Homework 6
Due Wed Apr 27, 5:00pm

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General Instructions: Same as in Homework 1.

Honor Principle: For Problem #12, you should work entirely on your own and not discuss with anyone. For Problem #13, the usual honor principle (as in Homework 1) applies.

12. For a string $x \in \{0,1\}^*$, let $N_1(x)$ denote the number of 1s in x. The *majority* function $\textsc{maj}_n : \{0,1\}^n \to \{0,1\}$ is defined as follows:

$$\textsc{maj}_n(x) = \begin{cases} 1, & \text{if } N_1(x) \geq n/2, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $\textsc{maj}_n$ can be computed using $O(n)$-sized circuits. [This is essentially Sipser's Problem 9.26 — if you use the approach suggested in the book, you need to first solve (in sufficient detail) any subproblems that come up, such as Sipser's Problem 9.24.] [2 points]

13. Prove that Shannon's lower bound is tight up to constant factors. That is, improve the upper bound we showed in class by proving that every function $f : \{0,1\}^n \to \{0,1\}$ has an n-input circuit of size $O(2^n/n)$. [2 points]

The second problem is hard. A hint is to consider the function f as being $f(y,z)$, where $y = \{x_1, \ldots, x_k\}$ and $z = \{x_{k+1}, \ldots, x_n\}$. Now, the truth table of f can be viewed as a $2^k \times 2^{n-k}$ matrix, with the rows indexed by all possible assignments to y. Each column of this matrix gives us a certain pattern in $\{0,1\}^{2^k}$. What if there aren't too many different patterns? Can we use that fact to cut down on the circuit size?