Homework 10
Due Wed May 27, 5:00pm

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General Instructions: Same as in Homework 1.

Honor Principle: Same as in Homework 1.

24. Recall our definition of the class $\mathsf{IP}_{\alpha,\beta}$: We say that a language $L \subseteq \{0,1\}^*$ is in this class if there is a polynomial-time verifier V that uses a random string r and has the following properties, where P is an arbitrarily powerful prover that interacts with V:

$$x \notin L \implies \forall\, P: \Pr_r[V * P(x,r) = 1] \le \alpha\,,$$
$$x \in L \implies \exists\, P: \Pr_r[V * P(x,r) = 1] \ge \beta\,.$$

We defined $\mathsf{IP} = \mathsf{IP}_{\frac{1}{3},\frac{2}{3}}$ and remarked that the choice of the constants isn't terribly important, as can be proven by suitable repetition and Chernoff bound analysis. We also remarked that β can be made equal to 1 (perfect completeness), though not by simple repetition. Finally, we remarked that α cannot be made zero (perfect soundness), because that would boil the underlying class to plain old NP.

Justify this last remark. Specifically, prove that $\mathsf{IP}_{0,\frac{2}{3}} = \mathsf{NP}$. [2 points]

25. Let p be a prime. This problem involves the group $\mathbb{Z}_p$, consisting of integers $\{1, 2, \ldots, p-1\}$ with multiplication performed mod p. At some point you will need to use the fact that every element of $\mathbb{Z}_p$ has a multiplicative inverse mod p (that's what makes it a group).

The *quadratic residuosity problem* asks whether a given integer is a square mod p. The brute force solution is to try out all elements of $\mathbb{Z}_p$ and compute the square of each, but it takes time proportional to p, which is exponential in the input length. But one can give interesting interactive proofs for this problem. To be precise, define the languages

$$\begin{aligned} \text{QR} &= \{\langle p, x\rangle : p \text{ is prime, } x \in \mathbb{Z}_p, \text{ and } \exists\, y \in \mathbb{Z}_p\ (y^2 \equiv x \pmod{p})\}\,, \\ \text{QNR} &= \{\langle p, x\rangle : p \text{ is prime, } x \in \mathbb{Z}_p, \text{ and } \forall\, y \in \mathbb{Z}_p\ (y^2 \not\equiv x \pmod{p})\}\,. \end{aligned}$$

The acronyms denote "quadratic residue" and "quadratic non-residue," respectively.

Prove that both these languages are in IP and that one of these is in fact in NP. [2 points]

Hint: Your protocol for one of the languages should mimic the one we gave in class for GNI (graph non-isomorphism). Suppose $\langle p, x\rangle \in$ QNR and $z \in \mathbb{Z}_P$. What can you say about $xz^2 \bmod p$?