Homework 2
Due Fri May 2, 5:00pm

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As usual, please think carefully about how you are going to organise your proofs *before* you begin writing. You shouldn't need more than a page for each solution, so reconsider your approach if you find yourself writing much more than that.

This homework is fairly challenging. Start early!

1. Treating an n-bit vector $\vec{x}$ as a string (so that the ordering of the variables is important) we can define a Boolean function $A_n^s : \{0,1\}^n \to \{0,1\}$ as follows: $A_n^s(\vec{x}) = 1$ iff $\vec{x}$ contains s as a substring. Here s is a fixed string.

   1.1. Show that A_5^{111} and A_6^{111} are not evasive, i.e., it is possible to compute these functions without ever having to look at every input bit. [5 points]

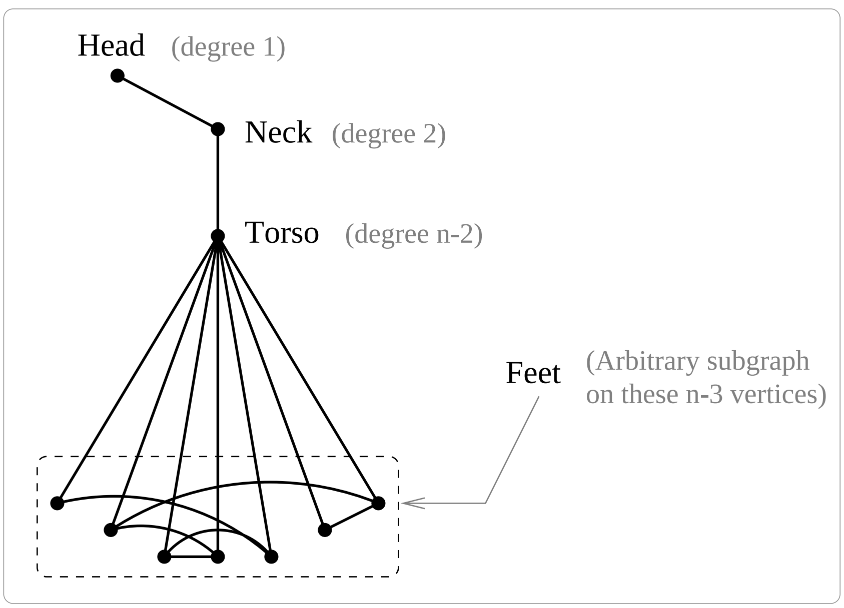
   1.2. Show that A_3^{111} and A_4^{111} are evasive. [5 points]

   1.3. Prove that A_n^{111} is evasive iff $n \equiv 0$ or $3 \pmod 4$. Hint: For the "if" direction, use a recurrence for the number of strings of length n that satisfy A_n^{111}, and prove that sometimes just looking at this number tells you that the function is evasive. For the "only if" direction, use the ideas from your solution to #1.1, plus induction. [10 points]

   1.4. Find all integers n for which A_n^{100} is evasive. Hint: Consider $n \bmod 3$. [10 points]

2. Even while working on lower bounds one often has to prove upper bounds, if only to provide counterexamples for plausible but false lower bound conjectures. In the early 1970s it was conjectured that any nontrivial graph property f_n on n-vertex graphs has $\mathrm{D}(f_n) = \Omega(n^2)$. The Rivest-Vuillemin theorem proves this for monotone f_n, but what about non-monotone properties?

   Call an n-vertex graph a *scorpion* if it has the structure shown in the following figure.



   Let f_n be the property of being a scorpion.

2.1. Argue that f_n is not monotone. [1 points]

2.2. Design an algorithm that computes f_n while querying at most $6n$ of the $\binom{n}{2}$ Boolean variables representing the possible edges of the n-vertex graph. This shows that far from having an $\Omega(n^2)$ lower bound, we have an upper bound: $\mathrm{D}(f_n) \leq 6n = O(n)$. [9 points]

Hint: If an input graph is indeed a scorpion, it is easy to verify this if an oracle tells you which vertex is the torso.

3. In class, we *almost* finished the proof of the Rivest-Vuillemin theorem. We proved that if $f : \{0,1\}^N \to \{0,1\}$ is a nonconstant monotone Boolean function invariant under a transitive group of permutations of the variables, then:

   (a) If N is a power of 2, then f is evasive.

   (b) If f is an n-vertex graph property (and so, $N = \binom{n}{2}$) and n is a power of 2, then $\mathrm{D}(f) \geq n^2/4$.

   This problem walks you through the last bit of the proof, where we handle n-vertex graph properties f for arbitrary $n \geq 2$. Let $k = \lfloor \log_2 n \rfloor$, so that $2^k \leq n < 2^{k+1}$. The basic idea is to identify a suitable subfunction g of f, note that $\mathrm{D}(f) \geq \mathrm{D}(g)$ and lower bound $\mathrm{D}(g)$ either directly, using one of facts (a) or (b) above, or indirectly, through an induction hypothesis.

3.1. Let the variables of f be named x_{ij}, with $1 \leq i < j \leq n$. Consider the two possible subfunctions of f obtained by setting $x_{1j} = b$ for all possible j, where $b \in \{0,1\}$. Show that if either of these subfunctions is nonconstant, then you can "make progress," according to the above plan. [3 points]

3.2. Give an example of a natural (and very common) nonconstant graph property that causes both the above subfunctions to be constant. [2 points]

3.3. Now suppose both the above subfunctions are constant. Partition the vertex set $[n]$ into disjoint parts A, B, C with $A < B < C$,[1] $|A| = |B| = 2^{k-1}$ and $|C| = n - 2^k$. Consider the subfunction of f obtained by setting

$$x_{ij} \;=\; \begin{cases} 0\,, & \text{if } i \in A \text{ and } j \in A \cup C\,, \\ 1\,, & \text{if } i, j \in B \cup C\,. \end{cases}$$

Prove that this subfunction is nonconstant. Identify a transitive permutation group under which it is invariant. [10 points]

3.4. Based on the above observations, conclude that $\mathrm{D}(f) \geq n^2/16$, thereby finishing the proof. [5 points]

4. The *extreme points problem* asks whether the convex hull of n given points in the plane has n vertices; note that this is an easier problem than computing the convex hull, so the convex hull lower bound does not apply to it directly.

   Model this problem as a set recognition problem for an appropriate set $W \subseteq \mathbb{R}^{2n}$. Prove that $\#W \geq (n-1)!$ and conclude that the algebraic computation tree complexity of the problem is $\Omega(n \log n)$. [10 points]

[1] This notation means that we have $a < b < c$ for all $a \in A, b \in B, c \in C$.

5. Let $\mathbf{a_1}, \ldots, \mathbf{a_k}$ and $\mathbf{b}$ be fixed nonzero vectors in $\mathbb{R}^n$ such that the system of inequalities $\{\mathbf{a_i} \cdot \mathbf{x} \geq 0, \, i = 1, \ldots, k\}$ is feasible and implies the inequality $\mathbf{b} \cdot \mathbf{x} \geq 0$. Then, it can be shown that $\mathbf{b}$ is a non-negative linear combination of the $\mathbf{a_i}$'s, i.e., $\mathbf{b} = \sum_{i=1}^{k} \lambda_i \mathbf{a_i}$ for some non-negative reals λ_i. This fact is sometimes known as Farkas's Lemma.

   Using Farkas's Lemma, prove the following two lower bounds in the *linear* decision tree model.

   5.1. The complexity of finding the largest of n given reals is $n - 1$. [10 points]

   5.2. The complexity of finding the second largest is at least $n - 2 + \log n$. [5 points]

   Hint: Once you have solved #5.1, use what you learnt along with a leaf counting argument to solve #5.2.