Further Primal-dual

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Facility Location problem

Primal

$$\min \sum_{i\in F} f_i y_i + \sum_{j\in C} d_{ij} x_{ij}$$

$$\forall j\in C: \quad \sum_{i\in F} x_{ij} = 1 \quad (\alpha_j)$$

$$\forall i,j: \quad y_i - x_{ij} \geq 0 \quad (\beta_{ij})$$

$$x, y \geq 0$$

Dual

$$\max \sum_{j\in C} \alpha_j$$

$$\forall i\in F: \quad \sum_{j\in C} \beta_{ij} \leq f_i$$

$$\forall i\in F, j\in C: \quad \alpha_j - \beta_{ij} \leq d_{ij}$$

$$\beta_{ij} \geq 0$$

$$\alpha_j \text{ free}$$

Methodology:

- Start with a feasible dual $(\alpha, \beta \equiv 0)$
- Feasible dual value $\leq$ opt
- "Raise duals" s.t. (a) it remains feasible, and (b) as we raise we get better & better lower bounds on opt.
- When we reach a hindrance in raising dual, that is, when some dual constraint becomes tight; at that point "make a primal decision."

FL primal-dual Algorithm

(1) Initialize $\alpha_j = 0, \forall j\in C$; $\beta_{ij} = 0, \forall i,j$

(1.b) Active clients : A; initially $A = C$.

(1.c) "Tentatively" open facilities : $\tilde{O} \subseteq F$. Initially $\tilde{O} = \emptyset$

(1.d) Tight edges :- $E \subseteq F \times C$; initially $E = \emptyset$

(1.e) "Tentative" Assignment : $\tilde{\sigma} : C \to \tilde{O} \cup \{\perp\}$

Initially, $\tilde{\sigma}(j) = \perp \ \forall j$

Invariants : (1) $\tilde{\sigma}(j) = i \Rightarrow (i,j) \in E$ & $i \in \tilde{O}$

(2) $\forall (i,j) \in E$: $\alpha_j = \beta_{ij} + d_{ij}$

(3) $\forall i \in \tilde{O}$: $\sum \beta_{ij} = f_i$

(2) • Raise α_j s of every $j \in A$, AND

Raise β_{ij} of every $(i,j) \in E$ where $j \in A$

@ uniform rate TILL

(2.a) constraint : "$\alpha_j - \beta_{ij} \leq d_{ij}$" becomes

tight for some $(i,j) \notin E$

- If $i \in \tilde{O}$ (ie, i is tentatively open)
  - Remove j from A
  - $\tilde{\sigma}(j) = i$
  - Don't add (i,j) to E
- If $i \notin \tilde{O}$,

In this case, add (i,j) to E and get back to step (2)

(2.b) Constraint : "$\sum_{j \in C} \beta_{ij} \leq f_i$" becomes tight for some $i \notin \tilde{O}$

In this case,

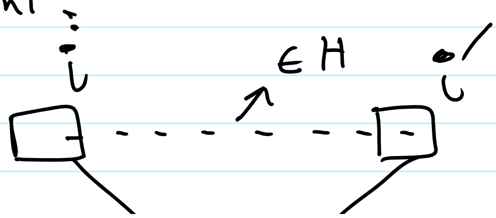
- "tentatively" open i ; $\tilde{O} = \tilde{O} + i$
- Remove all j st $(i,j) \in E$ from A
- "tentatively" assign j to i
  $\tilde{\sigma}(j) = i$

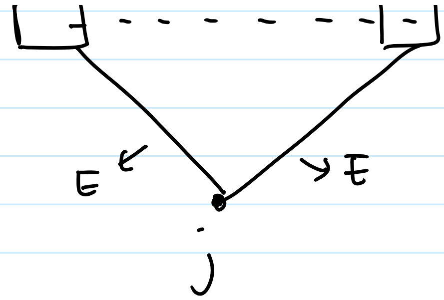
//Observe : All Invariants are satisfied.

(3) TERMINATION: Since any $j \notin A$ is tentatively assigned, i.e., $\tilde{\sigma}(j) \mapsto \tilde{O}$, @ the end we have $\tilde{\sigma} : C \longrightarrow \tilde{O}$ satisfying the 3 inv. given above.

(4) CLEANING UP

- Consider a graph with the vertices $\tilde{O}$ with an edge (i, i') in the graph if $\exists j \in C$ st. (i,j) & (i',j) are both tight.





- Let I be any MAXIMAL INDEPENDENT SET in H.

- Open all facilities in I
- Assign clients to the nearest facility in I.

Analysis :

- We need to bound both facility opening cost & connection costs.
- To argue about the latter, we explicitly define an assignment mapping $\sigma : C \to I$ & upper bound that.
- For any $j \in C$ st $\tilde{\sigma}(j) \in I$, define
$$\sigma(j) = \tilde{\sigma}(j)$$
- For any $j \in C$ s.t $\tilde{\sigma}(j) \notin I$
  - let $\tilde{i} = \tilde{\sigma}(j)$

- Since I is an MIS, $\exists\, i \in I$ s.t $(i, \tilde{i})$ is an edge in H
- $\sigma(j) = i$

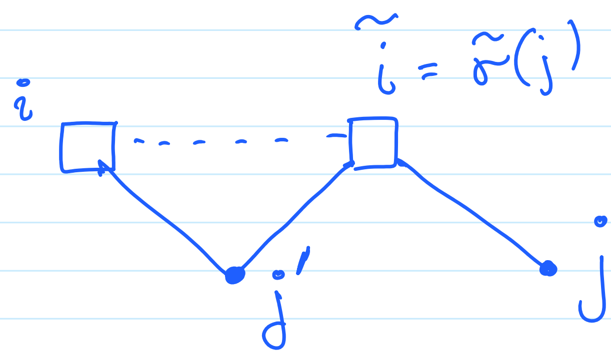
- Let $C = C_1 \,\dot\cup\, C_2$ where
  $C_1 = \{ j \in C : \tilde{\sigma}(j) \in I \}$ and
  C_2 the rest

- $\forall j \in C_1$: $d(\sigma(j), j) = d(\tilde{\sigma}(j), j)$
  $= \alpha_j - \beta_{\tilde{\sigma}(j), j}$

Claim :- For any $j \in C_2$, $d(j, \sigma(j)) \leq 3\alpha_j$.

Pf :- $j \in C_2 \Rightarrow \tilde{\sigma}(j) \notin I$ $\quad$ ($= \tilde{i}$, say)

$\Rightarrow \exists\, (i, i')$ edge in H



$\Rightarrow$

$\exists\, j' \in C$ st
$(j', \tilde{i}) \in E$
$(j', i) \in E$ ... note j' could be j.

$$d(j, i) \leq d(i, j') + d(\tilde{i}, j') + d(\tilde{i}, j)$$

$$= \alpha_{j'} - \beta_{ij'} + \alpha_{j'} - \beta_{\tilde{i}j'} + \alpha_j - \beta_{\tilde{i}j}$$
$$\leq 2\alpha_{j'} + \alpha_j$$

Sub-claim: $\alpha_{j'} \leq \alpha_j$

Why? Because j remains active at least as long as j'

Suppose not. Suppose j leaves A @ $t < t'$, the time at which j' leaves A.

But then @ time t, $\tilde{i}$ must be "tight" since $\tilde{\sigma}(j) = \tilde{i}$.

But then since $(\tilde{i}, j')$ is in E, @ time t j' would've been kicked out of A as well.

$\therefore d(i,j) \leq 3\alpha_j$ ∎

- Opening cost:

$$\sum_{i \in I} f_i = \sum_{i \in I} \sum_{j \in C} \beta_{ij}$$

$$= \sum_{i \in I} \sum_{j \in C_1} \beta_{ij}$$

$\forall j \in C_2,\ i \in I$ $\beta_{ij} = 0$ since there is no edge $(i,j) \in E$ ... o/w $j \in C_1$.

$$= \sum_{j \in C_1} \left(\sum_{i \in I} \beta_{ij} \right)$$

$$= \sum_{j \in C_1} \beta_{\sigma(j),j}$$

For any $j \in C_1$ there can be only one $i \in I$ s.t $\beta_{ij} > 0$

$$= \sum_{j \in C_1} \left(\alpha_j - d_{j,\sigma(j)} \right)$$

one

$\beta_{ij} > 0$

Suppose there are two, i & i', then there would be an (i,i') edge in H

$$\therefore \sum_{j \in C_1} \alpha_j = \sum_{i \in I} f_i + \sum_{j \in C_1} d_{j,\sigma(j)}$$

$$3 \sum_{j \in C_2} \alpha_j \geq \sum_{j \in C_2} d_{j,\sigma(j)}$$

$$\therefore 3 \cdot \sum_{j \in C} \alpha_j \geq 3 \cdot \sum_{i \in I} f_i + \sum_{j \in C} d(j,\sigma(j))$$

But (α, β) is a feasible dual, by design of the algorithm.

$$\Rightarrow \quad 3 \cdot F_{alg} + C_{alg} \leq 3LP$$

$\blacksquare$