Solutions to Quiz 1
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1. Let A and B be languages over some alphabet Σ such that $|A| = m$ and $|B| = n$. Does it follow that $|AB| = mn$? Justify your answer.

[5 points]

Solution: No, it does not follow. Here is a counterexample, with $\Sigma = \{0, 1\}$.

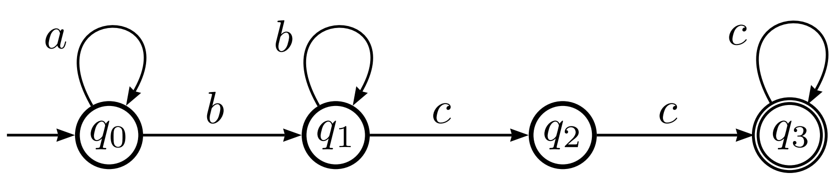
$$A = \{10, 100\}, \quad \text{so } |A| = 2,$$
$$B = \{01, 001\}, \quad \text{so } |B| = 2.$$

Then $AB = \{1001, 10001, 100001\}$, so $|AB| = 3 \neq 2 \times 2$.

2. Draw an NFA that recognizes the language $\{a^i b^j c^k : i \geq 0, j \geq 1, k \geq 2\}$. Keep it simple!

[10 points]

Solution:



3. Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be two DFAs over the same alphabet Σ. Write a formal description of a DFA that recognizes the language $\mathcal{L}(M_1) \cap \mathcal{L}(M_2)$. No proof of correctness required.

[10 points]

Solution: The following DFA, M, recognizes $\mathcal{L}(M_1) \cap \mathcal{L}(M_2)$.

$$M \;=\; (Q_1 \times Q_2,\ \Sigma,\ \delta,\ (q_1, q_2),\ F_1 \times F_2)\,,$$

where δ is given by

$$\delta((q,r),a) \;=\; (\delta_1(q,a), \delta_2(r,a)), \quad \forall\, q \in Q_1, r \in Q_2, a \in \Sigma\,.$$

4. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and let $x = a_1 a_2 \ldots a_n$ be a string with each $a_i \in \Sigma$. Write a formal mathematical definition of what we mean when we say "M accepts x."

[10 points]

Solution: "M accepts x" means that there exists a sequence $r_0, r_1, \ldots, r_n$ with each $r_i \in Q$ such that

- $r_0 = q_0$,
- $r_{i+1} = \delta(r_i, a_{i+1}),\ \forall\, i$ with $0 \le i < n$, and
- $r_n \in F$.

5. Write a regular expression for the language $\{x \in \{R, G, B\}^* : x \text{ has an odd number of } R\text{'s}\}$. You do not have to give a proof of correctness.

[10 points]

Solution: The intuition is that any string in the given language consists of

- an initial block of symbols ending in the string's first R, followed by
- zero or more blocks, each containing two R's and ending in the second R, followed by
- a final block of symbols containing no R's.

Thus, the following regular expression generates the language:

$$(G \cup B)^* R \left((G \cup B)^* R (G \cup B)^* R\right)^* (G \cup B)^* \, .$$

6. Consider the language $L = 0^*1^* \cap ((0 \cup 1)(0 \cup 1))^*$.

6.1. Why is the above expression for L not a regular expression?

[5 points]

Solution: Because it contains an intersection operator '$\cap$', whereas regular expressions may only contain union, concatenation and Kleene star operators.

6.2. Write a regular expression for L.

[10 points]

Solution: Observe that L consists of all even-length strings of the form $0^i 1^j$, where i and j are non-negative integers. In such a string, either i and j are both even or they are both odd. Therefore,

$$L \;=\; (00)^*(11)^* \cup 0(00)^*1(11)^* \, .$$

7. For a string $x \in \{0,1\}^*$, let $N_0(x)$ and $N_1(x)$ denote the number of 0's and 1's (respectively) in x. Prove that *at least one* of the following two languages is regular.

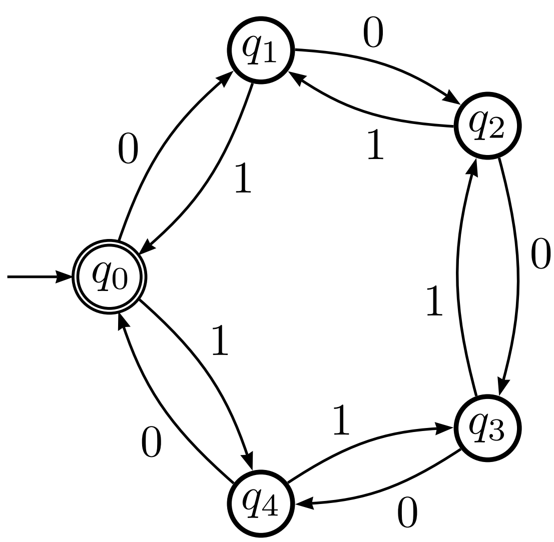
$$L_1 = \{x \in \{0,1\}^* : |N_0(x) - N_1(x)| = 5\} ,$$
$$L_2 = \{x \in \{0,1\}^* : |N_0(x) - N_1(x)| \text{ is divisible by } 5\} .$$

Obviously you should start by picking one of the two languages. Remember, you don't need to prove anything about the other language.

Note on notation: If n is an integer, $|n|$ denotes the absolute value of n.

[10 points]

Solution: The language L_2 is regular, because it is recognized by the following DFA.



The language L_1 is not regular. You did not have to prove this, but you should be able to do so by using the pumping lemma to try to pump the string $0^{p+5}1^p \in L_1$, where p is the hypothetical pumping length of L_1.