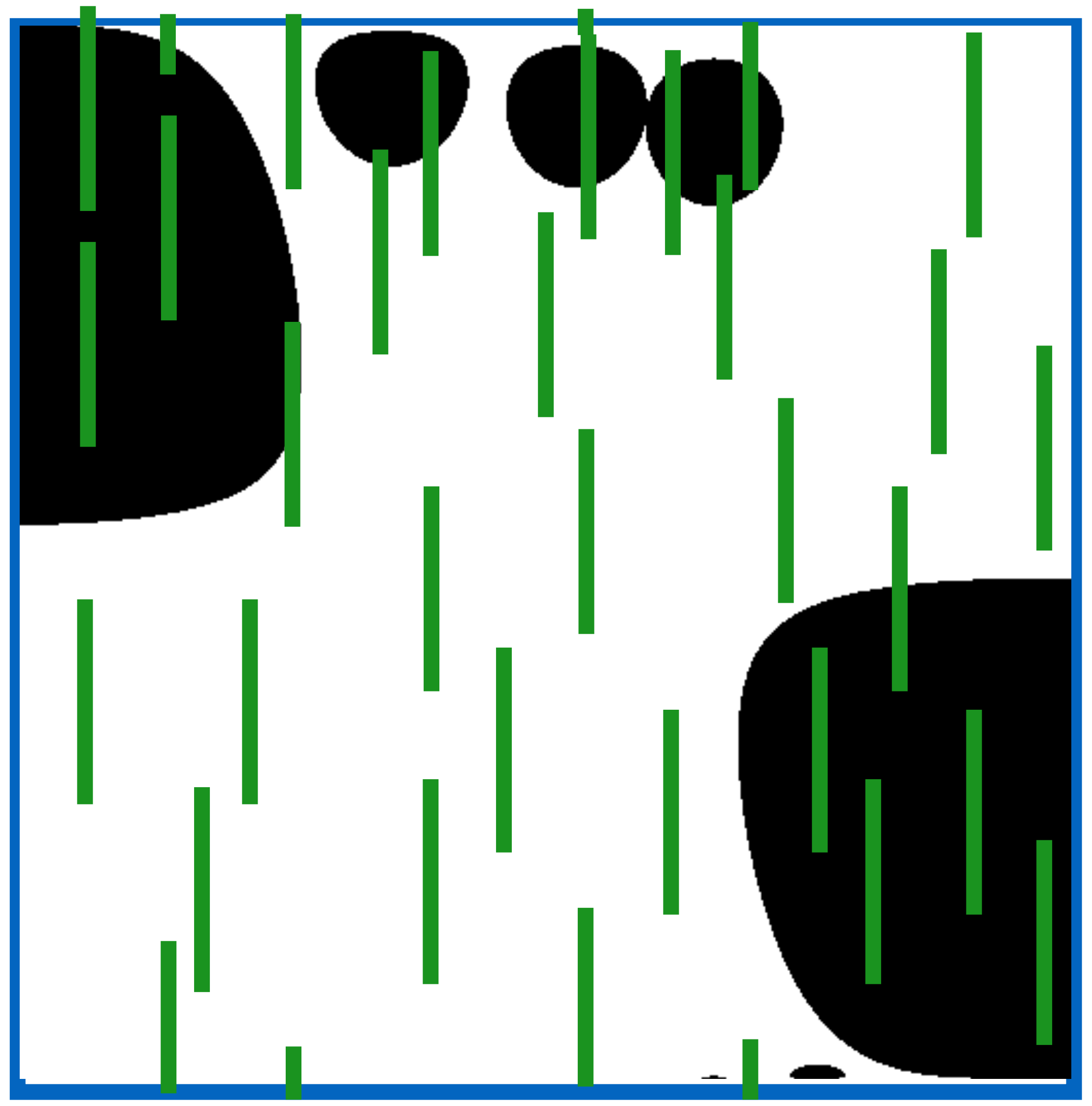
Original Integrand

Segments in d-dimensions

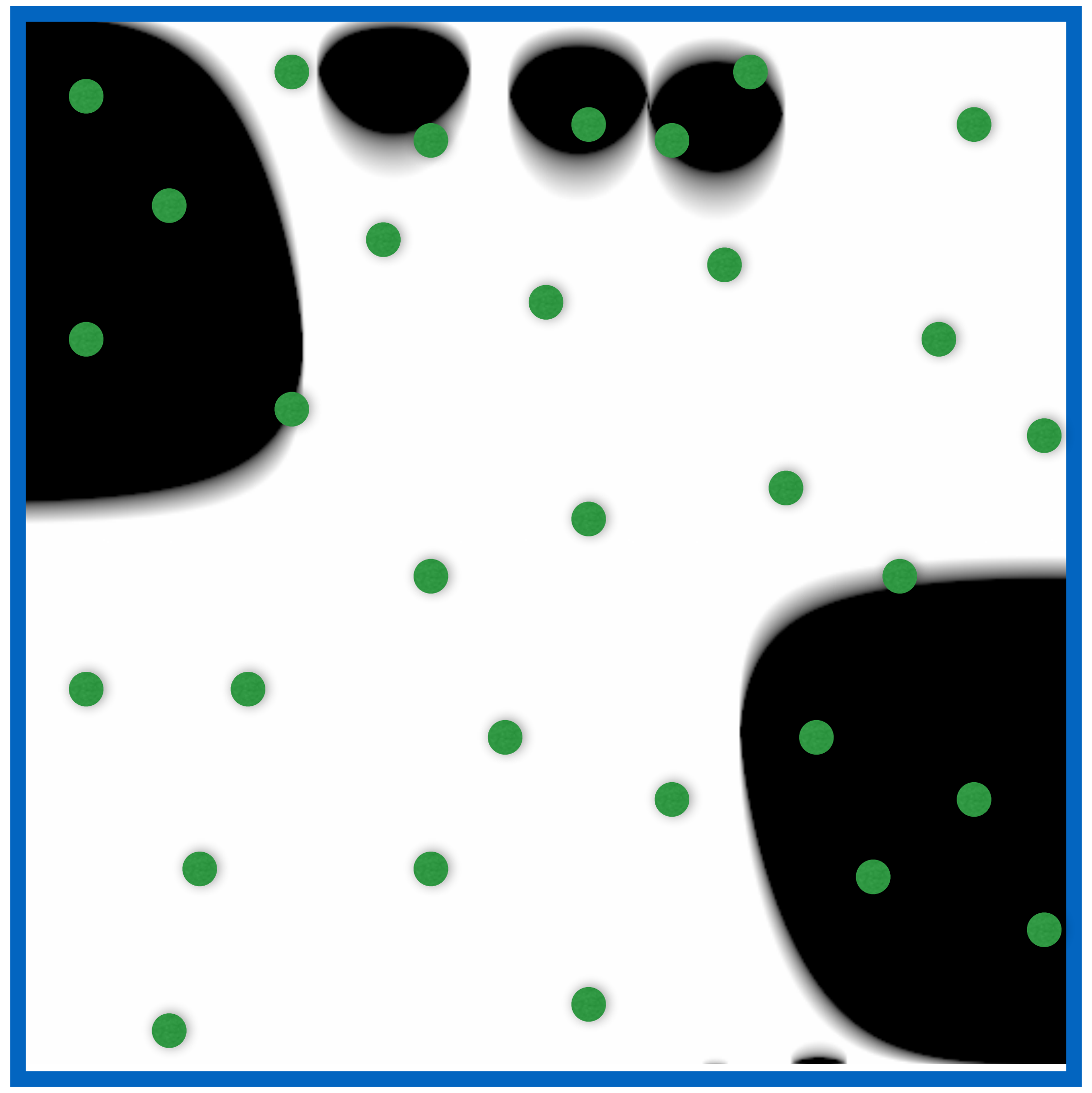
$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k)\mathcal{K}_S(\lambda, \nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Convolved Integrand

Points in d-dimensions



First Interpretation



Second Interpretation

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Monte Carlo Estimator in Frequency Domain for Line Samples

Original Integrand

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp})\mathcal{K}_L(\nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Lines in d-dimensions

First Interpretation

Convolved Integrand

$$I_N = \int_\Omega \frac{1}{N}\sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp})\mathcal{K}_L(\nu_k^{||})\overline{\mathcal{F}_f(\nu)}d\nu$$

Points in d-1 dimensions

Second Interpretation

Monte Carlo Estimator in Frequency Domain for Line Samples

Original Integrand

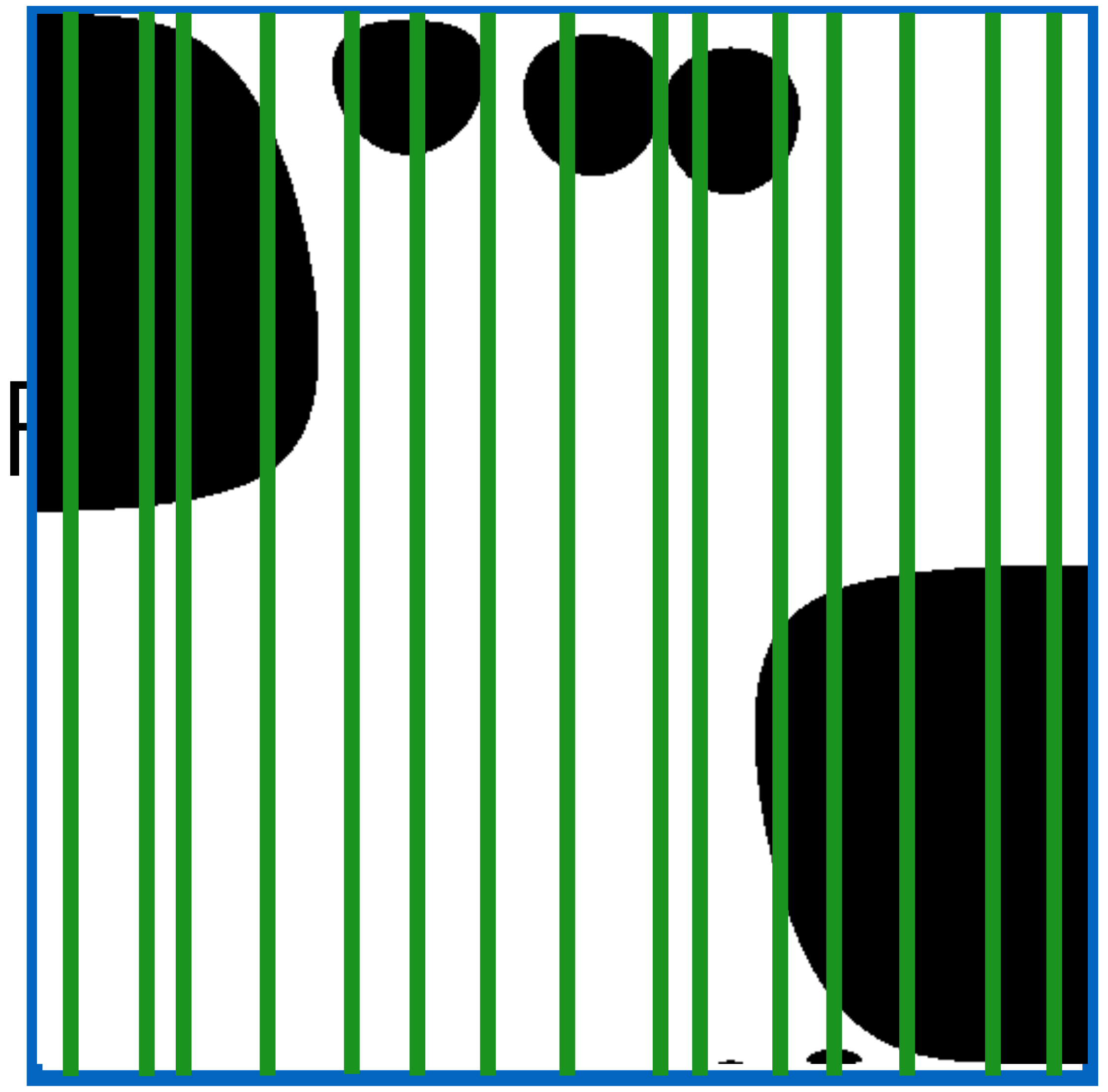
$$I_N = \int_\Omega \frac{1}{N} \sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp}) \mathcal{K}_L(\nu_k^{||}) \overline{\mathcal{F}_f(\nu)} d\nu$$

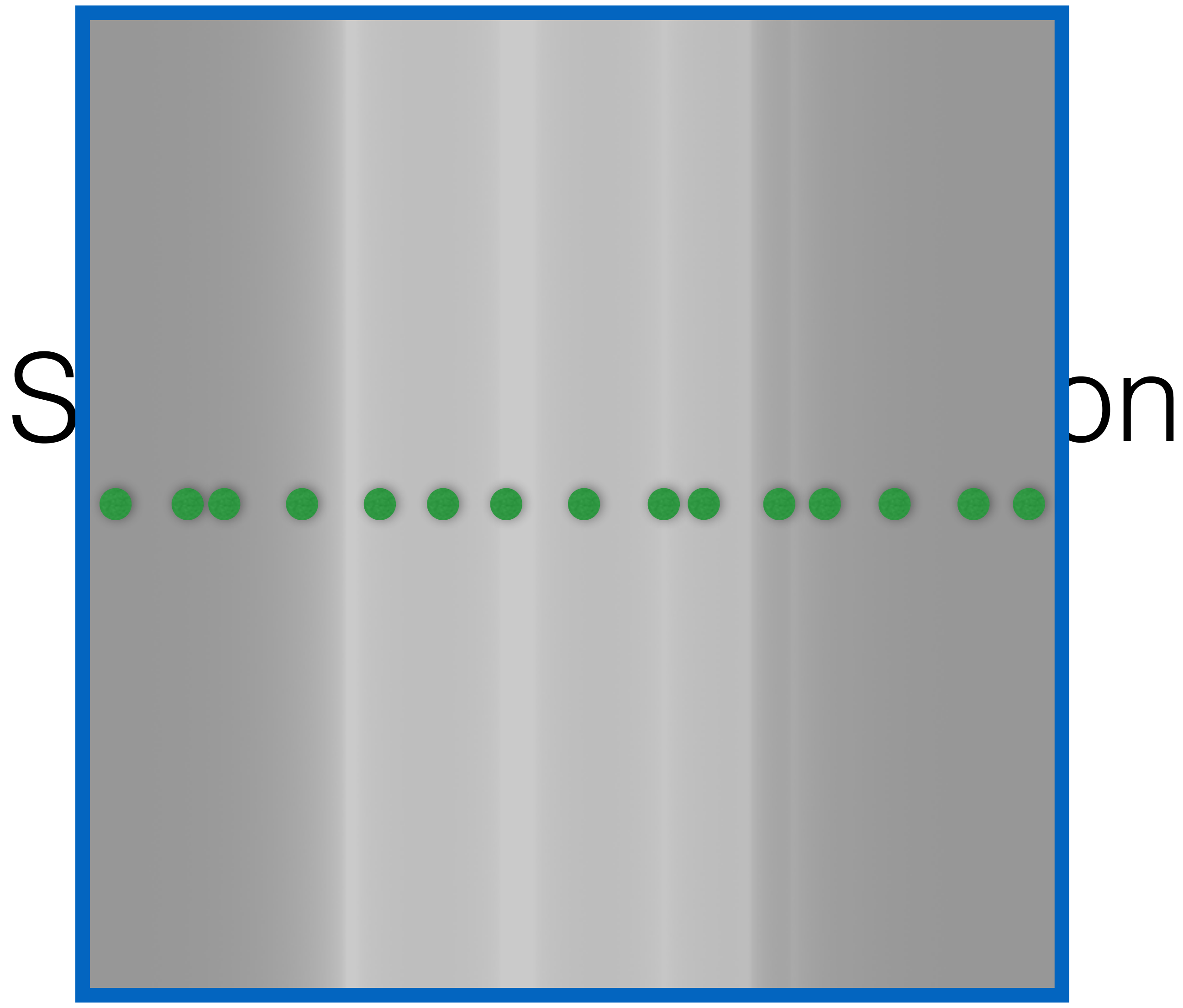
Lines in d-dimensions

Convolved Integrand

$$I_N = \int_\Omega \frac{1}{N} \sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp}) \mathcal{K}_L(\nu_k^{||}) \overline{\mathcal{F}_f(\nu)} d\nu$$

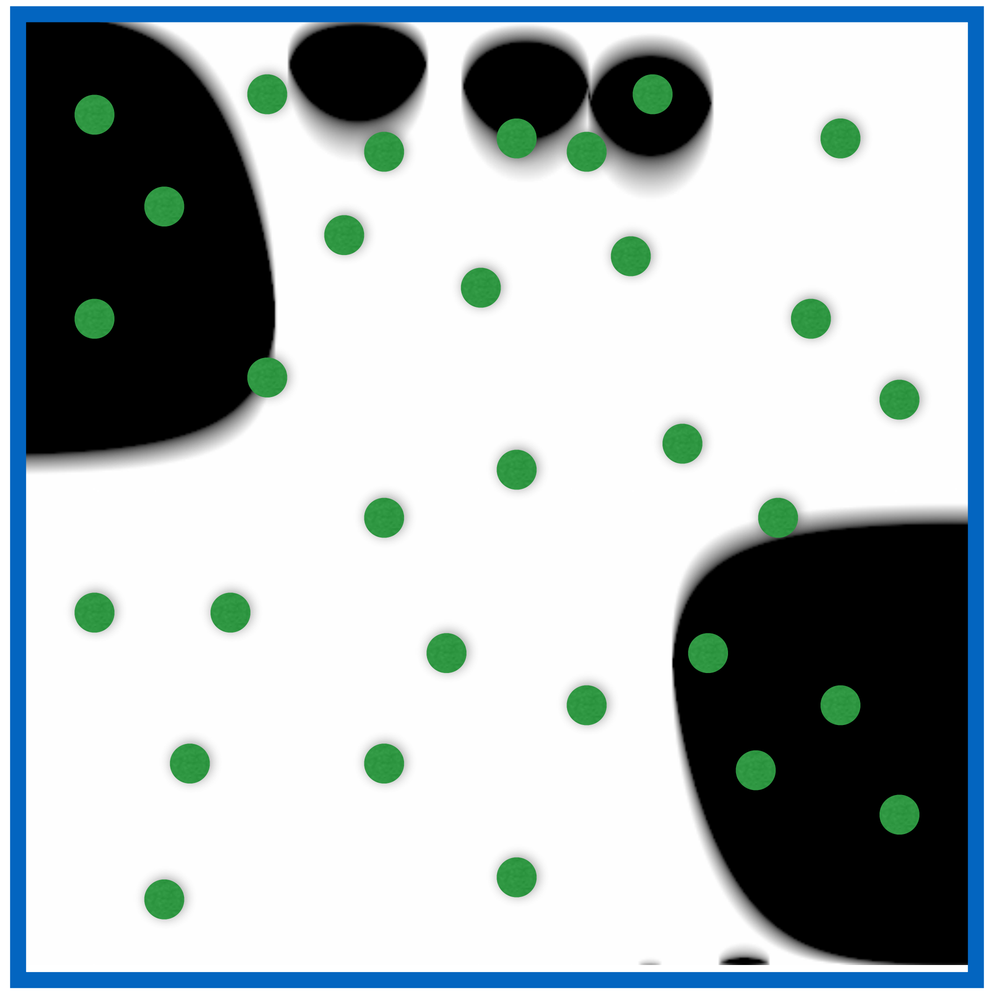
Points in
d-1 dimensions





Second Interpretation

Segments

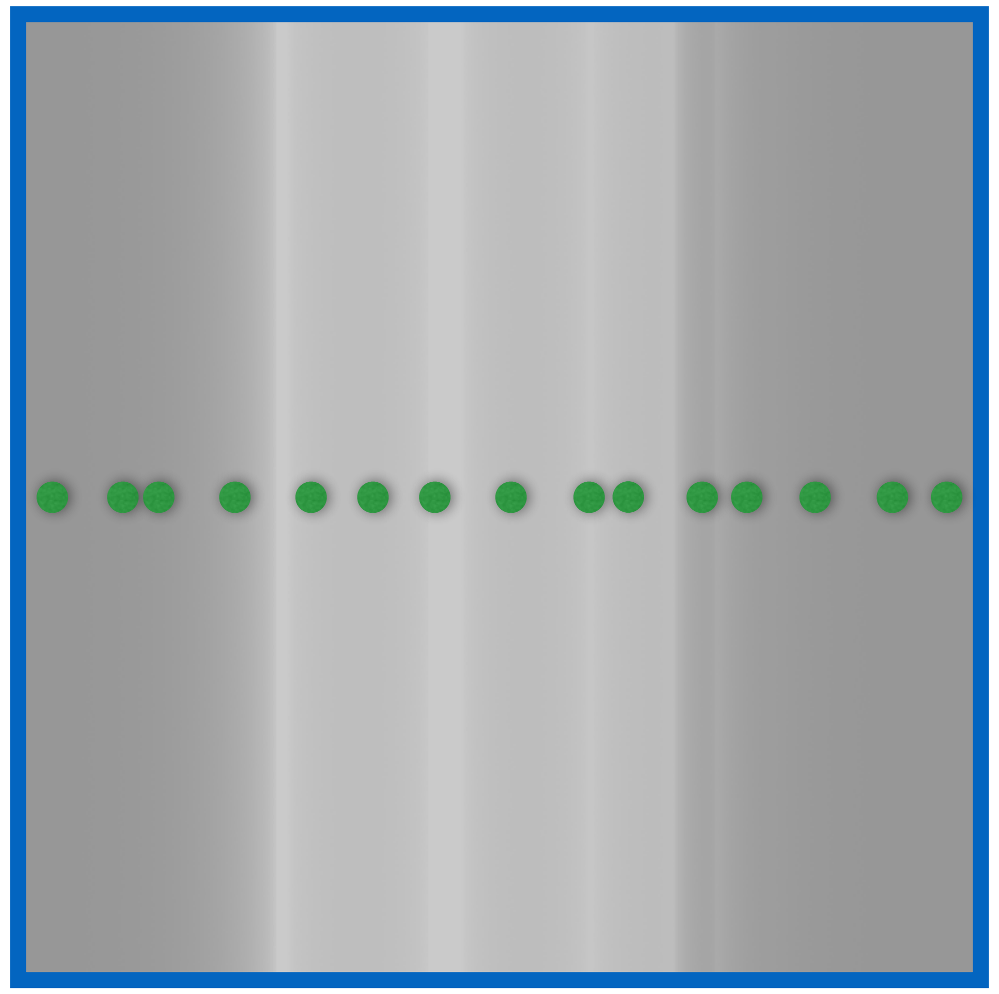


$$I_N = \int_\Omega \frac{1}{N} \sum_{k=1}^{N} \mathcal{F}(\nu_k) \mathcal{K}_S(\lambda, \nu_k^{||}) \overline{\mathcal{F}_f(\nu)} d\nu$$

Convolved Integrand

Points in
d-dimensions

Lines



$$I_N = \int_\Omega \frac{1}{N} \sum_{k=1}^{N} \mathcal{F}(\nu_k^{\perp}) \mathcal{K}_L(\nu_k^{||}) \overline{\mathcal{F}_f(\nu)} d\nu$$

Convolved Integrand

Points in
d-1 dimensions

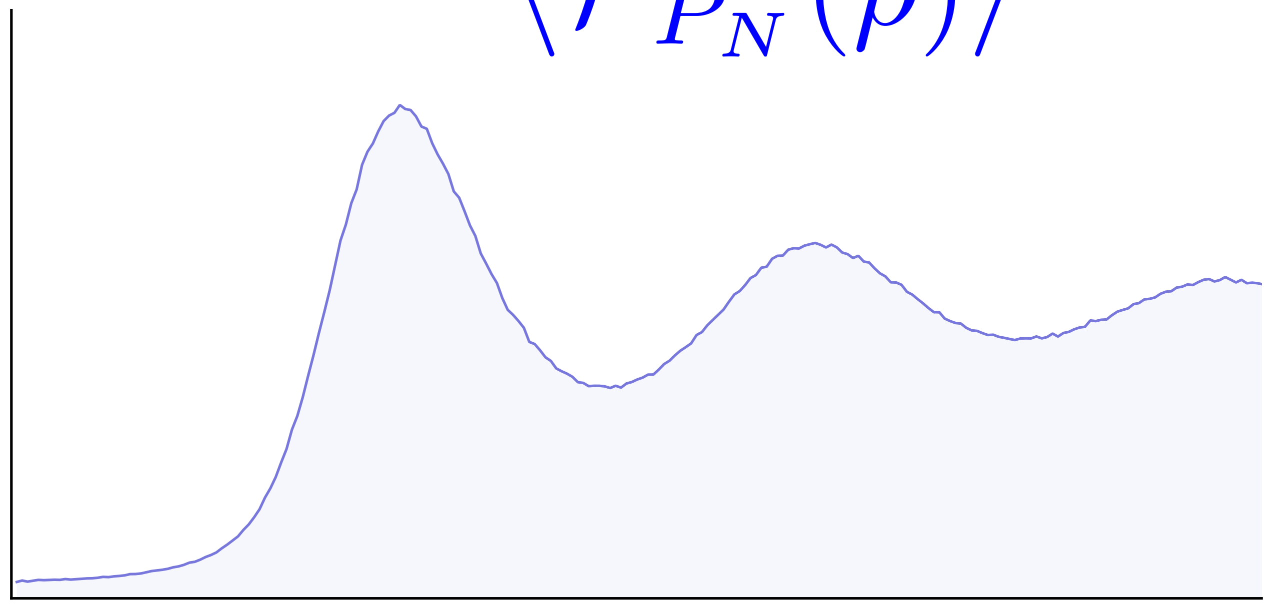
Variance formulation in Radial Form

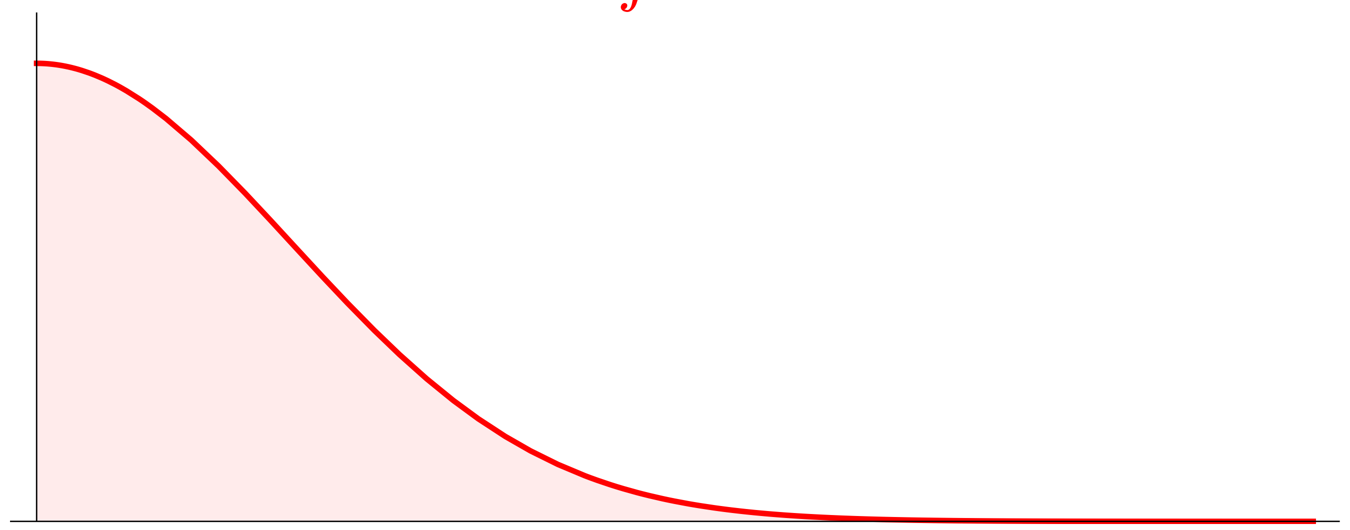
Segments

Poisson Disk

Integrand

$$\mathrm{Var}(I_N) = \int_0^\infty \langle \mathcal{P}_{P_N}(\rho) \rangle \times \mathcal{P}^S_{\tilde{f}}(\rho) \, d\rho$$





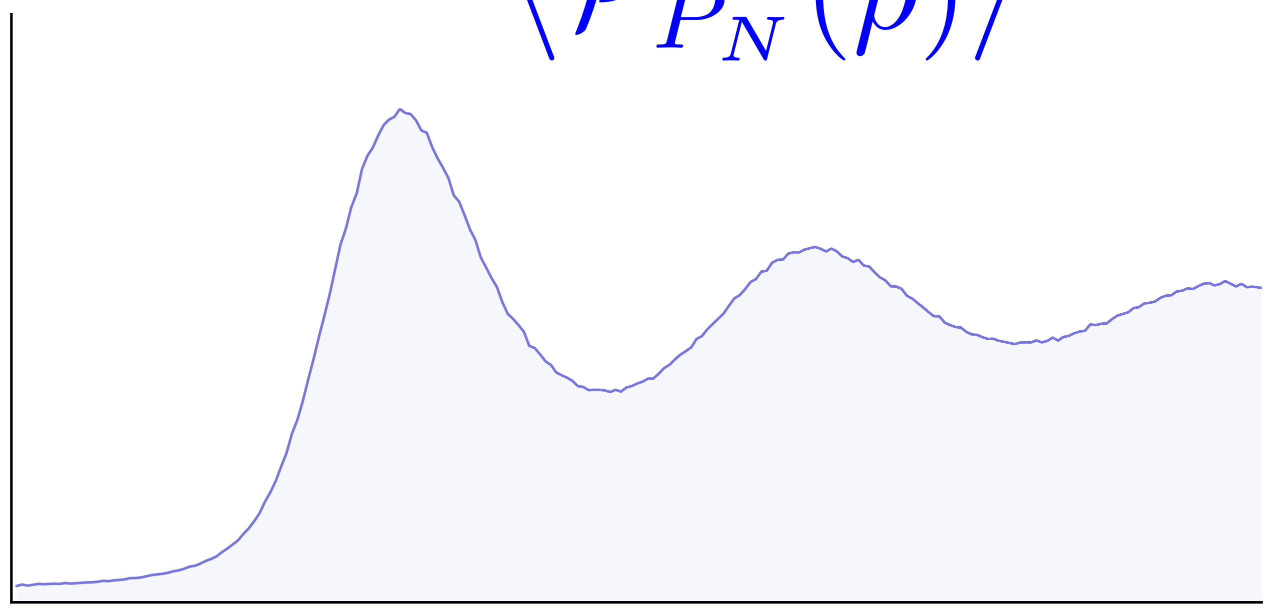
Pre-filtered integrand in d-dimensions

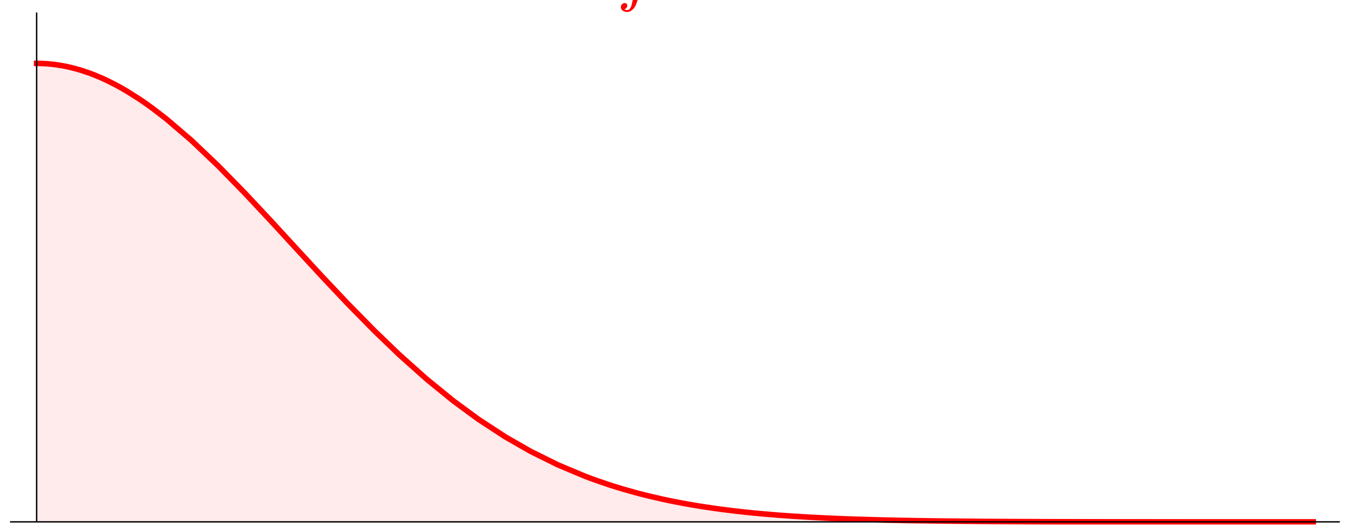
Variance formulation in Radial Form

Poisson Disk | Integrand

Segments

$$\mathrm{Var}(I_N) = \int_0^\infty \langle \mathcal{P}_{P_N}(\rho) \rangle \times \mathcal{P}^S_{\tilde{f}}(\rho) \, d\rho$$

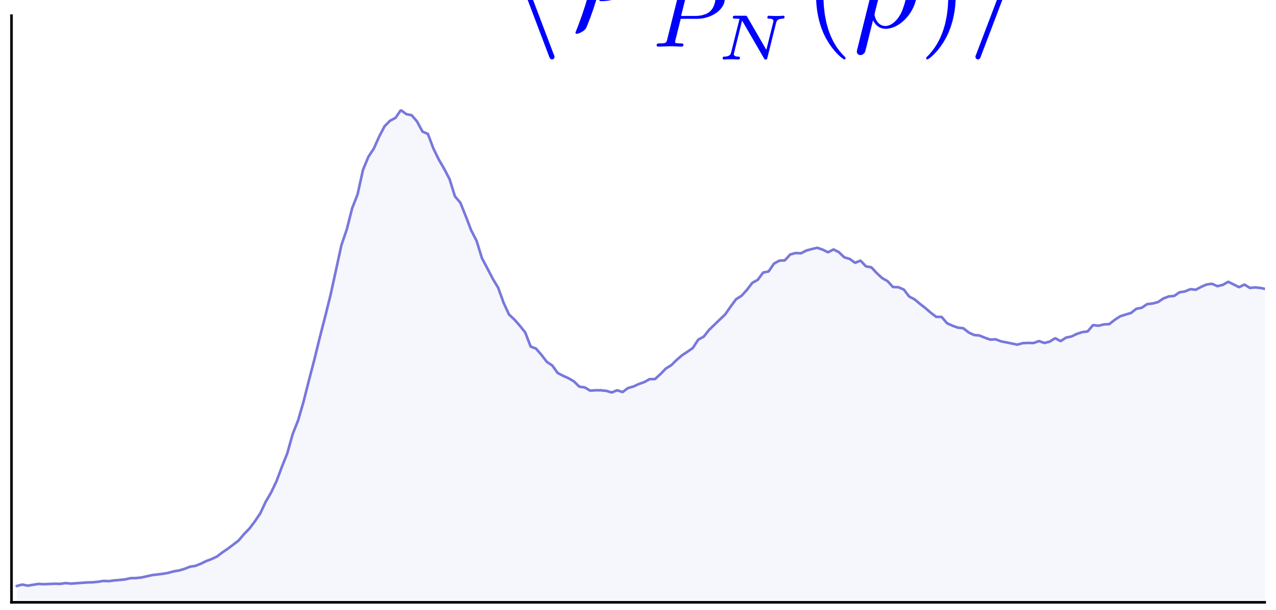


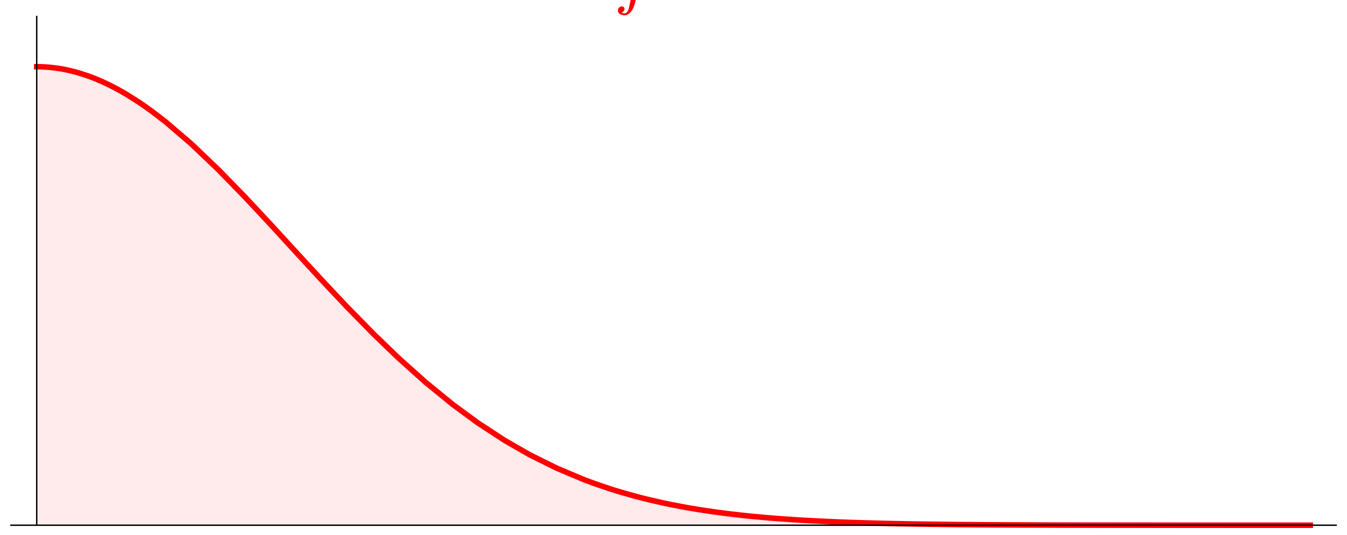


Pre-filtered integrand in d-dimensions

Lines

$$\mathrm{Var}(I_N) = \int_0^\infty \langle \mathcal{P}_{P_N}(\tilde{\rho}) \rangle \times \mathcal{P}^L_{\tilde{f}}(\tilde{\rho}) \, d\tilde{\rho}$$





Pre-filtered integrand in d-1 dimensions

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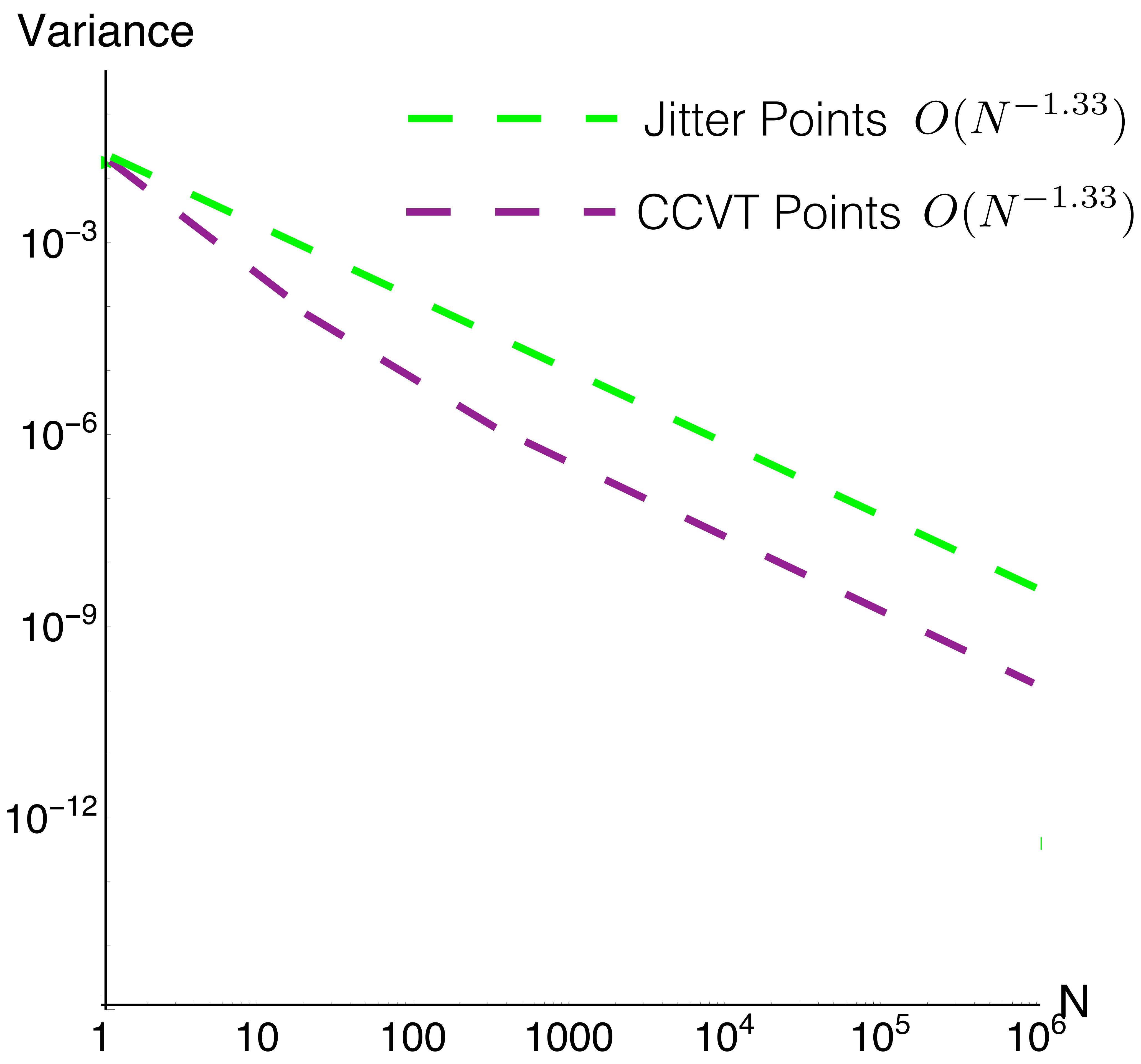
- Monte Carlo Integration
- Variance Formulation for Points
- Variance Formulation for Segment and Line Samples
- Experimental Verification

Theoretical Variance Convergence Rates for Point Samples

Samplers	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

Unidirectional Line Sampling Convergence Rates

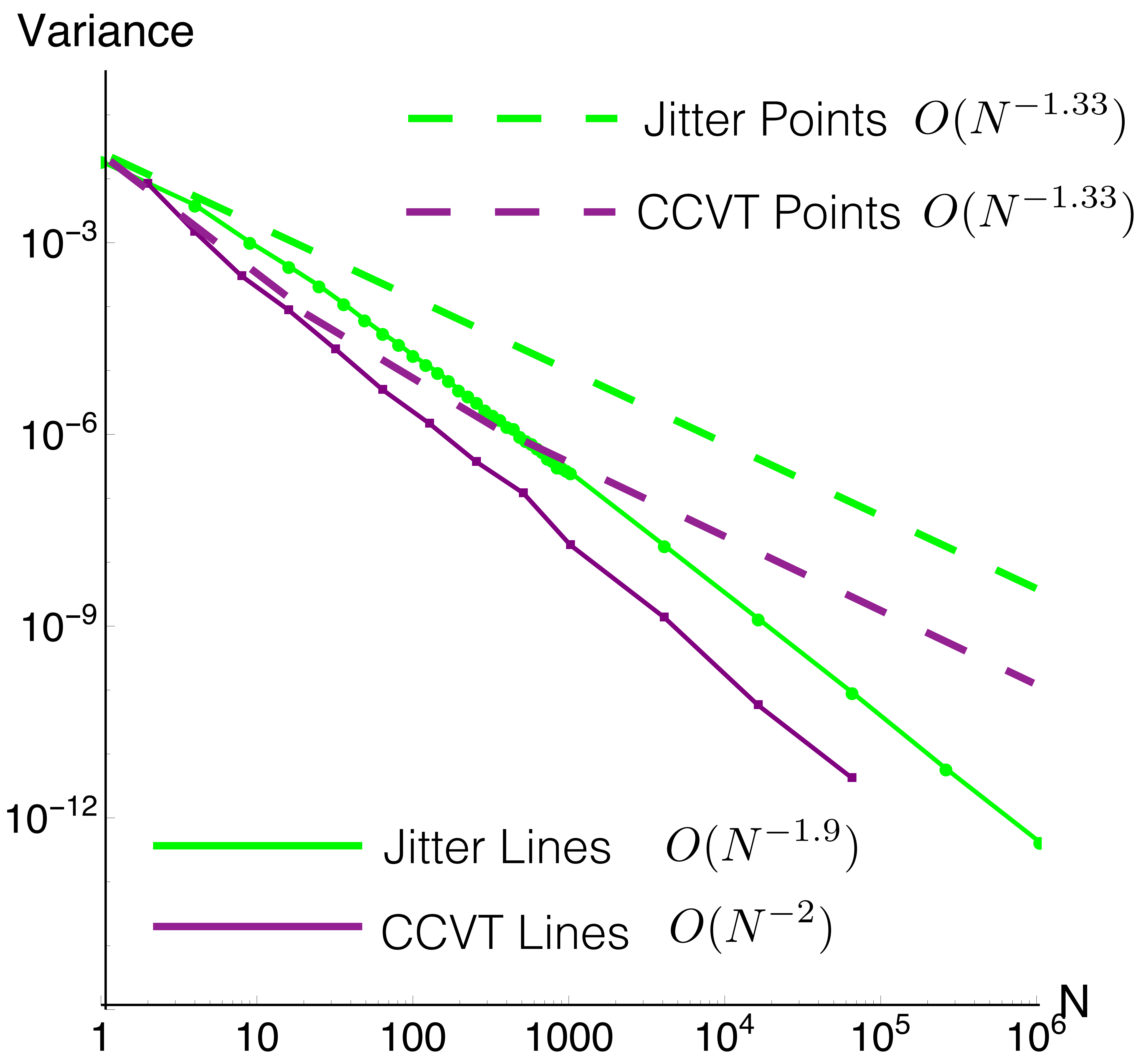


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = \begin{cases} 1 & x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Sphere 3D

Unidirectional Line Sampling Convergence Rates

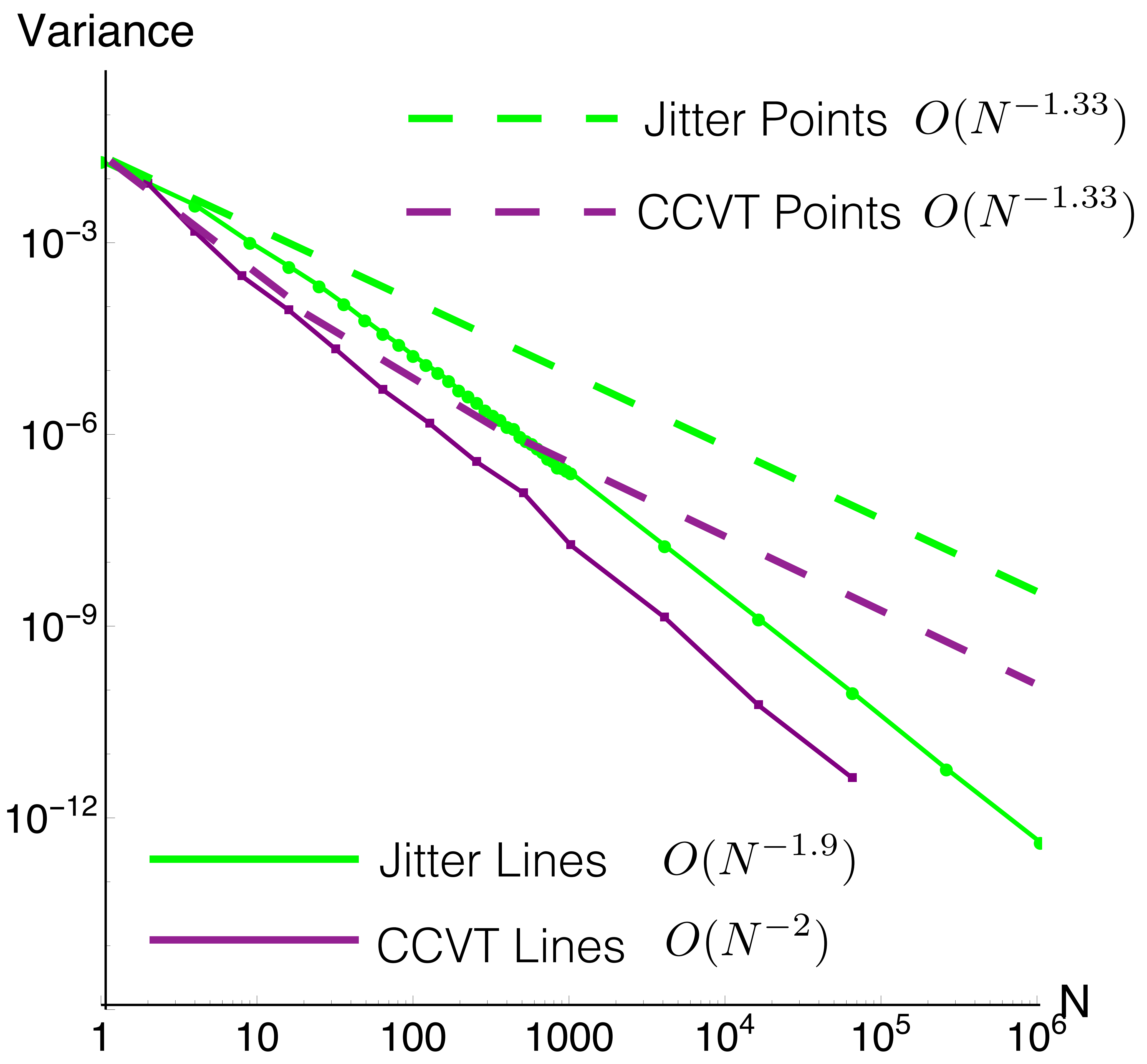


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = \begin{cases} 1 & x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Sphere 3D

Unidirectional Line Sampling Convergence Rates

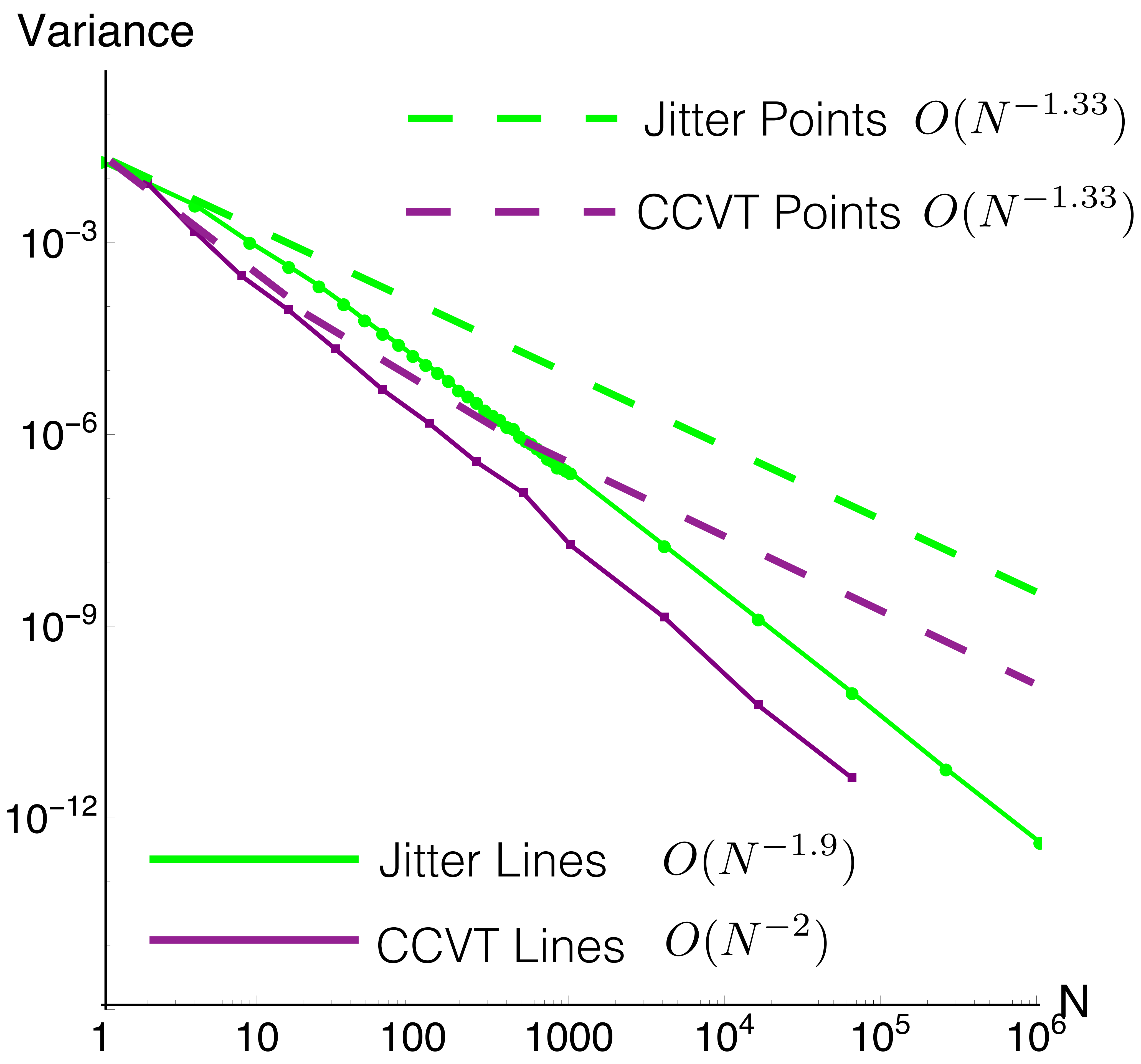


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = \begin{cases} 1 & x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Sphere 3D

Unidirectional Line Sampling Convergence Rates

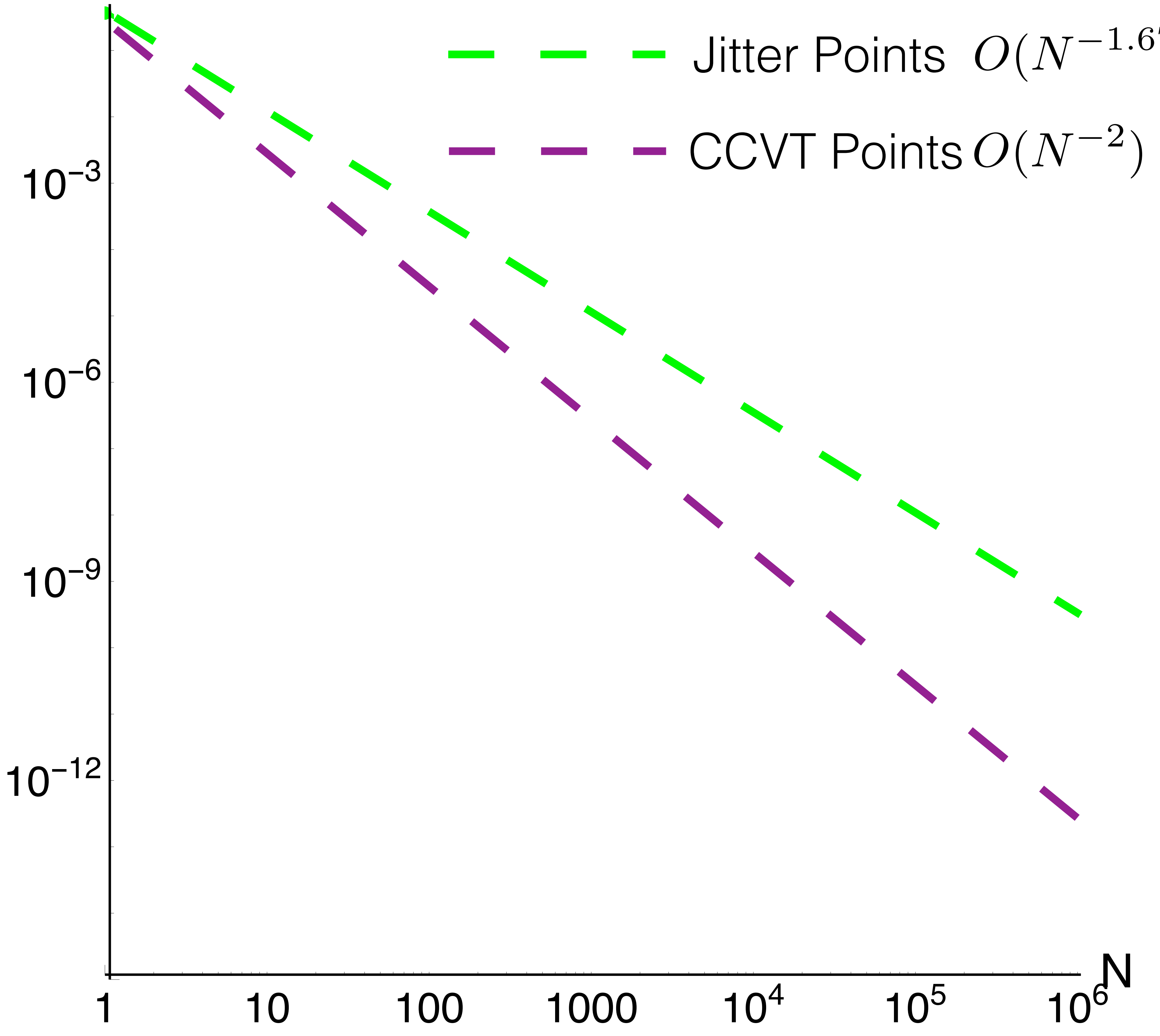


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = \begin{cases} 1 & x^2+y^2+z^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Sphere 3D

Unidirectional Line Sampling Convergence Rates

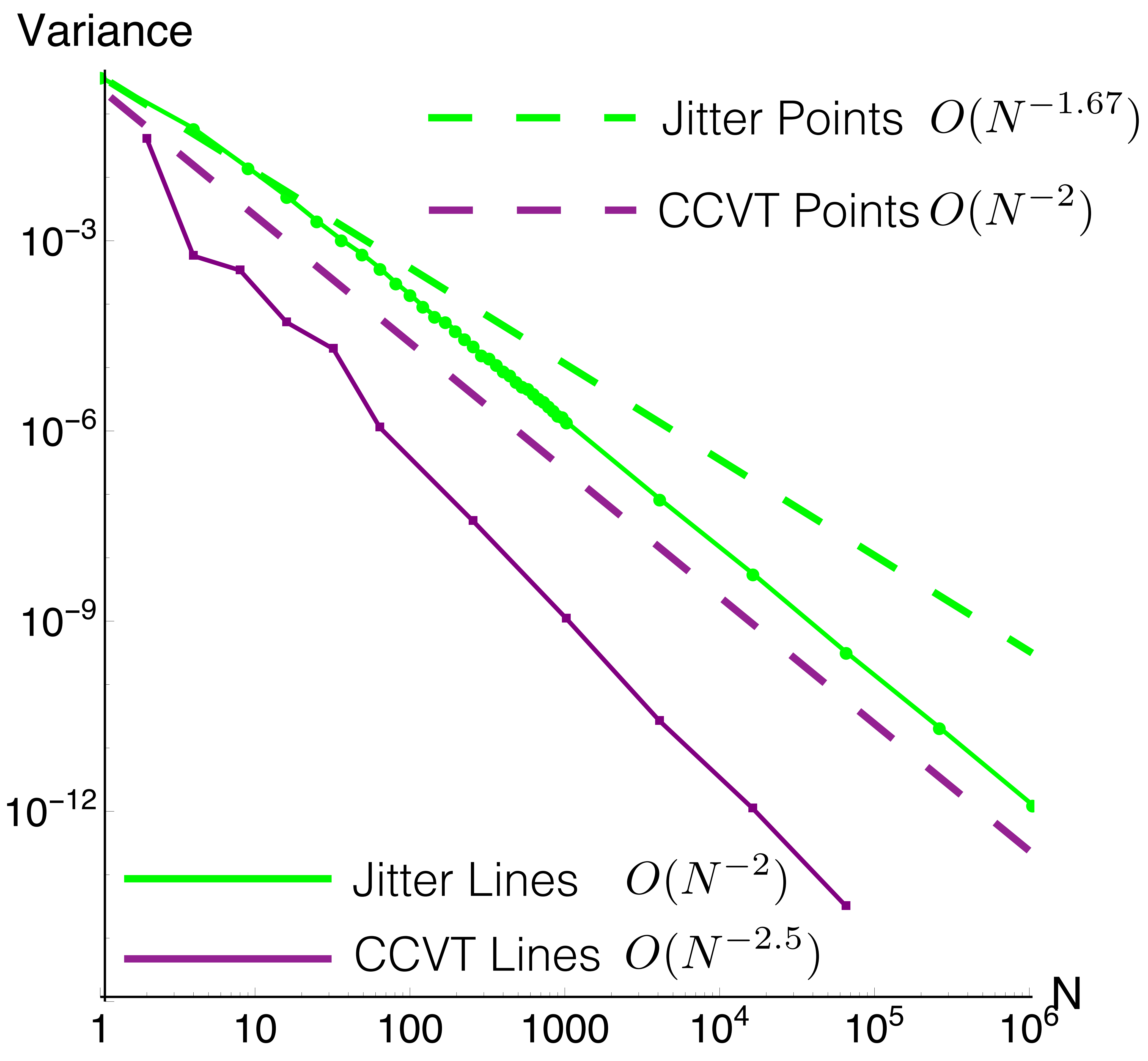


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}$$

Gaussian 3D

Unidirectional Line Sampling Convergence Rates

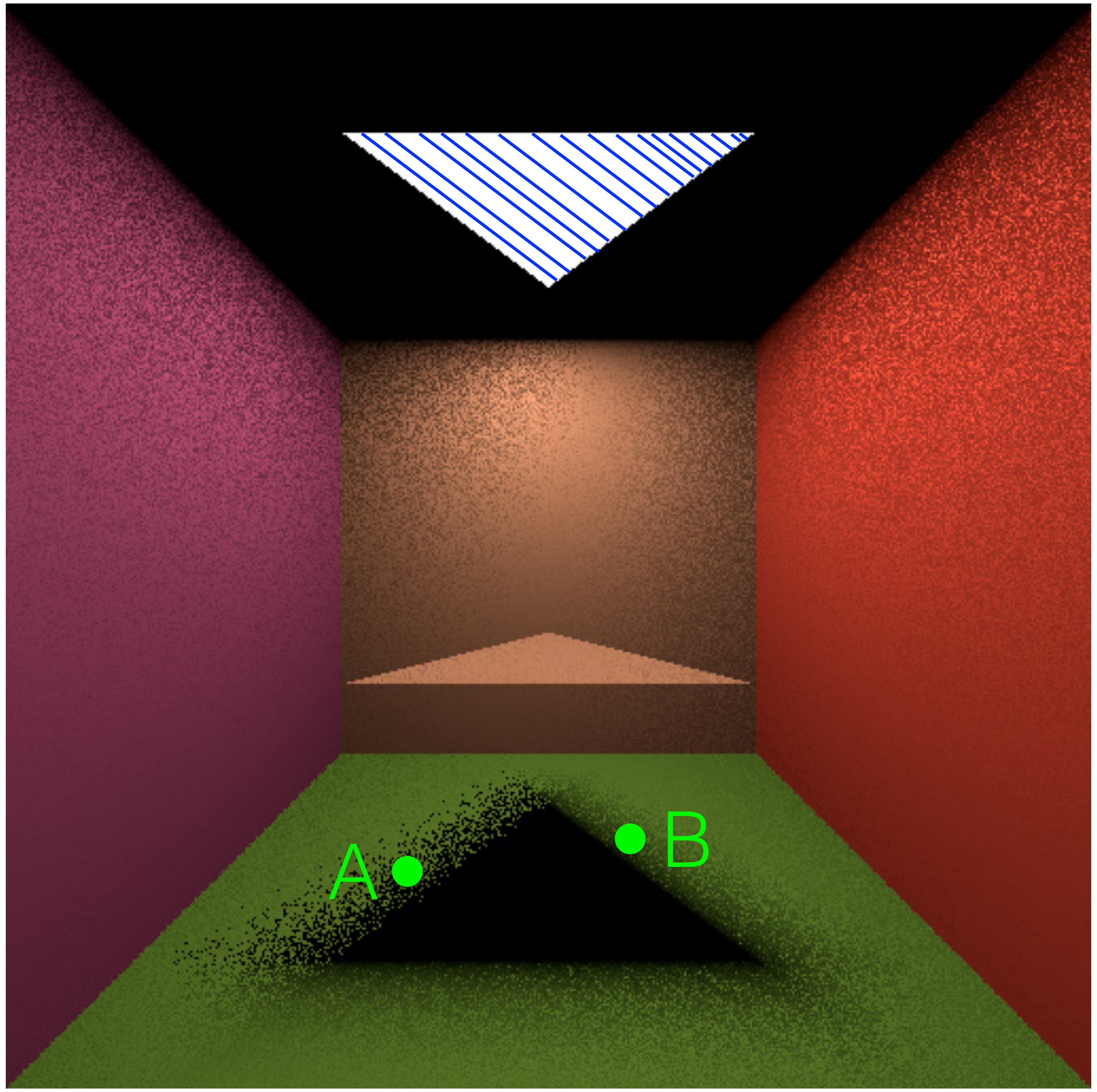


Sampler	d	d = 1	d = 2	d = 3
Jittered (Best Case)	$\mathcal{O}\left(N^{-1-\frac{2}{d}}\right)$	$\mathcal{O}(N^{-3})$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.67})$
Jittered (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$
CCVT (Best Case)	$\mathcal{O}\left(N^{-1-\frac{3}{d}}\right)$	$\mathcal{O}(N^{-4})$	$\mathcal{O}(N^{-2.5})$	$\mathcal{O}(N^{-2})$
CCVT (Worst Case)	$\mathcal{O}\left(N^{-1-\frac{1}{d}}\right)$	$\mathcal{O}(N^{-2})$	$\mathcal{O}(N^{-1.5})$	$\mathcal{O}(N^{-1.33})$

$$f(x,y,z) = e^{-\frac{x^2+y^2+z^2}{2\sigma^2}}$$

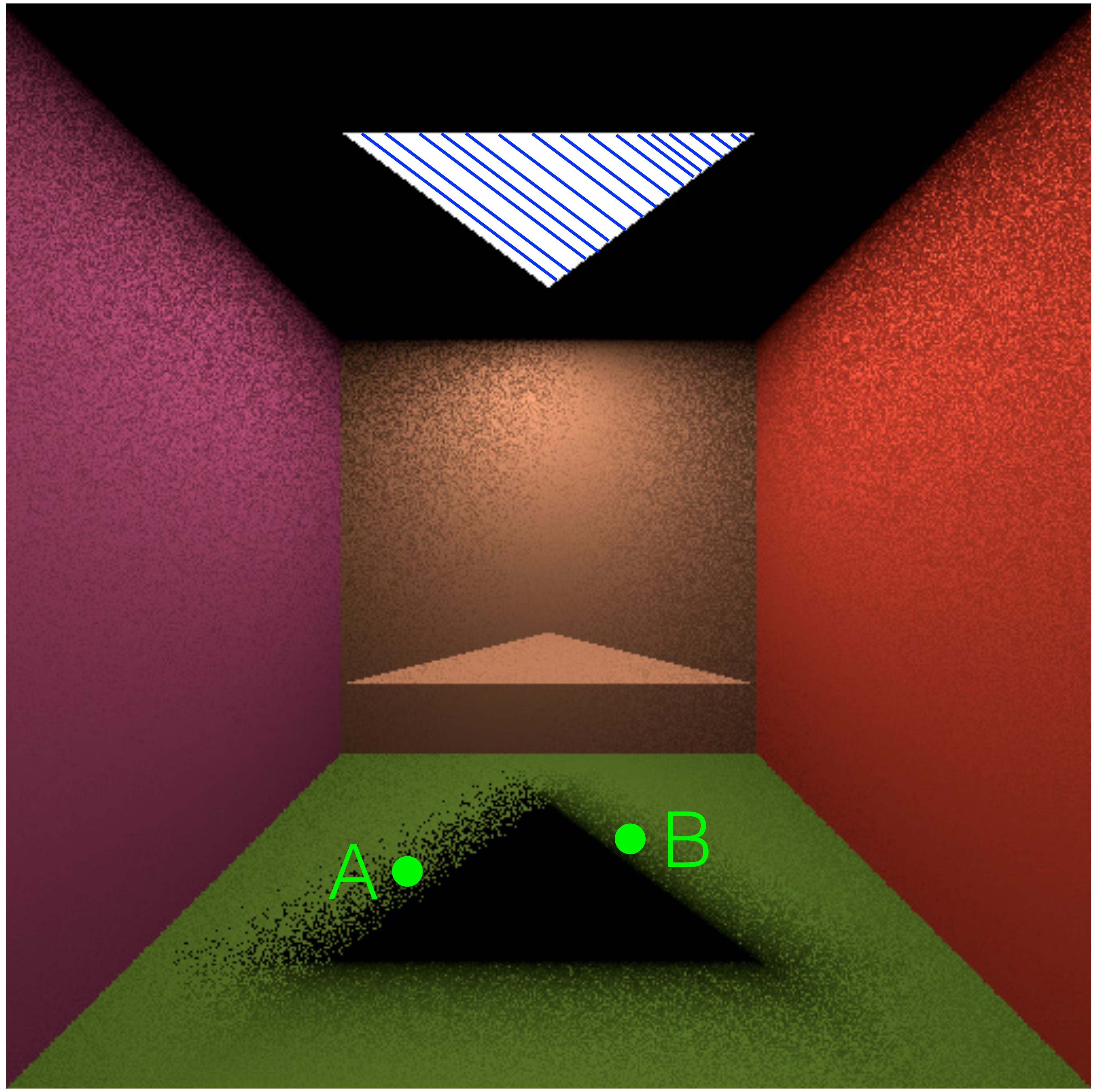
Gaussian 3D

Cornell box Scene: Line Orientations



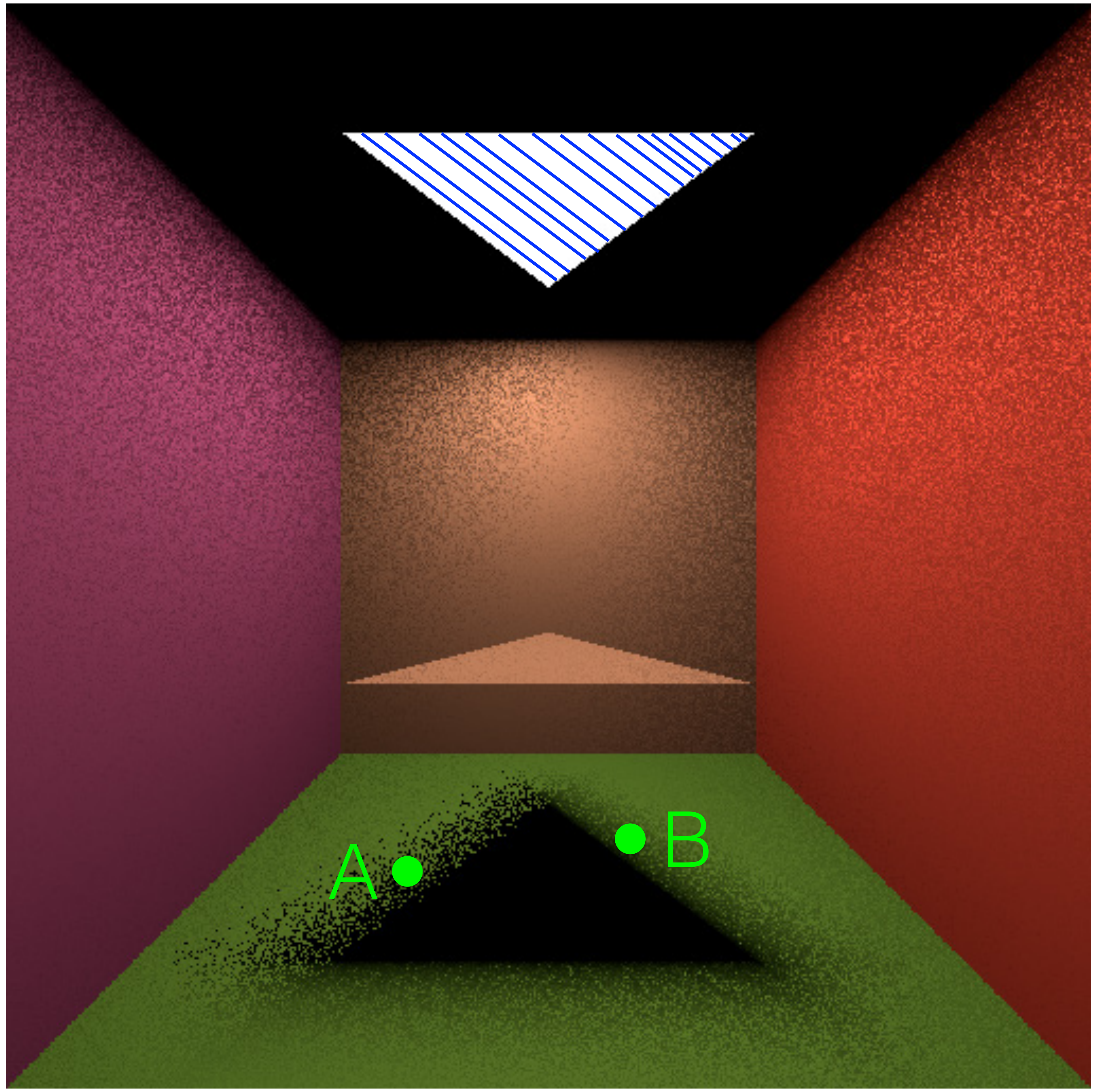
Billen & Dutré 2016

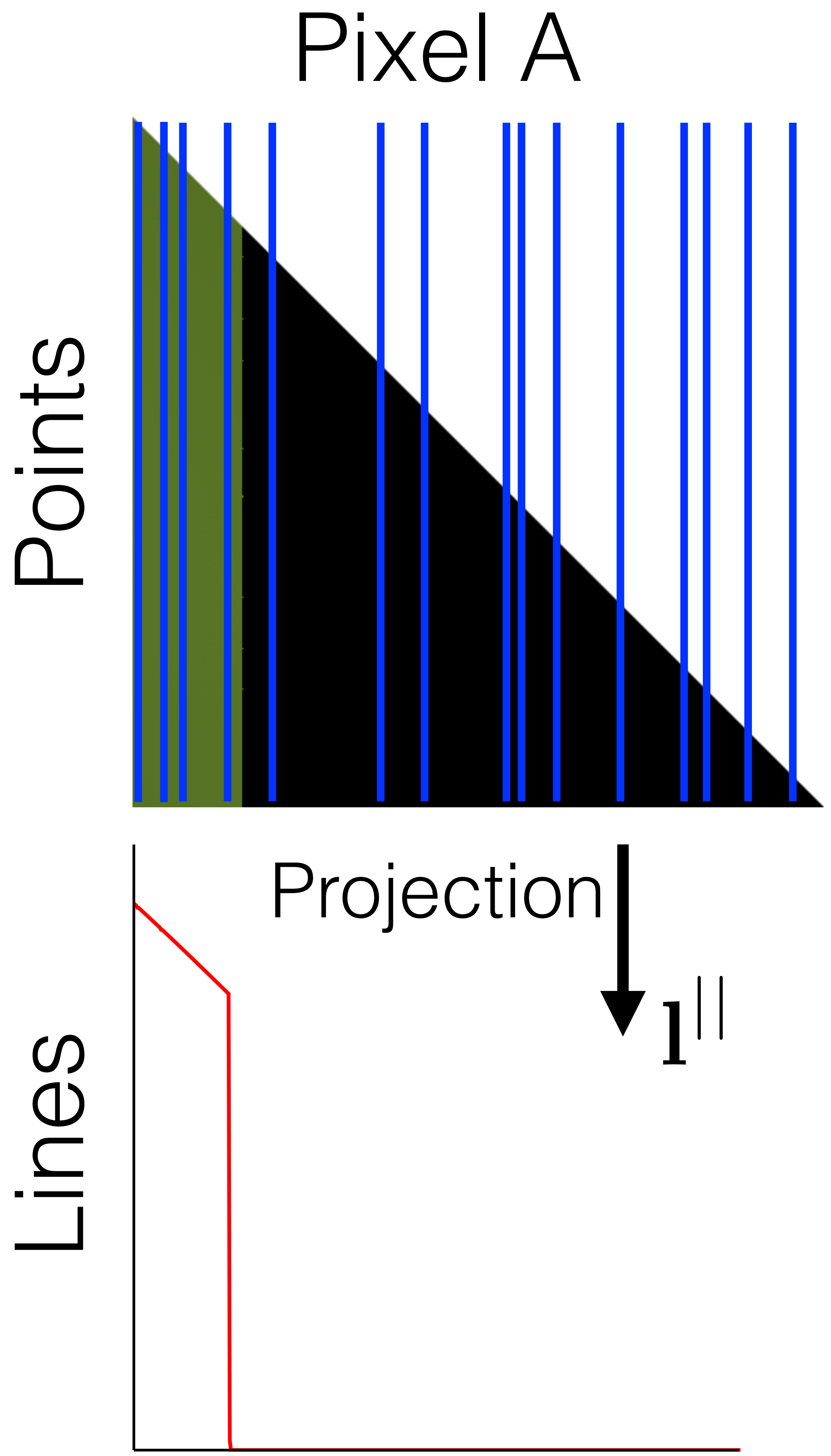
Cornell box Scene: Line Orientations



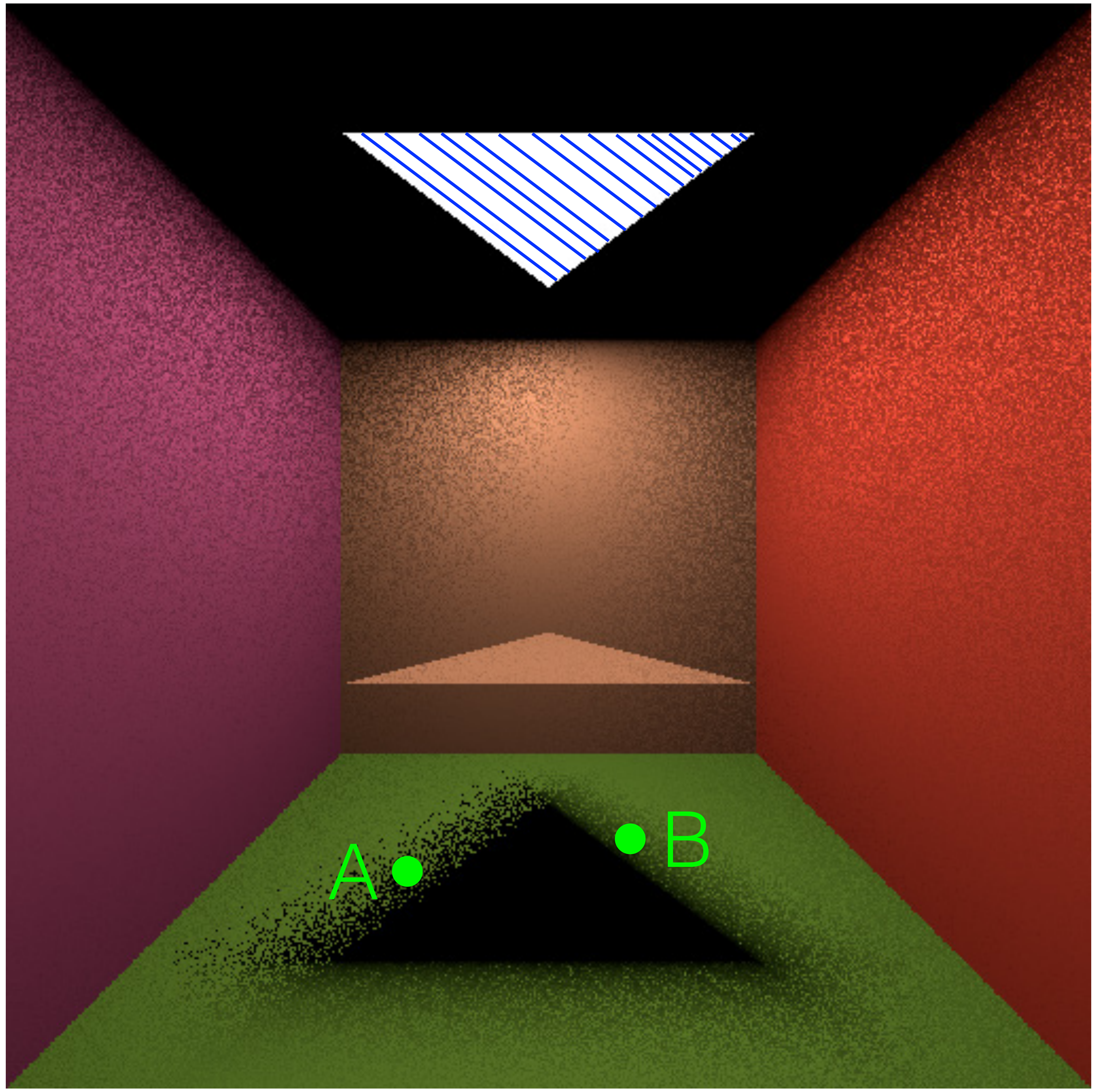
Pixel A

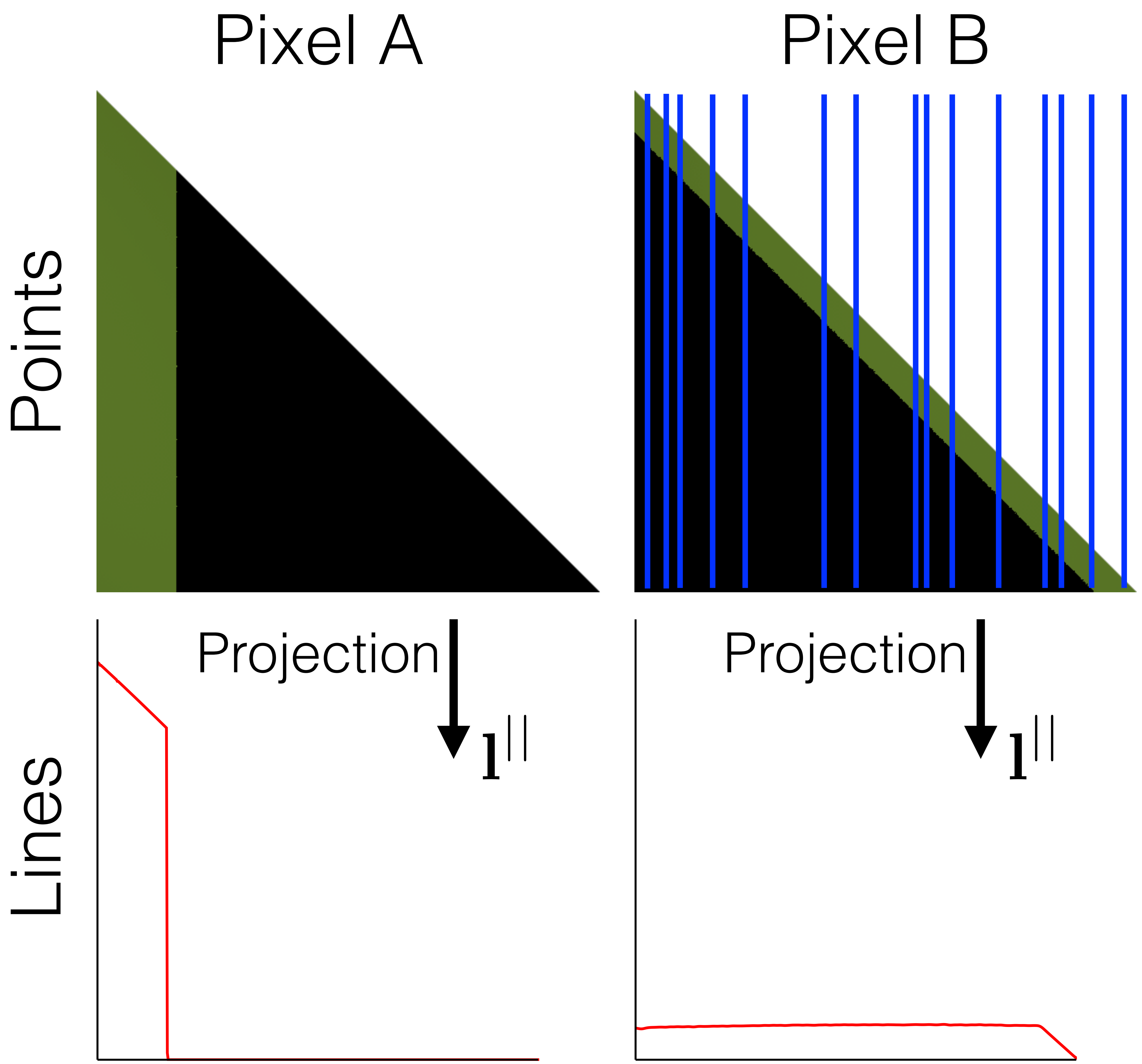
Cornell box Scene: Line Orientations



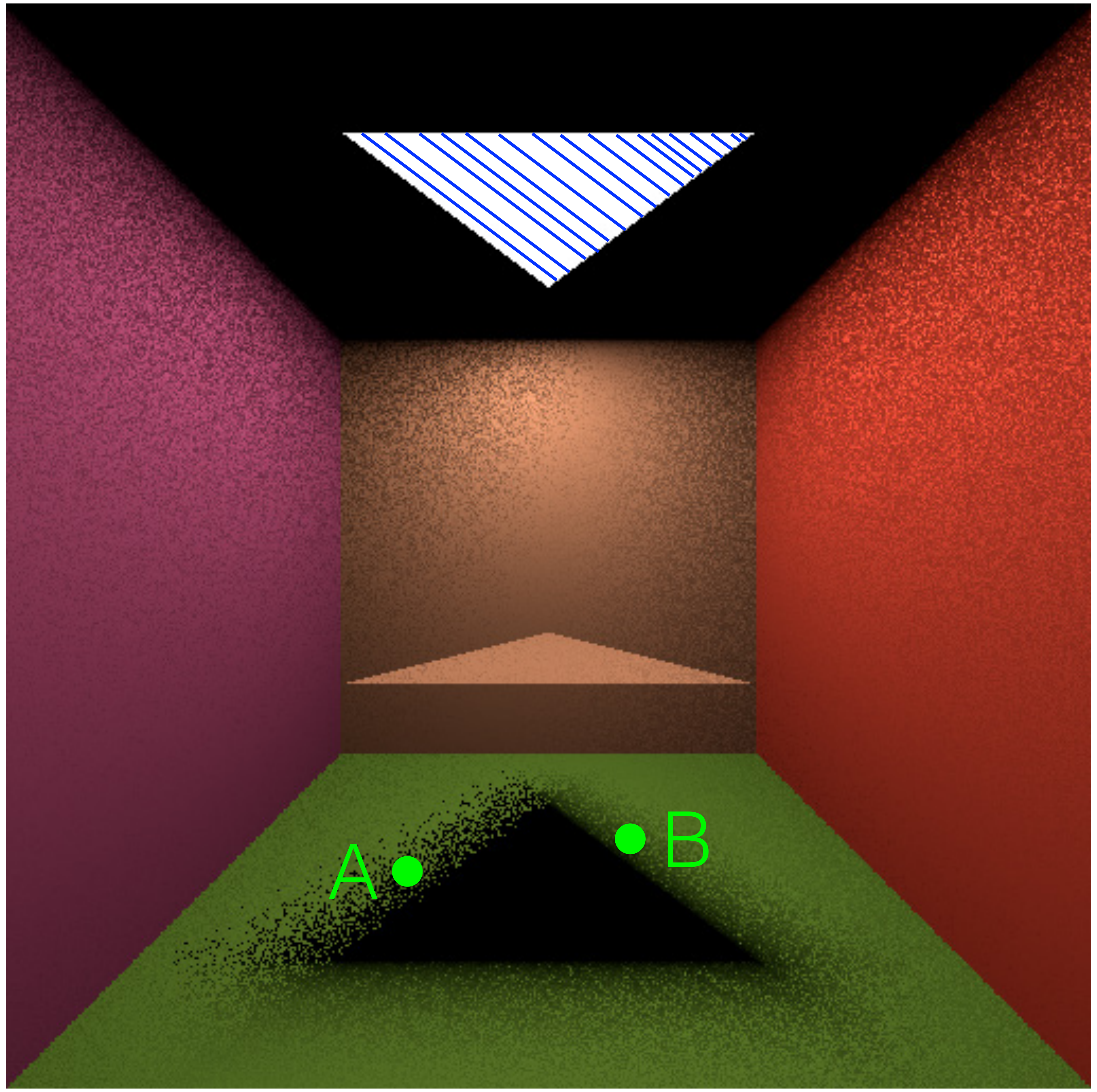


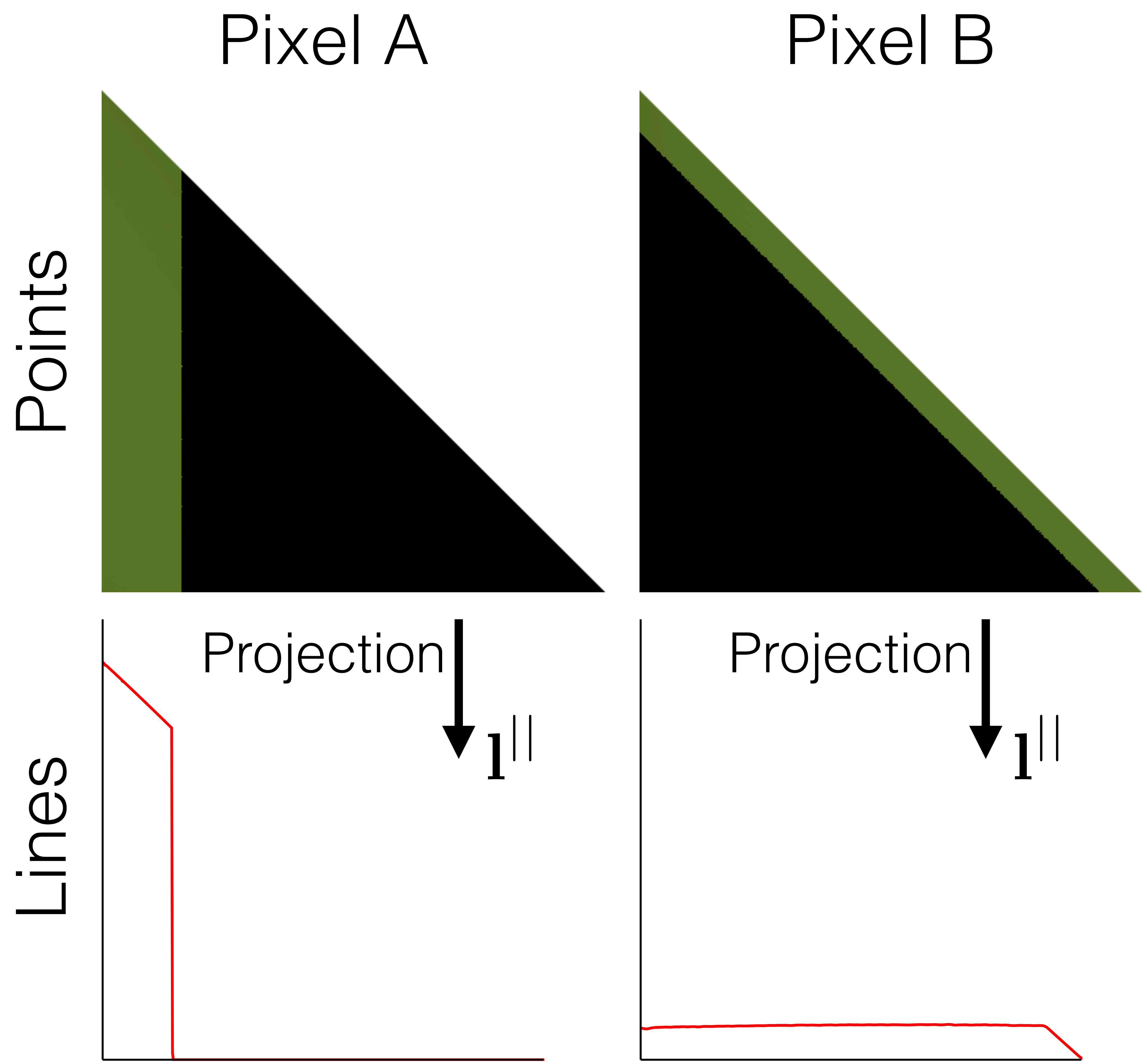
Cornell box Scene: Line Orientations



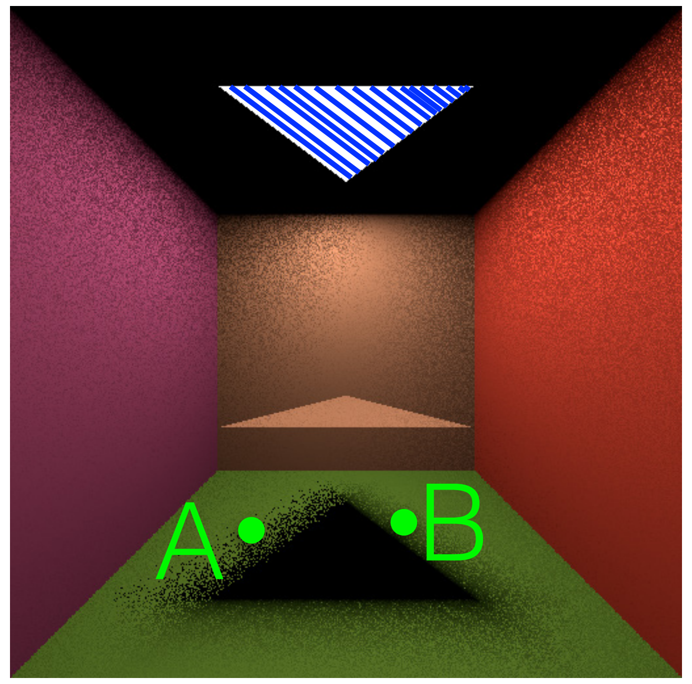


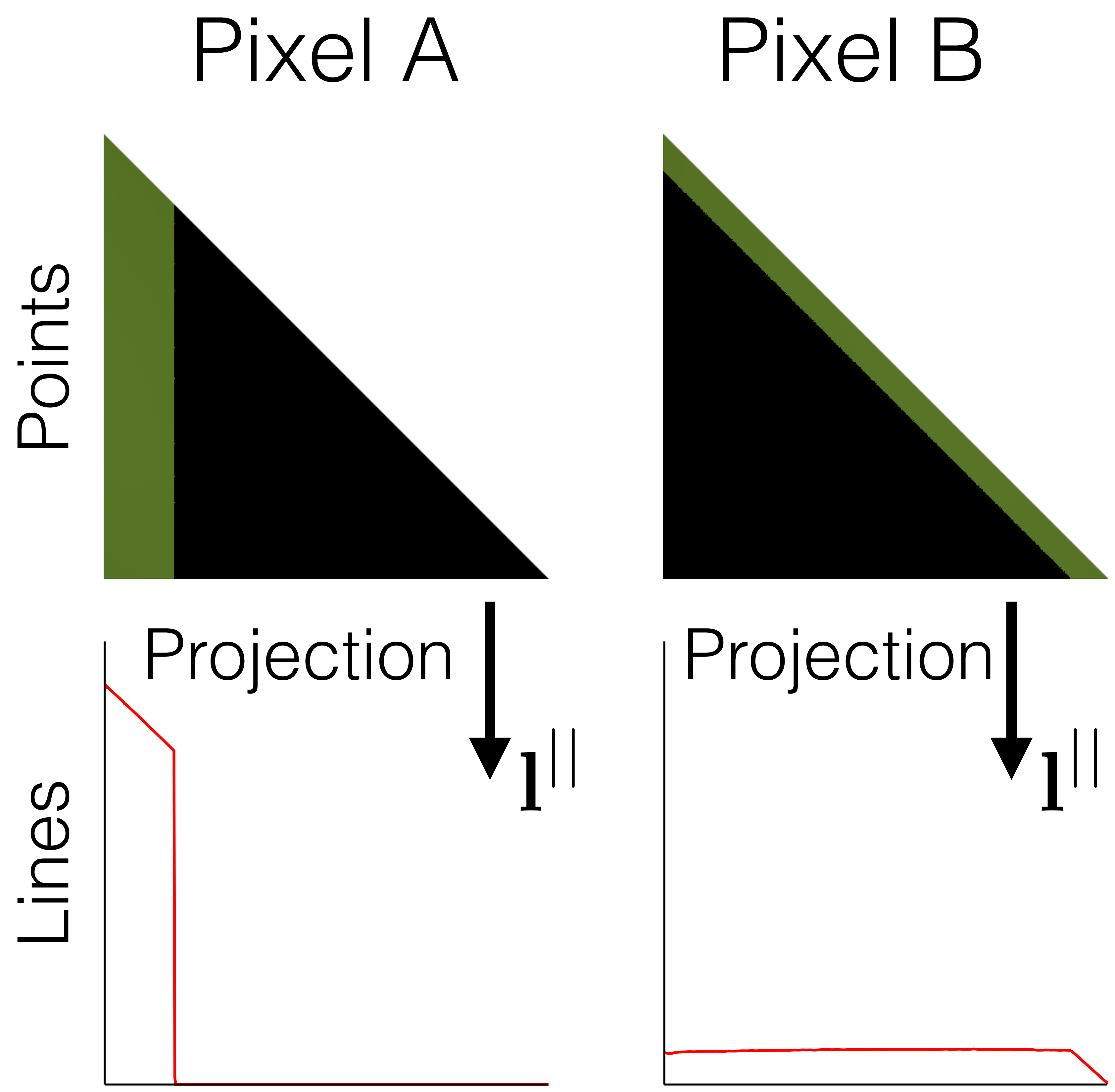
Cornell box Scene: Line Orientations

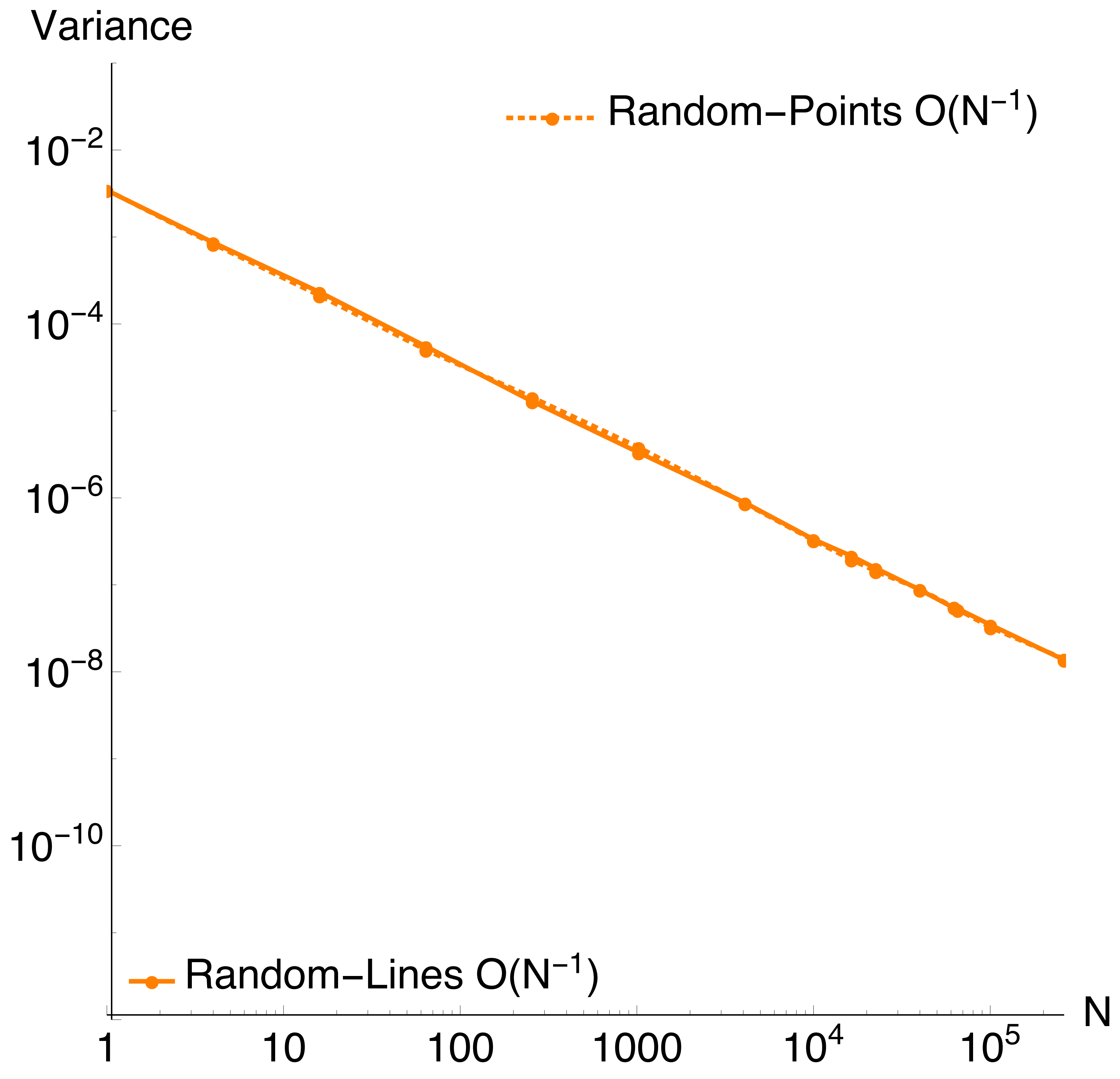


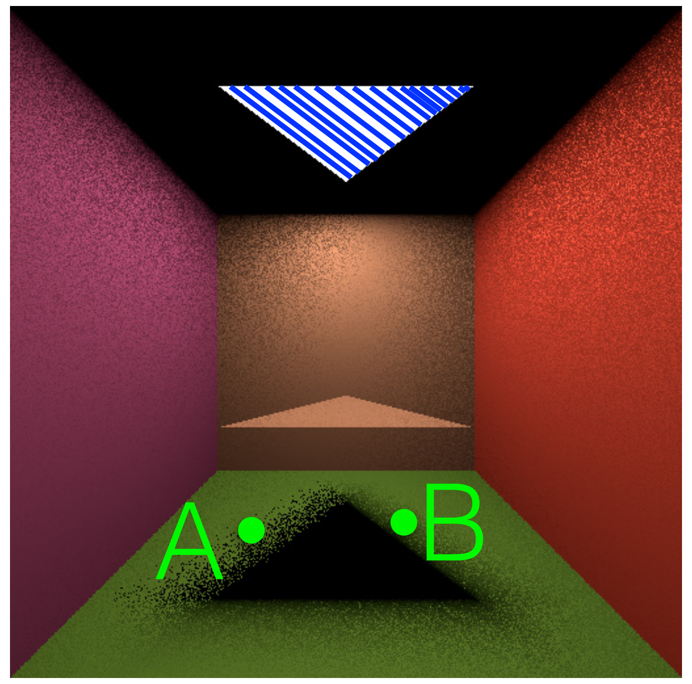


Cornell box Scene: Line Orientations

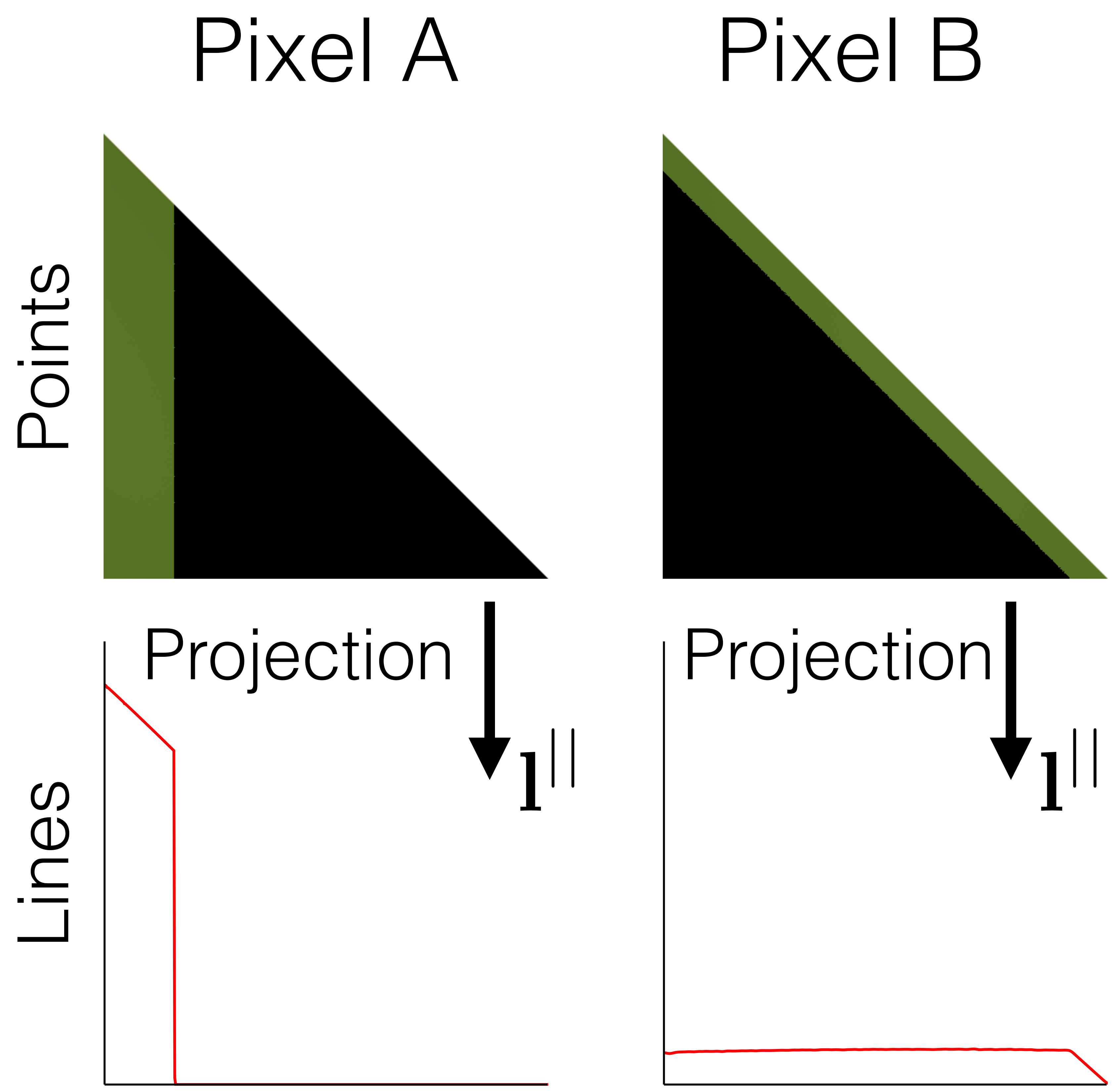


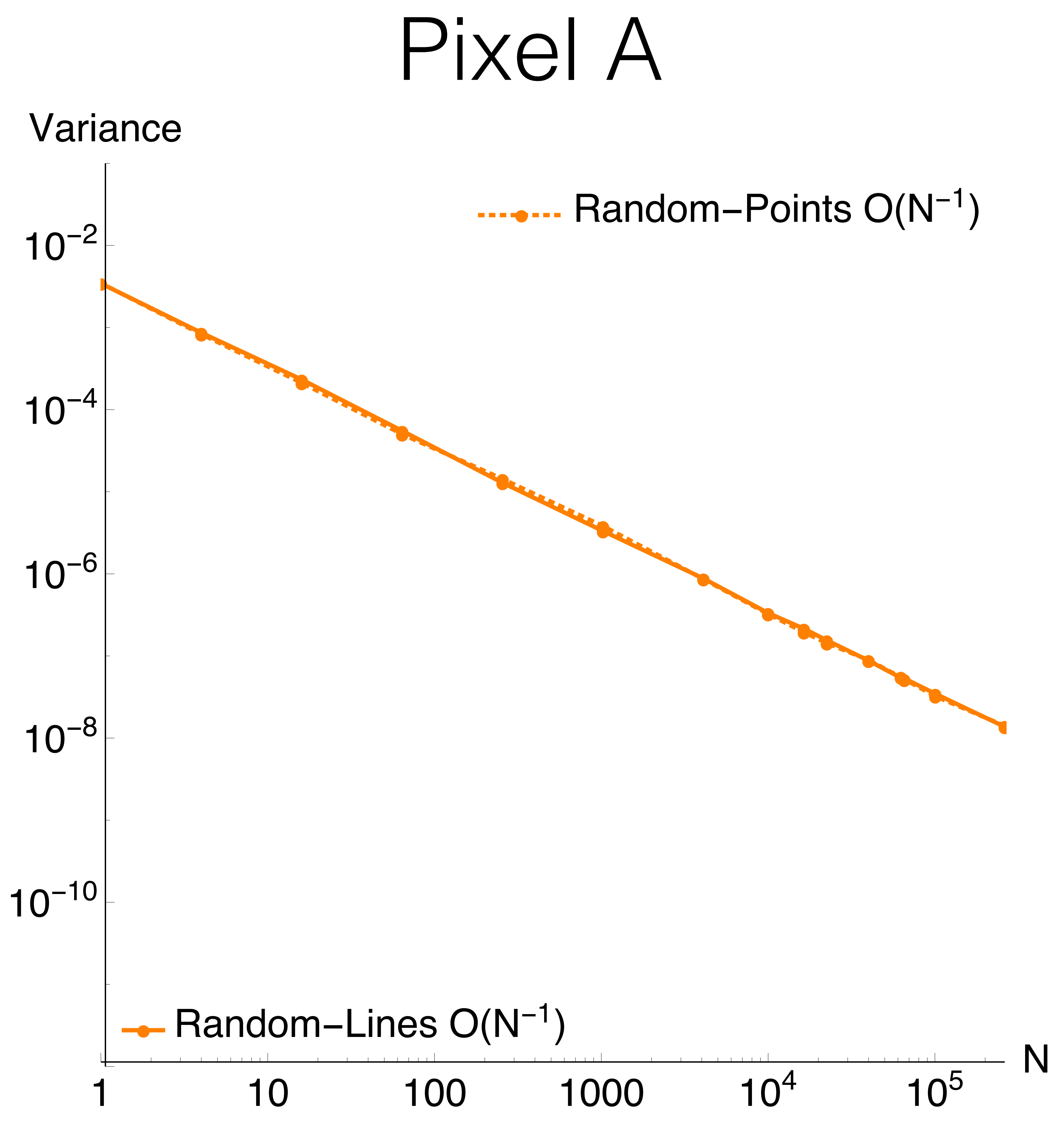


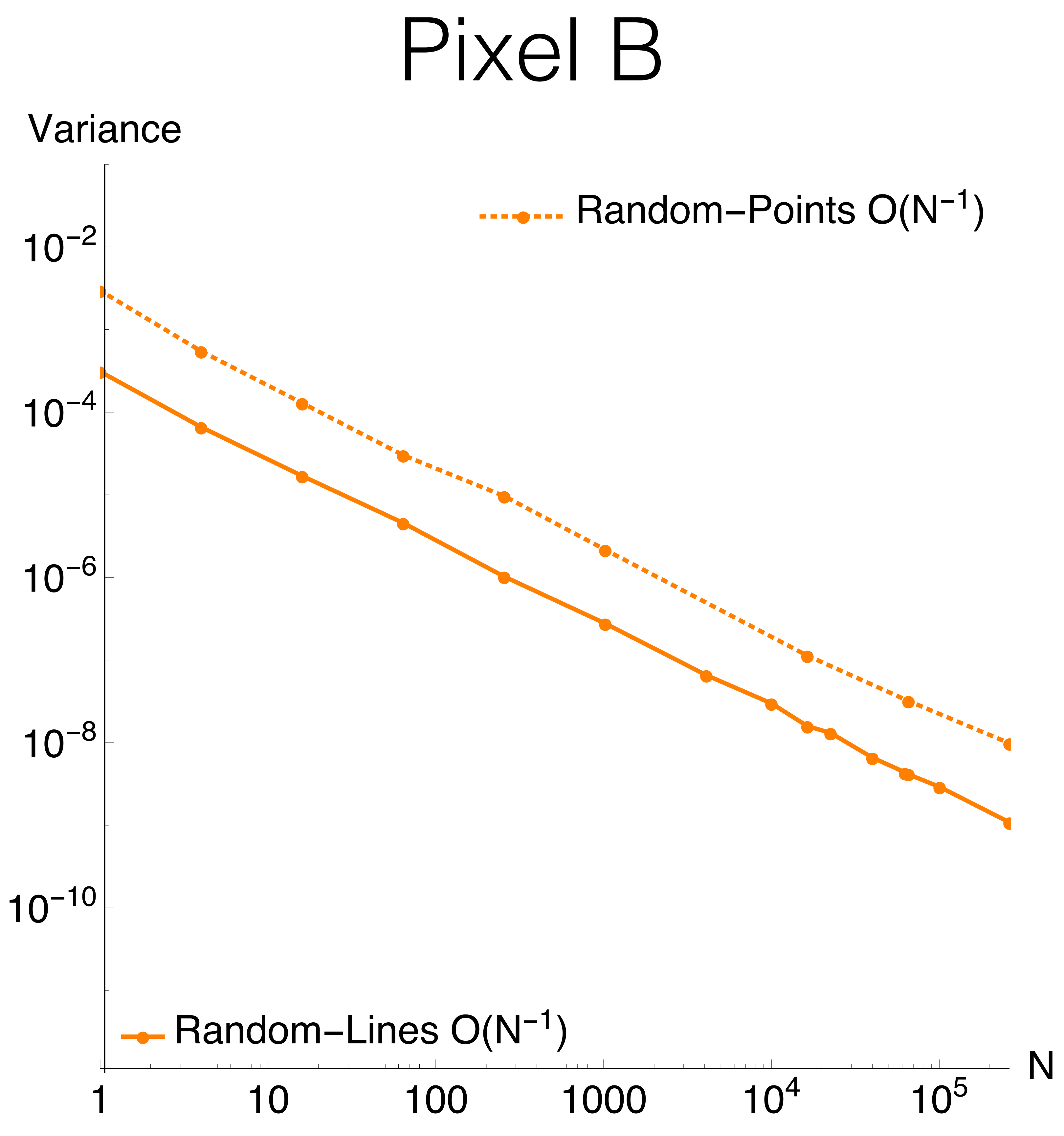


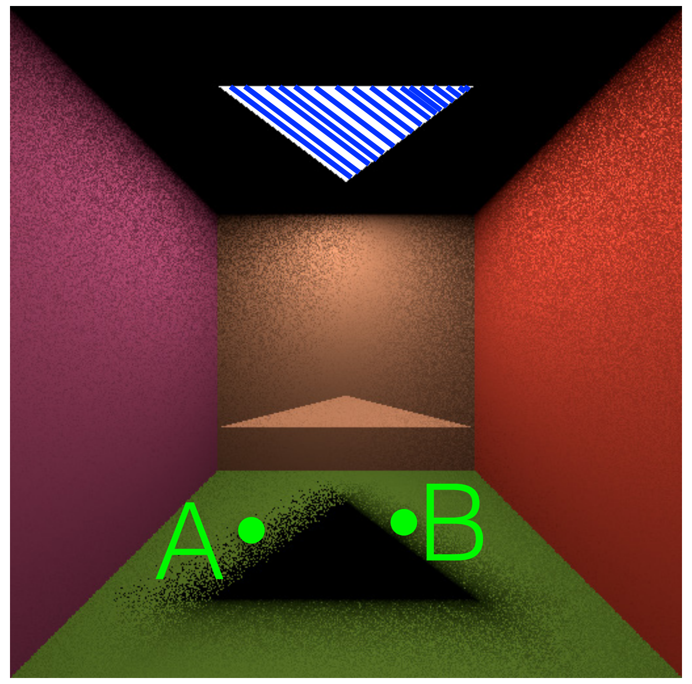


Cornell box Scene: Line Orientations

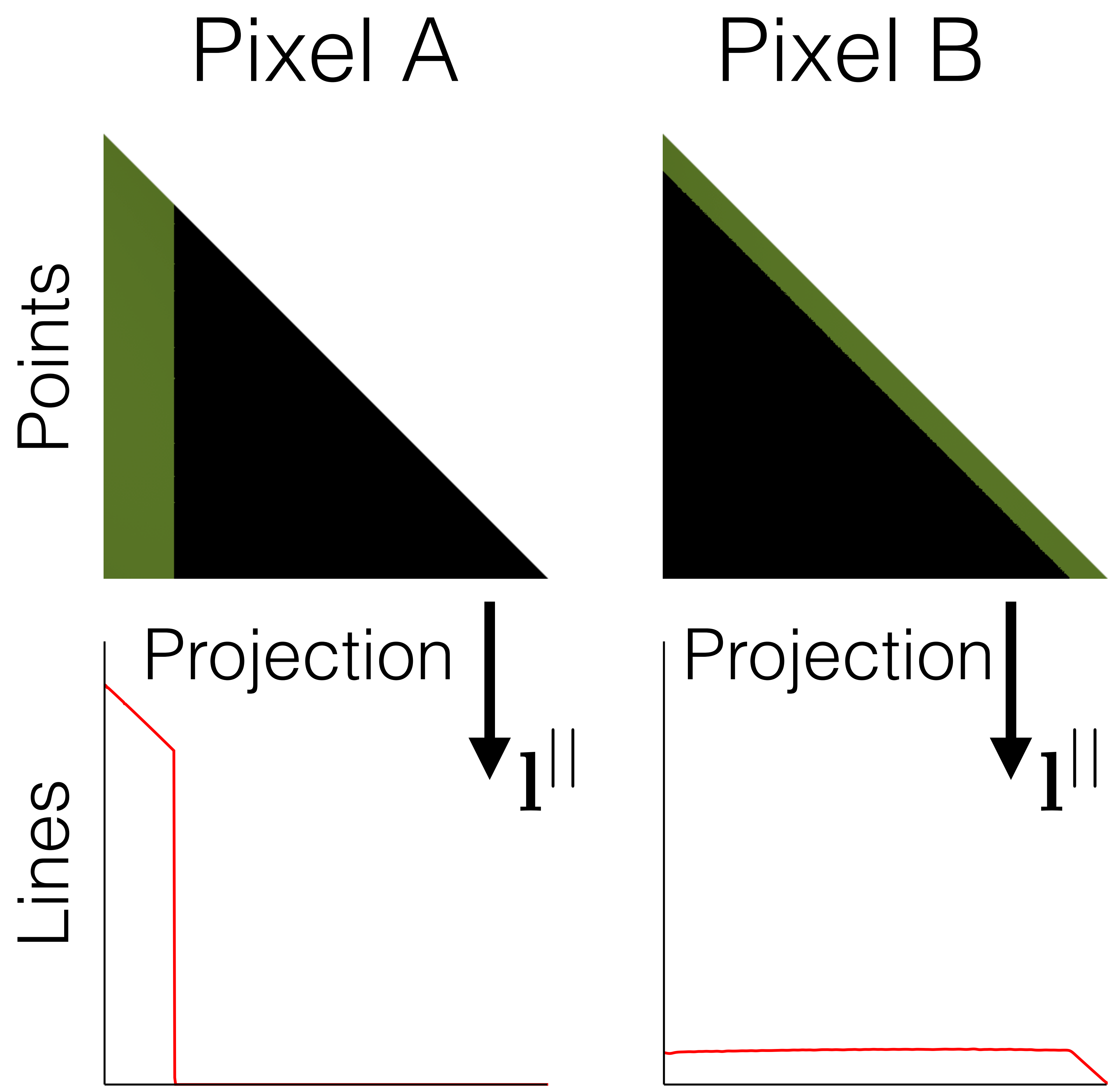


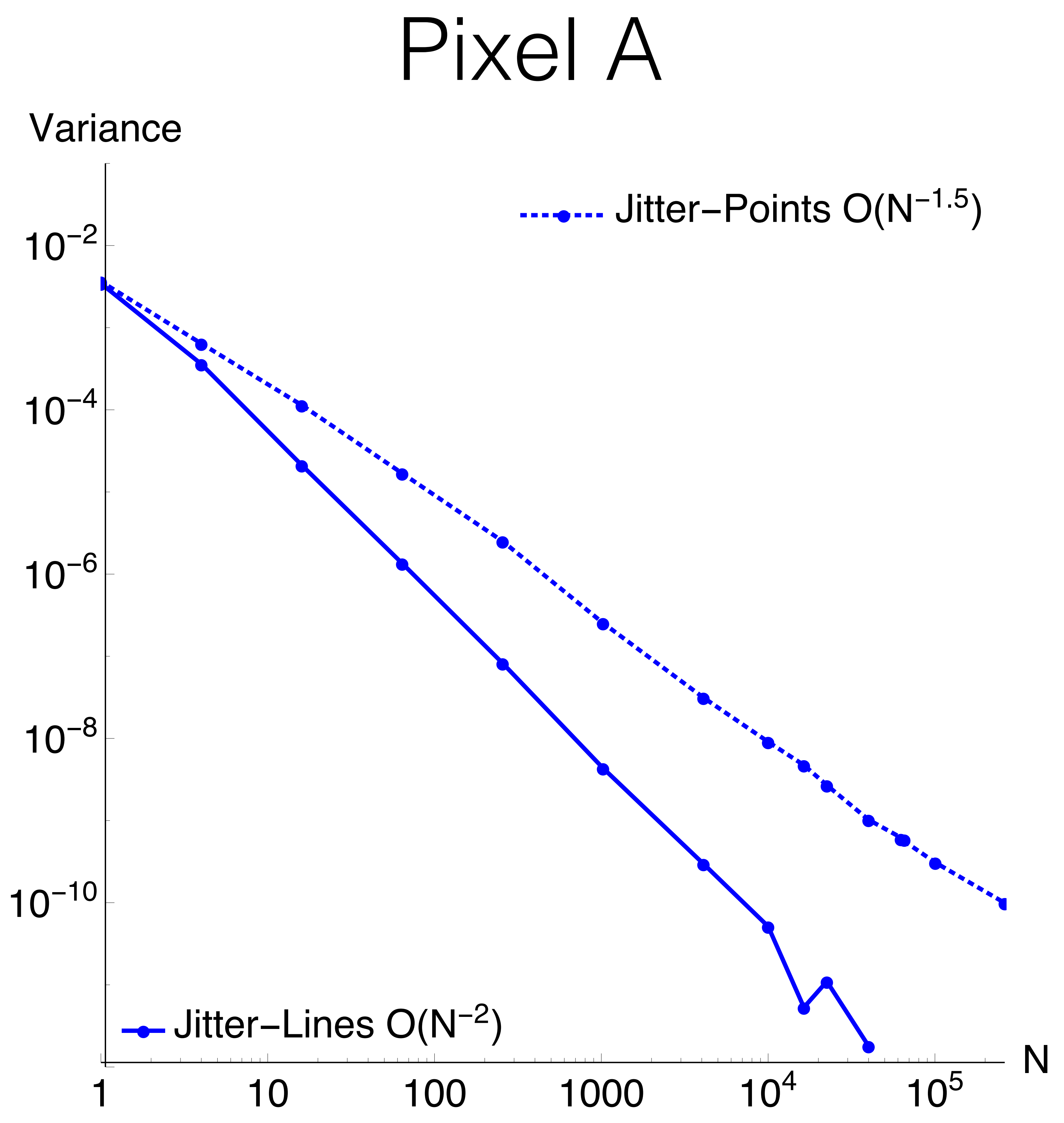


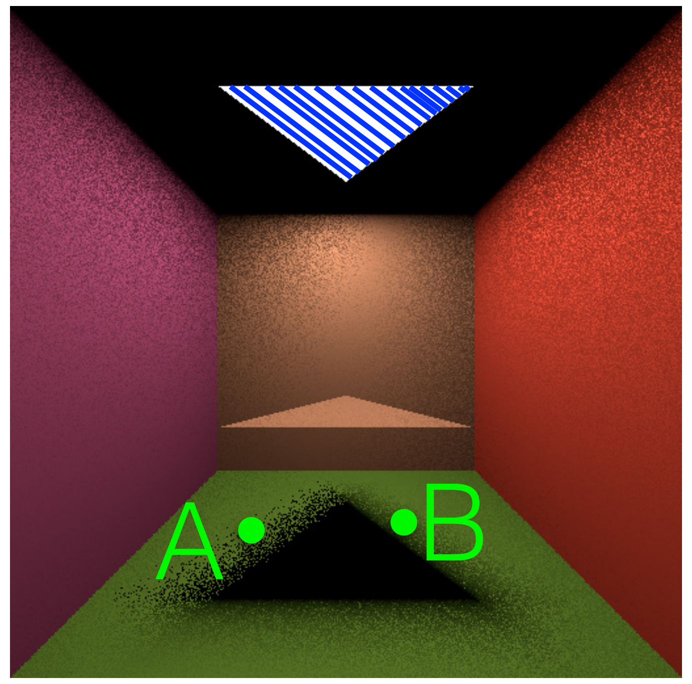




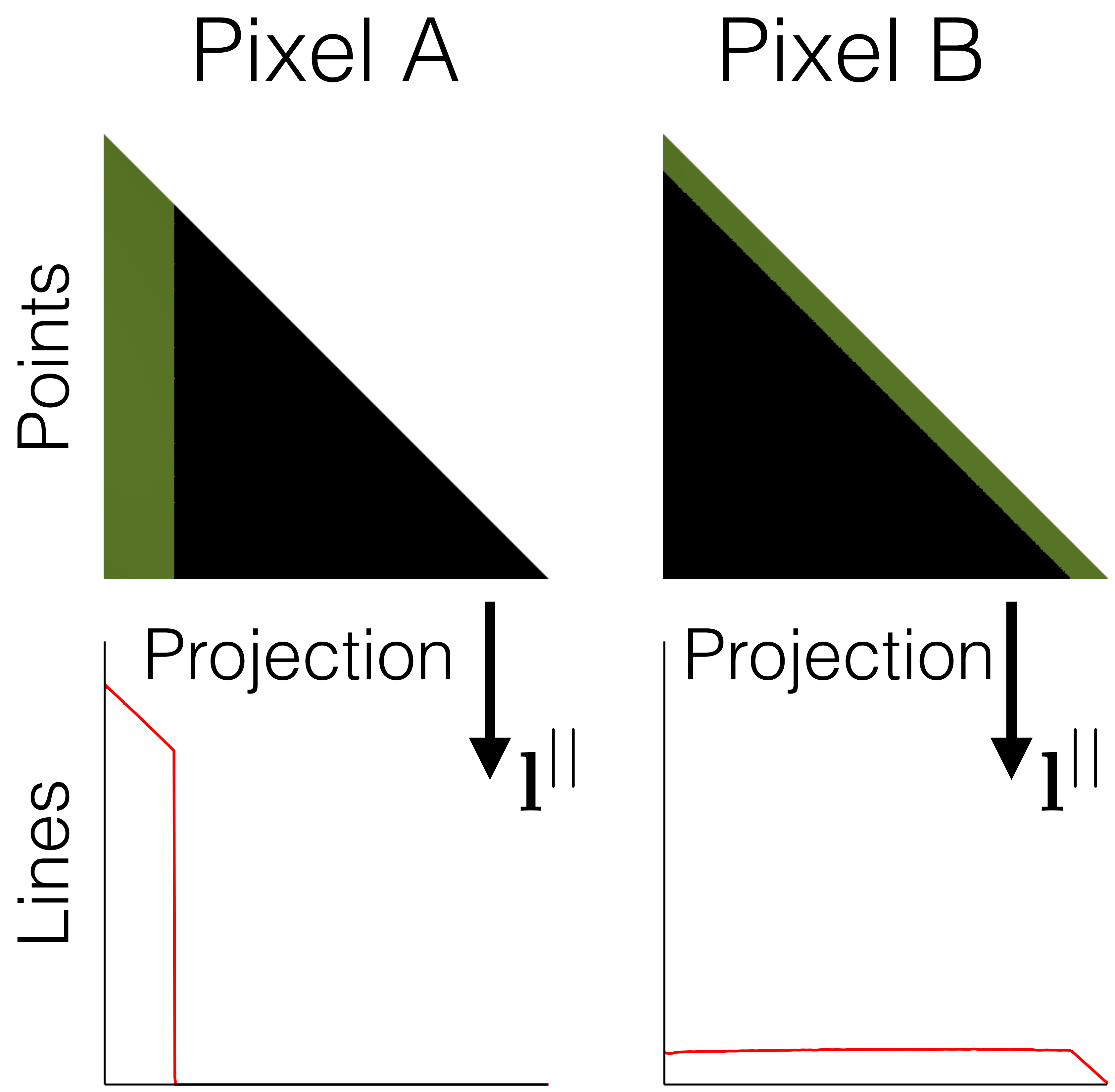
Cornell box Scene: Line Orientations

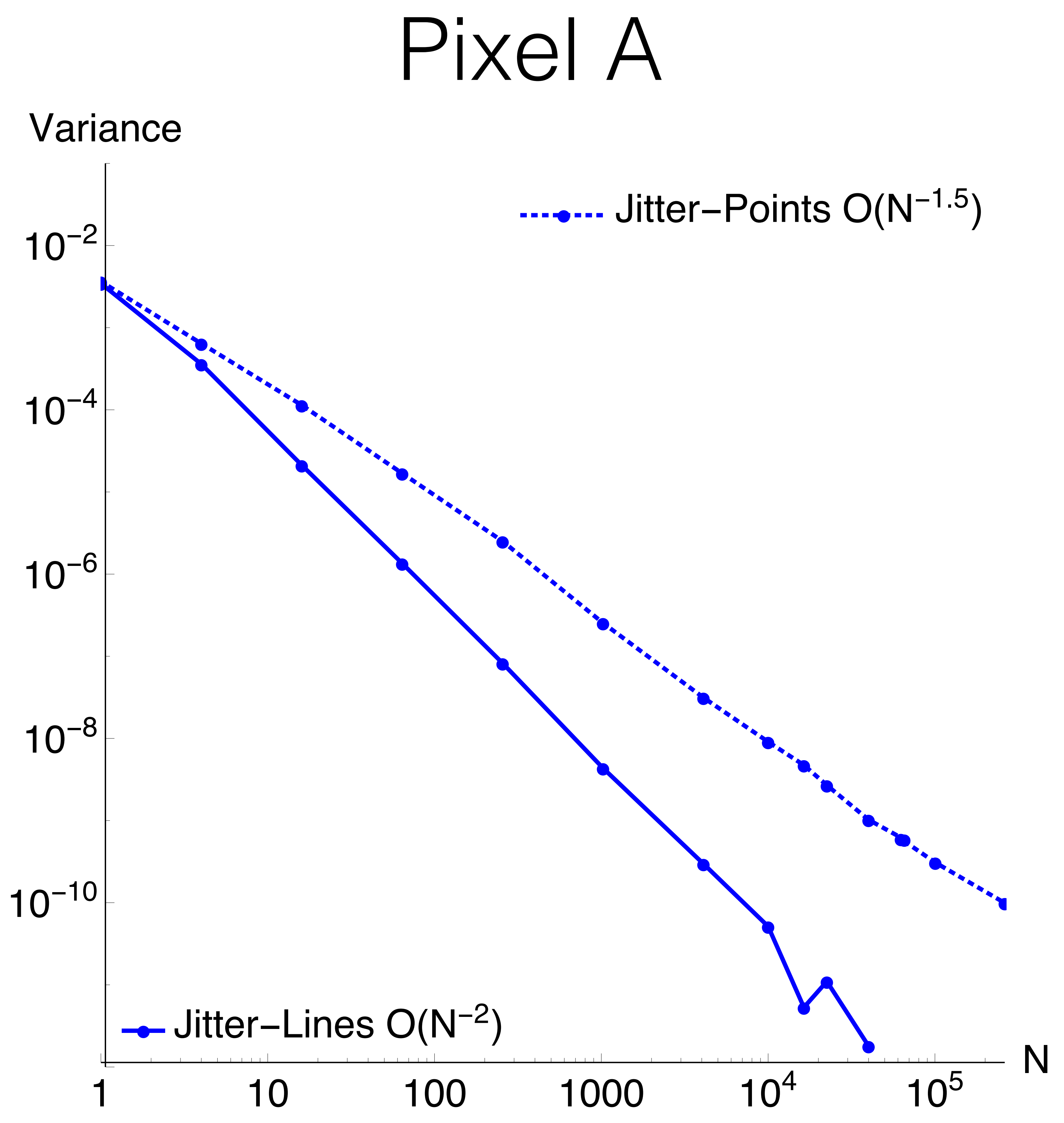


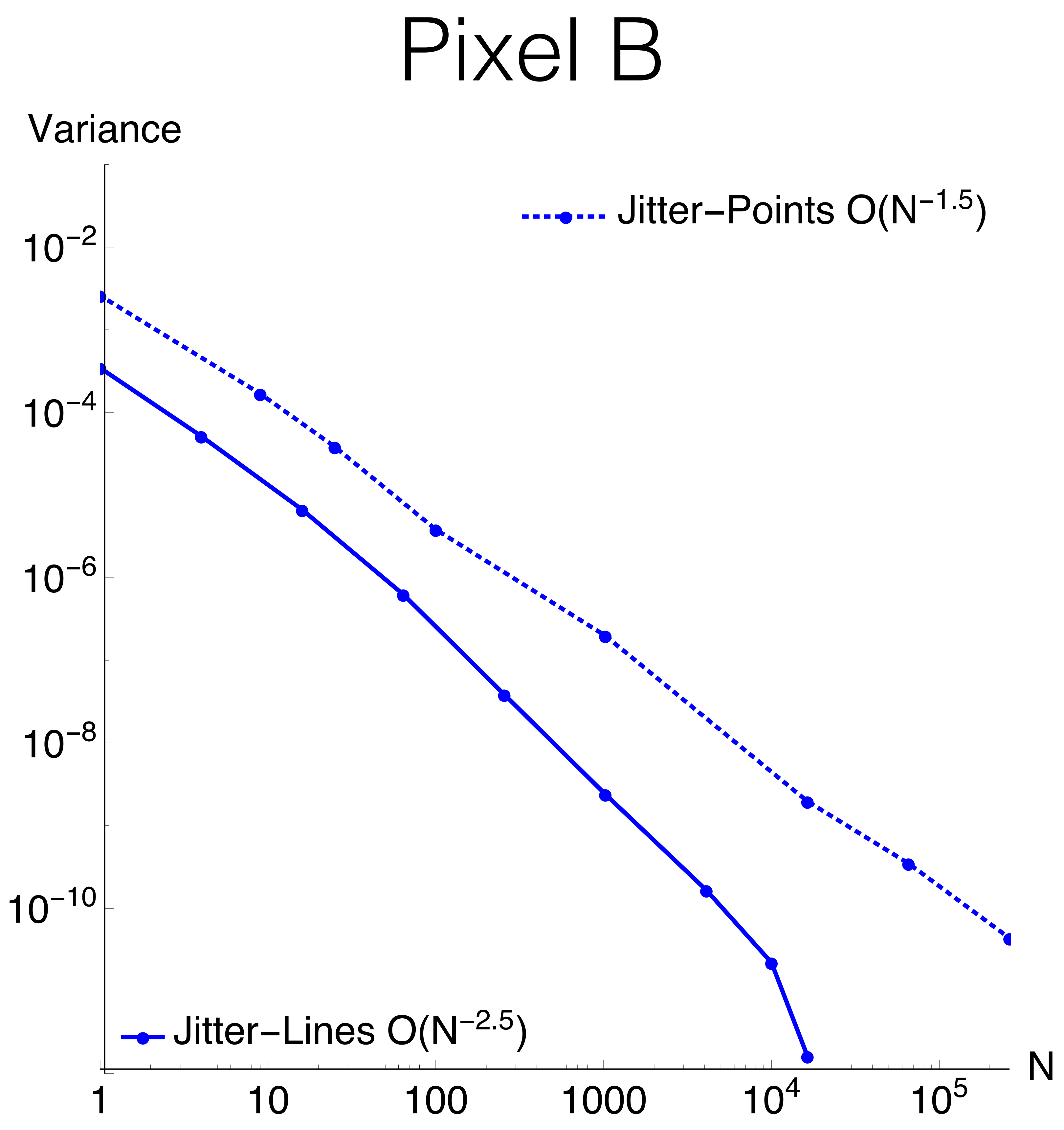




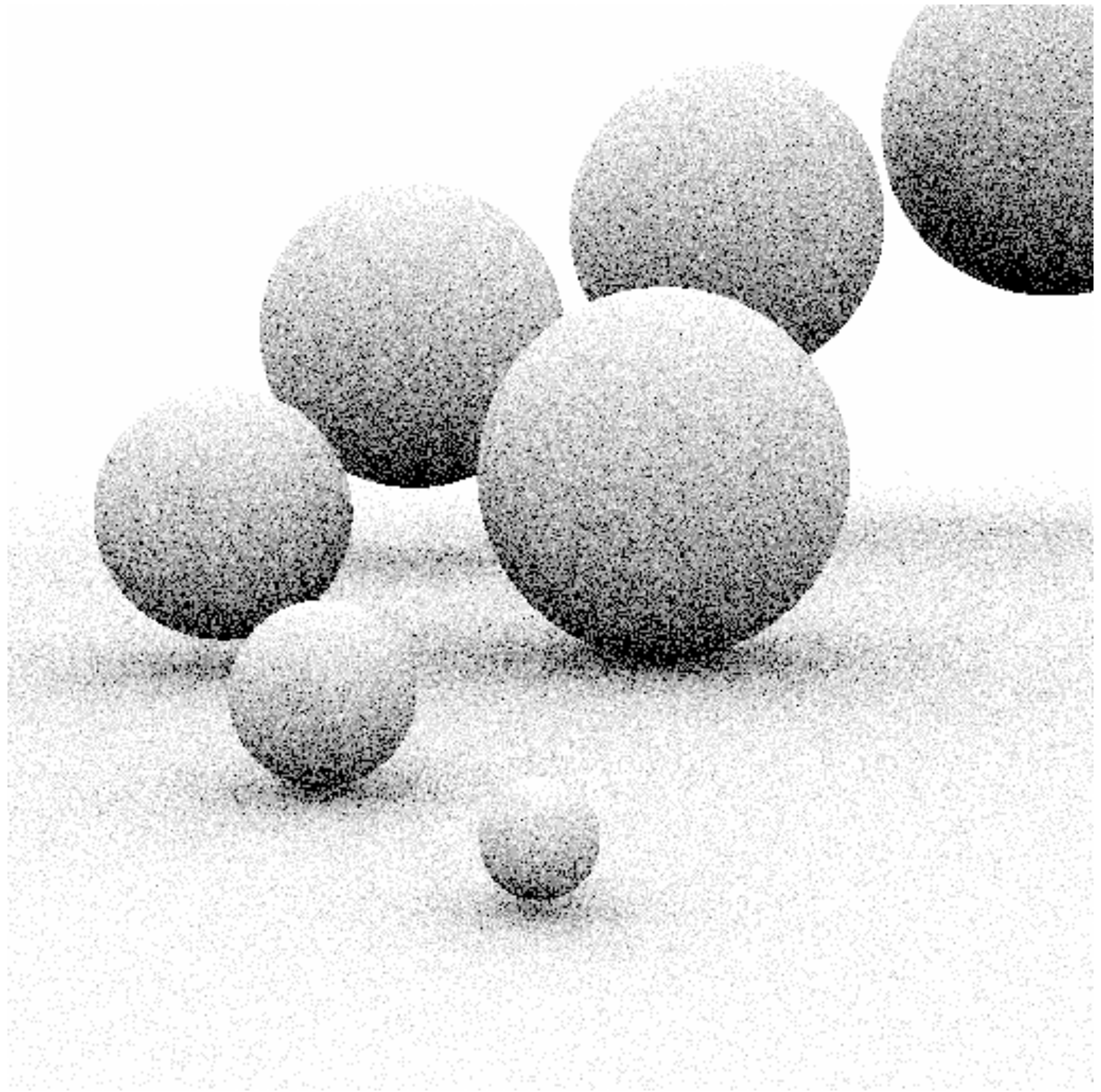
Cornell box Scene: Line Orientations



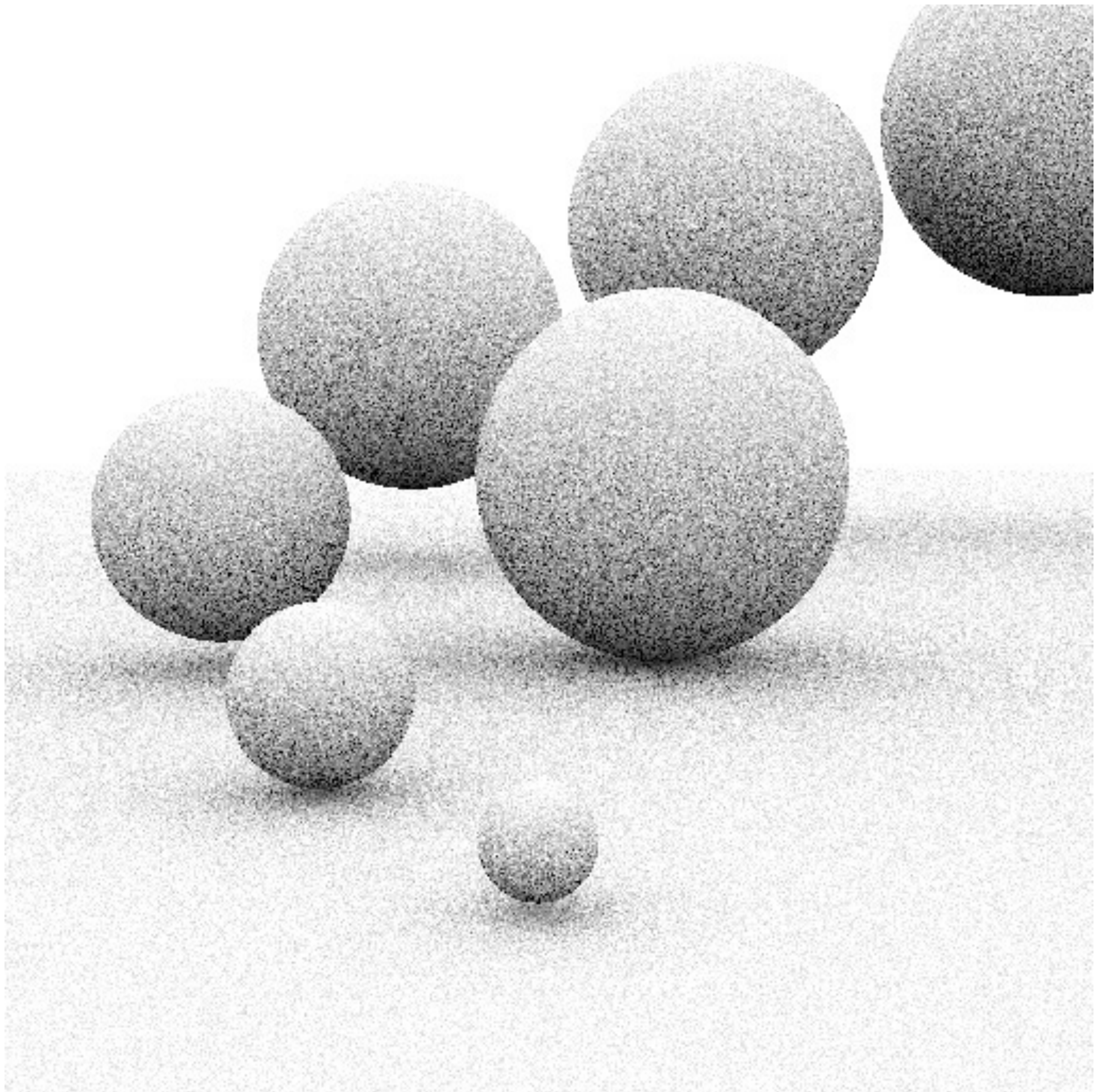




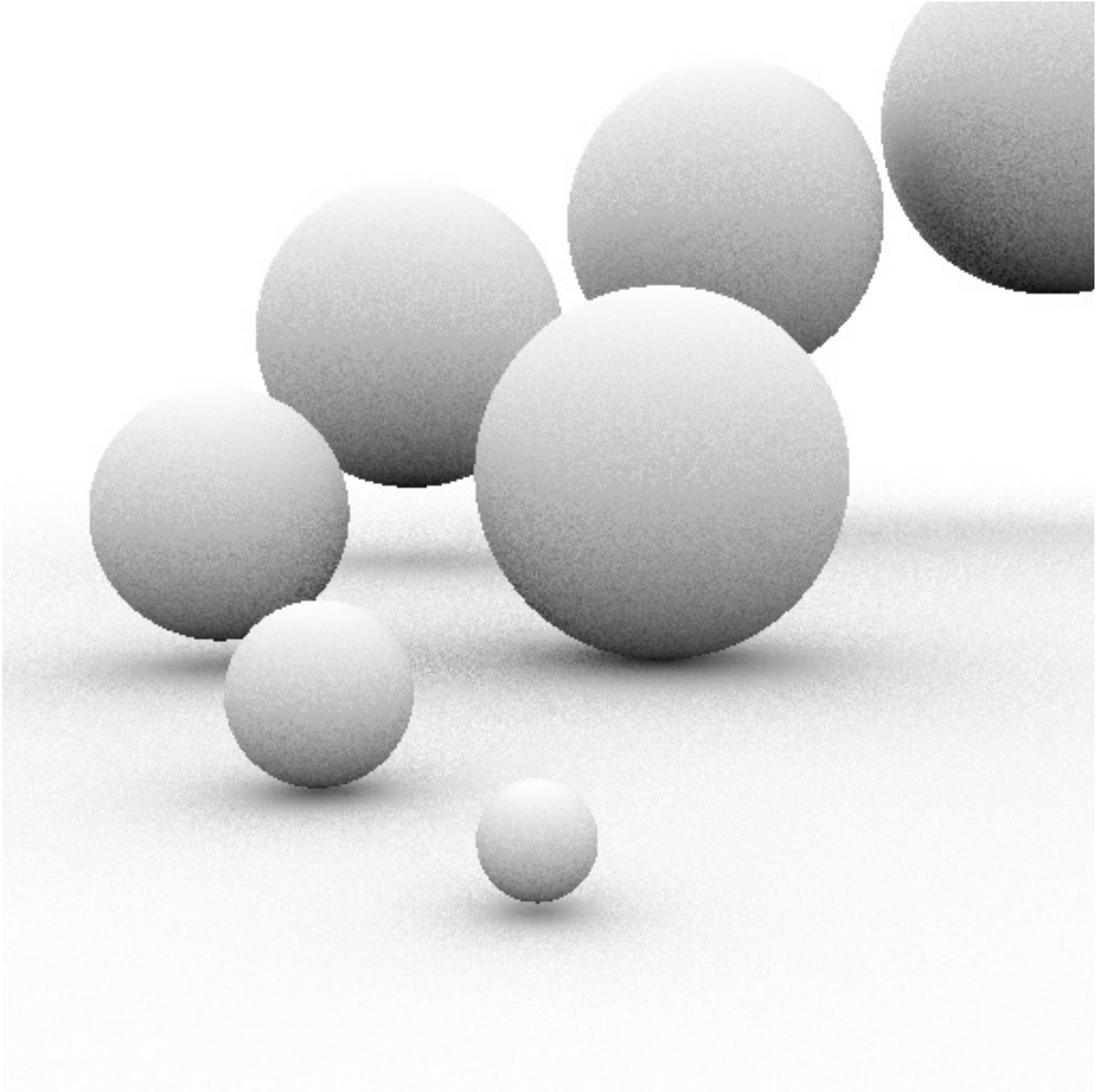
Ambient Occlusion: Points, Segments and Lines



Points

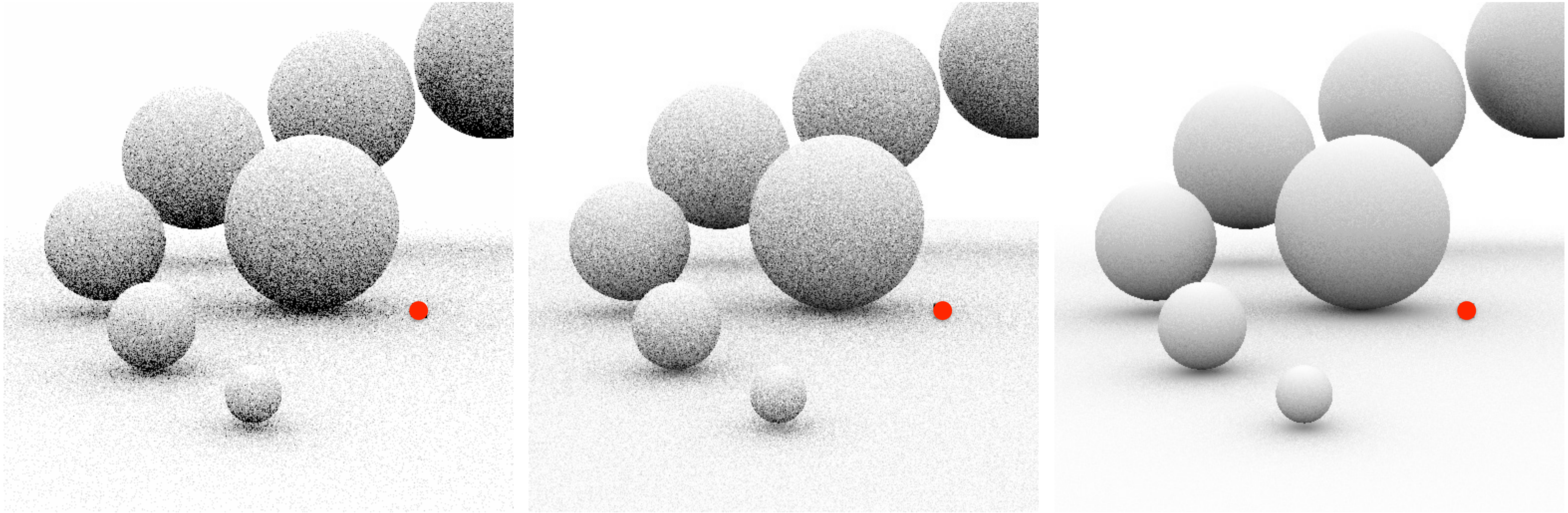


Segments



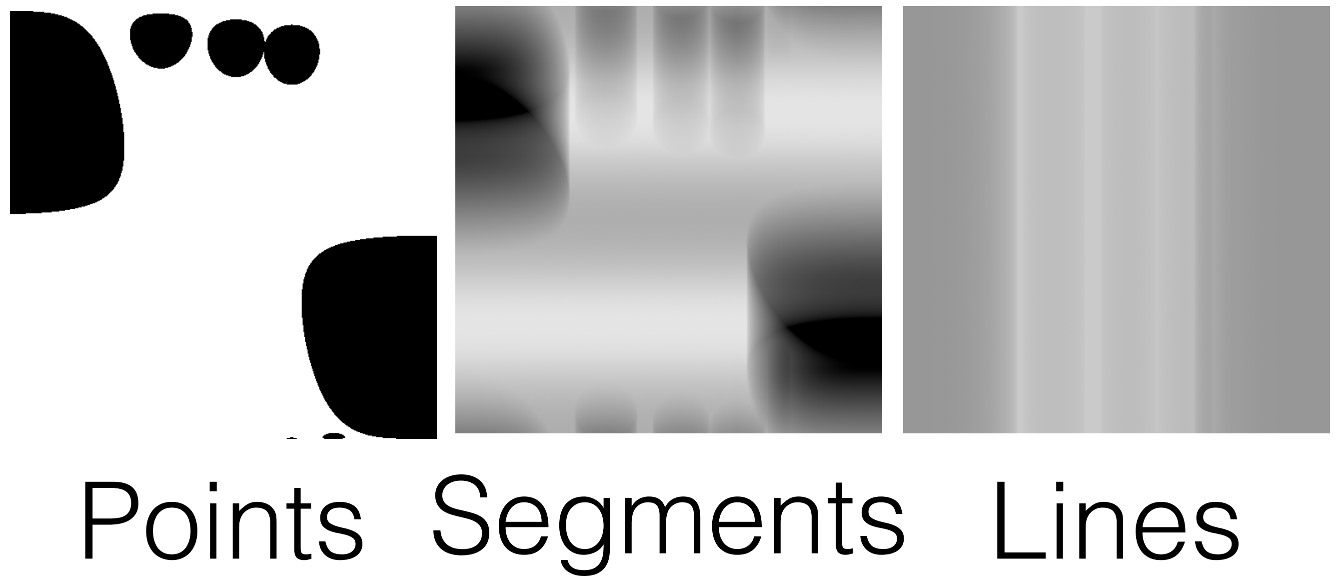
Lines

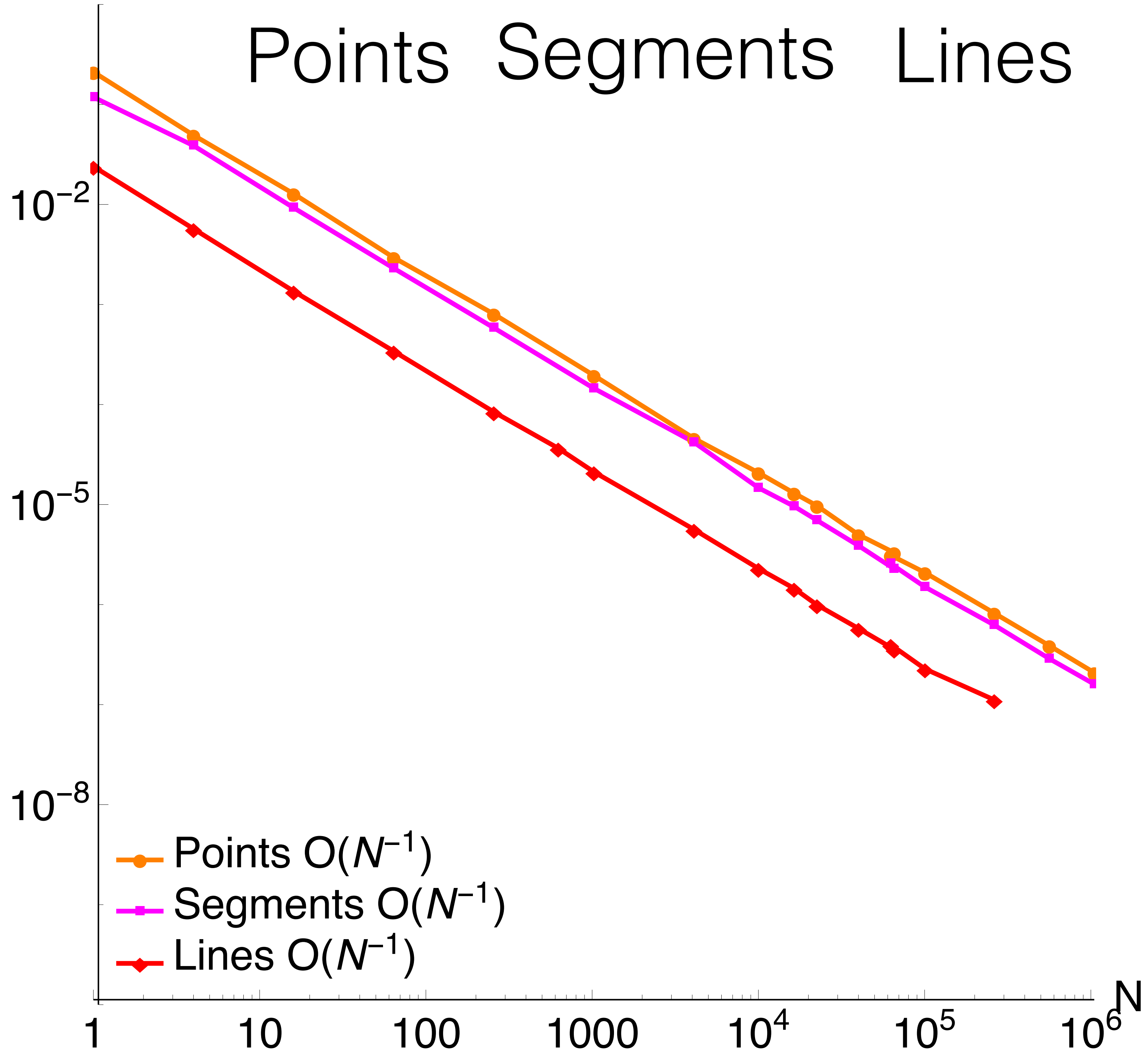
Ambient Occlusion: Points, Segments and Lines



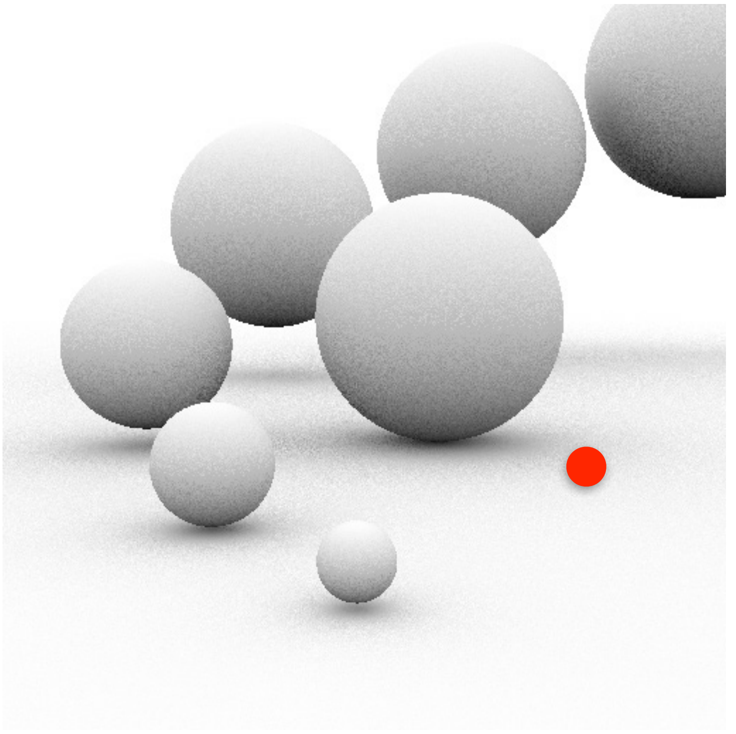
Points Segments Lines

Ambient Occlusion: Points, Segments and Lines

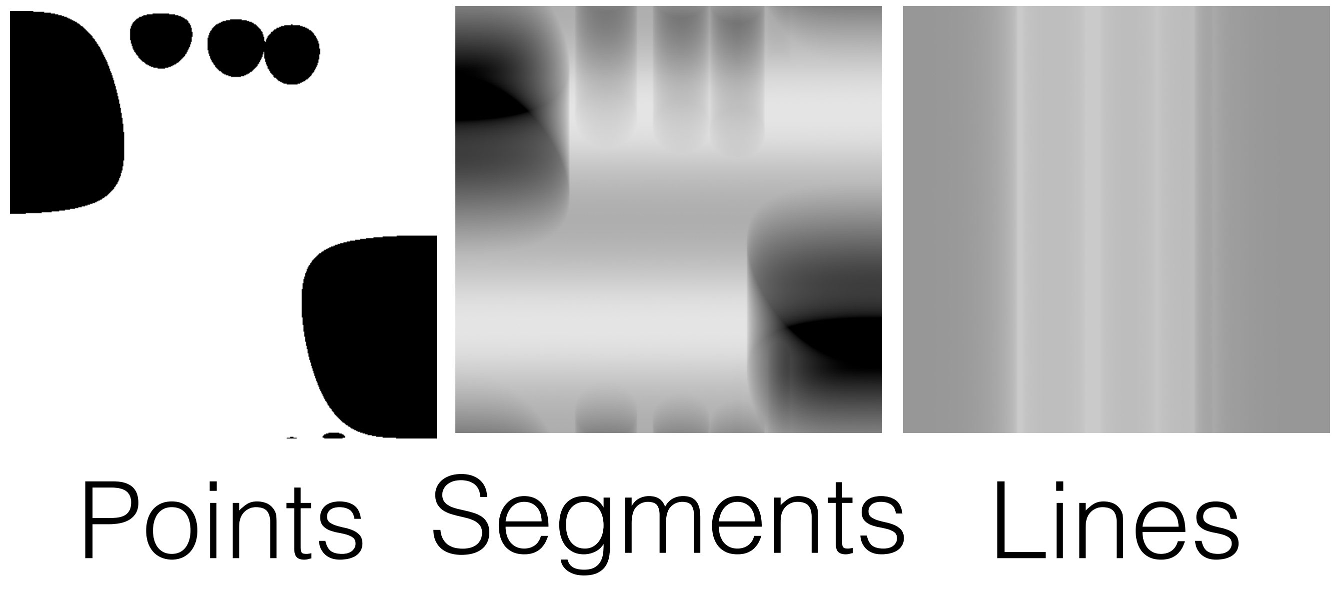


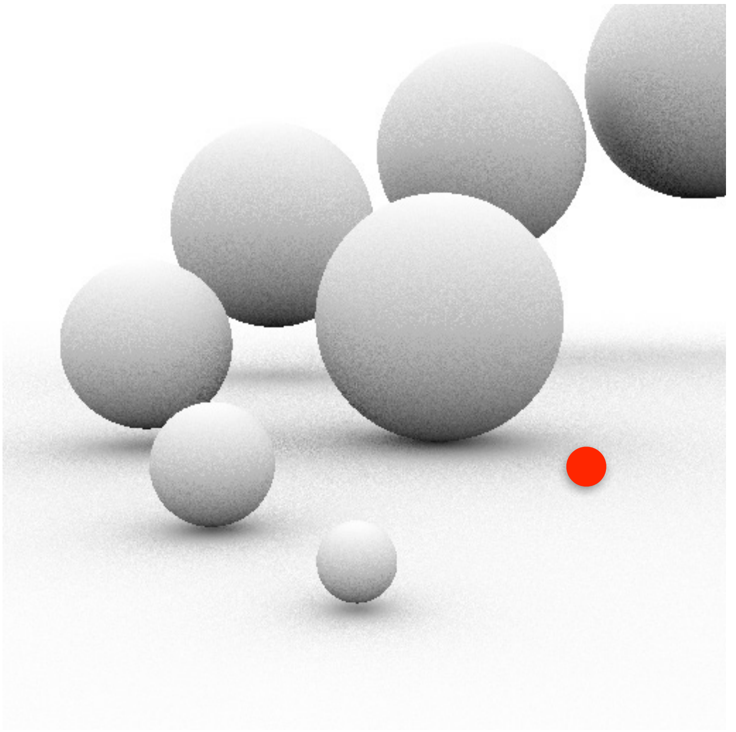


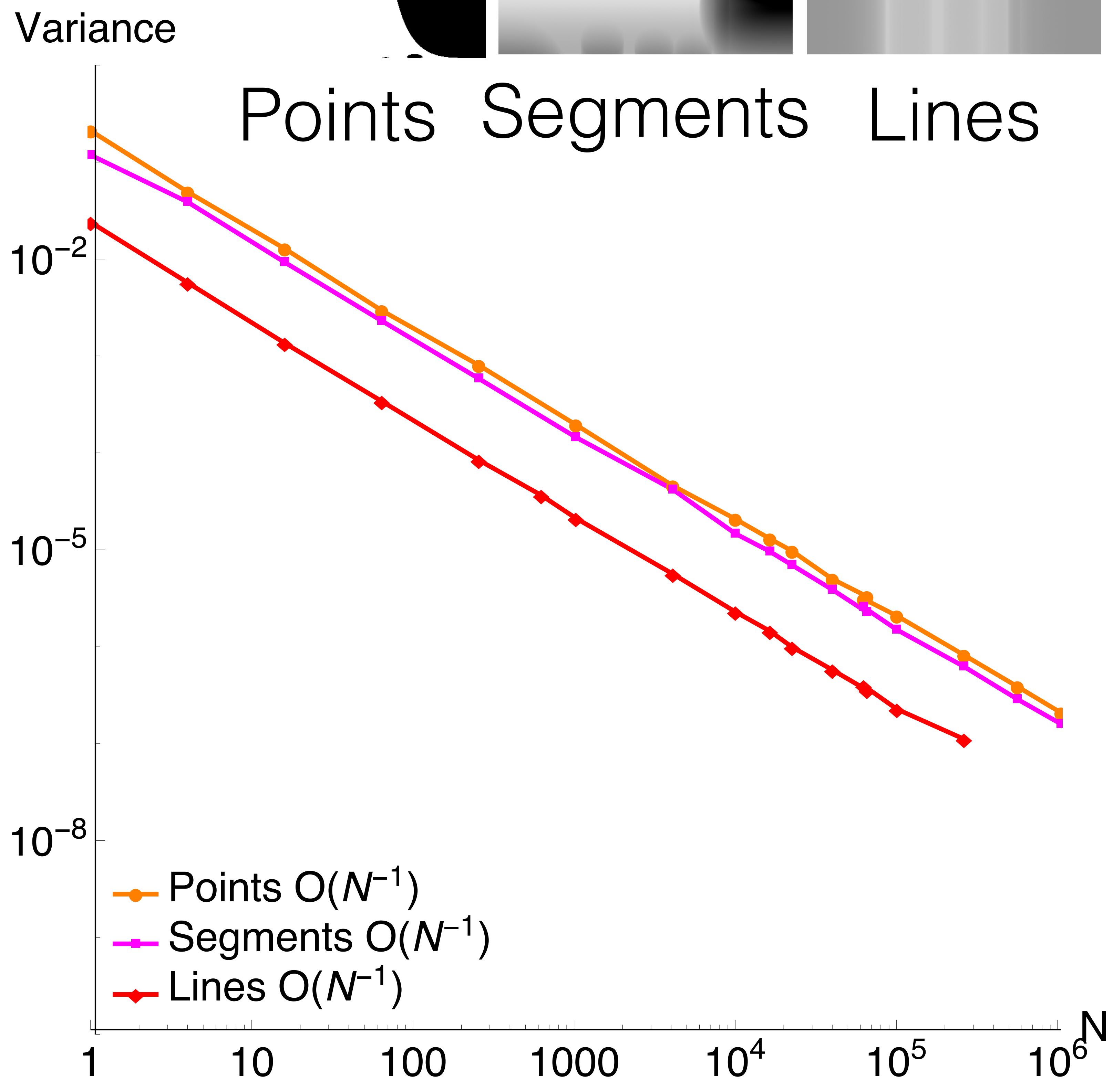
Random



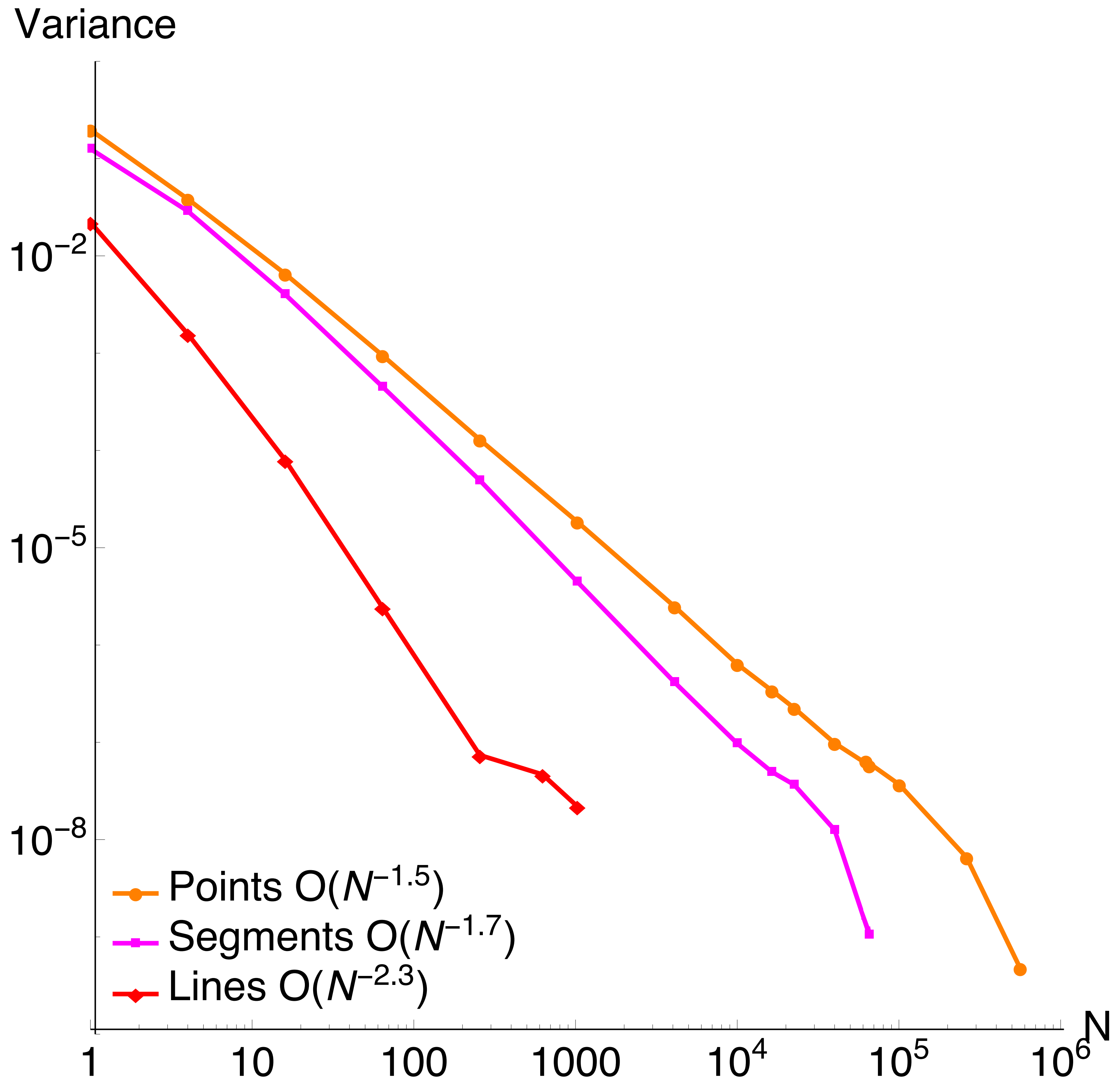
Ambient Occlusion: Points, Segments and Lines





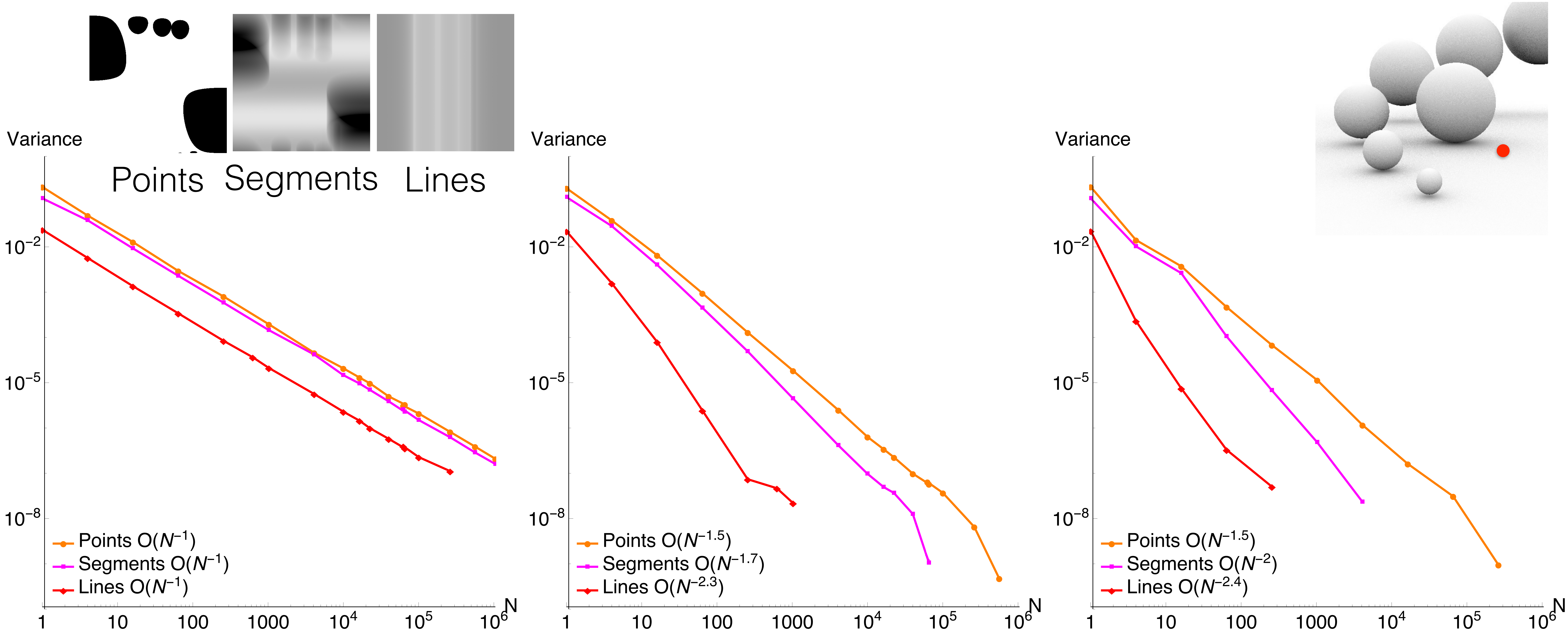


Random



Jitter

Ambient Occlusion: Points, Segments and Lines



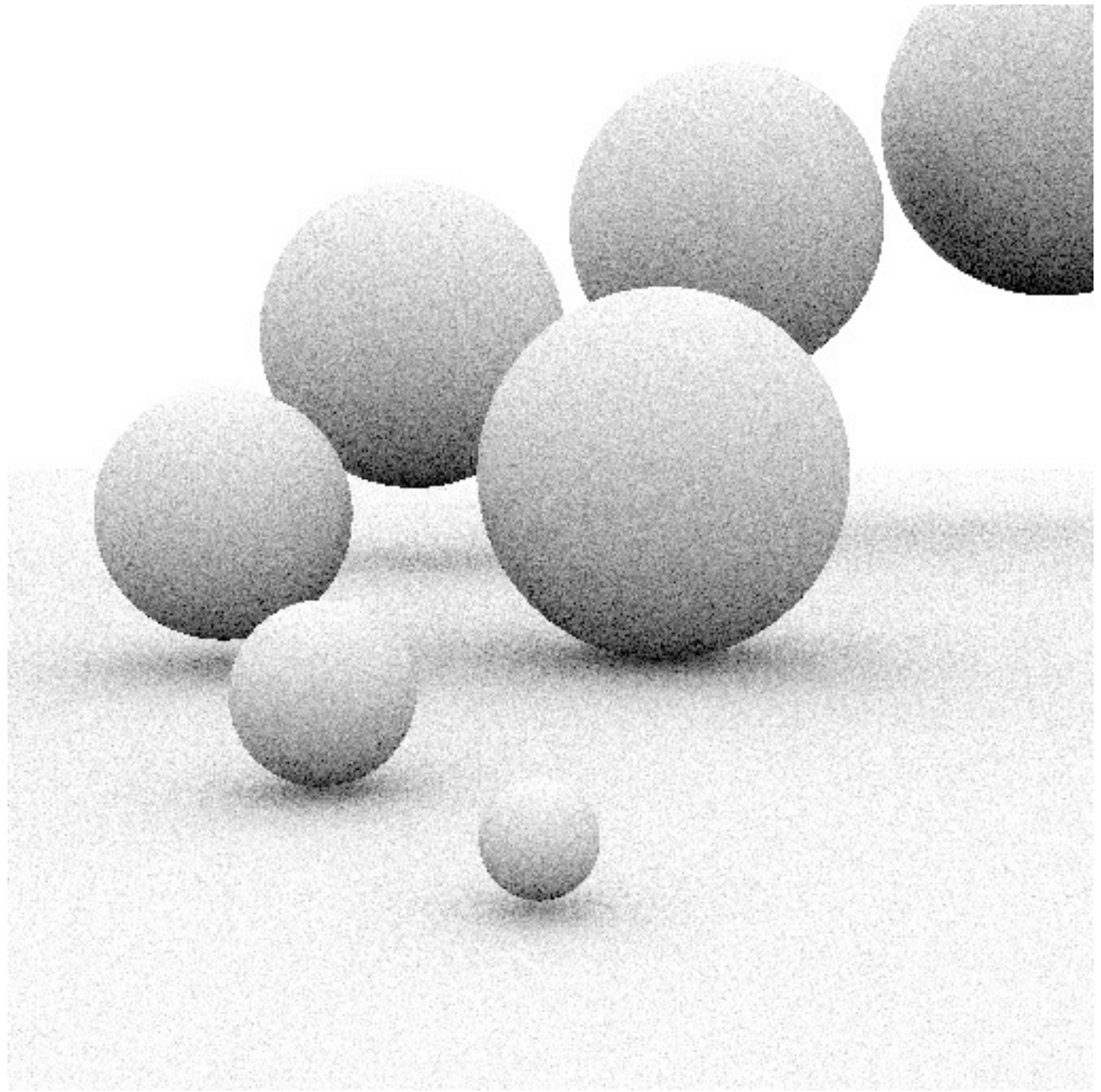
Random

Jitter

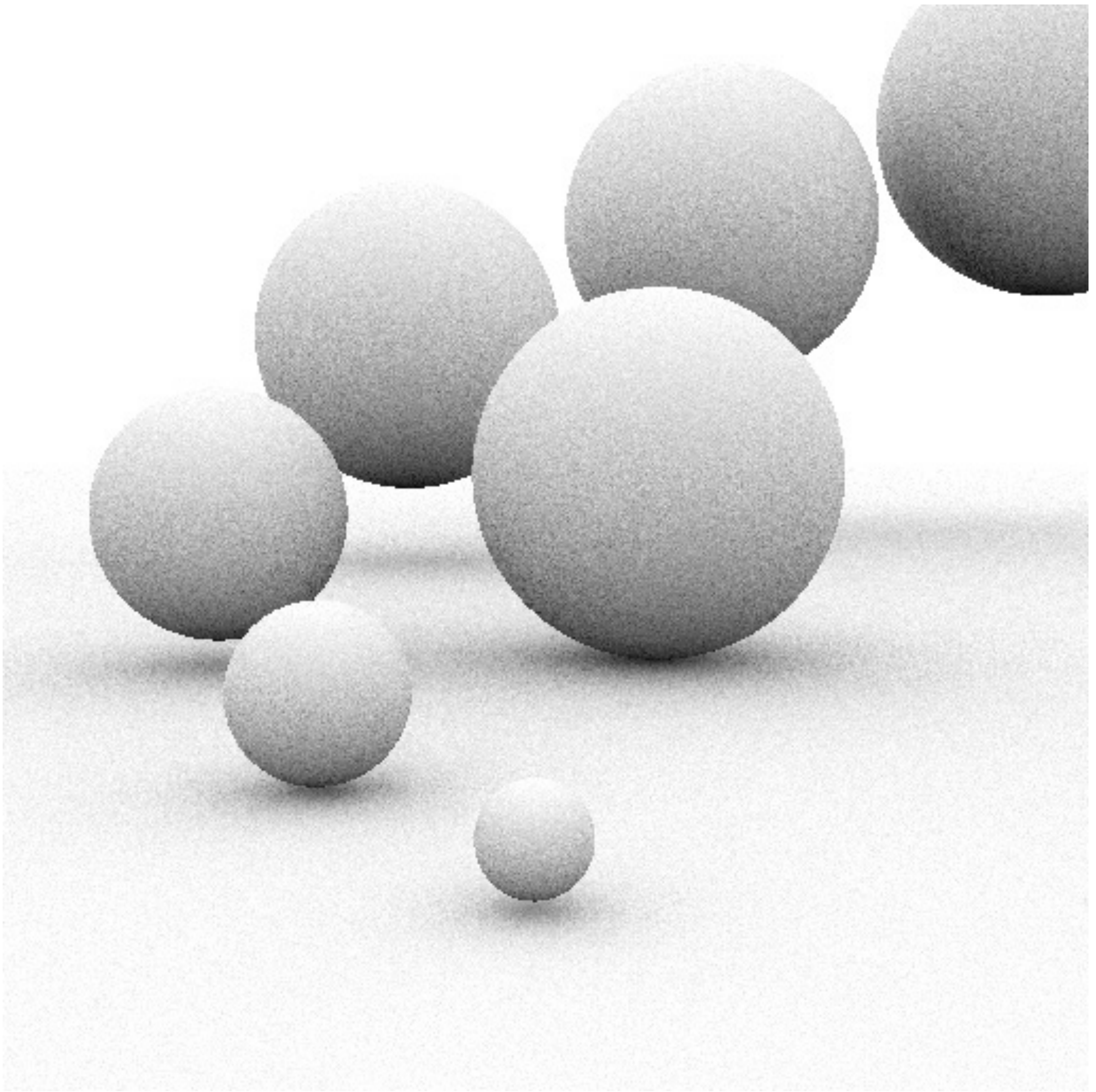
Low discrepancy

Ambient Occlusion: Segments Varying Length

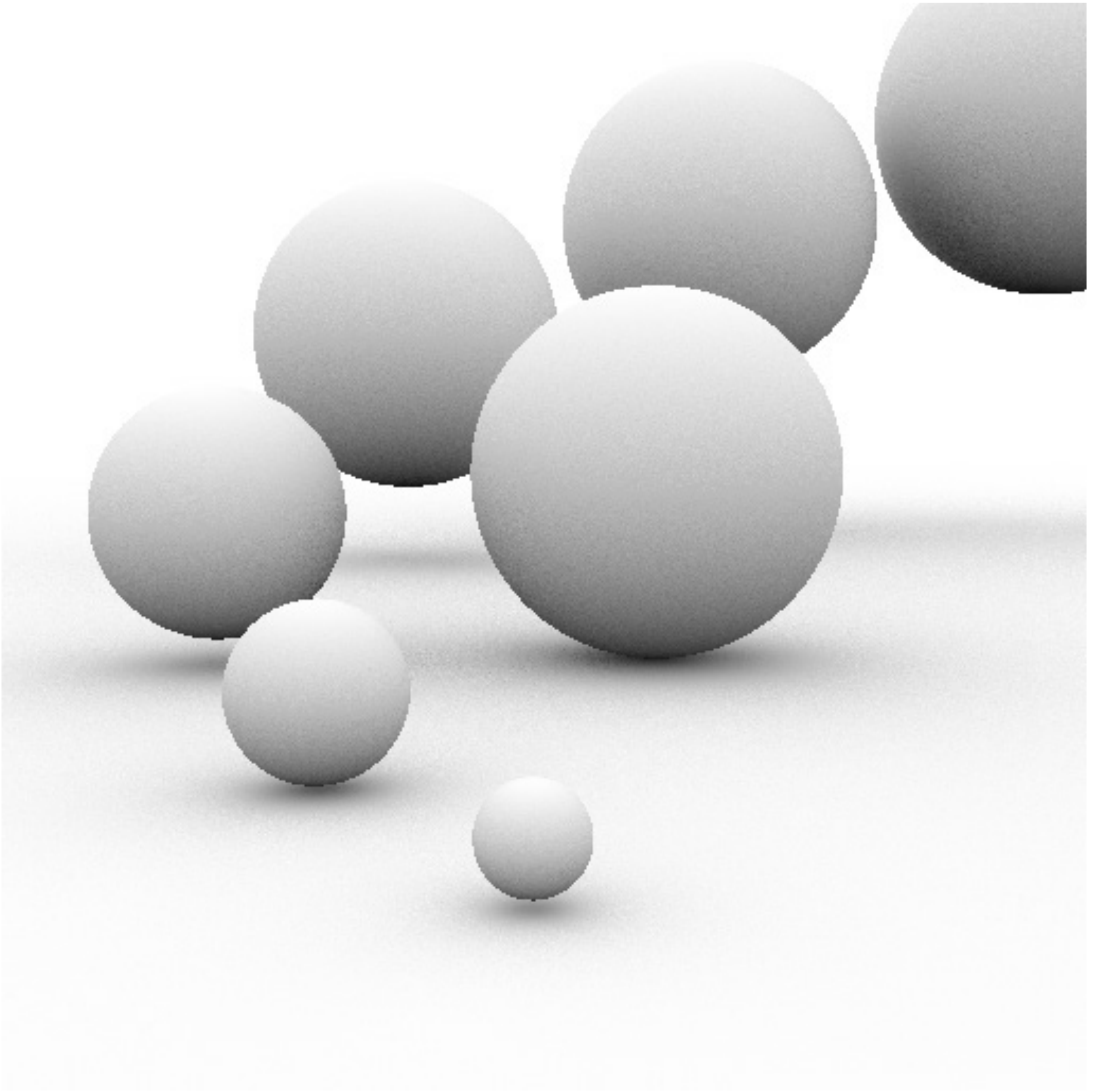
Segment length = 0.001



Segment length = 1



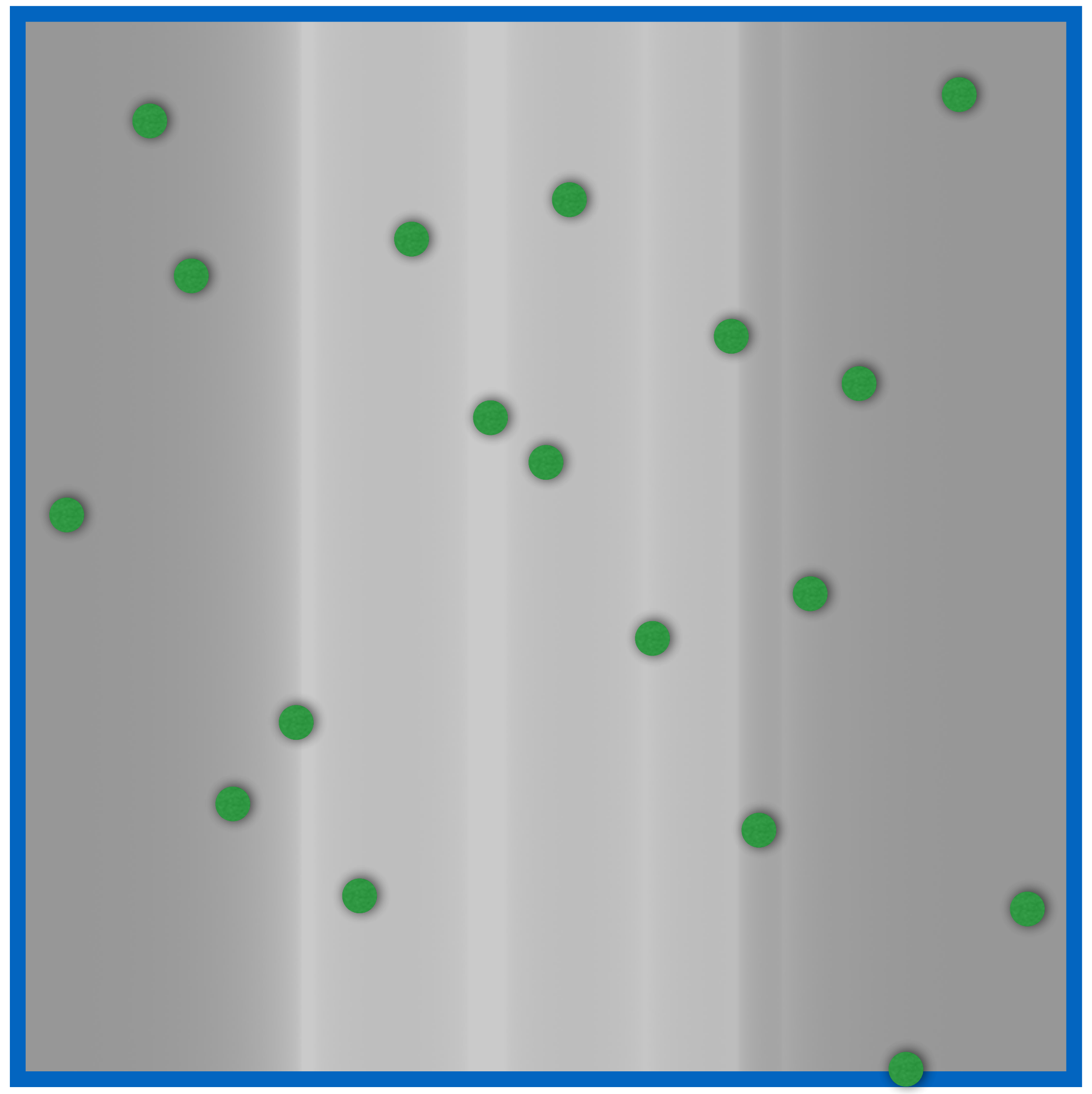
Segment length = π

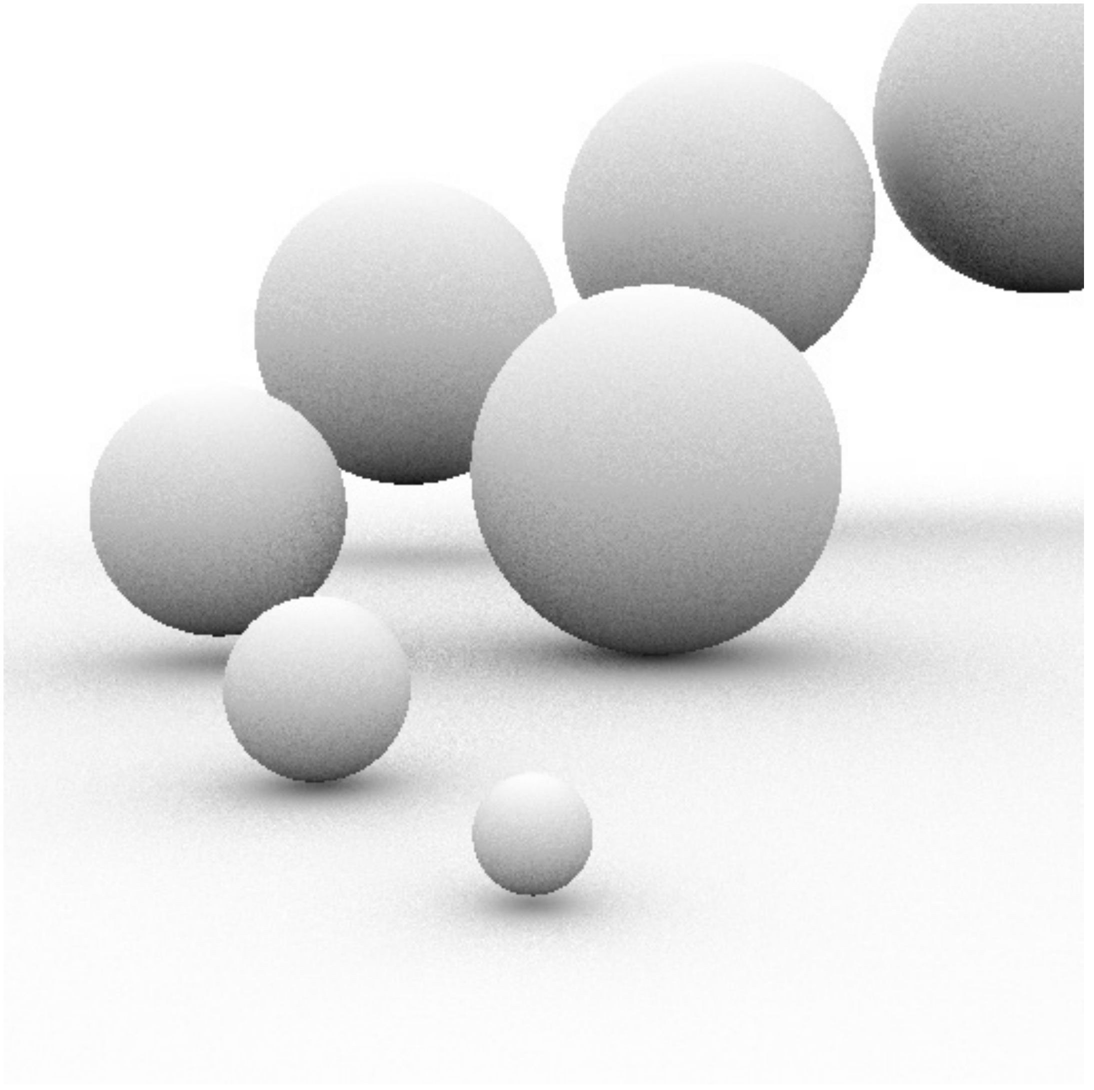


Jittered Samples

Segment Offsets vs Line Offsets

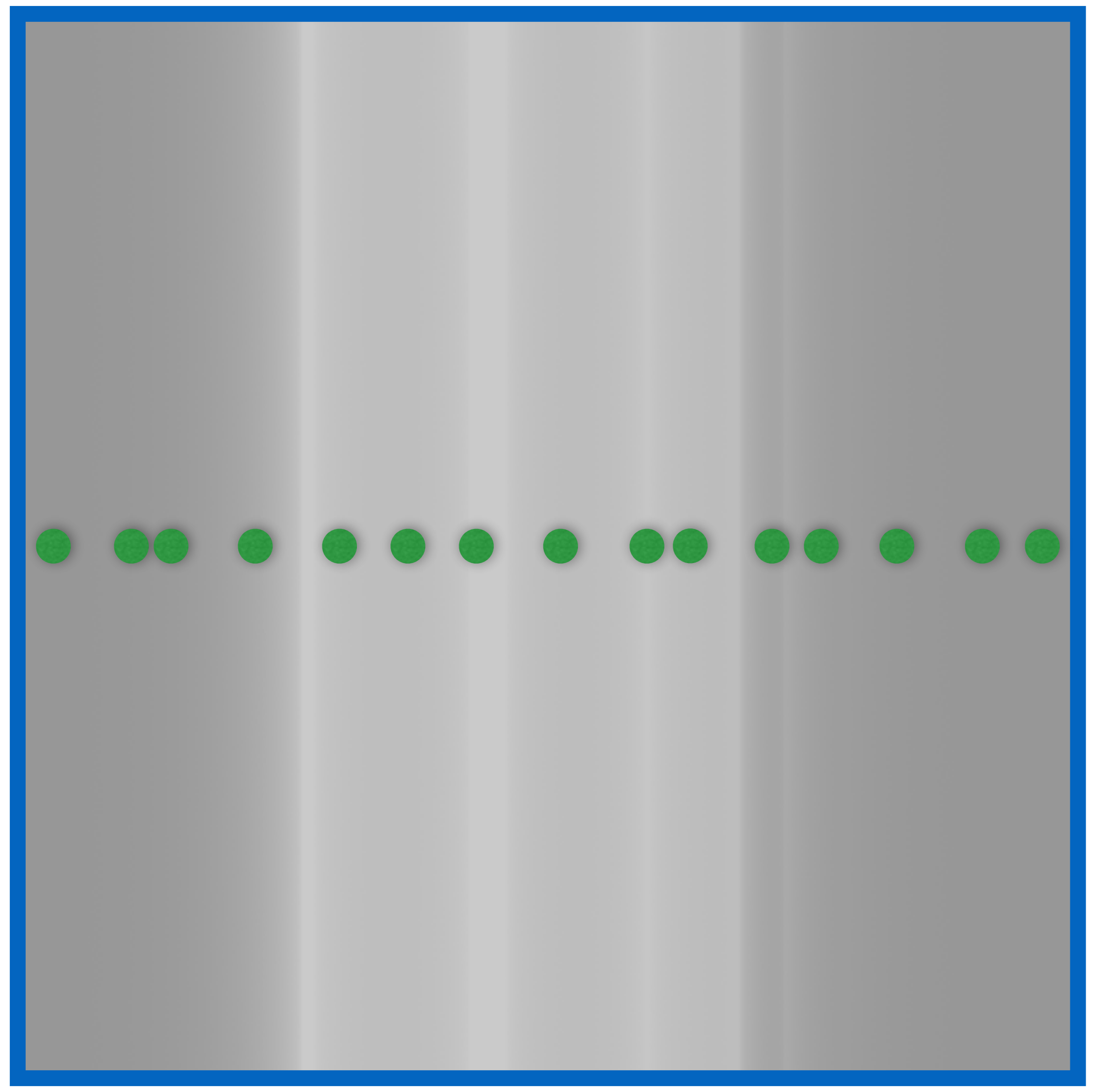
Segments

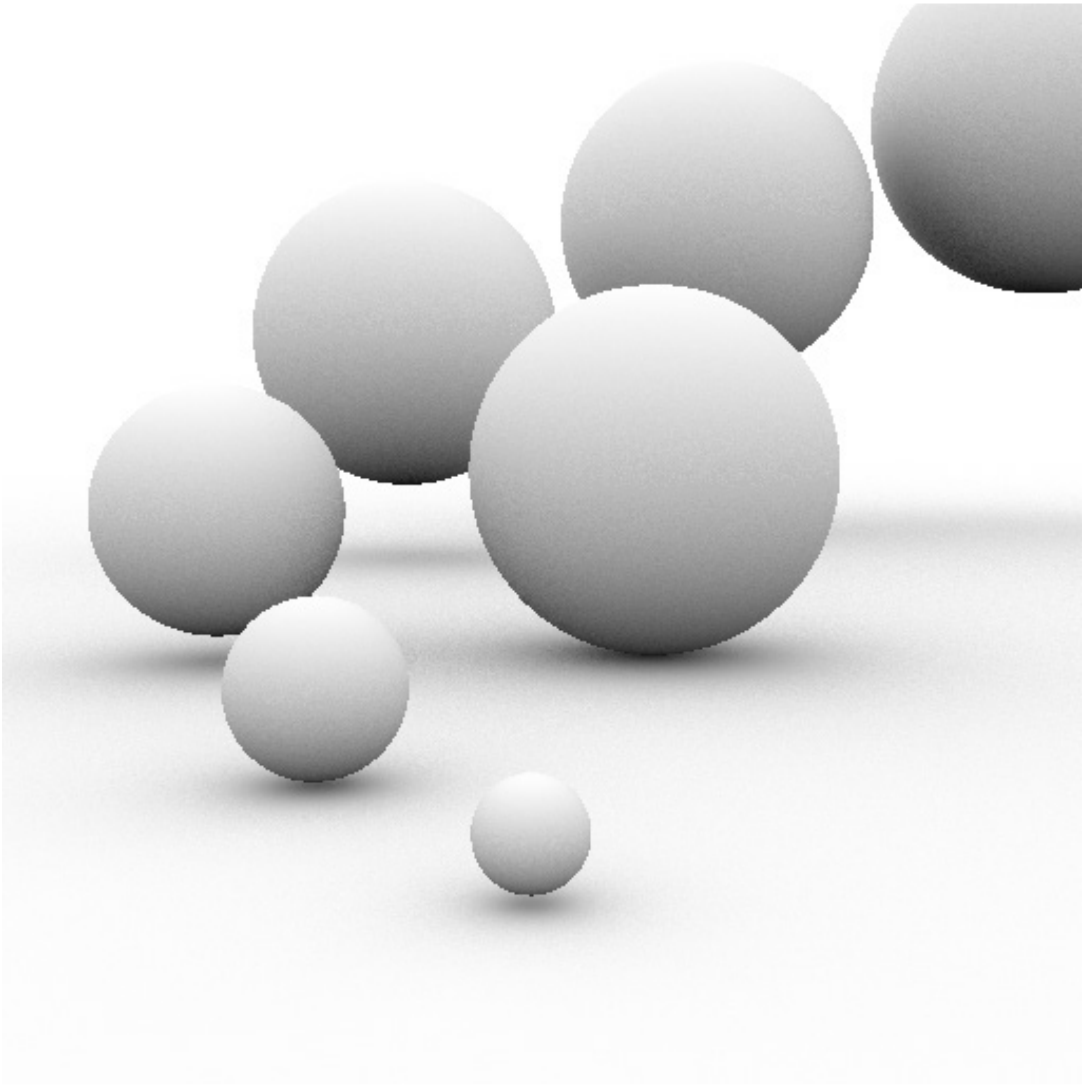




length = π

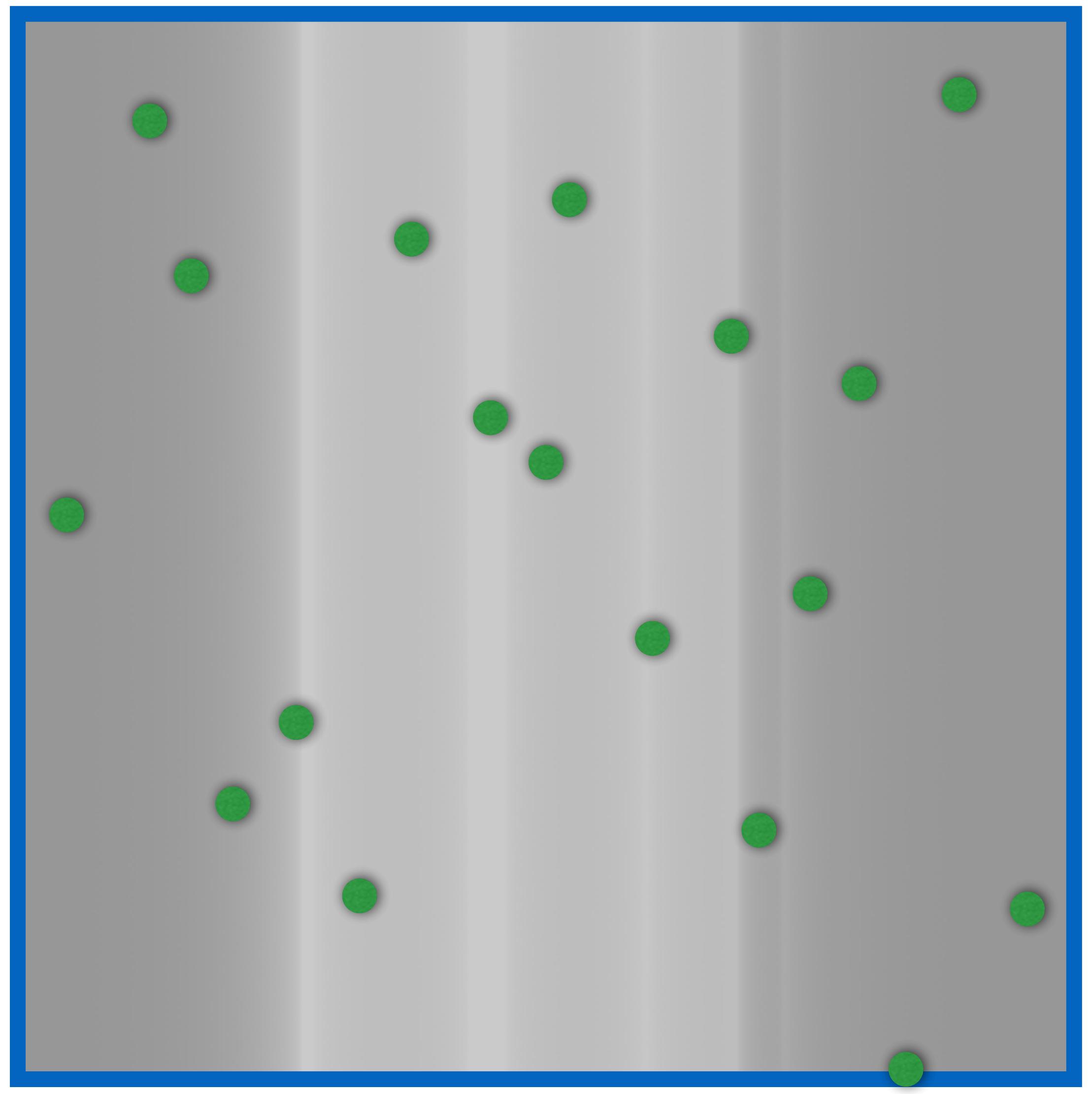
Lines

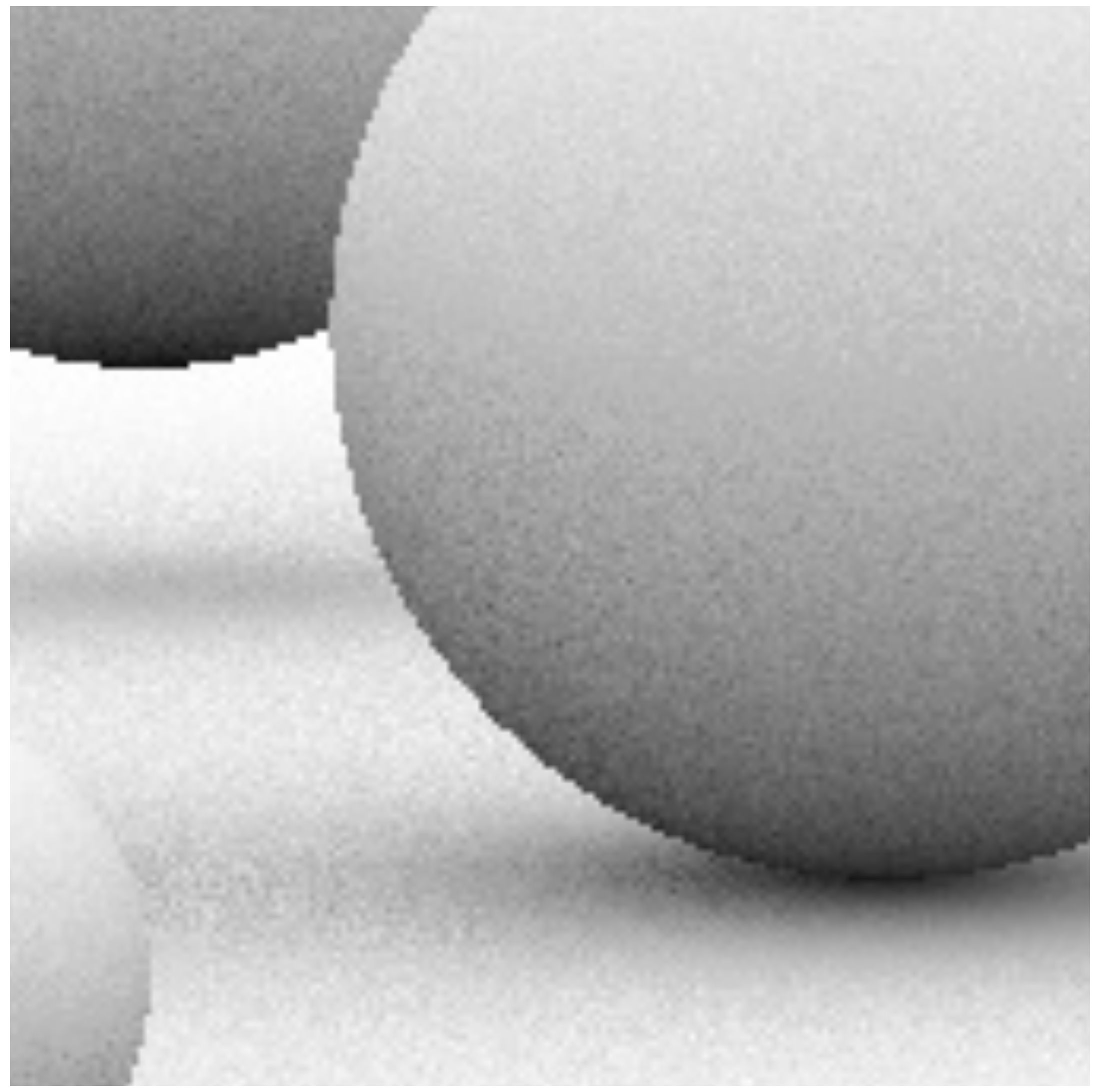




Segment Offsets vs Line Offsets

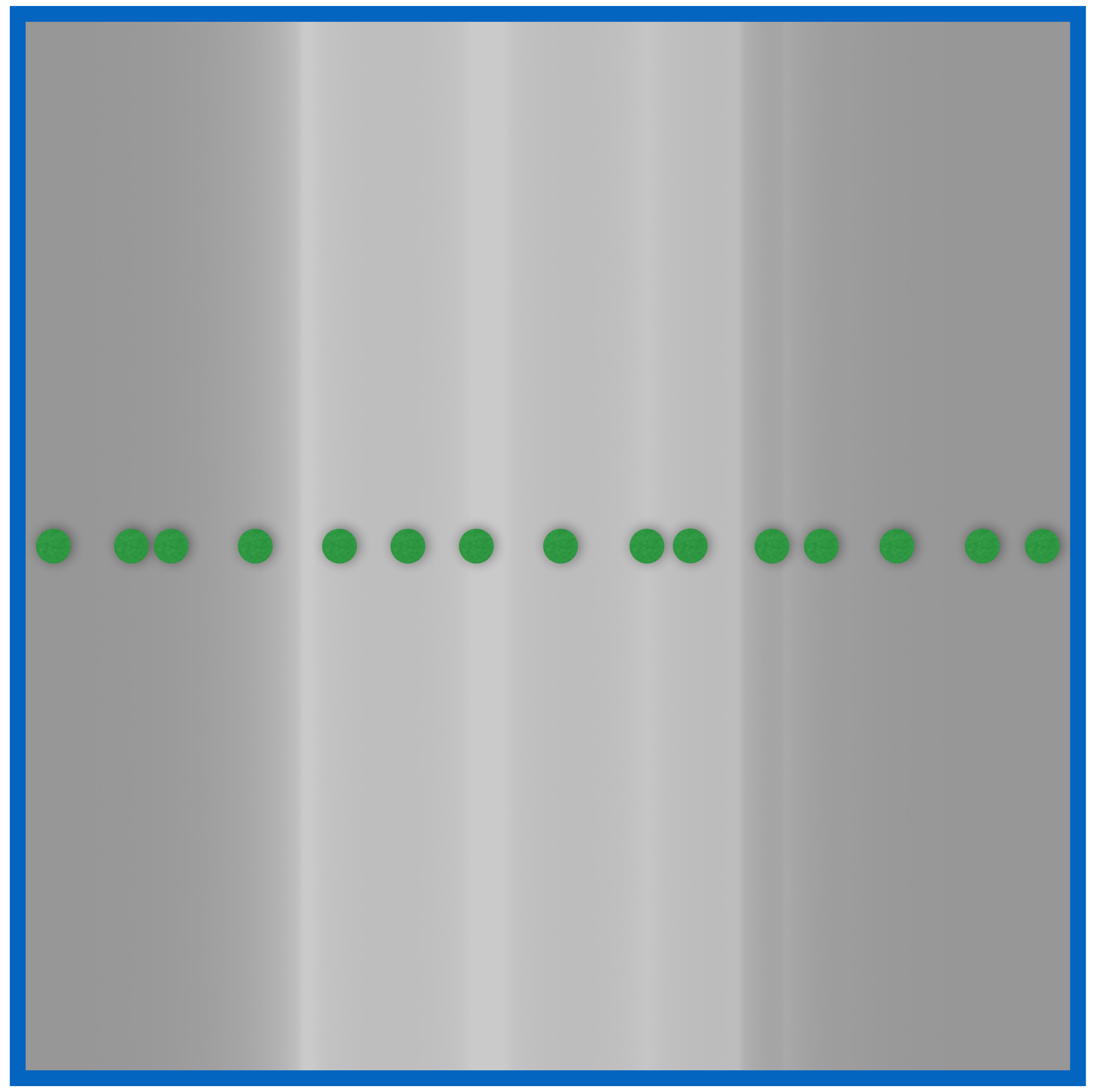
Segments

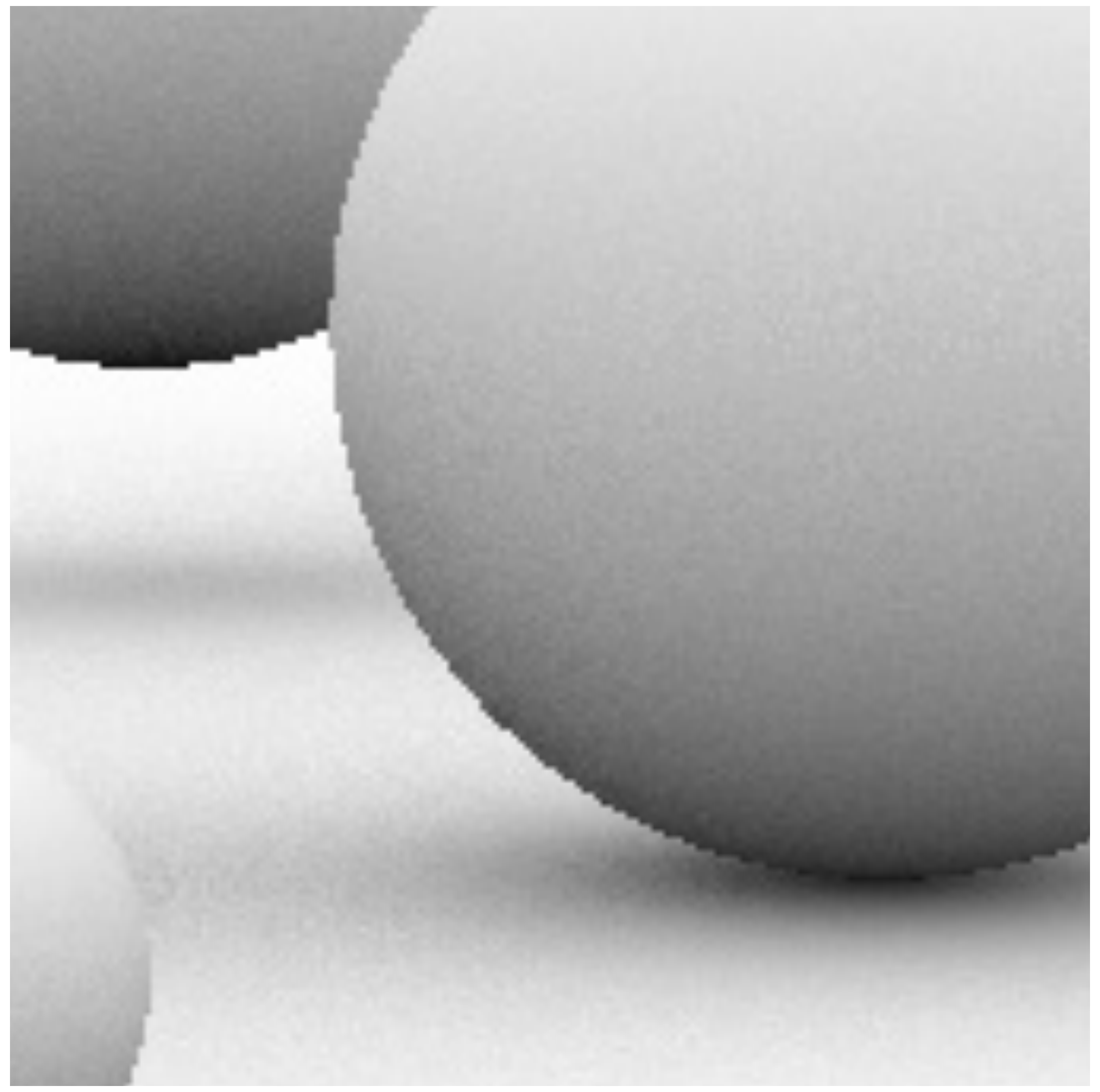




length = π

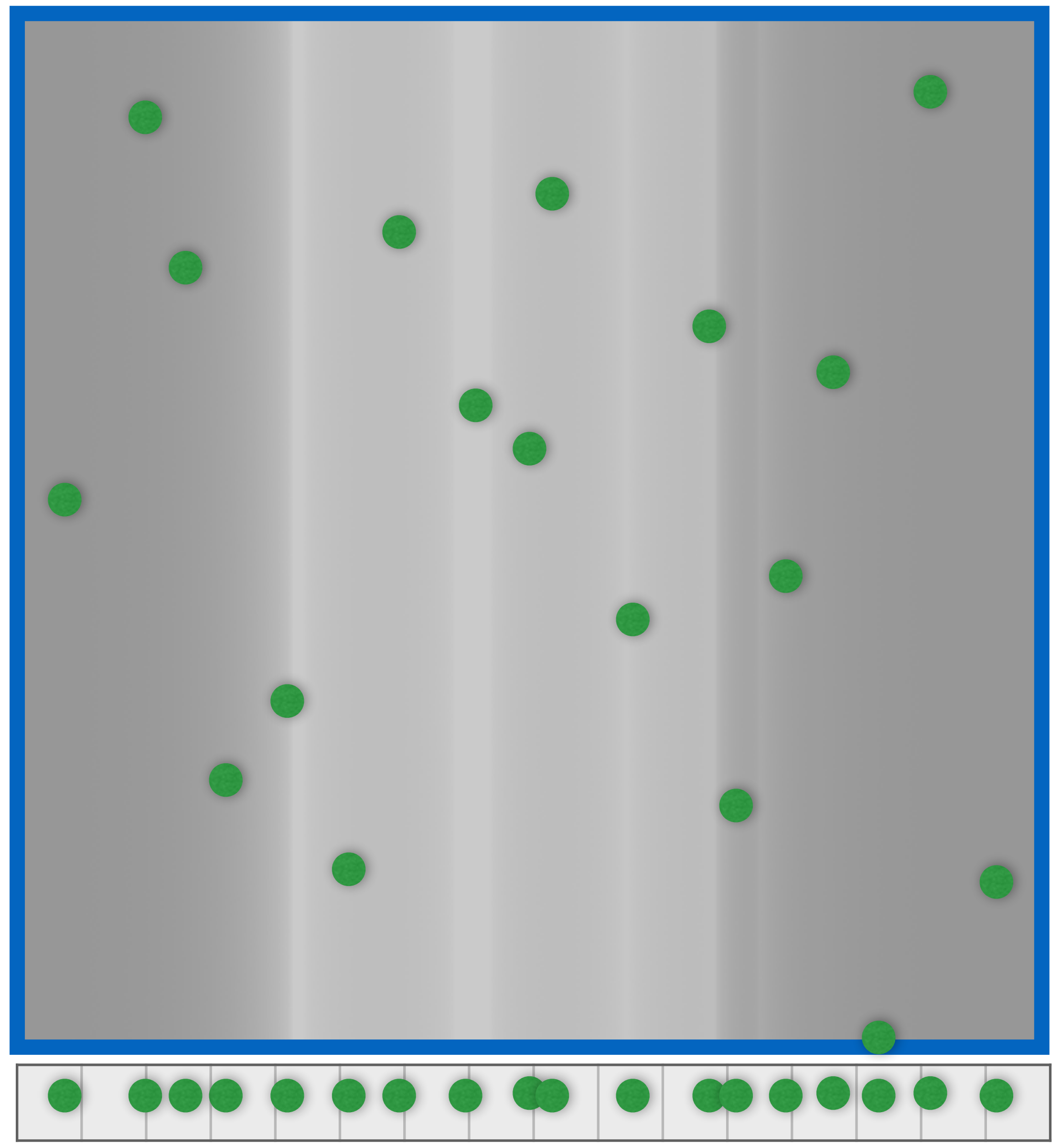
Lines



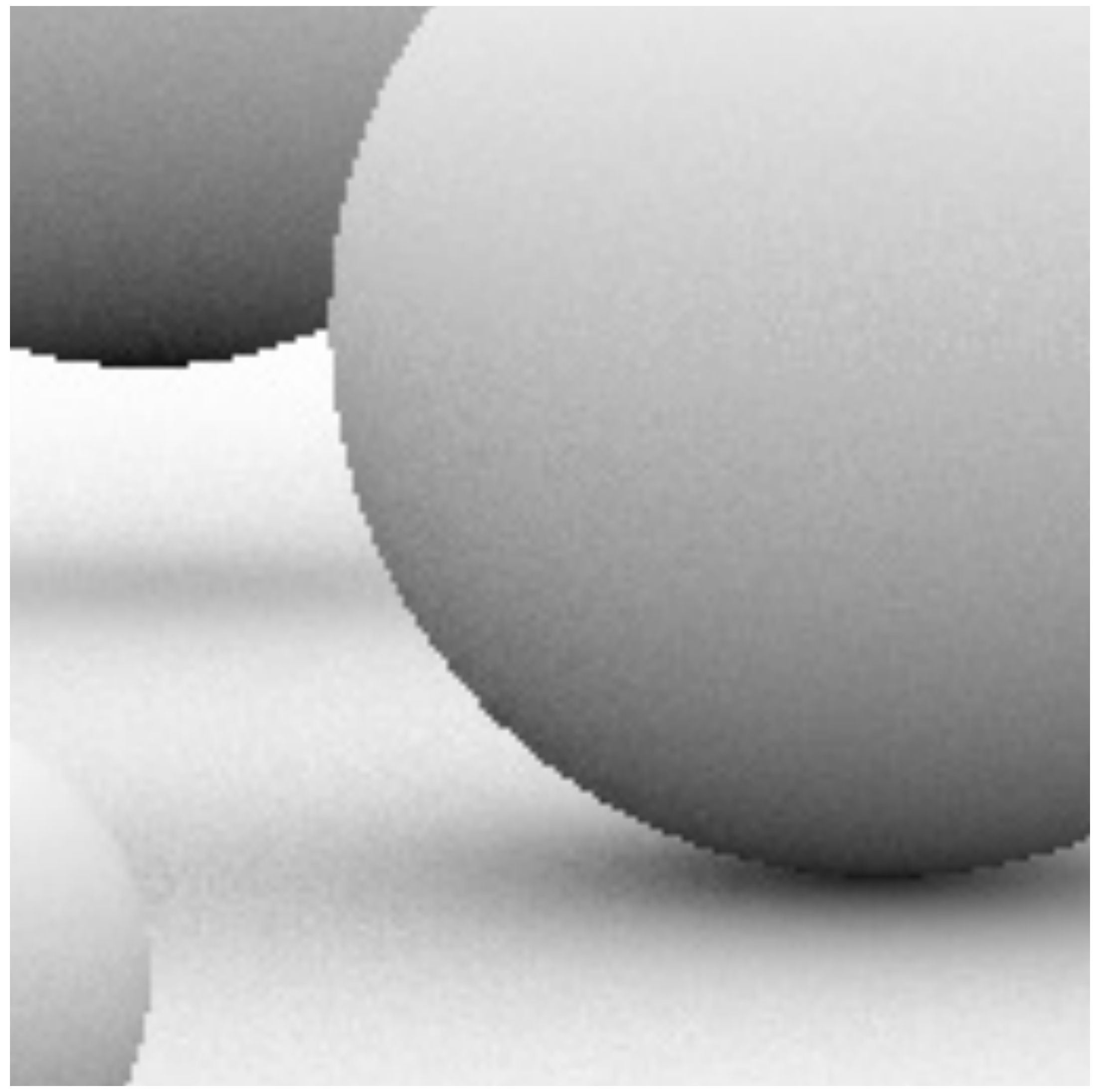


Segment Offsets with both 1D and 2D Stratifications

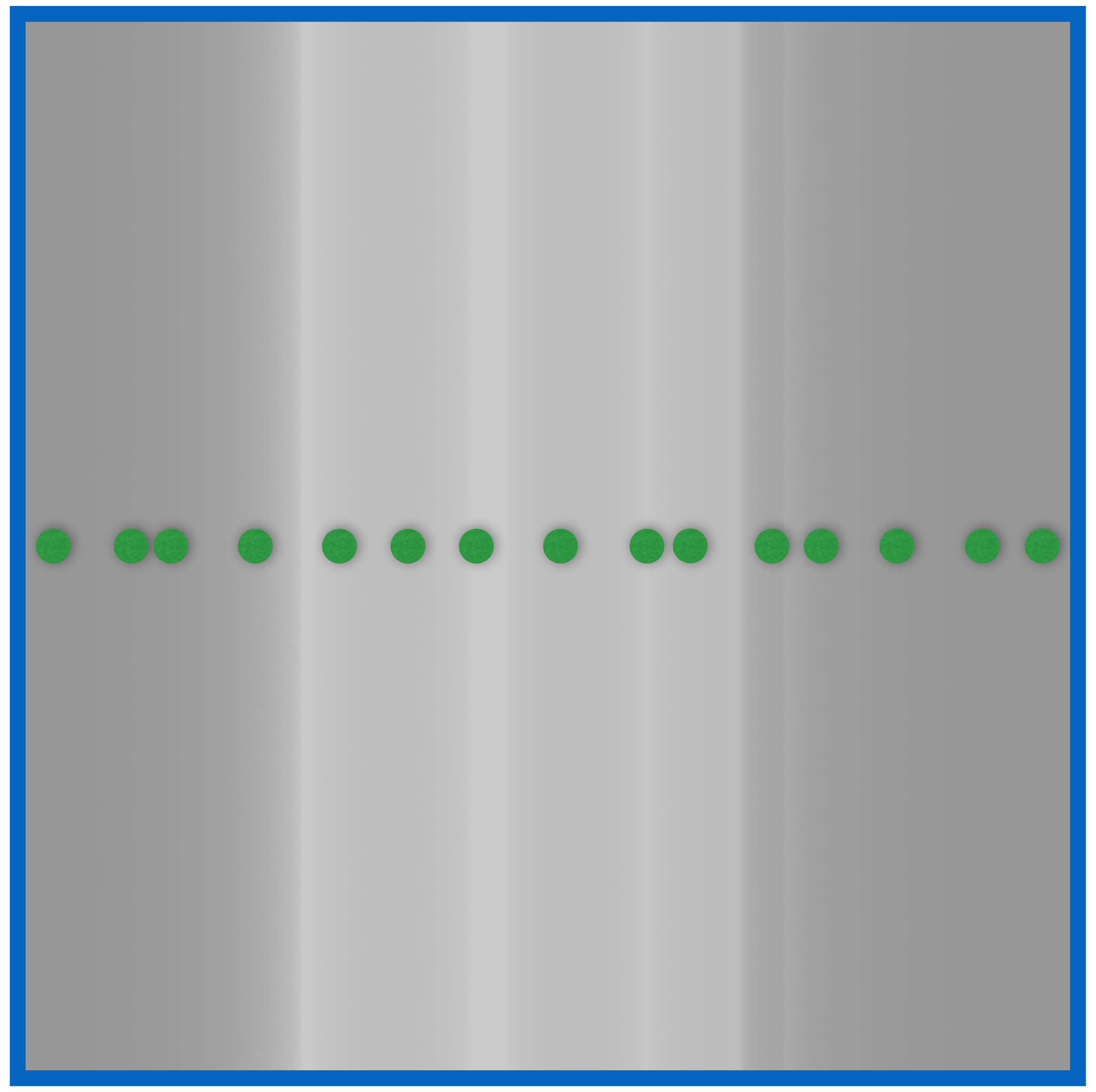
Segments

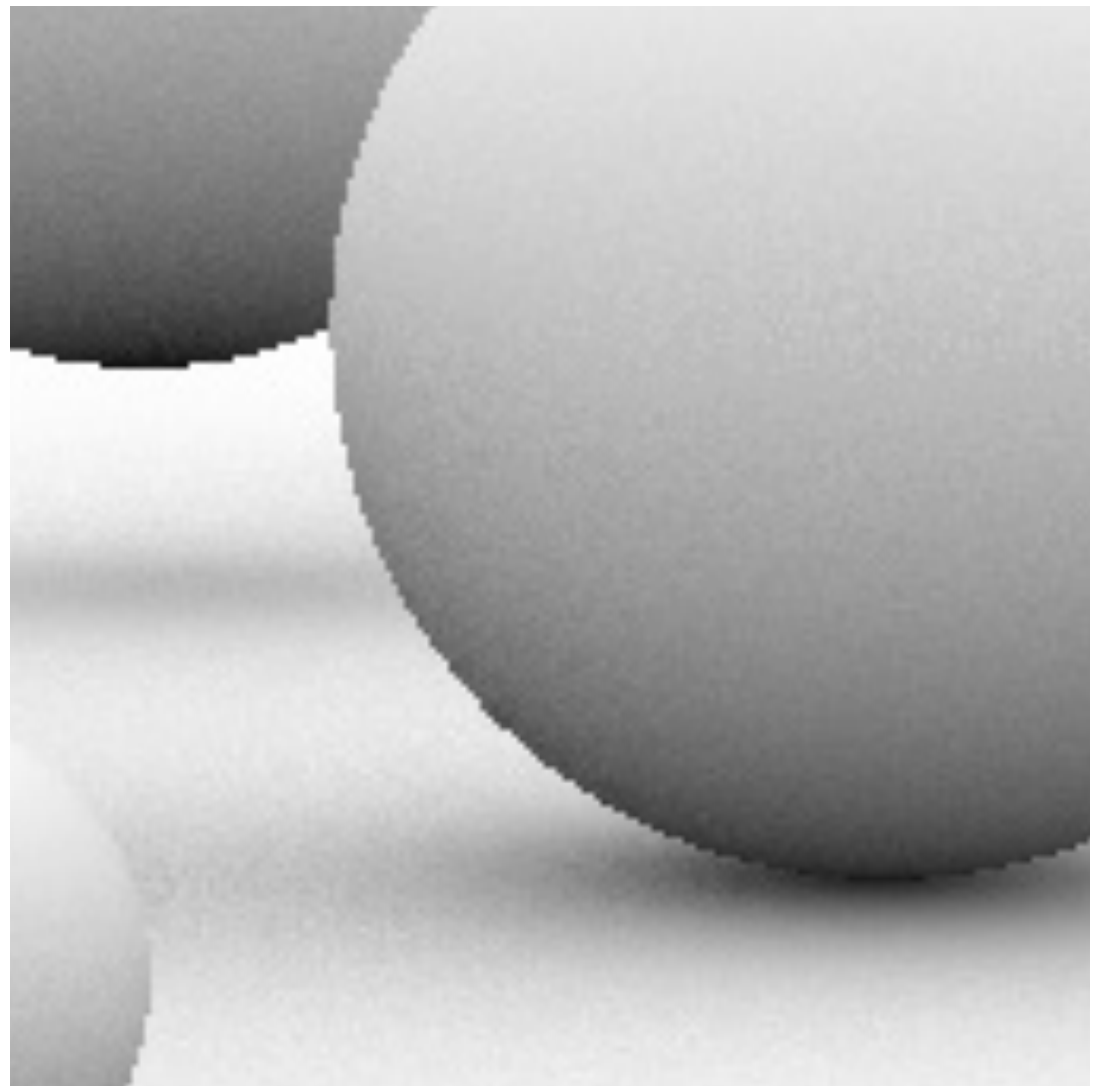


length = π



Lines

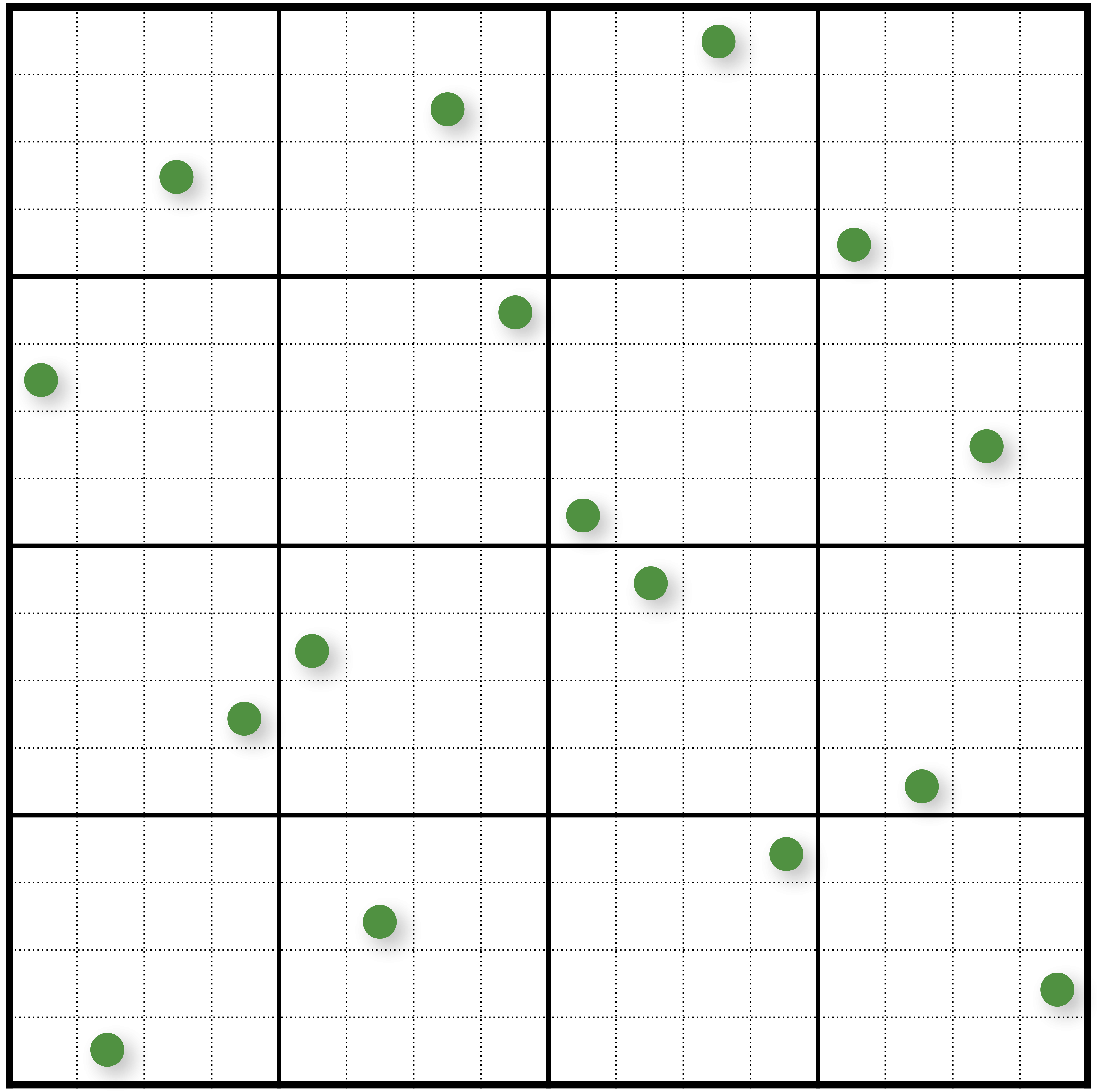




Both 1D and 2D stratification

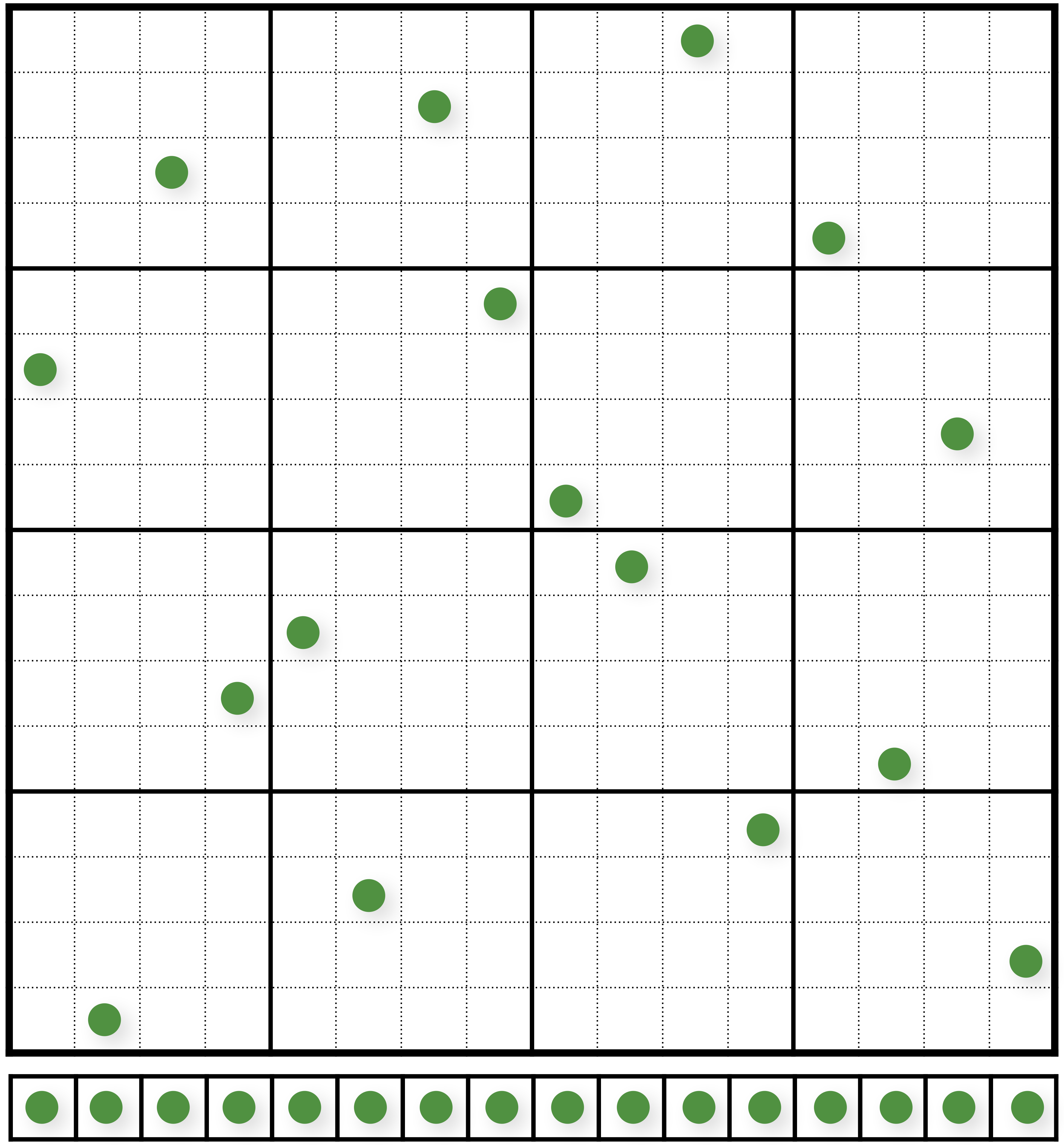
Segment Offsets with both 1D and 2D Stratifications

Multi-jitter



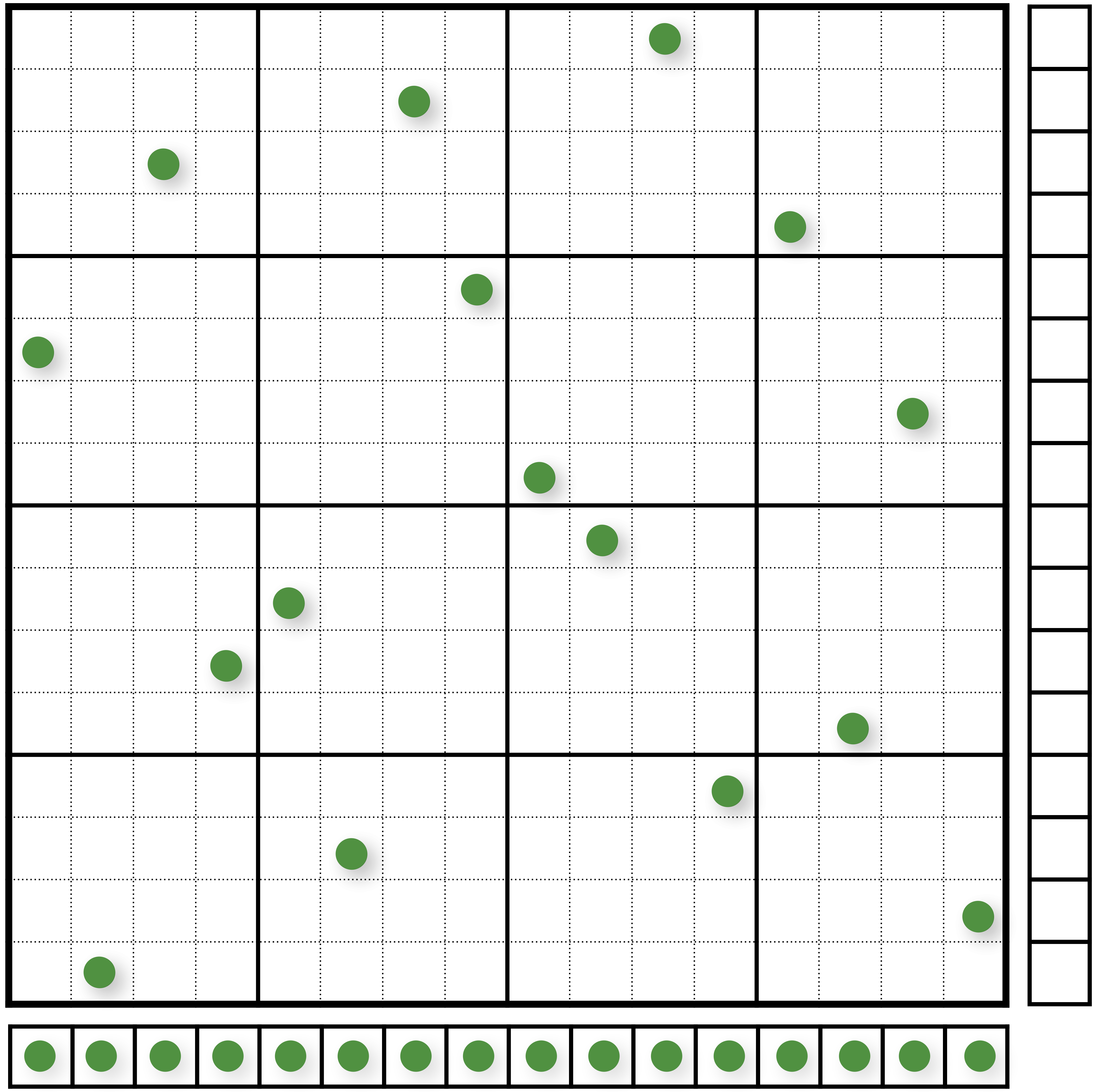
Segment Offsets with both 1D and 2D Stratifications

Multi-jitter



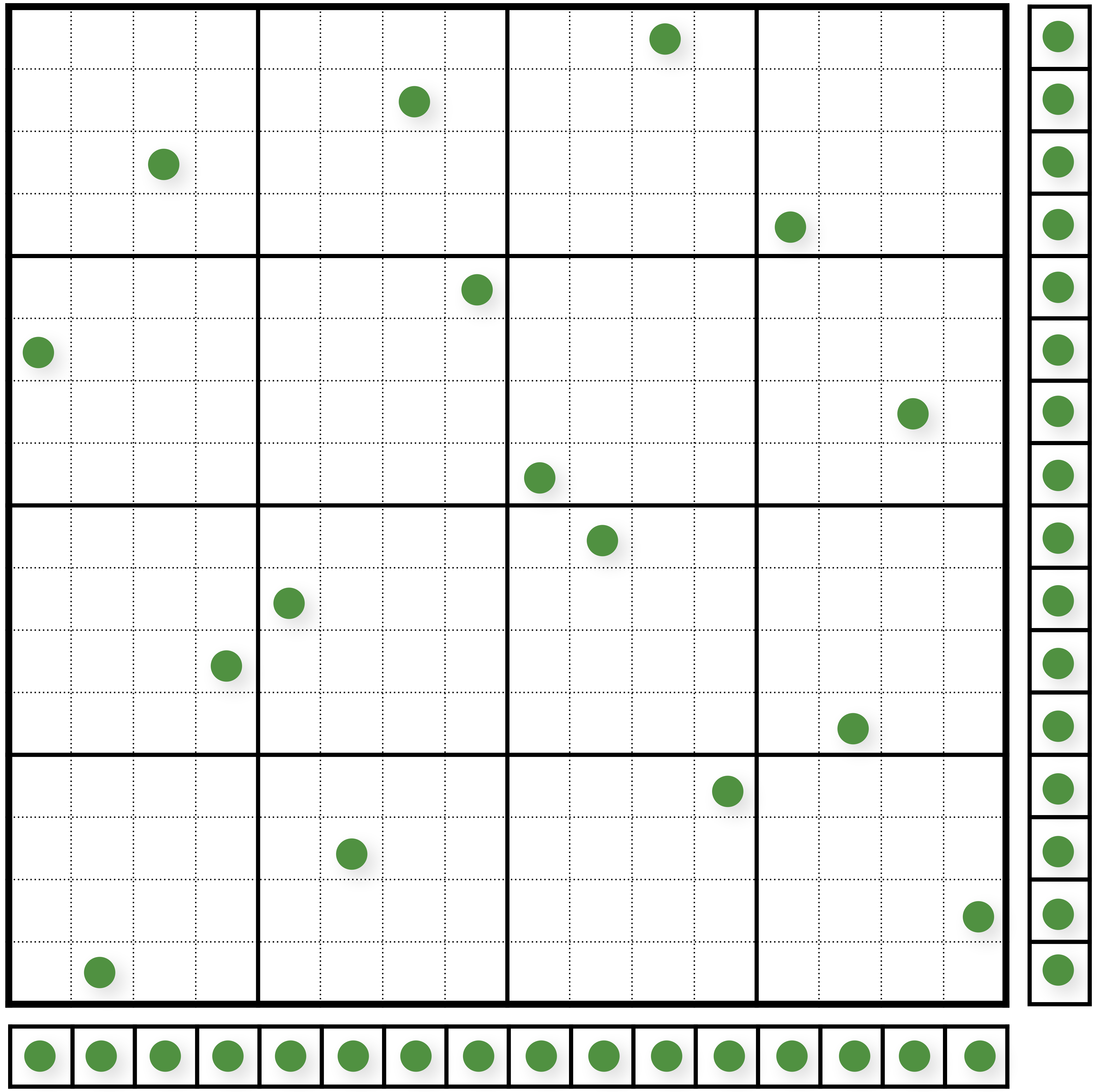
Segment Offsets with both 1D and 2D Stratifications

Multi-jitter



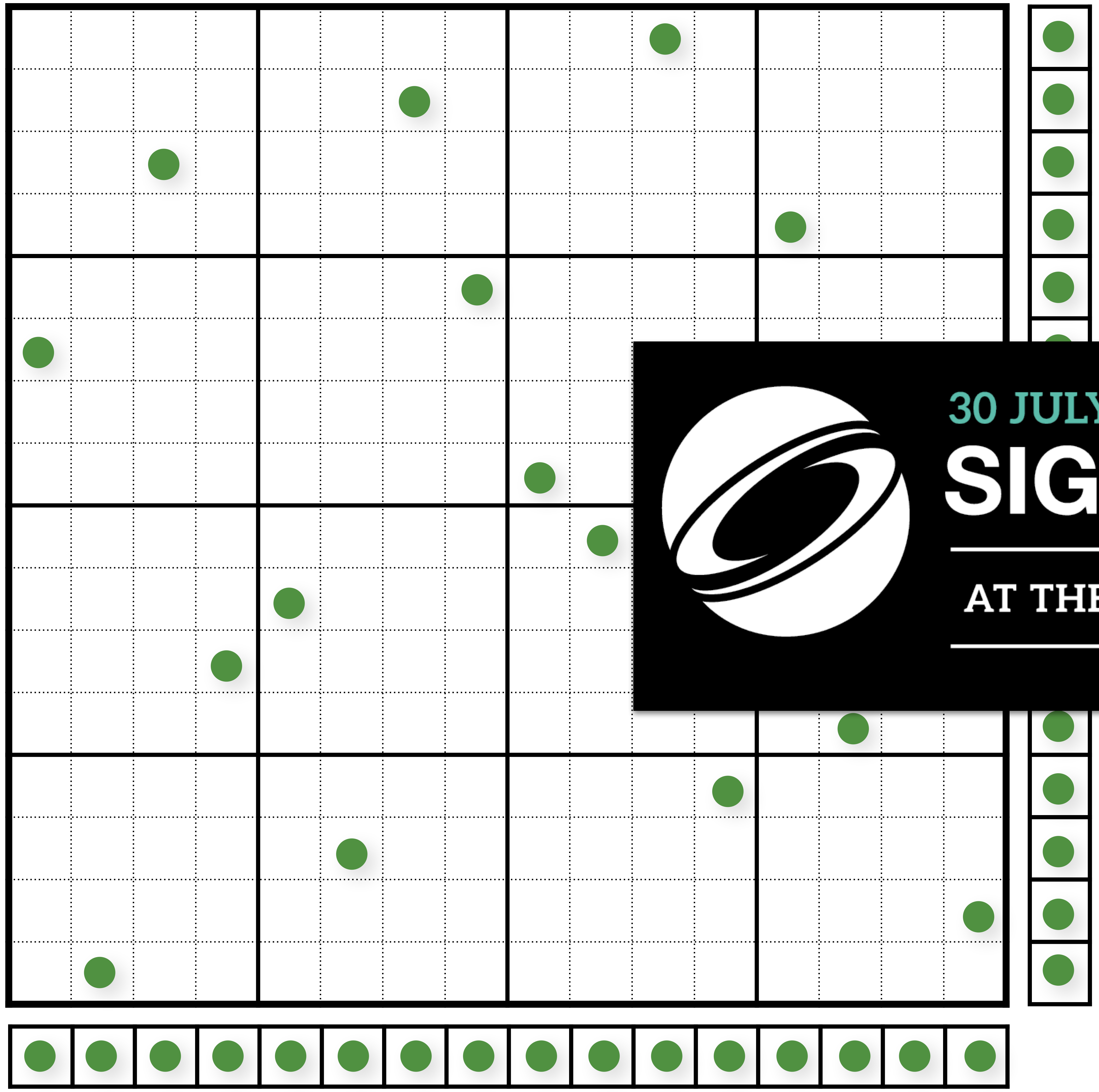
Segment Offsets with both 1D and 2D Stratifications

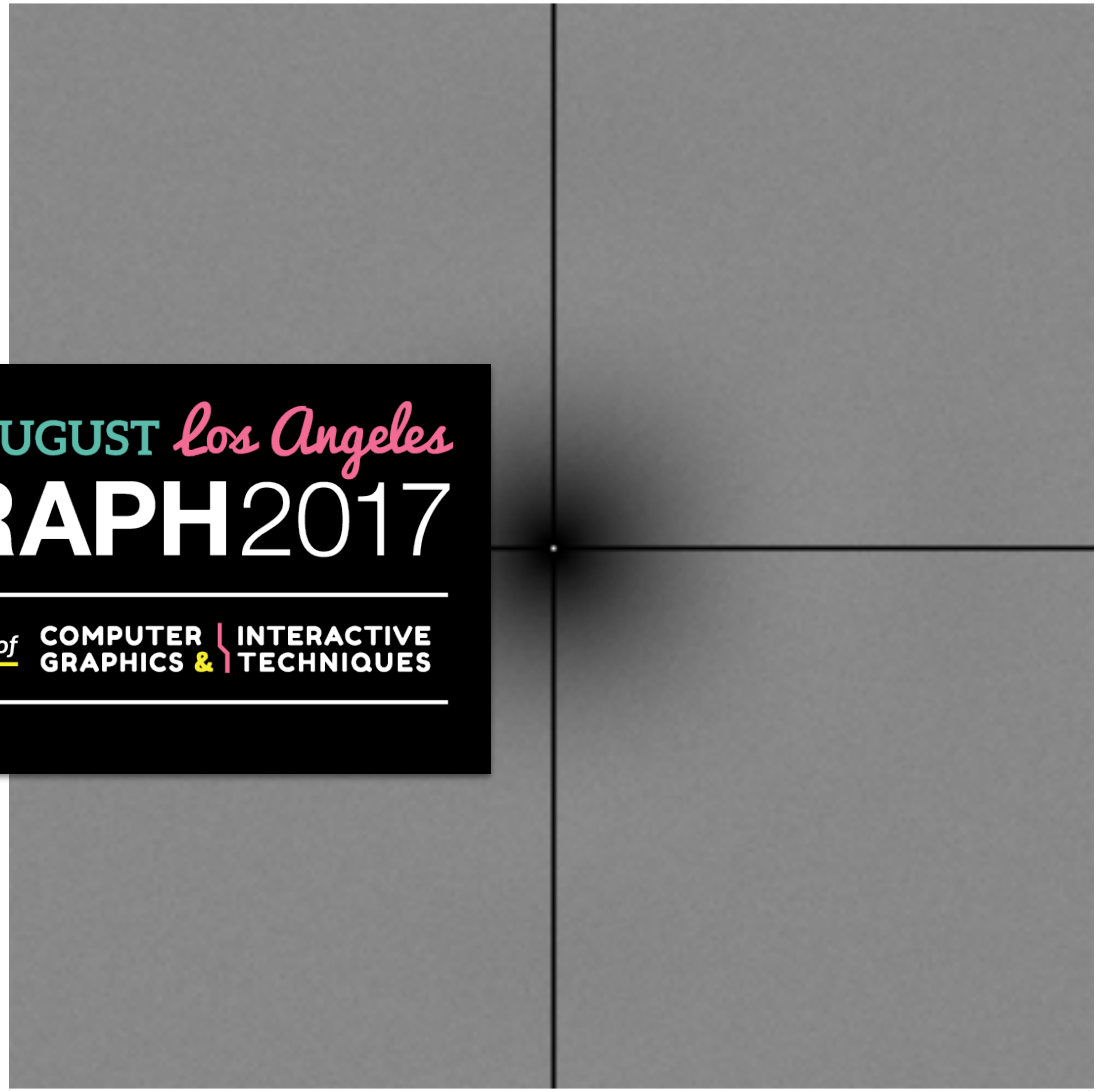
Multi-jitter



Segment Offsets with both 1D and 2D Stratifications

Multi-jitter





Power spectrum

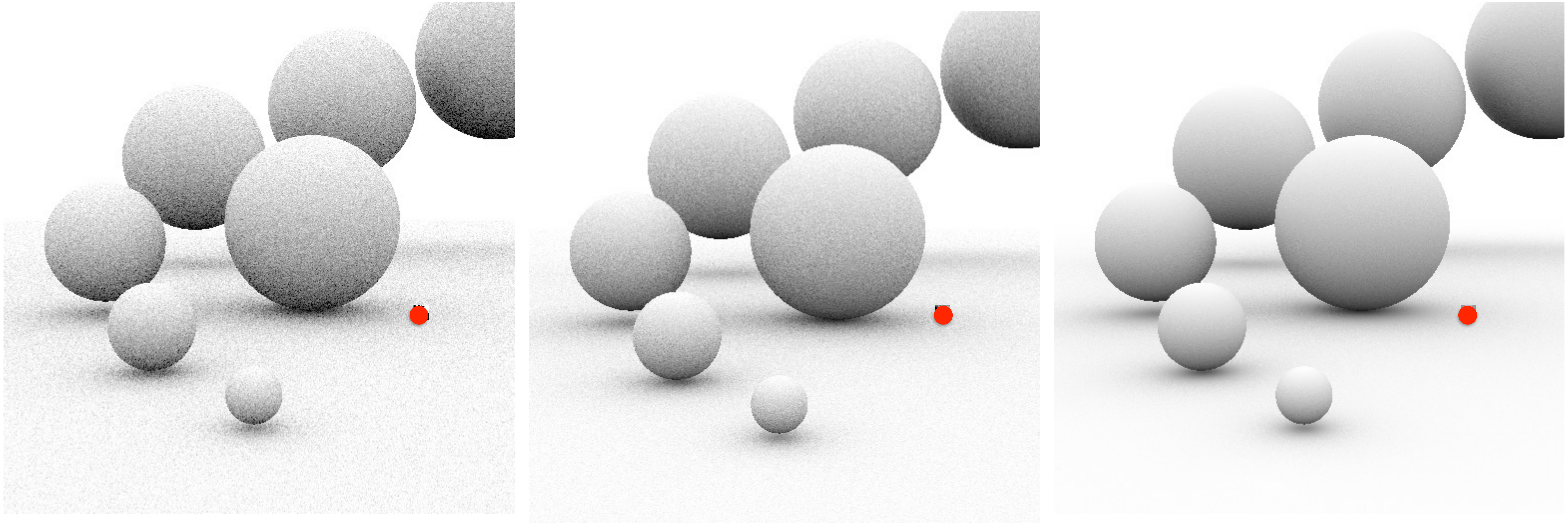
Future Work

- Develop sampling strategies that can generate samples according to the pre-filtered integrand
- Analyze multi-directional samples with correlations e.g. multi-class blue noise samples
- Choosing sample orientations for maximum benefits

Thank you for your attention!

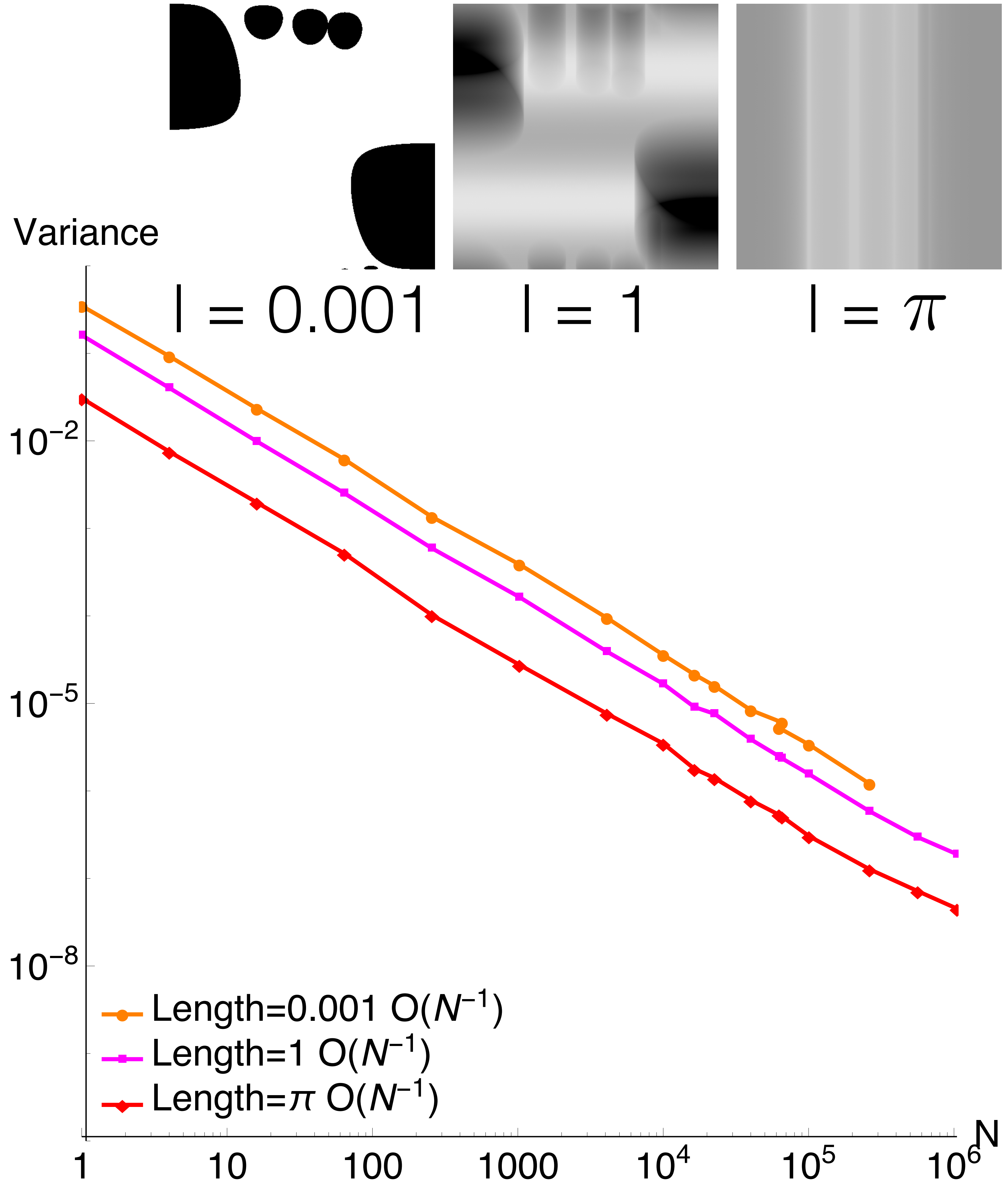
Ambient Occlusion: Segments Varying Length

Segment length = 0.001 | Segment length = 1 | Segment length = π

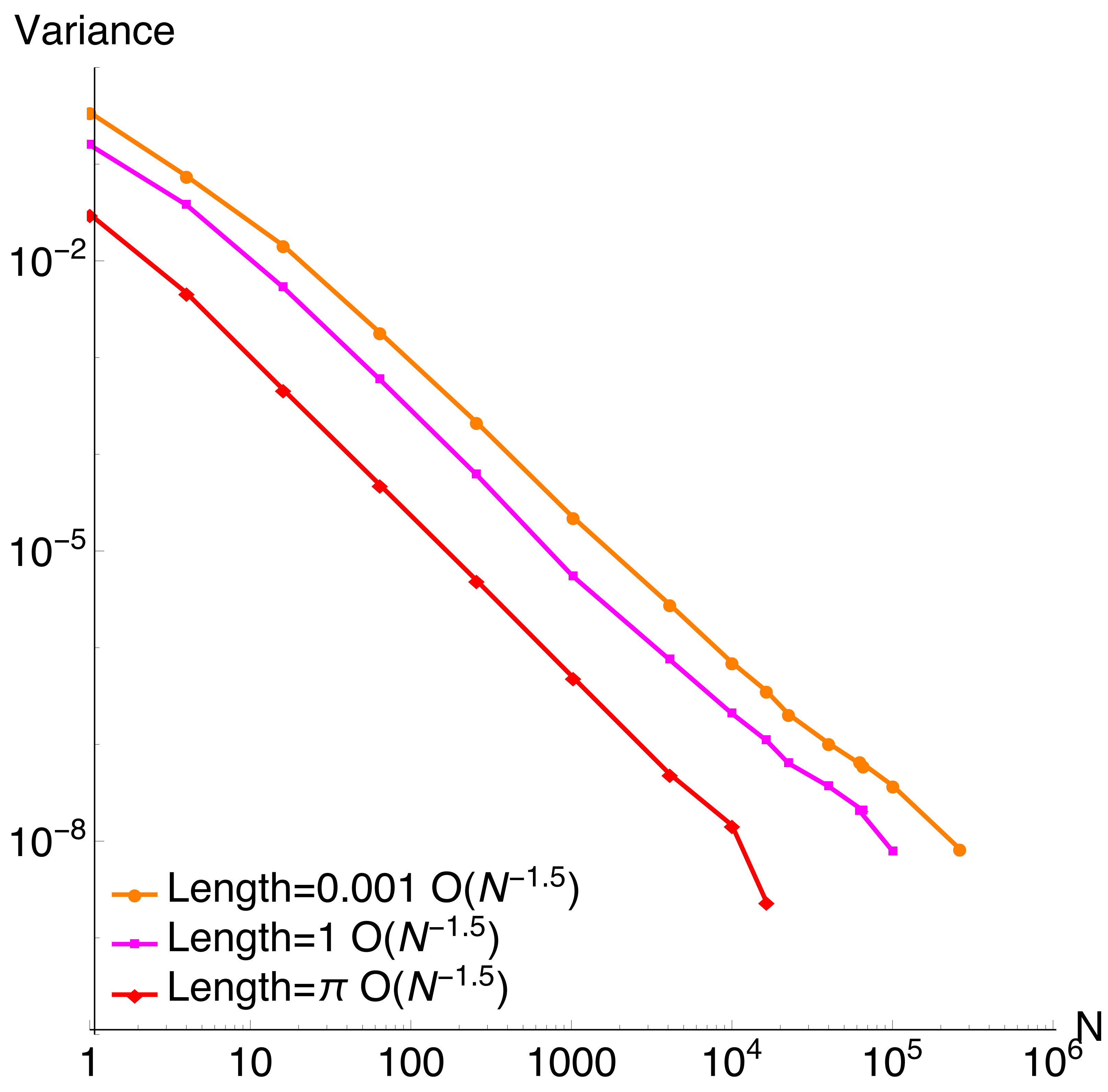


Jittered Samples

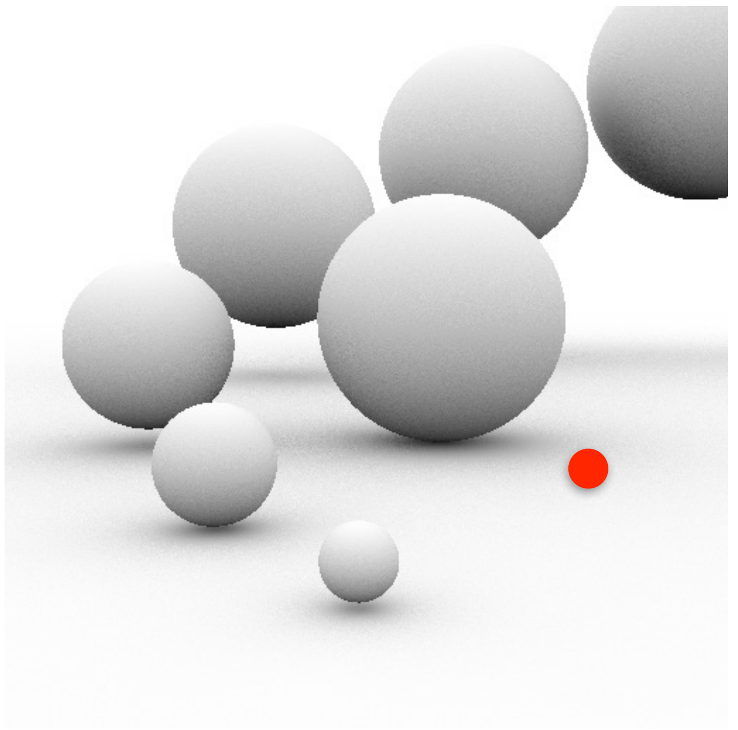
Ambient Occlusion: Segments Varying Length



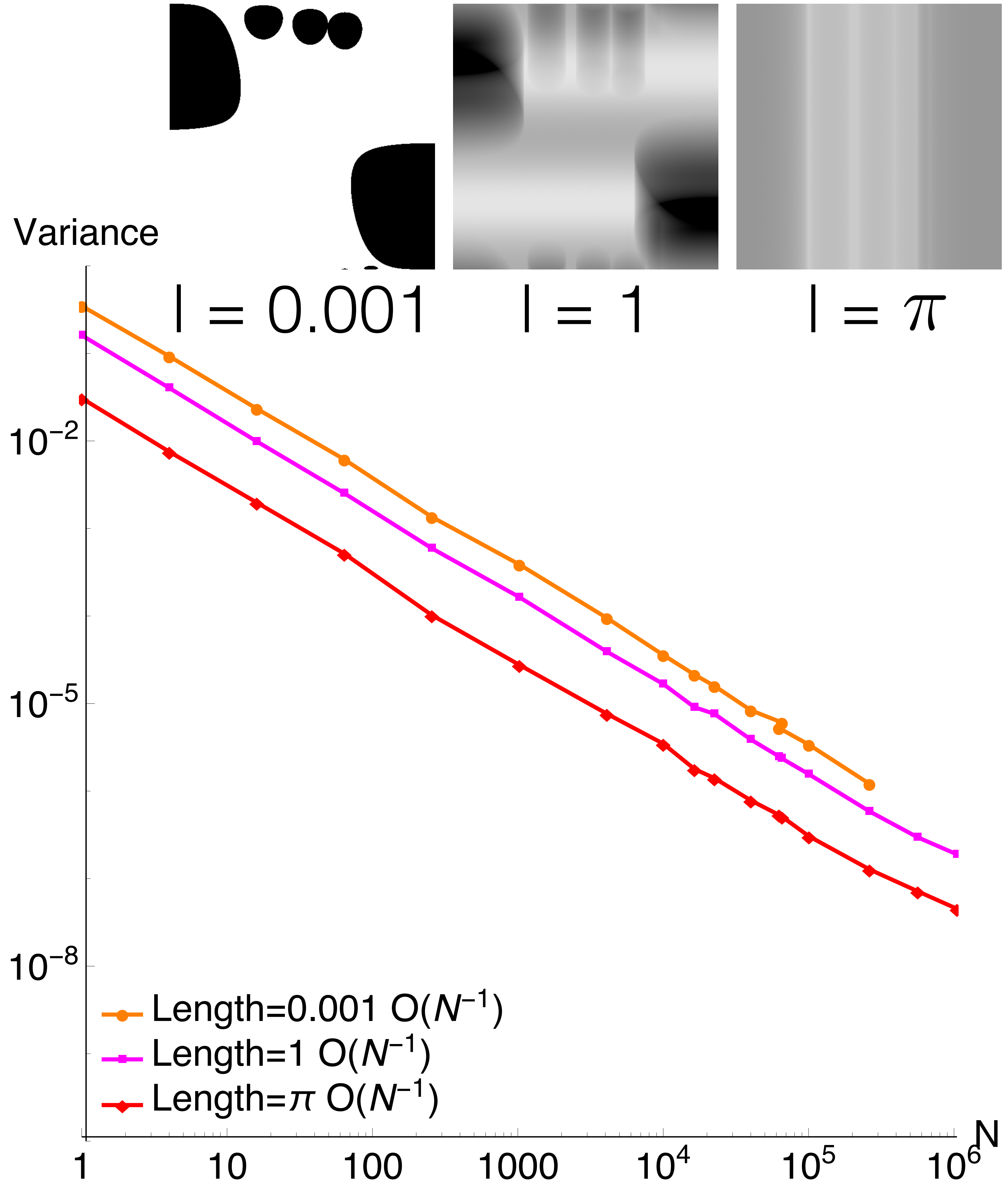
Random



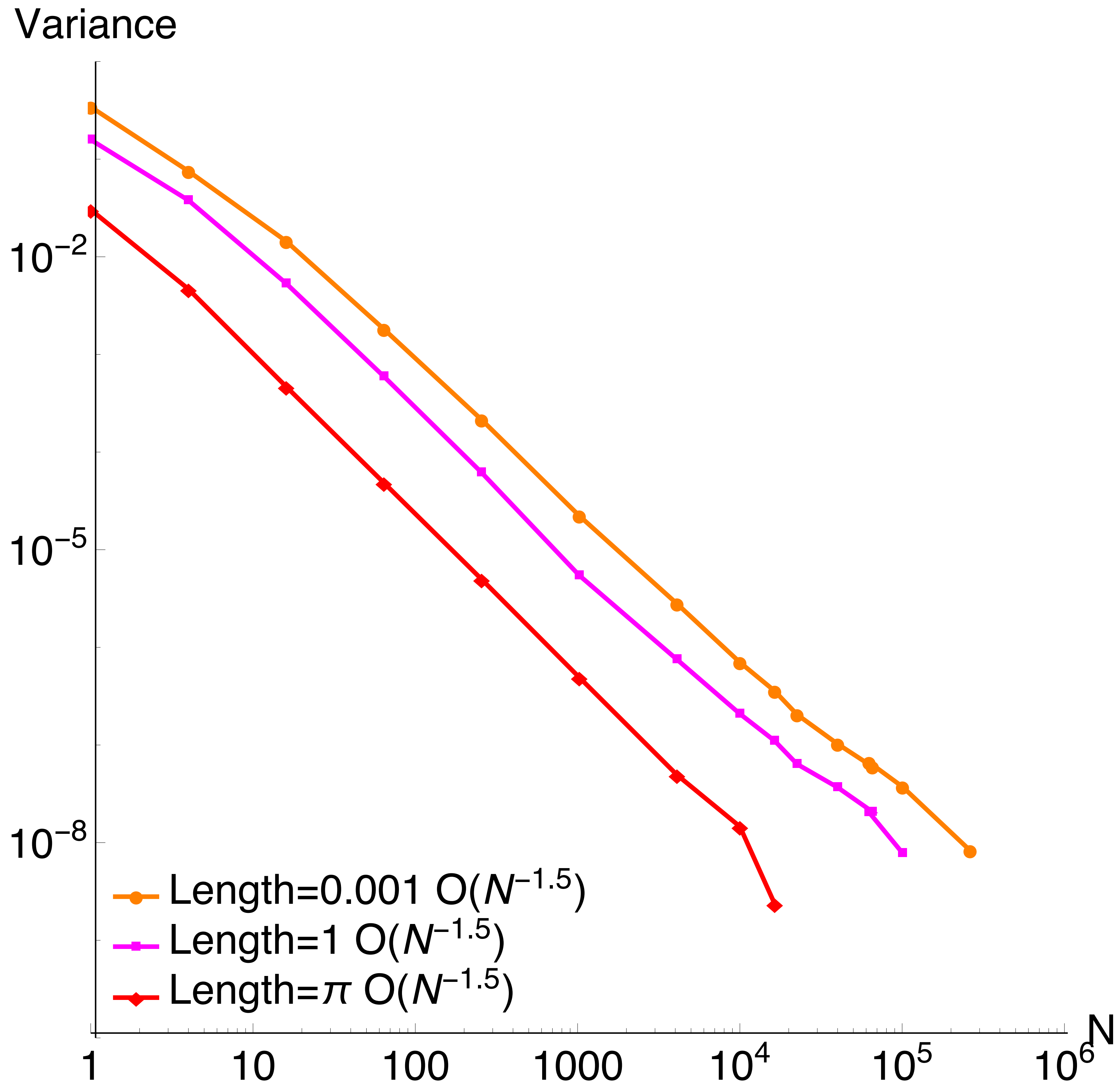
Jitter



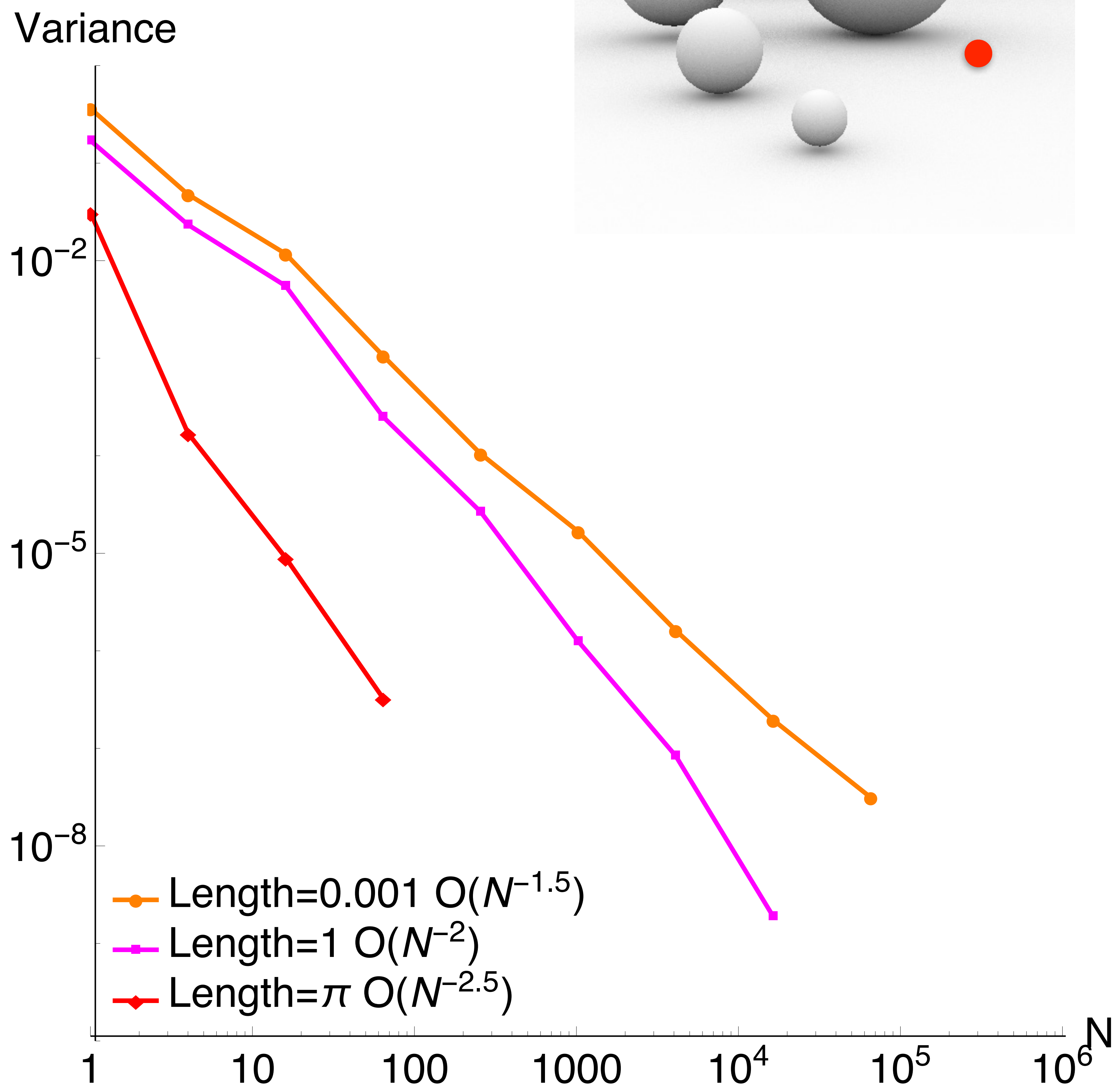
Ambient Occlusion: Segments Varying Length

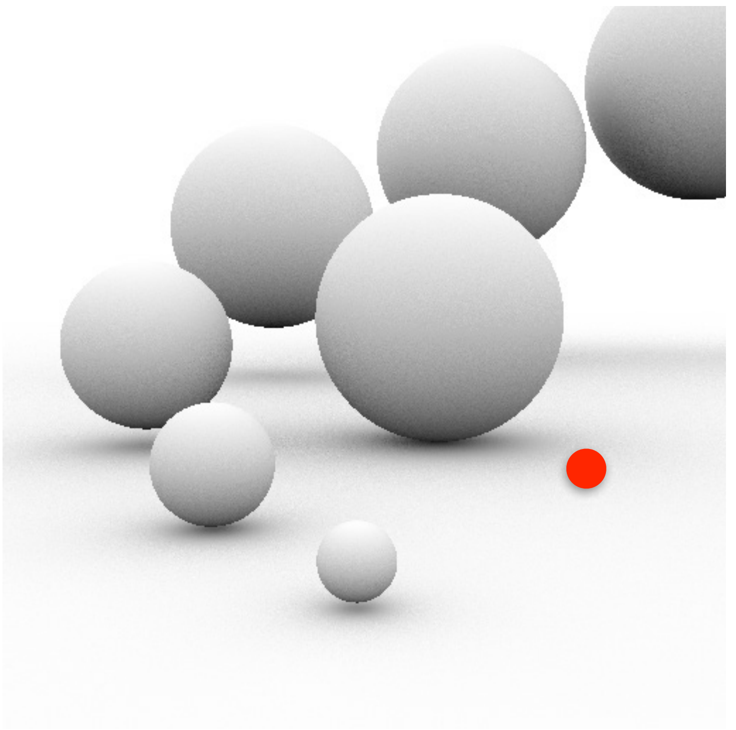


Random



Jitter

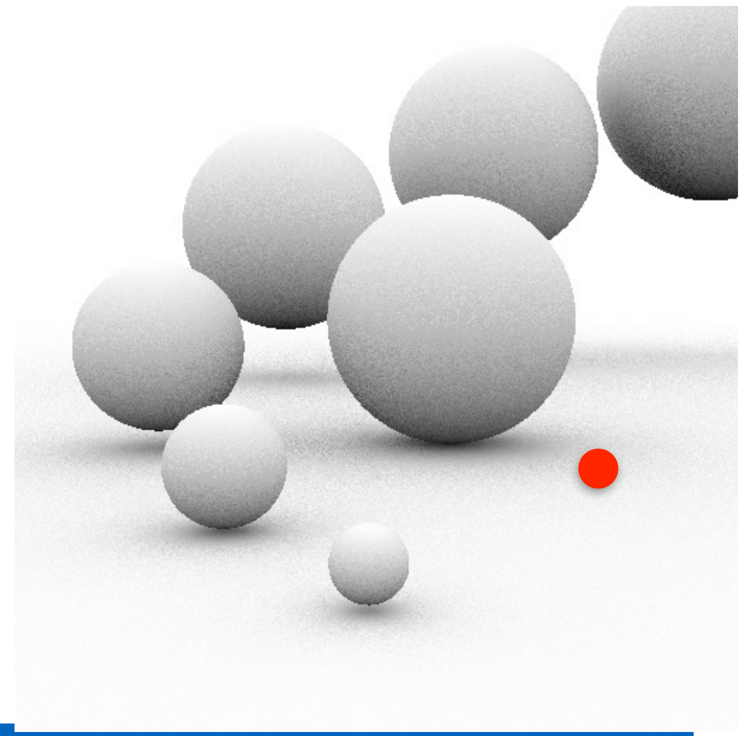


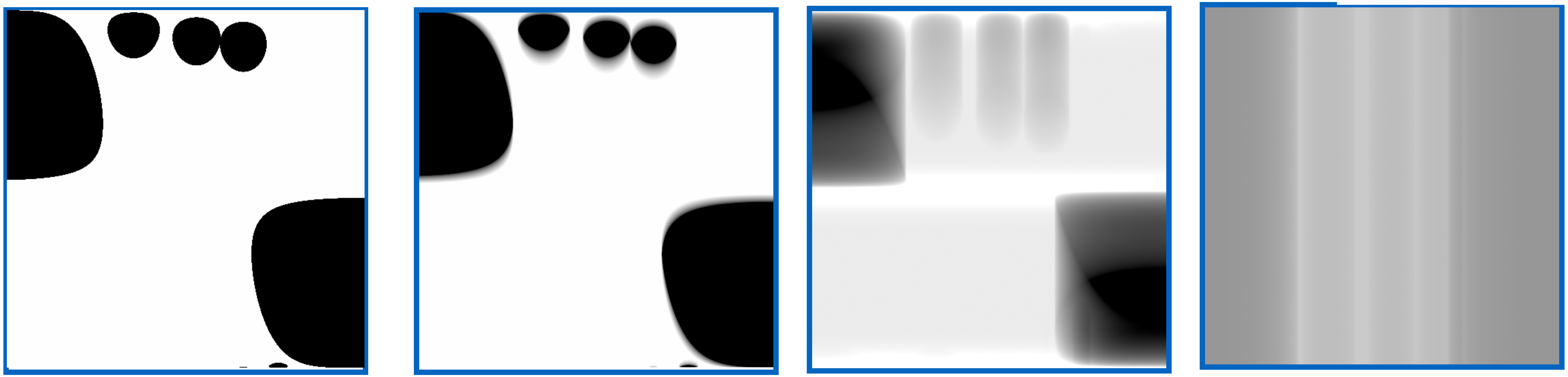


Low discrepancy

Ambient Occlusion: Segments Varying Length

Uniform spherical coordinate sampling

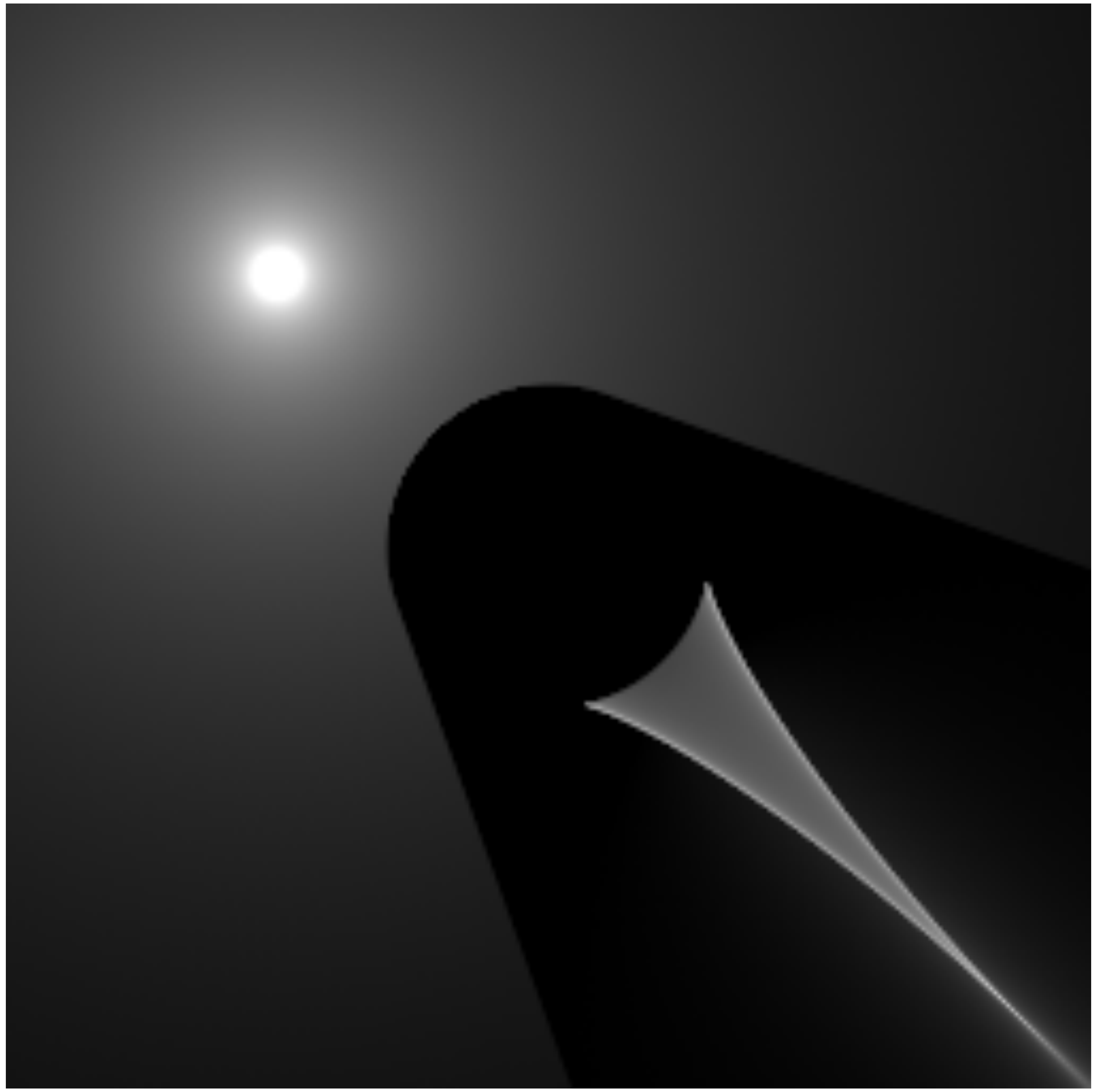




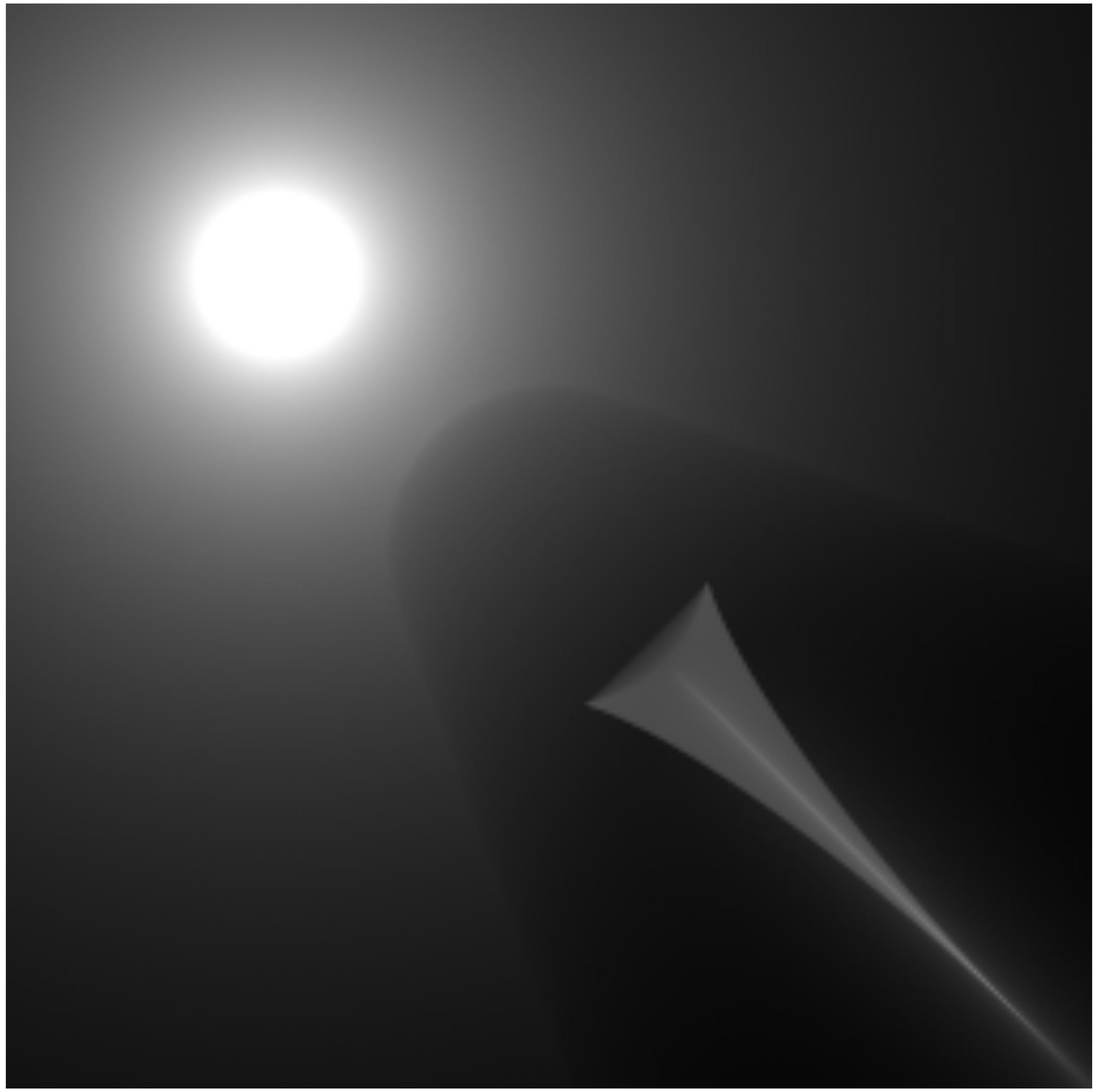
Length = 0.001 | Length = 0.1 | Length = 1 | Length = π

Sphere caustics: Points vs Line Samples

2D



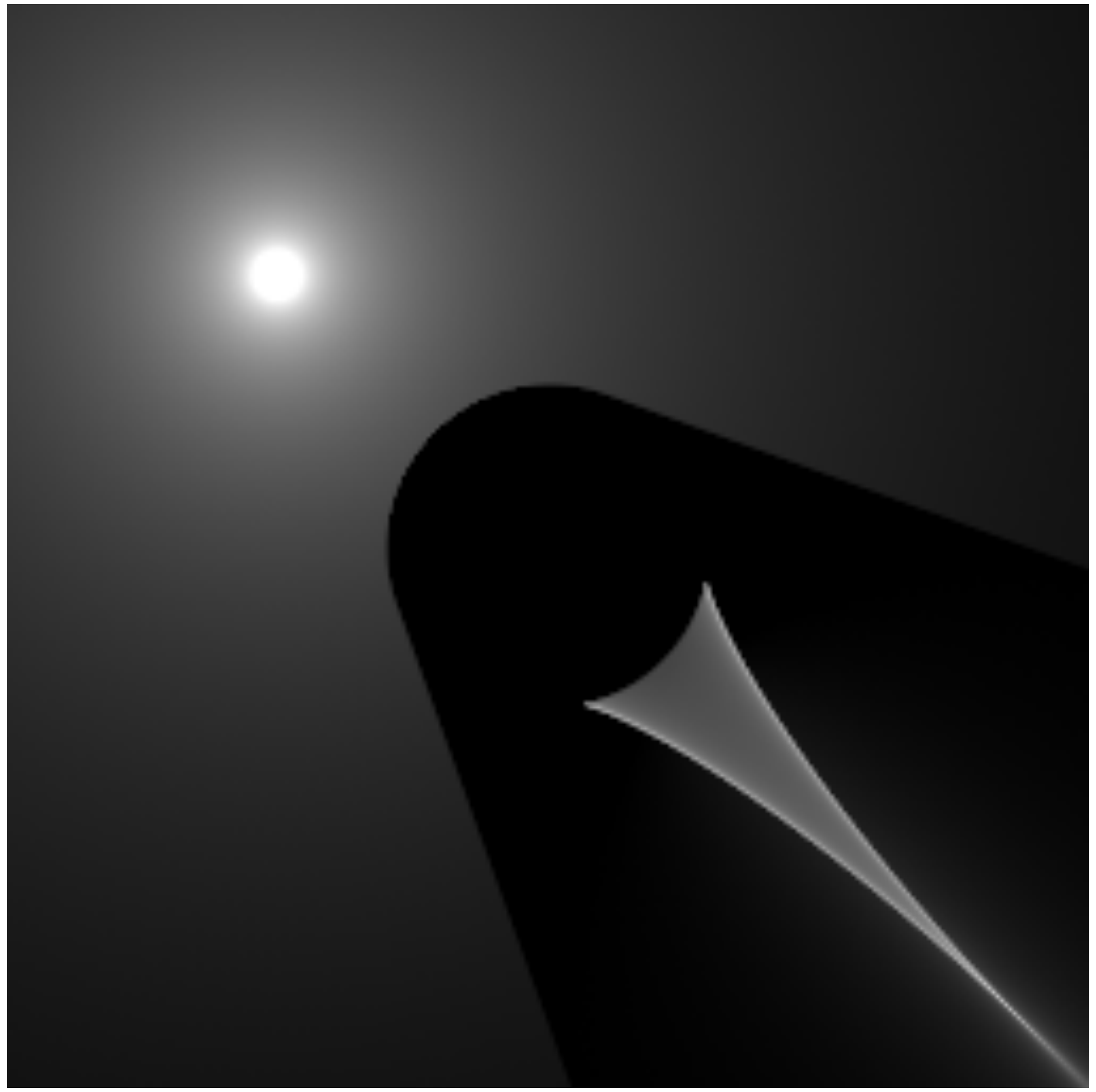
3D



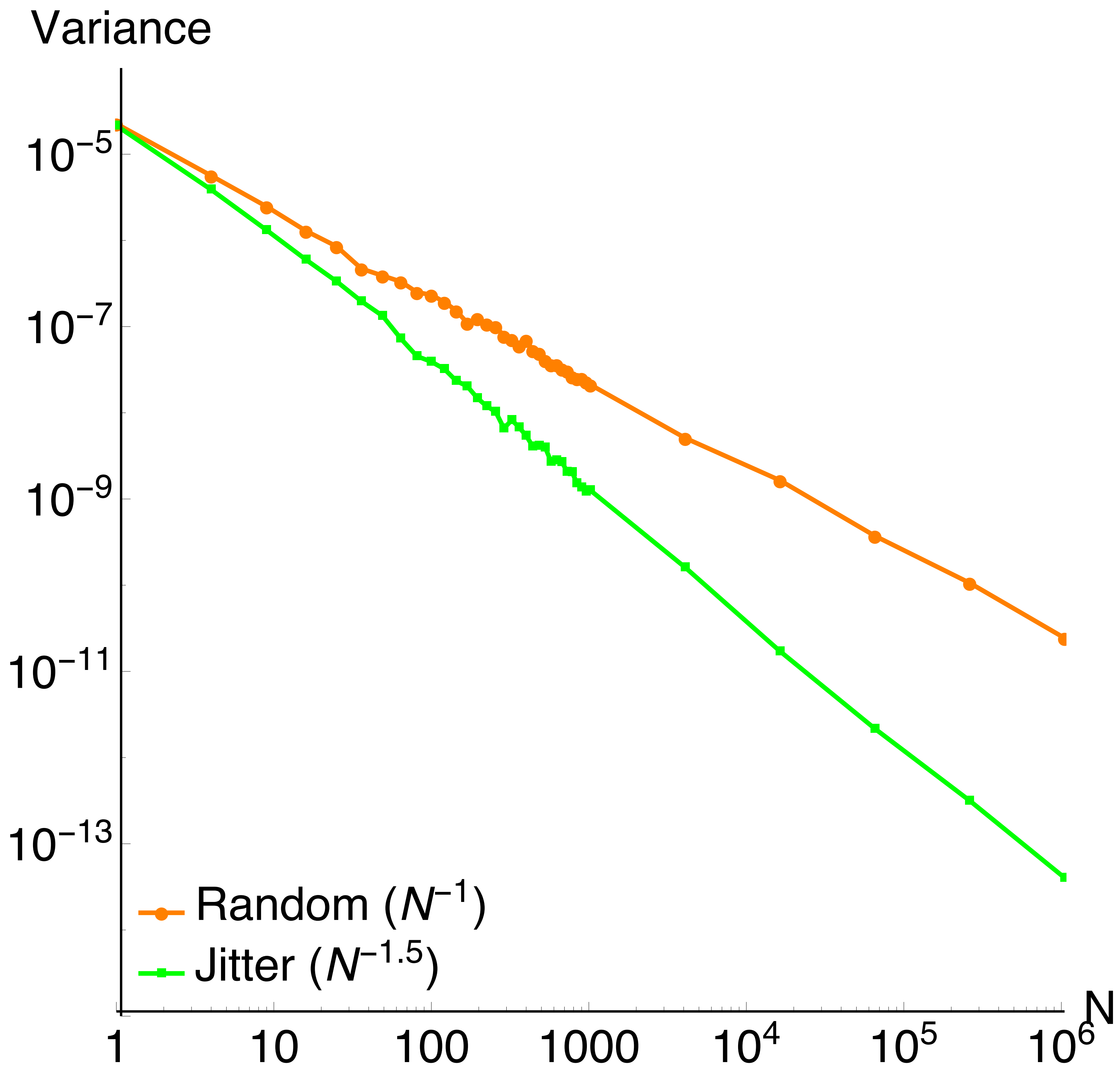
Sphere caustics 2D: Points vs Line Sampling

1D angular + 1D distance

2D

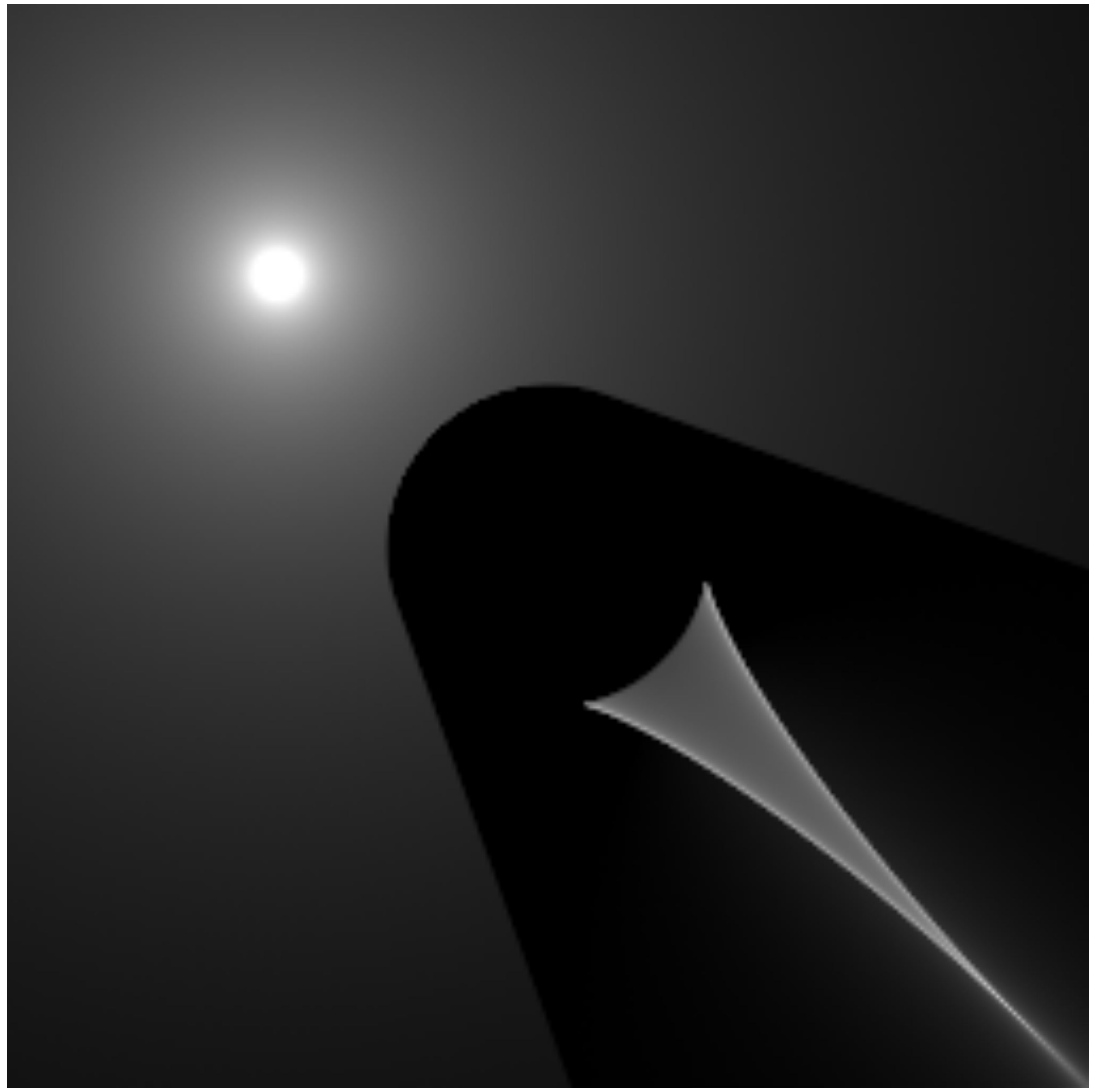


Points (photons)

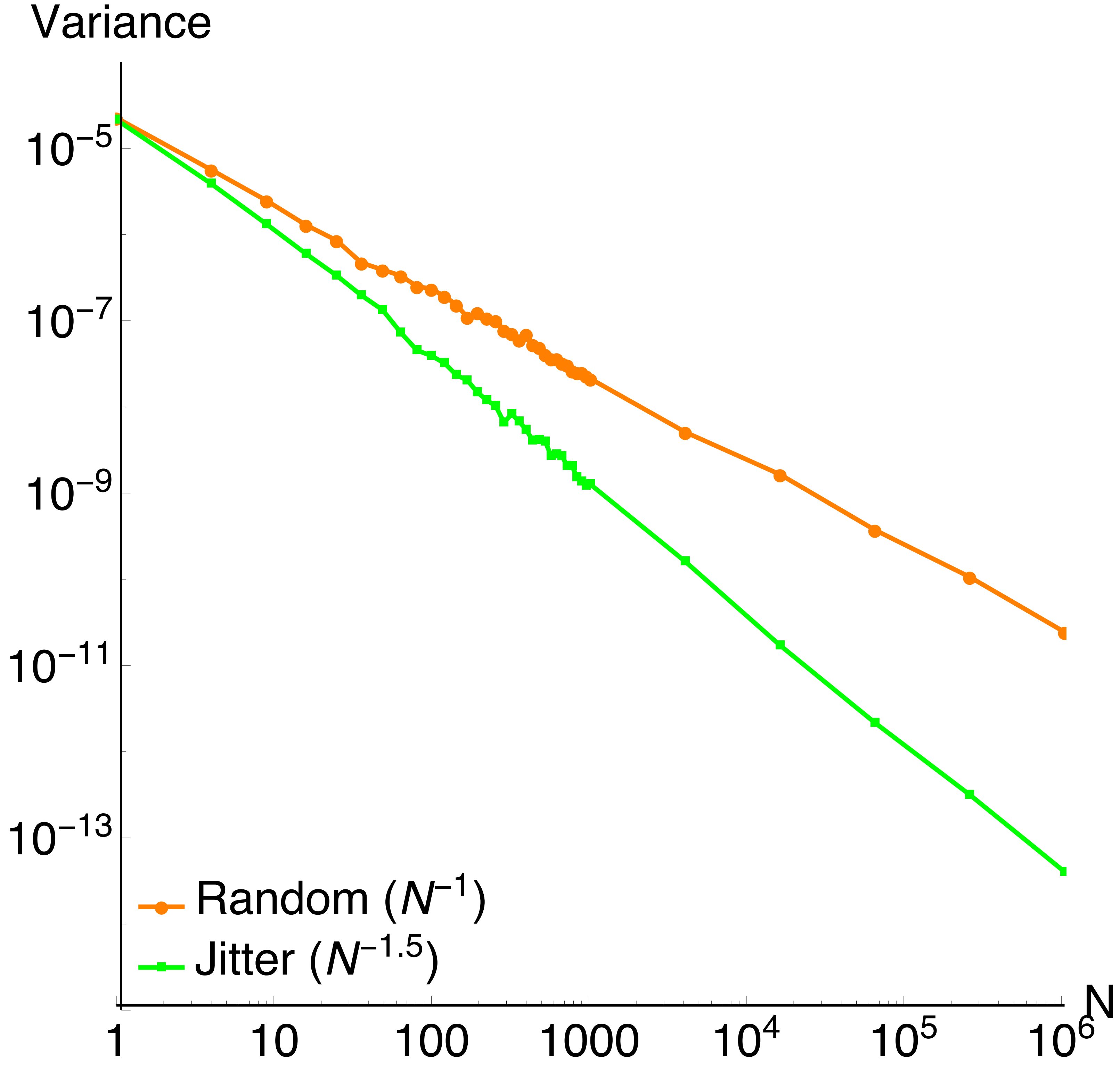


Sphere caustics 2D: Points vs Line Sampling

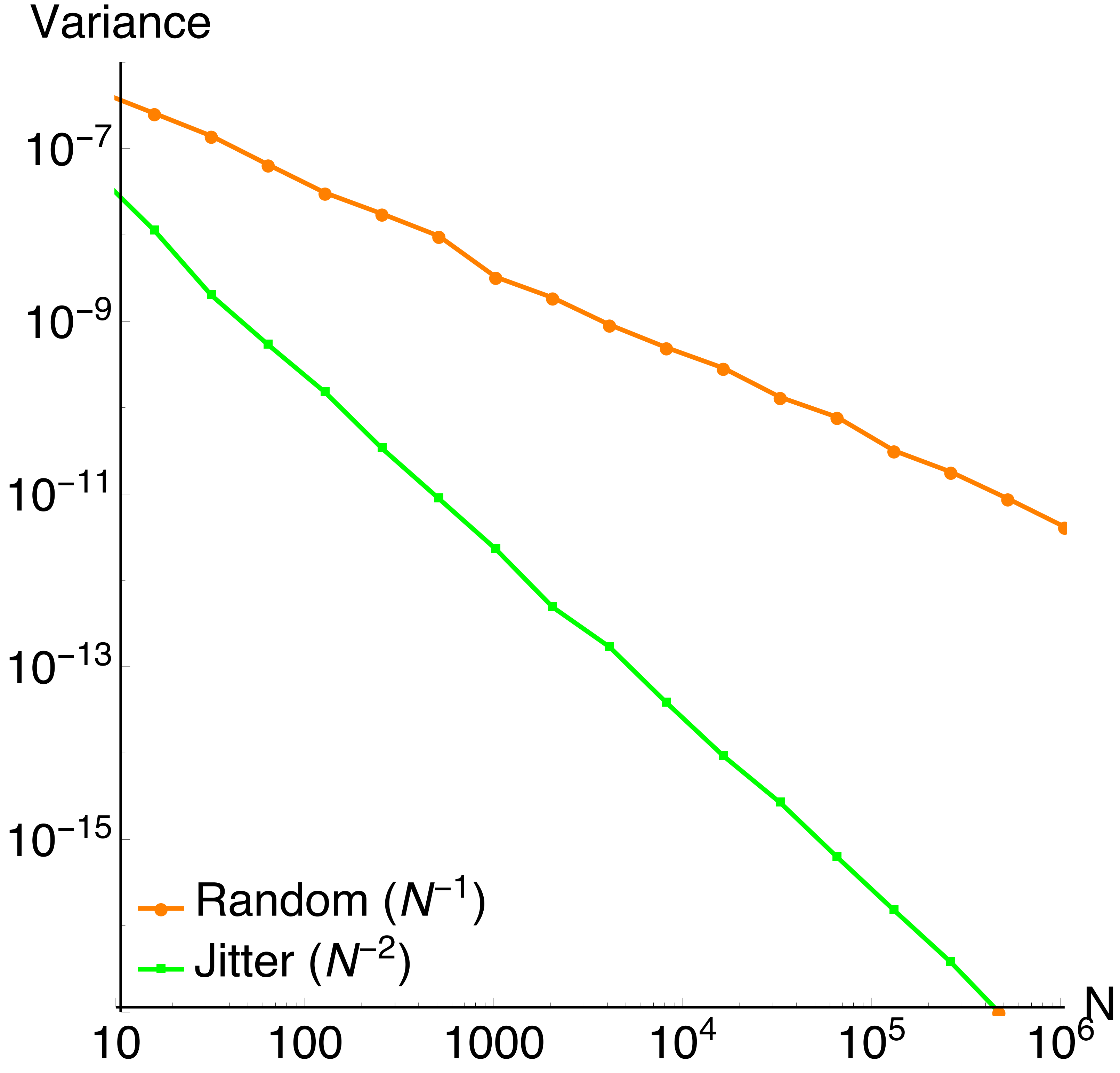
2D



1D angular + 1D distance
Points (photons)

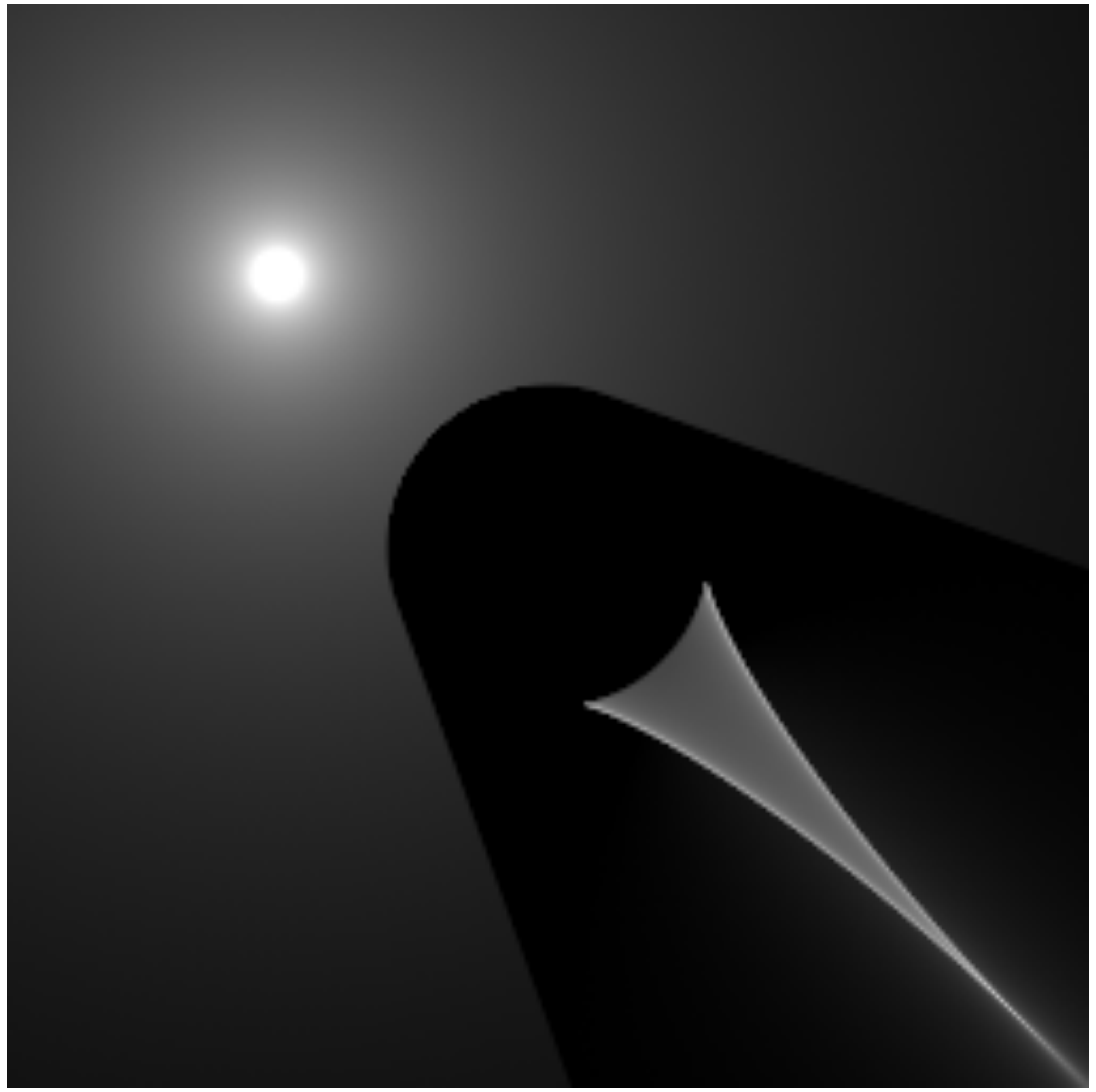


1D angular
Lines (beams)



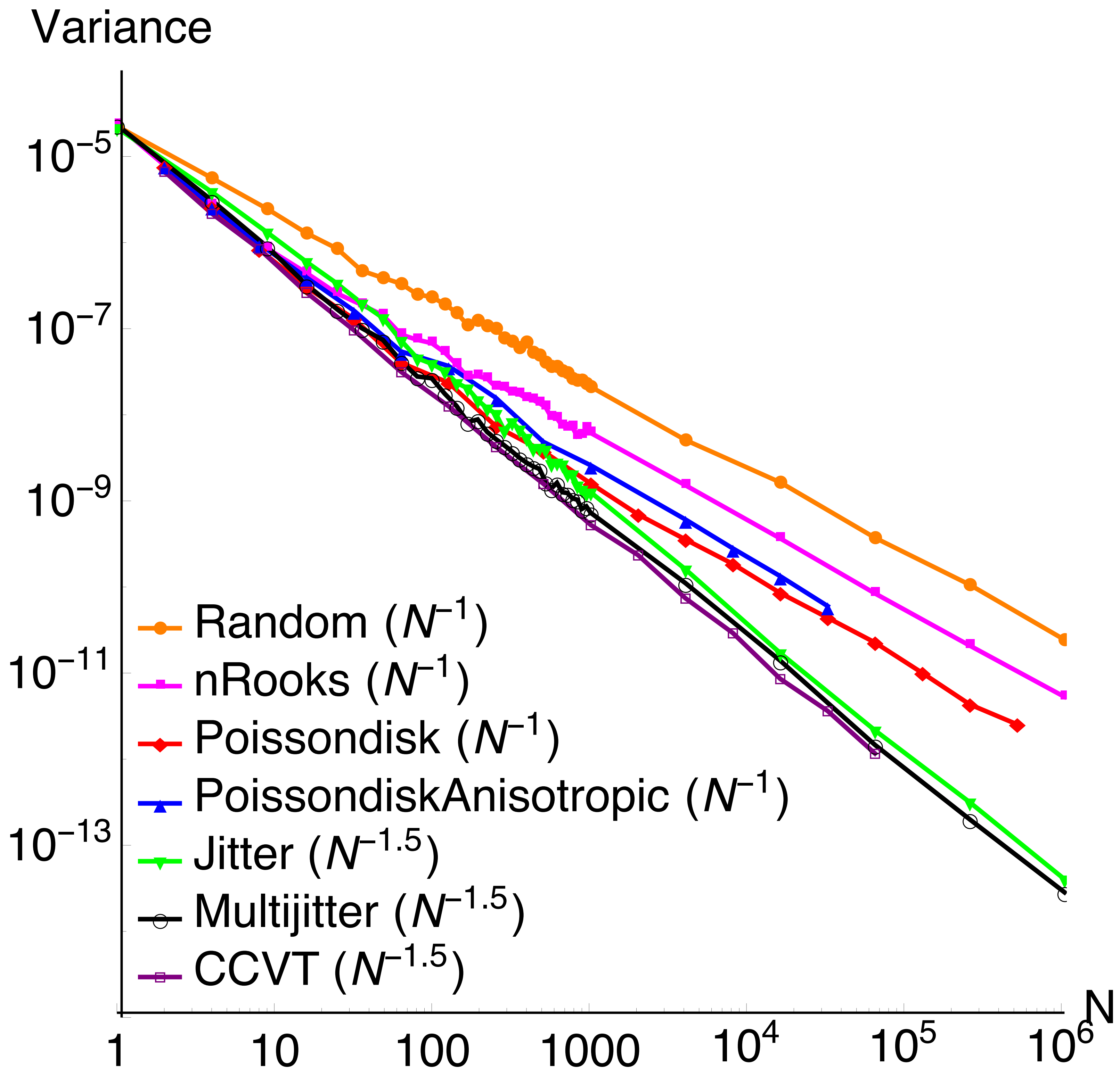
Sphere caustics 2D: Points vs Line Sampling

2D



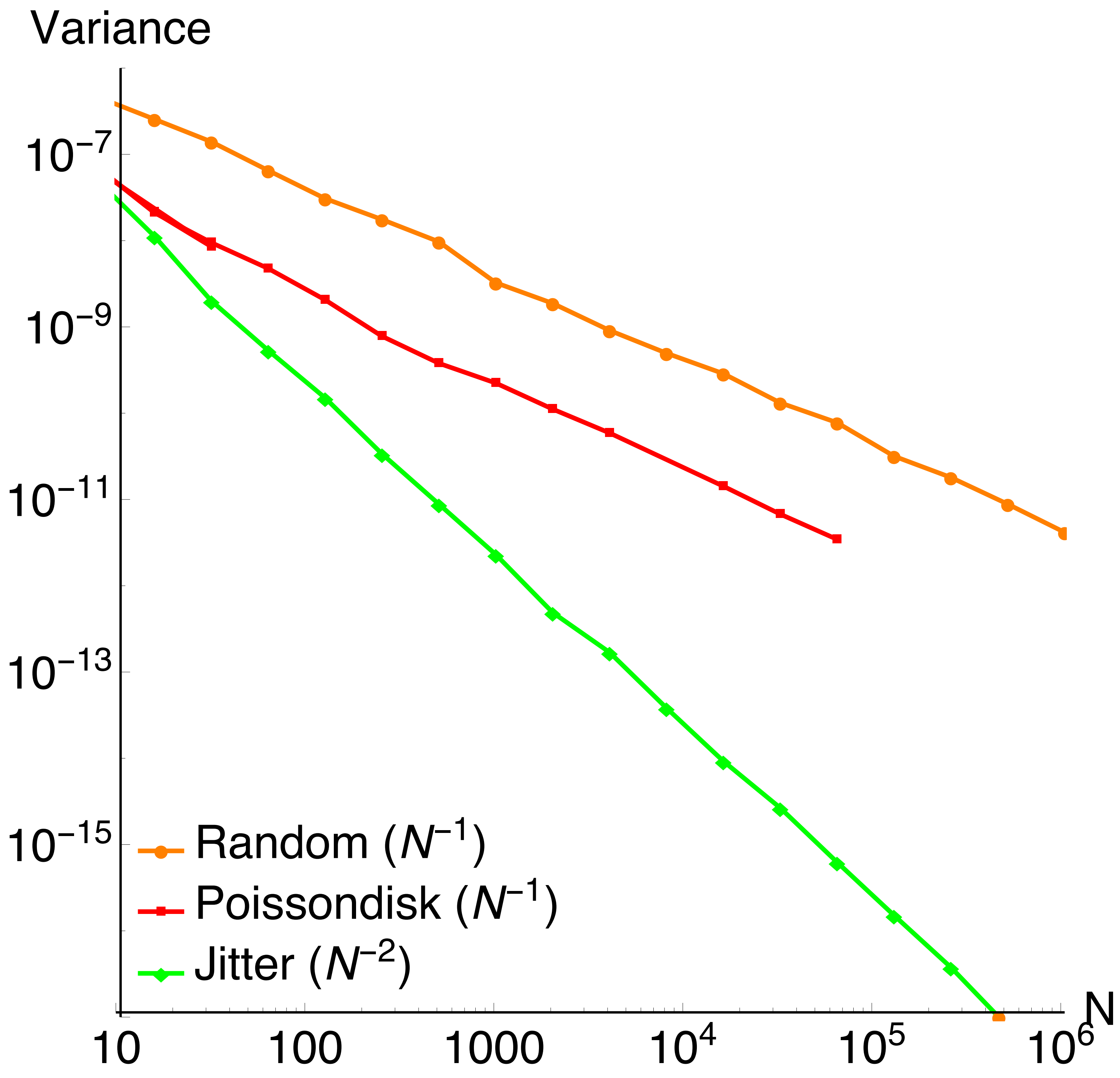
1D angular + 1D distance

Points (photons)



1D angular

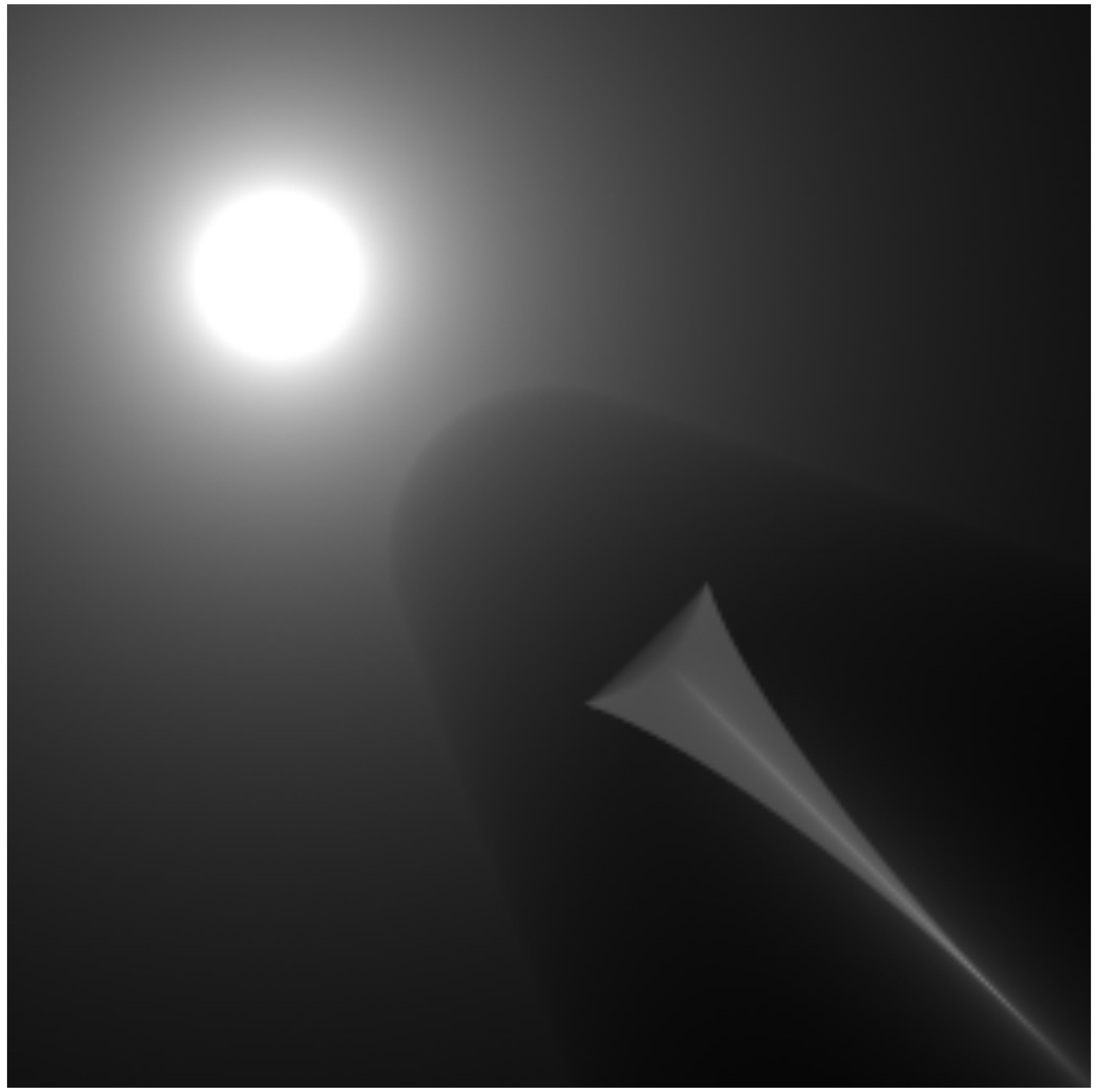
Lines (beams)



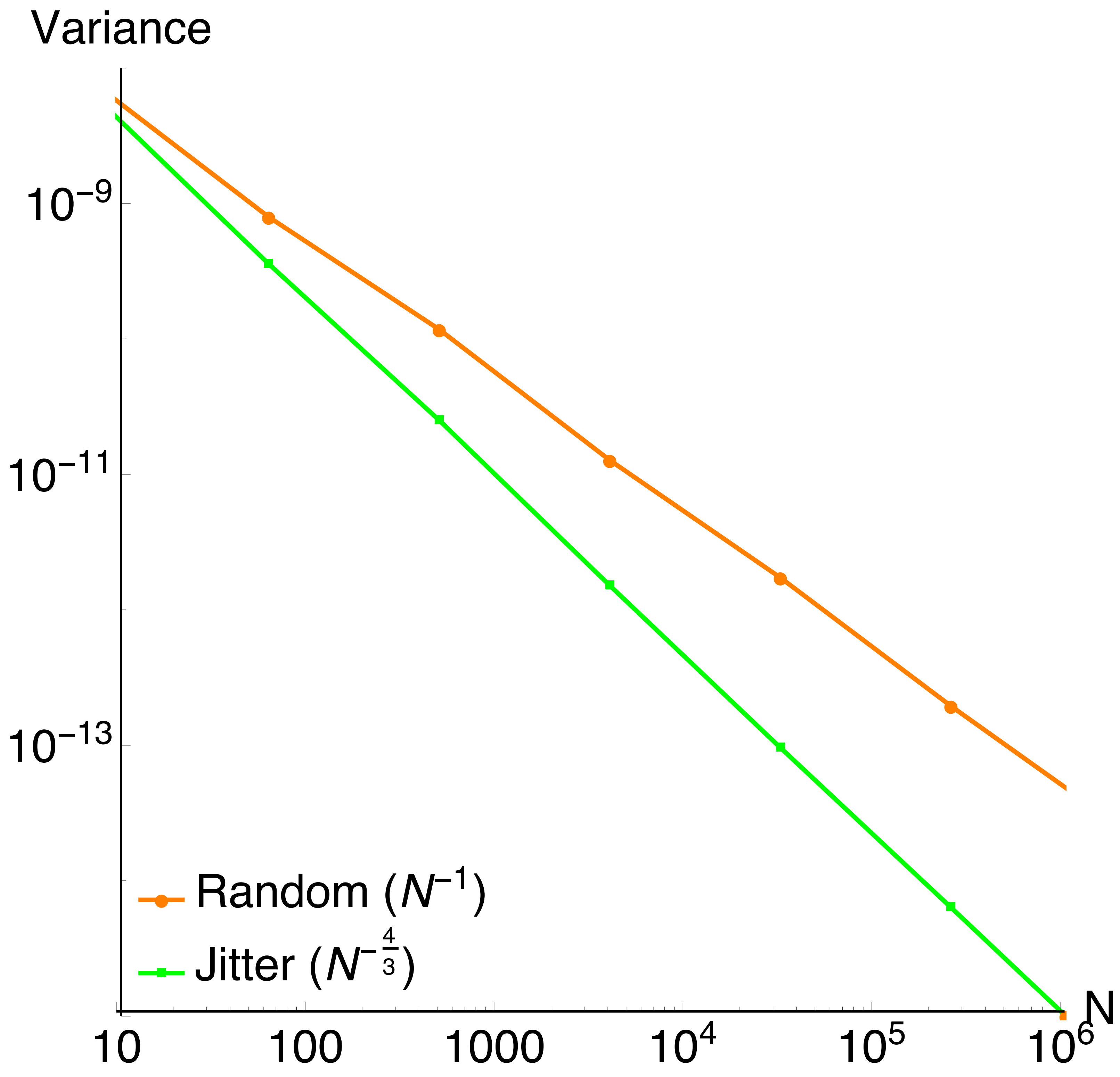
Sphere caustics 3D: Points vs Line Sampling

2D angular + 1D distance

3D

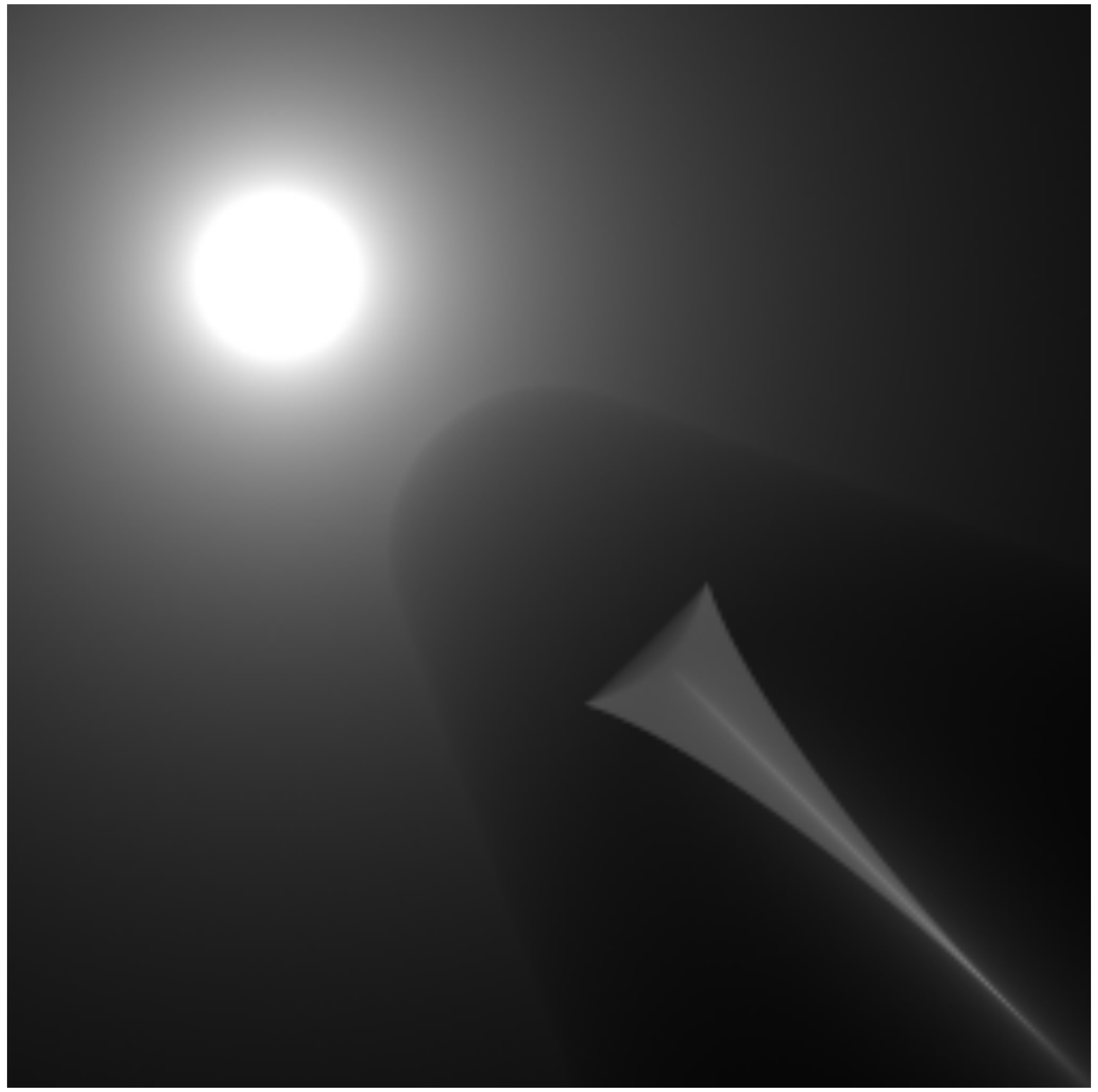


Points (photons)

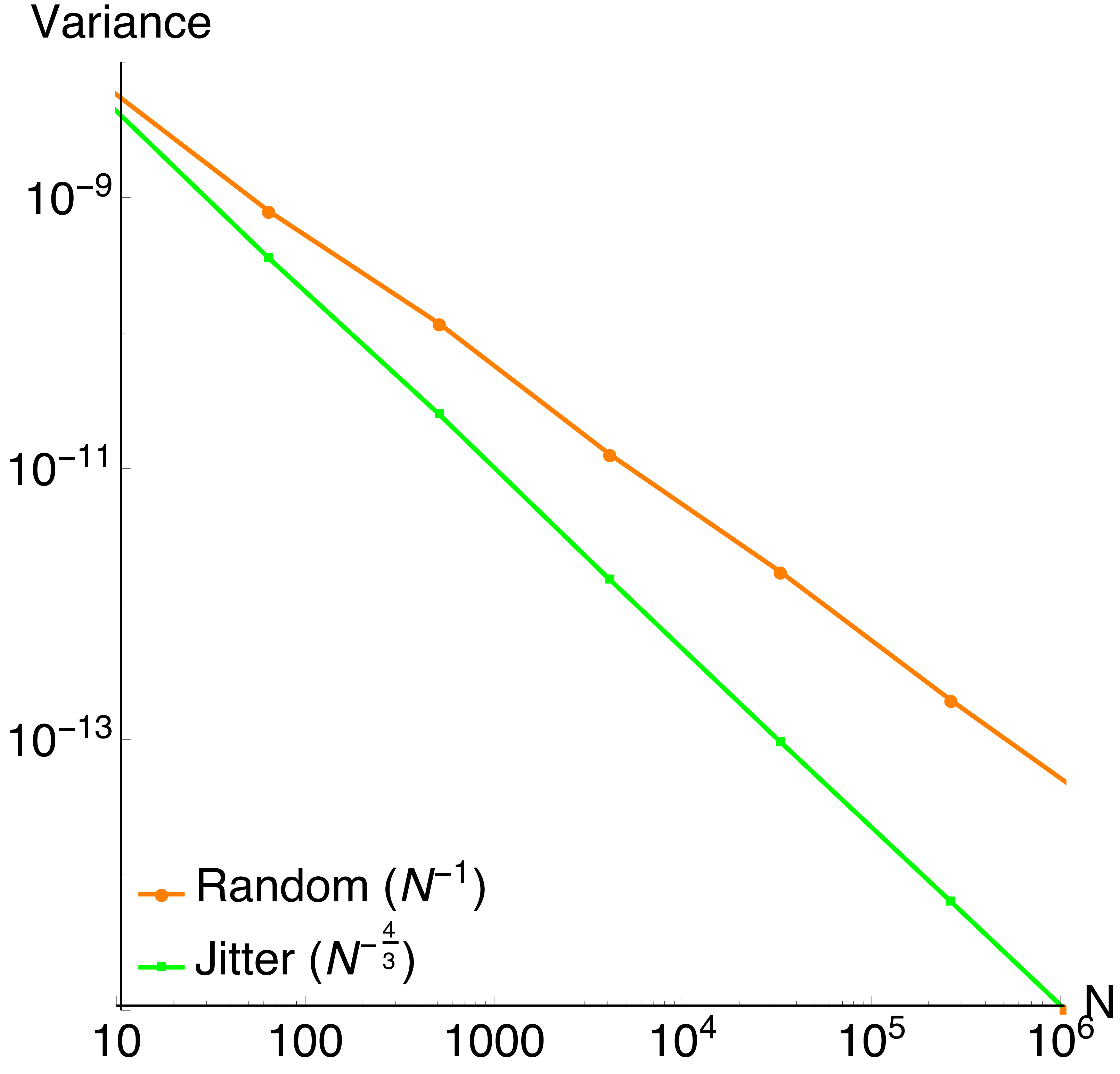


Sphere caustics 3D: Points vs Line Sampling

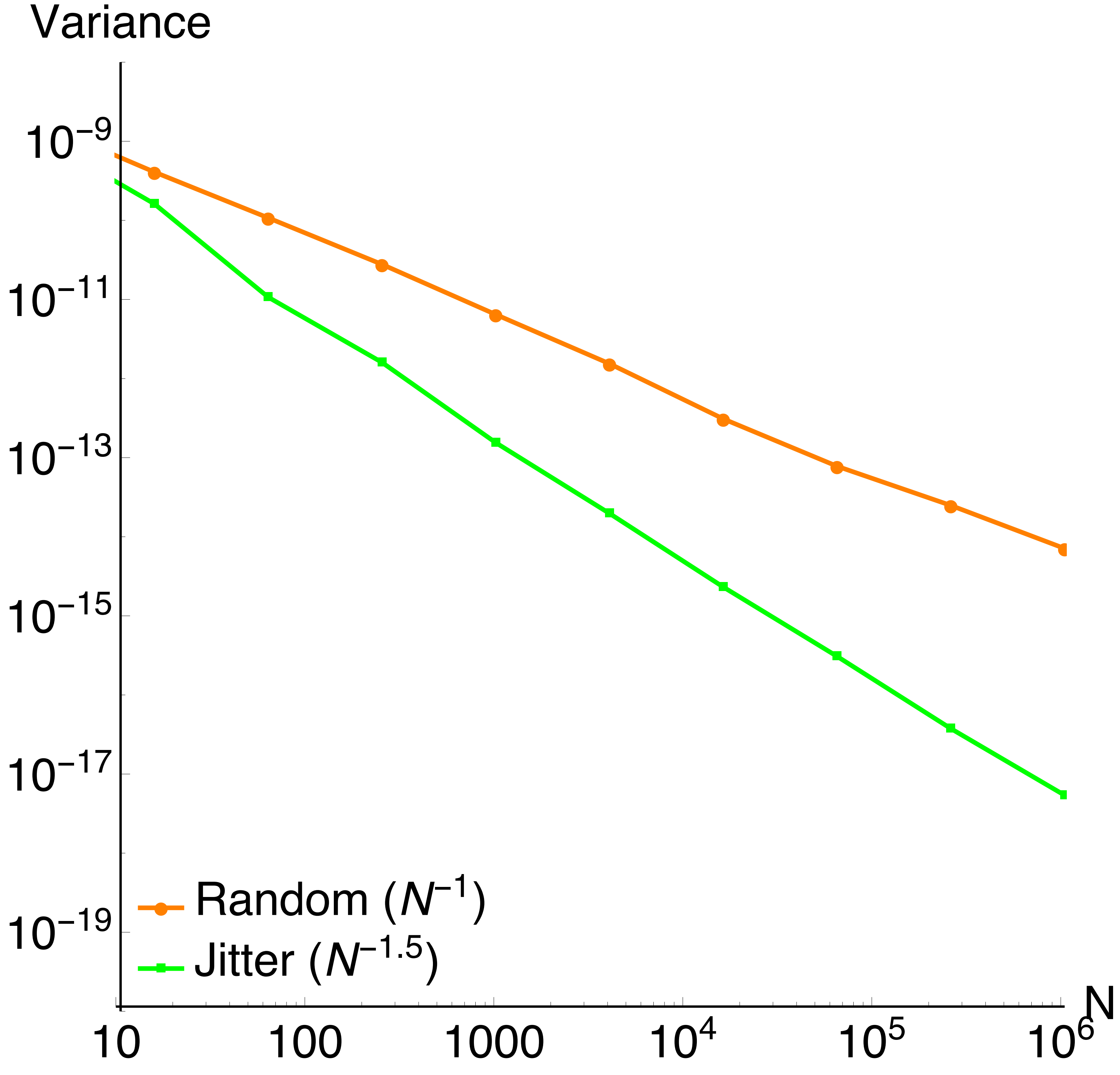
3D



2D angular + 1D distance
Points (photons)

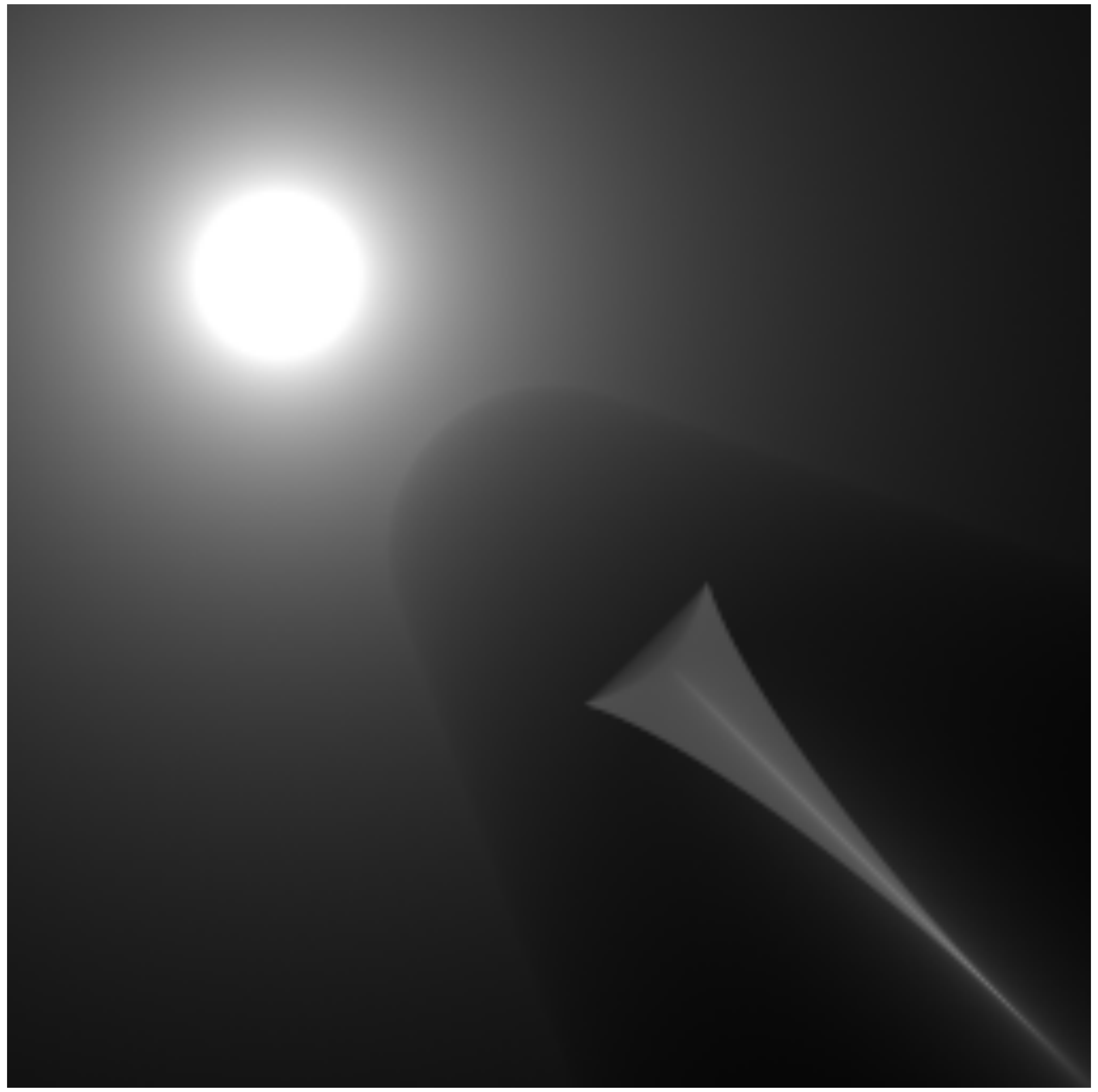


2D angular
Lines (beams)



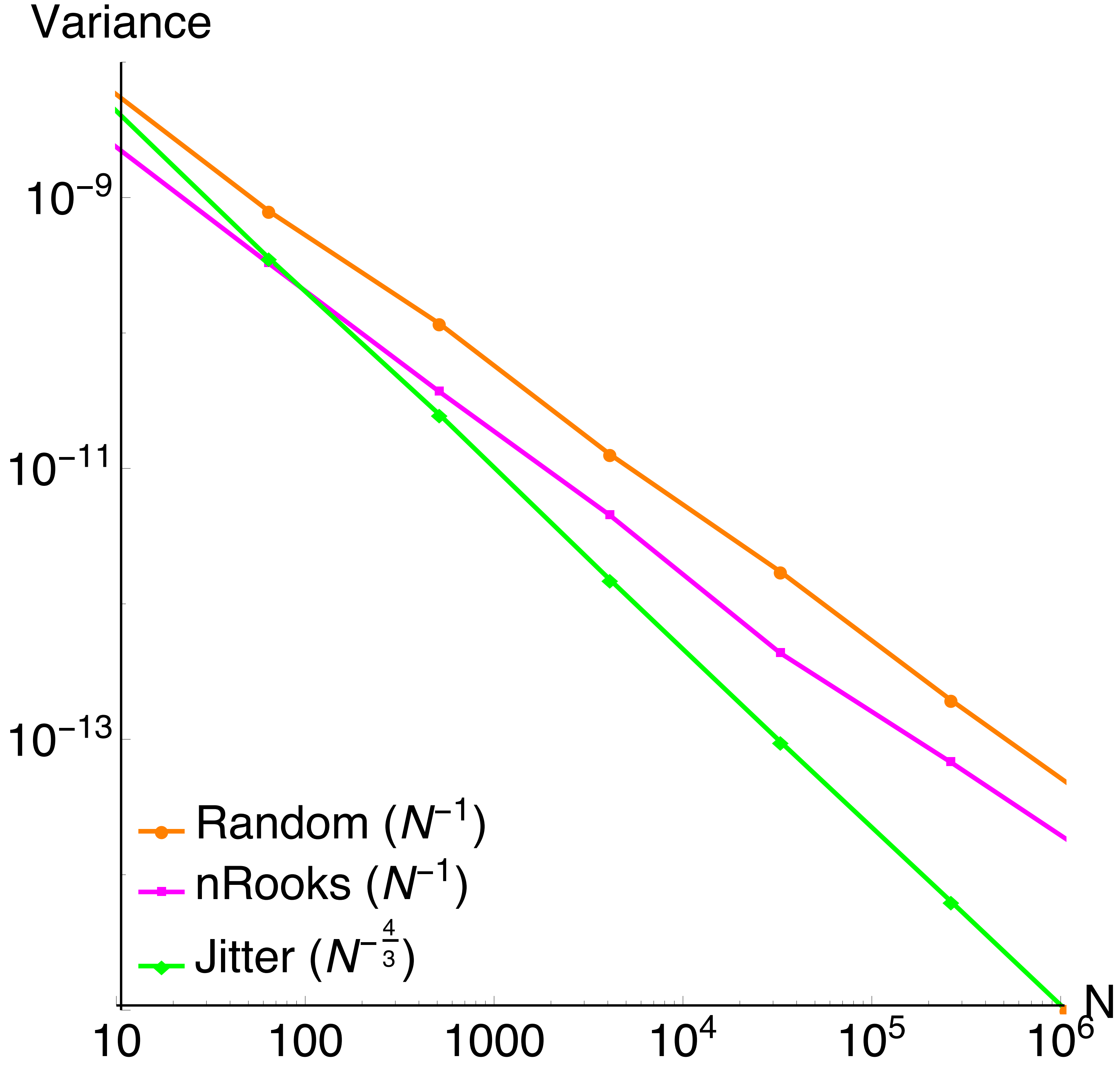
Sphere caustics 3D: Points vs Line Sampling

3D



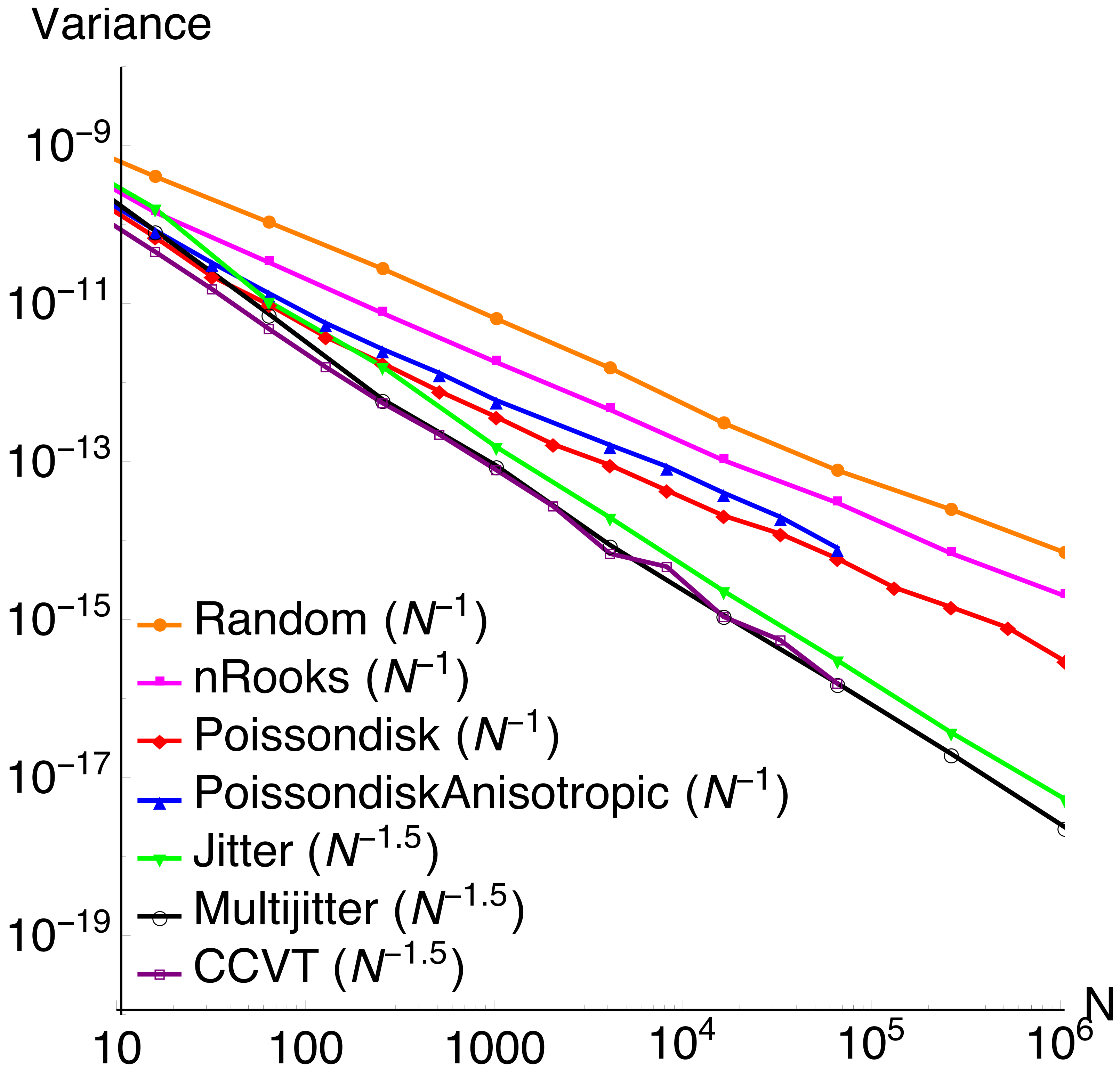
2D angular + 1D distance

Points (photons)



2D angular

Lines (beams)



Monte Carlo estimator for Segment Samples

$$I_N = \frac{1}{\pi N} \sum_{i=1}^{N} \frac{\int_{\theta_i^-}^{\theta_i^+} \mathrm{V}(\theta, \phi_i) \cos\theta |\sin\theta| d\theta}{\mathrm{pdf}(\theta_i, \phi_i)\, \Delta\theta_i}$$

Monte Carlo estimator for Segment Samples

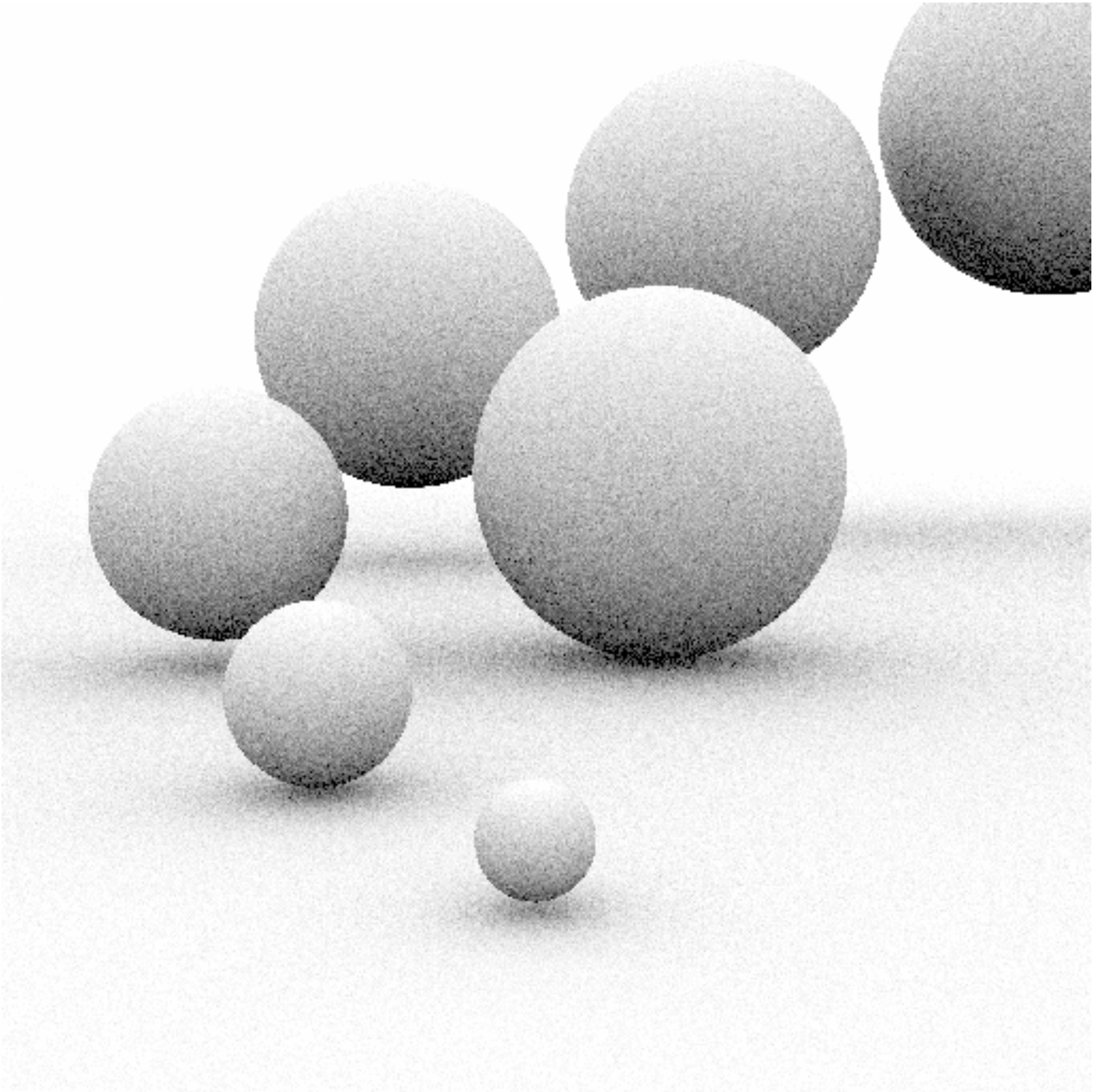
$$I_N = \frac{1}{\pi N} \sum_{i=1}^{N} \frac{\int_{\theta_i^-}^{\theta_i^+} \mathrm{V}(\theta, \phi_i) \cos\theta |\sin\theta| d\theta}{\mathrm{pdf}(\theta_i, \phi_i)\, \Delta\theta_i}$$

$$\mathrm{pdf}(\theta_i, \phi_i) = \frac{\cos\theta_i \sin\theta_i}{\pi}$$

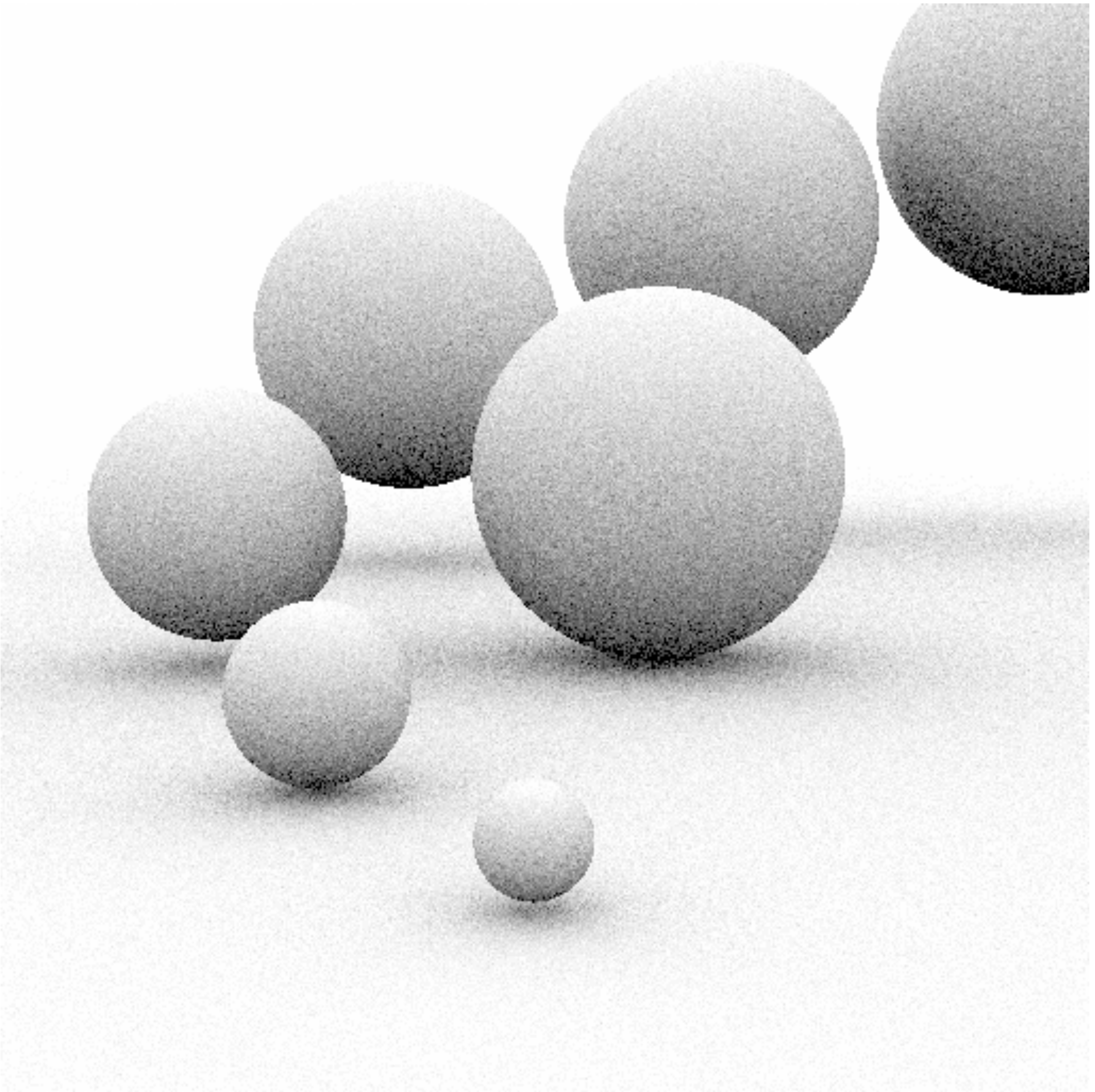
Monte Carlo estimator for Segment Samples

$$I_N = \frac{\pi 1}{\pi N} \sum_{i=1}^{N} \frac{\int_{\theta_i^-}^{\theta_i^+} \mathrm{V}(\theta, \phi_i) \cos\theta |\sin\theta| d\theta}{\cos\theta_i \sin\theta_i \Delta\theta_i}$$

Point Sampling with different PDFs



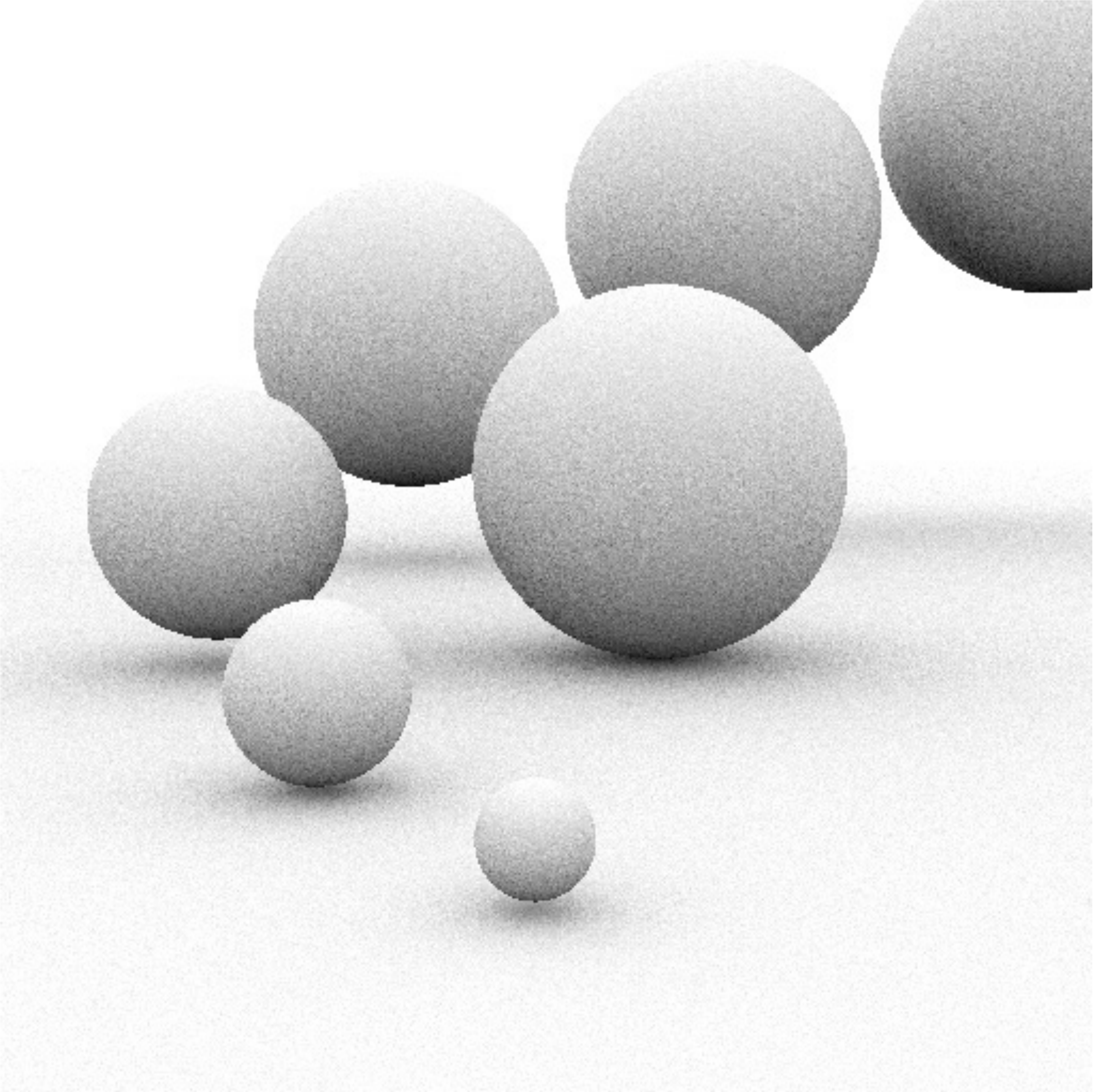
Uniform theta/phi



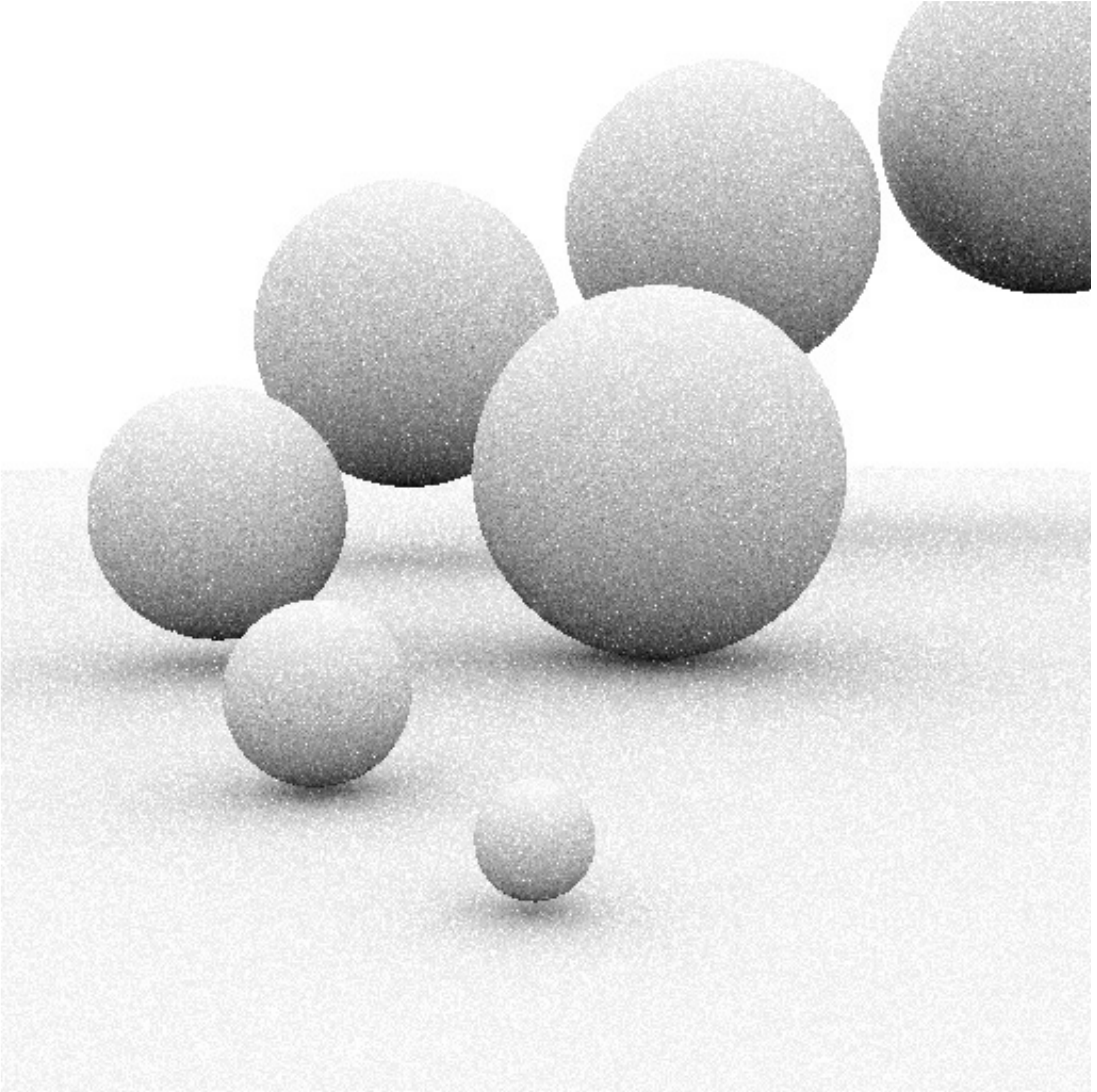
Cosine Weighted

EGSR 2017
Helsinki, Finland, June 19–21

Segment Sampling with different PDFs



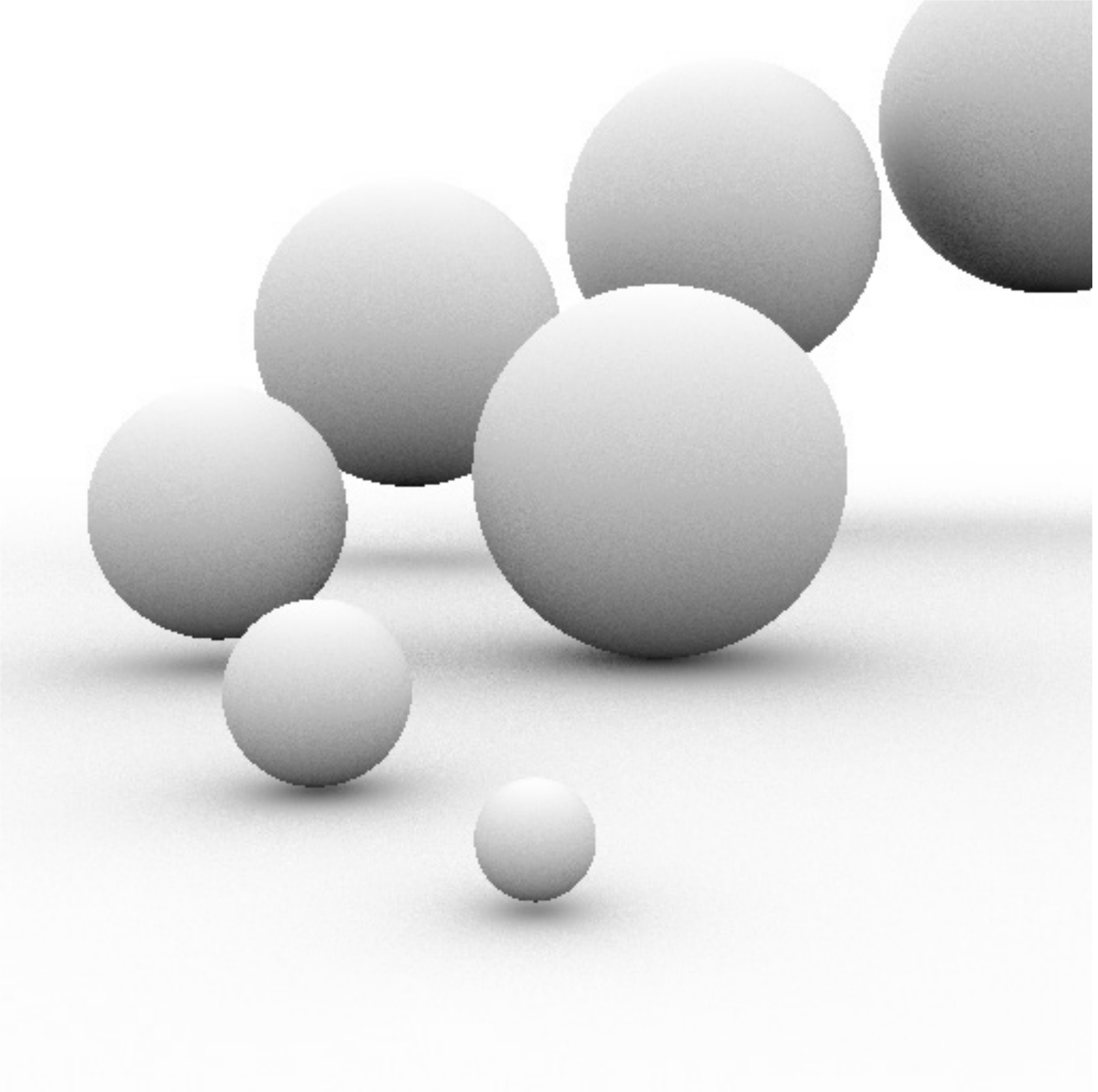
Uniform theta/phi



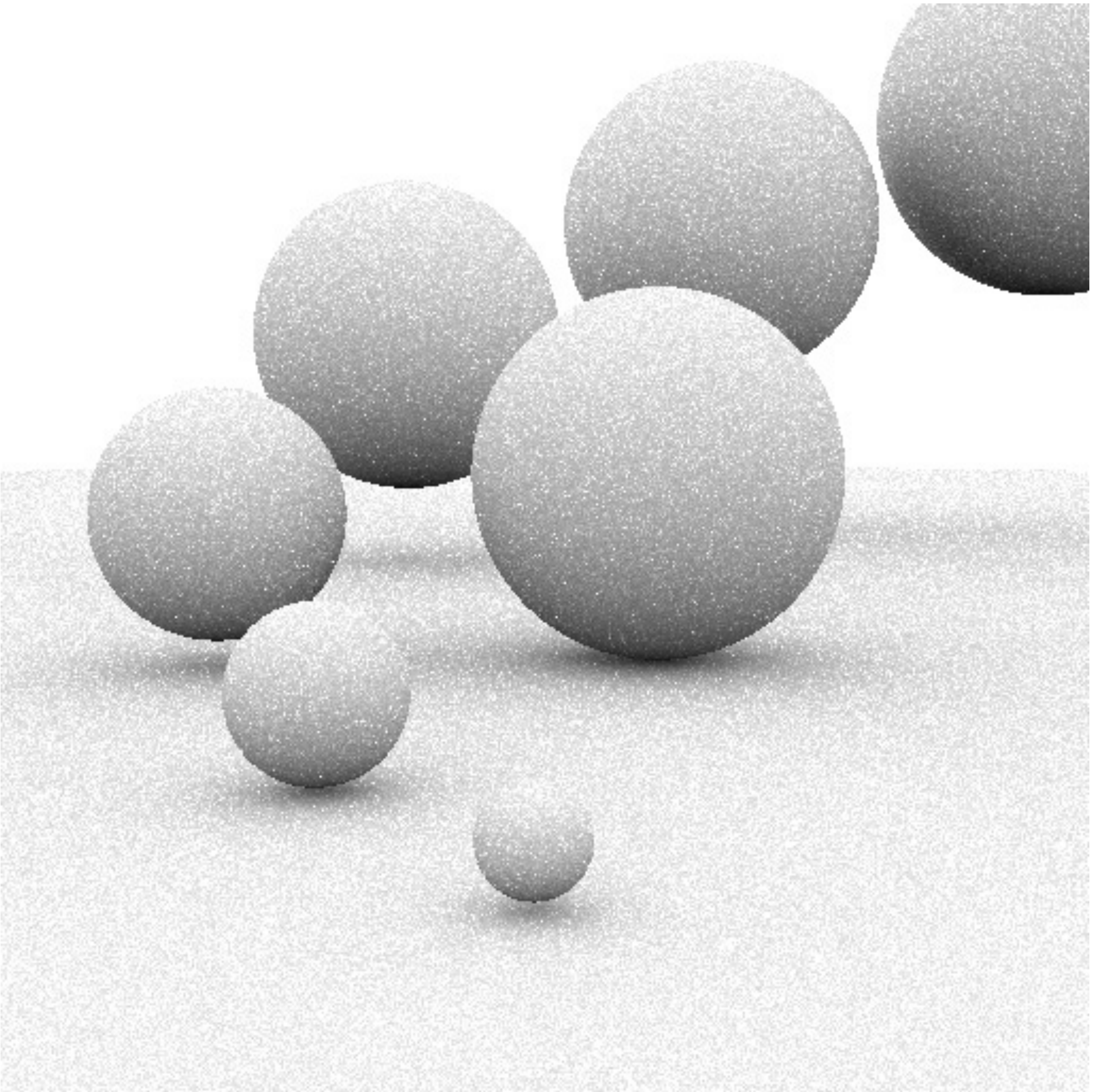
Cosine Weighted

length = 0.1

Segment Sampling with different PDFs



Uniform theta/phi

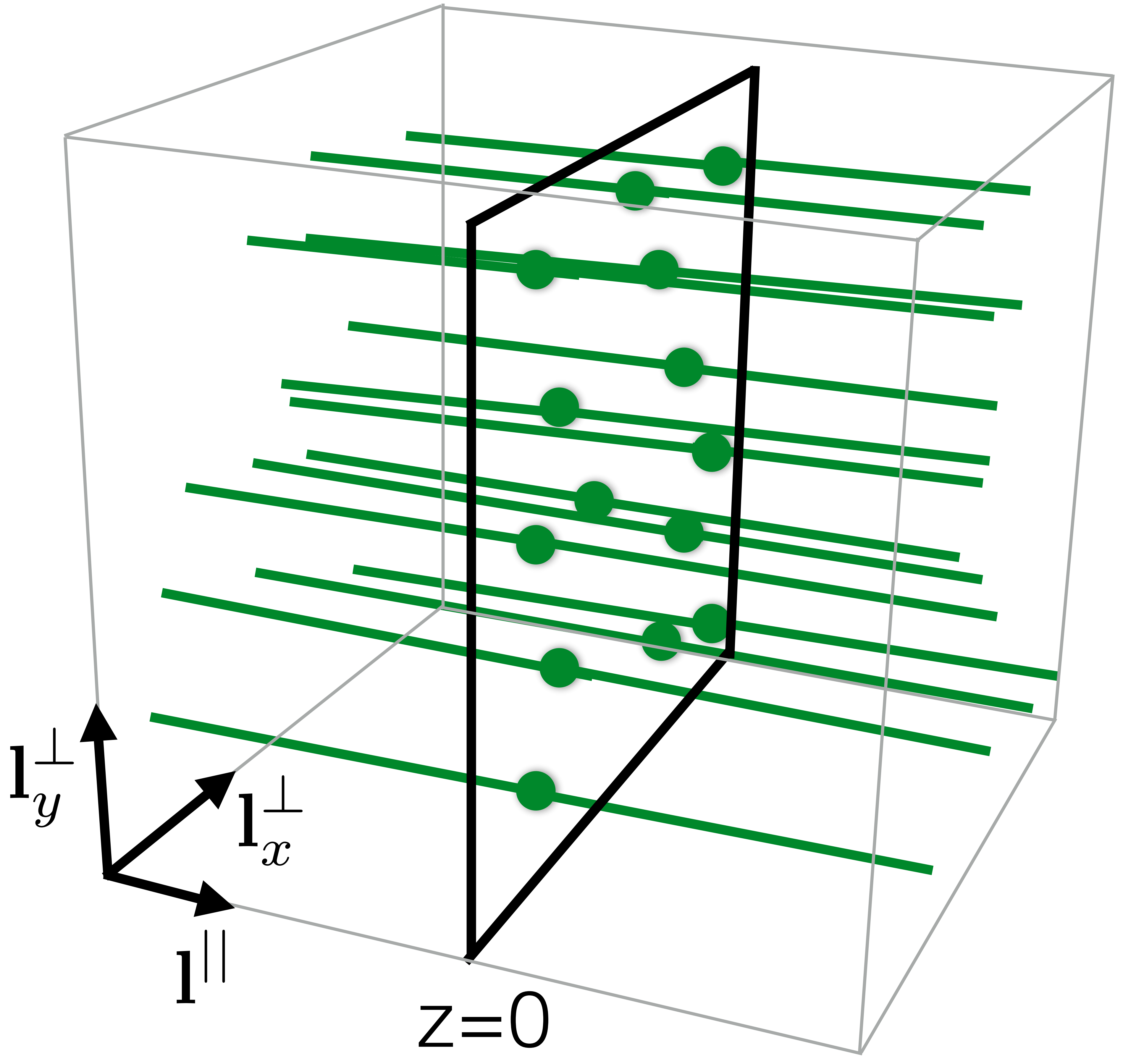


Cosine Weighted

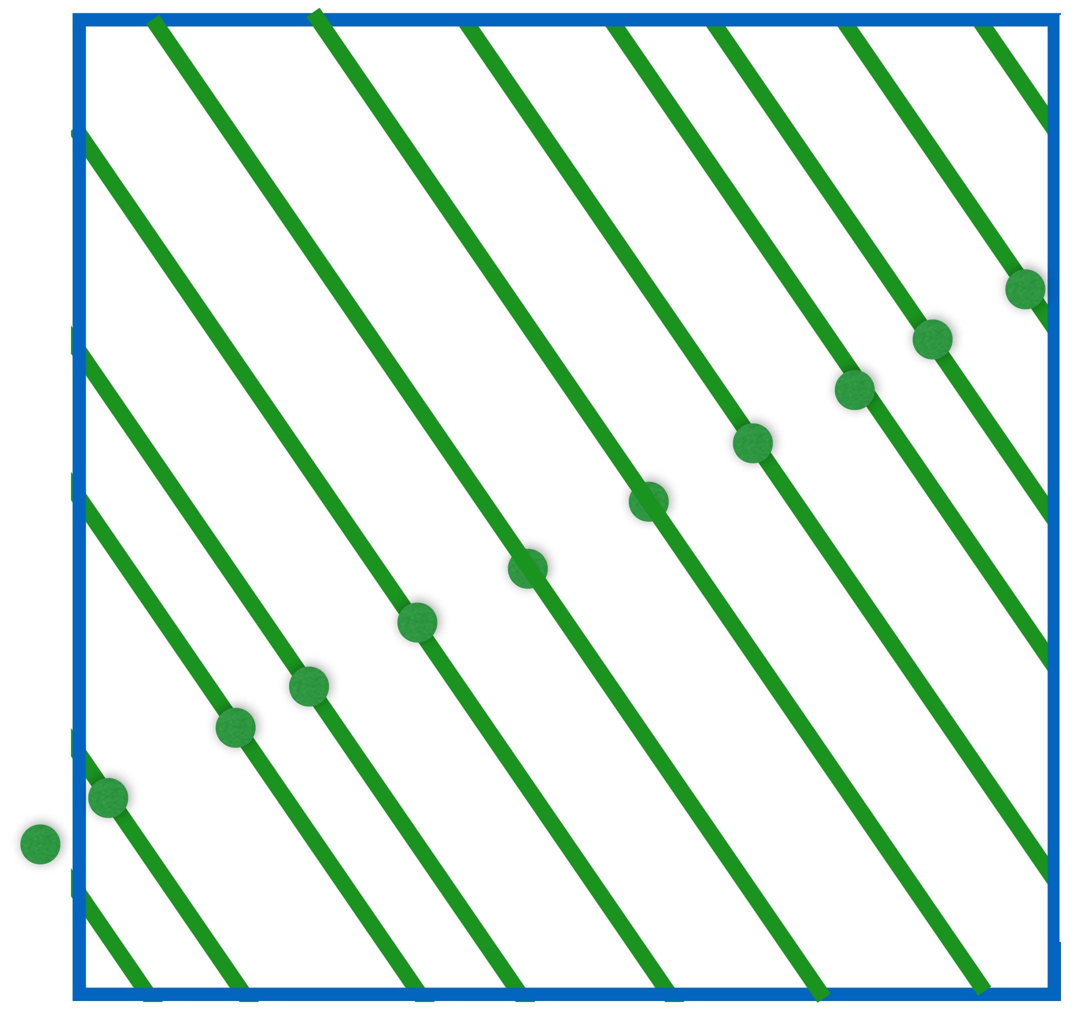
length = π

Line sampling with Poisson Disk line offsets

3 dimensions (3D)

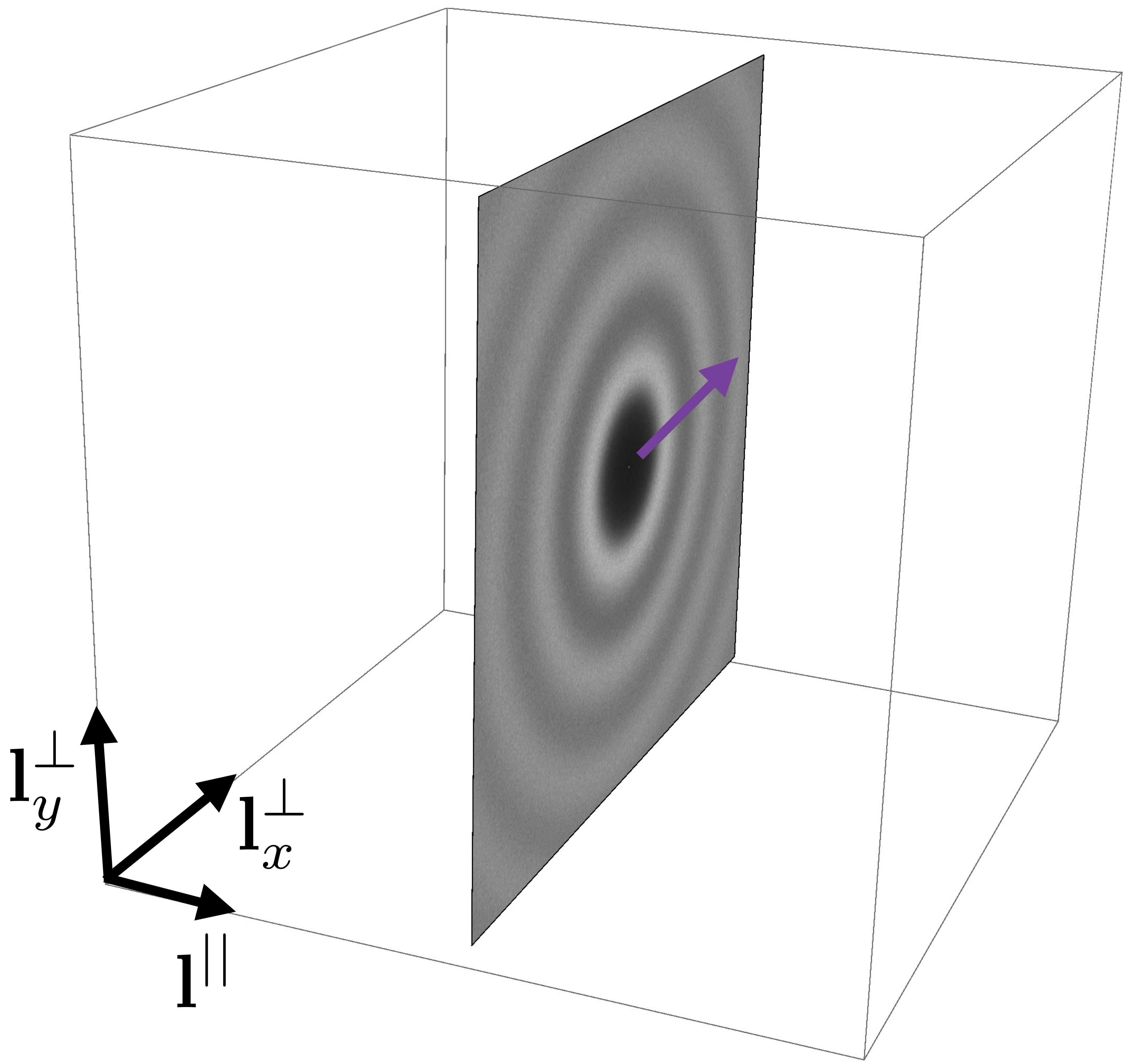


2 dimensions (2D)

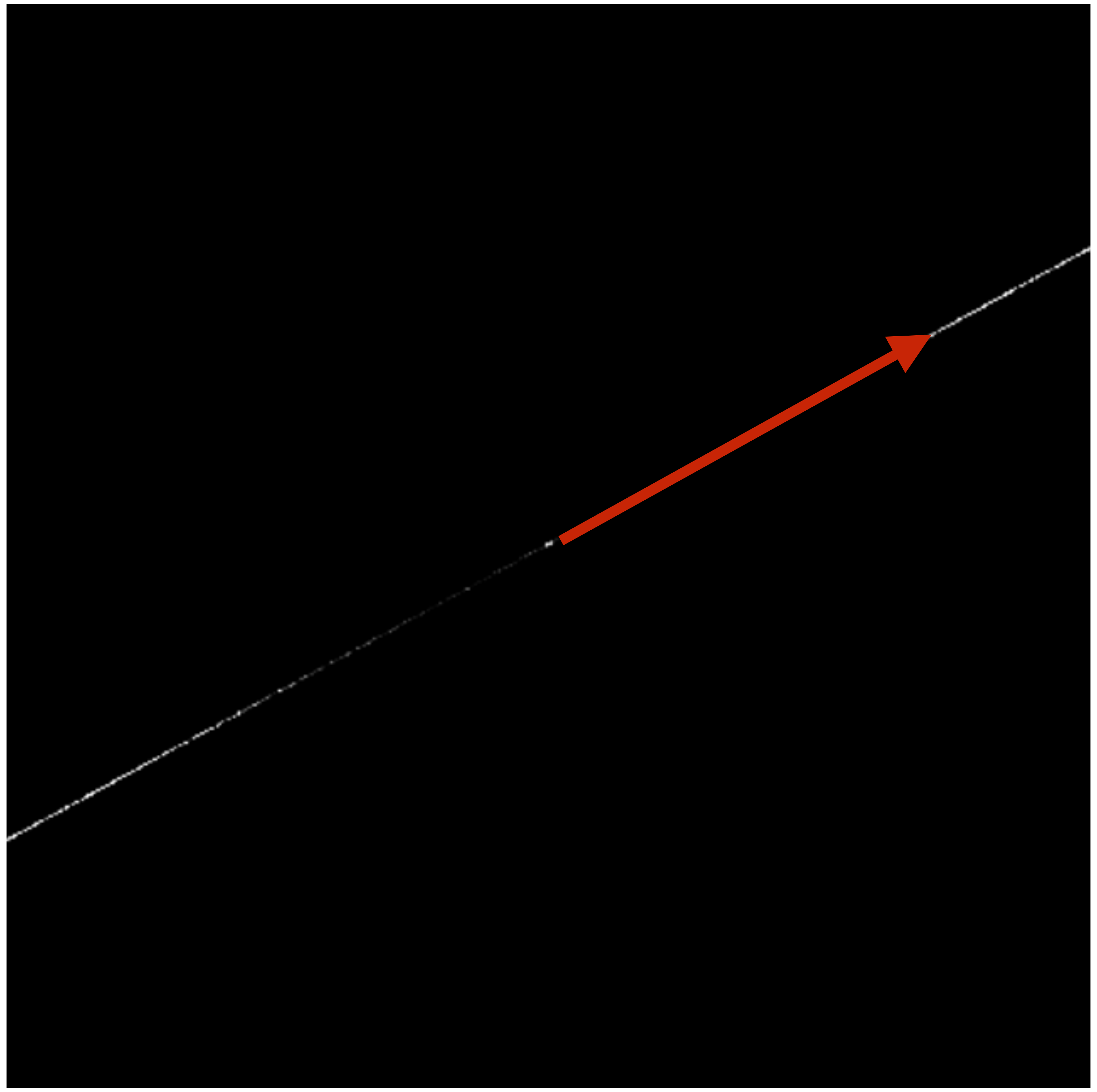


Line sampling with Poisson Disk line offsets

3 dimensions (3D)



2 dimensions (2D)



Expected Power Spectrum