CS31 (Algorithms), Spring 2020 : Lecture 1

Topic: Algorithms with numbers, Intro to Algorithms Analysis

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

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In this lecture, we are going to look at some algorithms involving numbers.

1 Addition

All of us see an algorithm as early as in elementary school. Addition. What? Yes. Addition.

Given two numbers, we add them by putting them one below the other and then adding digit-by-digit, and taking care of carries, etc. It is a step-by-step method (in fact a for-loop). But why does it work? That is, why is $17 + 13$ when written down using the above algorithm gives the same answer as counting the total number of sticks if I have 17 sticks in one hand and 13 sticks in the other? Have you wondered this?

Indeed, let us first formalize the computational problem. The first question is : how are these numbers represented? One could use the decimal notation, where the number "17" is used to indicate the concept of seventeen. Computers use the binary representation, and we will use that way of representing for these notes. This is simply a choice we are making; all we say below (except for one or two exceptions) will make sense even if you have the decimal representation in your head. Let us start by recalling the *binary representation*.

Remark: *An n-bit number a is represented by a bit-array*[a] $a[0:n-1]$ *where each* $a[i]$ *is* 0 *or* 1*, and*

$$a = \sum_{i=0}^{n-1} a[i] \cdot 2^i$$

So, for example, the number 37*, whose binary representation is* 100101 *is represented by the bit-array* $[1,0,1,0,0,1]$ *which is the reverse of the binary representation. Only when we talk about numbers will it be convenient to read arrays right to left.*

[a]As you can see, I have indexed the bit-arrays above from 0 instead of 1 (like I will usually do). Again, this is just a convenience, and doesn't change what we want to understand.

Now that we have refreshed our memory about bits and the binary representation, we can define the spec of the addition problem.

ADDITION
Input: Two n-bit numbers a,b expressed as bit-arrays $a[0:n-1], b[0:n-1]$.
Output: The number $c = a + b$ expressed as a bit-array.
Size: The number of bits n.

Given the definition, we are now ready to actually spell out the algorithm we learned in grade school as a pseudocode. However, we will need one "subroutine". We will need the ability to add three bits. More precisely, we will assume that we have access to a subroutine BIT-ADD which takes inputs three bits (b_1, b_2, b_3) and returns (c, s) where the number (c, s) interpreted as a binary number (that is $2c+s$) is indeed $b_1 + b_2 + b_3$. See the supplement for a precise definition.

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procedure ADD($a[0:n-1], b[0:n-1]$):    ▷ The two numbers are $a$ and $b$
    Initialize carry ← 0.
    Initialize $c[0:n]$ to all zeros    ▷ $c[0:n]$ will finally contain the sum
    for $i = 0$ to $n-1$ do:
        (carry, $c[i]$) ← BIT-ADD($a[i], b[i]$, carry)
    $c[n]$ ← carry
    return $c$
```

What are the elementary operations for this problem? We shall say that adding any three single bits forms one elementary operation (that is, a call to BIT-ADD) which takes *one unit of time*. At some level this choice is arbitrary but also natural. Very soon, we will use the "Big-Oh" notation very nicely to sweep these distractions cosily under the rug to focus on the bigger picture. For today's class, however, let this be our definition.

With this definition, we can easily see that the number of BIT-ADDs is exactly n. Why? Because there is a for-loop which runs for n iterations, and in each iteration there is exactly one call to BIT-ADD. We cast this in the following theorem.

Theorem 1. The algorithm ADD has worst case running time $T_{\text{ADD}}(n) = n$.

✍

Exercise: *Can you modify the above pseudo-code to add an n-bit number with an m-bit number (where $n \geq m$)? If $T(n,m)$ is the worst-case running time, what is this as a function of n and m?*

I don't know about you, but it is not utterly trivial to me that the above algorithm, when input two numbers a and b, actually returns a bit-array c which contains the sum $a+b$. That is, the number obtained by incrementing a exactly b times. Once again, this is taught to us in grade-school, but why it works does need a proof. We will not cover this in the course, but it is provided in the supplement to this lecture.

Note that the trivial algorithm that implements the definition of addition, has running time $T(n) = 2^n - 1$ as the number b can be as large as $2^n - 1$. Thus, the above ADD algorithm is a *remarkable* algorithm. Indeed, the "place-value-system" of numbers is one of the most remarkable inventions of human kind. If you doubt this, think Roman numbers and how you would add them. Or perhaps consider multiplying them.

2 Multiplication

MULTIPLICATION
Input: n-bit number x, m-bit number y expressed as bit-arrays $x[0:n-1], y[0:m-1]$.
Output: The number $z = x \cdot y$ expressed as a bit-array.
Size: The number of bits $n+m$.

On to multiplication. Many grade-school method of multiplication "reduces" multiplying two n-bit/digit numbers into adding n different numbers ranging from $n+1$ to $2n+1$ bits. Today, we see a different *recursive* algorithm for multiplication. Recursion is one essence of algorithm design which you should try to get in your blood. It is also a great philosophy and applicable to most things in life.

Remark: *Break the problem into smaller subproblems and let recursion take care of the smaller subproblems. Remember to solve the smallest subproblem.*

Correctness of recursive algorithms often will follow from the design. Mathematical Induction is often involved; good place to brush it up. Let us illustrate with multiplication. The proof of correctness can be found in the supplement.

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1:  procedure MULT($x, y$):
2:      ▷ The two numbers are input as bit-arrays; $x$ has $n$ bits, $y$ has $m$ bits. $n \geq m$.
3:      if $y = 0$ then: ▷ Base Case
4:          return 0 ▷ An all zero bit-array
5:      $x' \leftarrow (2x)$; $y' \leftarrow \lfloor y/2 \rfloor$ ▷ How much time does this take? See remark below.
6:      $z \leftarrow$MULT$(x', y')$
7:      if $y$ is even then:
8:          return $z$
9:      else:
10:         return ADD$(z, x)$ ▷ Time taken is the total number of bits in $z$ and $x$.
```

Remark: *"Hold on!," I hear you say, "Above, you seem to have multiplied by* 2 *and divided by* 2 *and taken floors. How do we do that?" Indeed. Note that when x is expressed as a bit-array,* $(2x)$ *is just a left-shift. That is, we take all of x and add a zero at the end. Similarly,* $\lfloor y/2 \rfloor$ *is a right-shift. That is, we just drop the last bit. For simplicity, we assume this takes* 0 *time. These are "easy" operations and today we won't even count them in our running time. In decimal notation, this would correspond to multiplying and dividing by* 10.

Define $T(n, m)$ to be the maximum time (that is, BIT-ADDS for now) MULT takes to multiply x, y where x is an n-bit number and y is an m-bit number. Recall, we assume adding an n bit number with an m bit number takes $\leq n$ time. We next write a *recurrence inequality* for $T(n, m)$.

Base Case: When $y = 0$, let us say[1] $m = 0$, and define $T(n, 0) = 0$.

Now, let us figure out how much time each step takes.

- As remarked above, Line 5 doesn't need any BIT-ADDS. So this costs **0**.
- How much does Line 6 cost? This is important. It is a recursive call on the input x' and y'. What is the size of this input? We see that $x' = 2x$ has $n+1$ bits and $y' = \lfloor y/2 \rfloor$ has $m-1$ bits. By the *definition* of the *worst case running time*, we can therefore conclude that this step takes *at most* $T(n+1, m-1)$ time. Observe that the pessimistic definition of worst-case-running-time really helped here. In sum, this step costs $\mathbf{T(n+1, m-1)}$.
- Now, if y was even, this would be the end of the algorithm.

[1]Ok, so 0 is 1 bit, but let's just say the number 0 requires 0-bits.

- However, we are in the worst-case land, and y could be odd. In this case we need to add z and x. Now we use the fact that $z \le x \cdot y$ has at most $(n+m)$ bits (see Fact 1 below). Therefore, the operation $\textsc{Add}(z, x)$ takes at most $(\mathbf{n}+\mathbf{m})$ time by Theorem 1.

Fact 1. If x has n bits and y are m bit integers, then $x \cdot y$ has at most $(n+m)$ bits.

Proof. Since x is an n-bit integer, we get $x \le 2^n - 1$. Similarly, $y \le 2^m - 1$. Therefore, $x \cdot y \le (2^n - 1) \cdot (2^m - 1) < 2^{n+m} - 1$, where we have used $2^n + 2^m - 1 > 1$. Therefore, xy has at most $(n+m)$ bits. □

Putting all these together, we see that the running time $T(n, m)$ for $\textsc{Mult}$ can be captured by the following ***recurrence inequality***.

$$\begin{aligned} T(n,0) &= 0 \\ T(n,m) &\le T(n+1, m-1) + (n+m) \quad \forall n, m > 0 \end{aligned} \tag{1}$$

Recurrence Inequalities: The heart of analyzing running times of recursive algorithms. Equation (1) is called a *recurrence* inequality; it is expressing the running time $T(n, m)$ as a function $3n$ plus $T()$ of something "smaller". Why is $(n+1, m-1)$ smaller than (n, m)? Because the lesser of the two numbers is becoming strictly smaller.

Recurrence inequalities form the bedrock of analyzing the efficiency of recursive algorithms, and in this class we will see how to solve a general class of them. The general way I like to think of it is the following picture (shown in Figure 1), which I call the *kitty method*[2].

Once we know the answer from the picture above, we can formally prove it as well. This is shown below.

Theorem 2. $\textsc{Mult}$ takes $T(n, m) \le m(n+m)$ time (i.e. $\textsc{Bit-Add}$s/elementary operations) to multiply an n-bit number with an m-bit number ($n \ge m$).

Proof. See Figure 1 to see how to "open up" the brackets. That is,

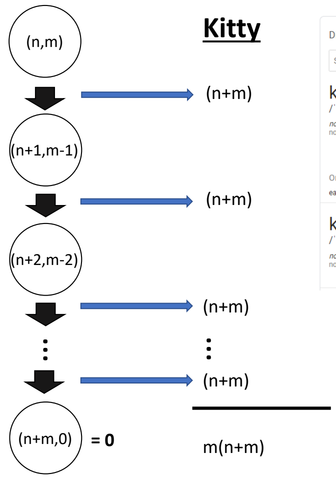
$$\begin{aligned} T(n,m) &\le T(n+1, m-1) + (n+m) \\ &\le T(n+2, m-2) + (n+m) + (n+m) \\ &\le T(n+3, m-3) + (n+m) + (n+m) + (n+m) \\ &\vdots \\ &\le T(n+m, 0) + m(n+m) \end{aligned}$$

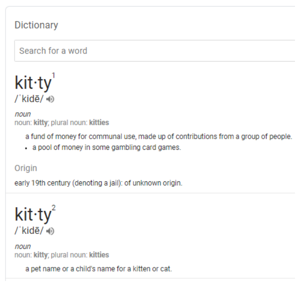
Now, we use that $T(n+m, 0) = 0$, the base case which we knew how to handle. This proves the theorem. □

3 Division

<u>$\textsc{Division}$</u>
Input: n-bit number x, m-bit number y expressed as bit-arrays $x[0:n-1], y[0:m-1]$.

[2] No one else (except students who have taken the class with me in the past) calls it by this name.




Dictionary

Search for a word

kit·ty[1]
/ˈkidē/

noun
noun: **kitty**; plural noun: **kitties**

a fund of money for communal use, made up of contributions from a group of people.
- a pool of money in some gambling card games.

Origin

early 19th century (denoting a jail): of unknown origin.

kit·ty[2]
/ˈkidē/

noun
noun: **kitty**; plural noun: **kitties**

a pet name or a child's name for a kitten or cat.

Figure 1: *The circles contain the various sizes. As we go down the sizes become smaller and smaller till we reach a small enough size for which we can figure out the running time directly. In particular, when the size becomes* 0*, the running time becomes* 0*. However, breaking the problem is not free. To break every circle you need to pay some in the kitty. This is given by the extra terms other than the* $T(\cdot)$*'s. In this case, each break "costs"* $(n+m)$*. In the end, we just add everything in the kitty to get the final answer.*

Output: The quotient-remainder pair (q, r) such that $x = qy + r$ where $r < y$.
Size: The number of bits $n + m$.

Our final course of the day is integer division. We want to take input two numbers x, y, and return the quotient and remainder obtained when x is divided by y. That is, we want to find non-negative integers (q, r) such that $x = qy + r$ and $r < y$.

Once again, we define a recursive algorithm to do the same. First we identify the base cases. If $x < y$, then we know that the quotient is 0 and remainder is x. If $x = y$, then the quotient is 1 and remainder is 0. Now suppose $x > y$.

Case 1. $x = 2k$ is even. Then if (q', r') is what we obtain *recursively* when we divide the smaller number k by y, that is, $k = q'y + r'$, then $x = 2q' \cdot y + 2r'$. Therefore, we should return $(2q', 2r')$, except $2r'$ may be bigger than y. In which case, we should return $(2q' + 1, 2r' - y)$. This suffices since $r' < y'$ and so $2r' < 2y$ and so $2r' - y < y$.

Case 2. $x = 2k + 1$ is odd. Again, suppose (q', r') is obtained *recursively* when we divide k by y. Then we get $x = 2k + 1 = 2q'y + 2r' + 1$. Once again we repeat the same as above.

```
1: procedure DIVIDE($x, y$):
2:     ▷ The two numbers are input as bit-arrays; $x$ has $n$ bits, $y$ has $m$ bits. $n \geq m$.
3:     ▷ Returns $(q, r)$ where $x = qy + r$ and $0 \leq r < y$.
4:     if $x < y$ then:
5:         return $(0, x)$
6:     if $x = y$ then:
7:         return $(1, 0)$
8:     $x' \leftarrow \lfloor x/2 \rfloor$   ▷ Obtained by right shifts
9:     $(q', r') \leftarrow$ DIVIDE$(x', y)$
10:    $q \leftarrow 2q'; r \leftarrow 2r'$   ▷ Obtained by left shifts
11:    if $x$ is odd then:
12:        $r \leftarrow r + 1$   ▷ Obtained by ADD$(r, 1)$.
13:    if $r \geq y$ then:
14:        $q \leftarrow q + 1$   ▷ Obtained by ADD$(q, 1)$.
15:        $r \leftarrow r - y$   ▷ Subtraction is just addition with the "complement"
16:    return $(q, r)$.
```

Once again, let us work to figure out the recurrence inequality for the running time. Let $T(n, m)$ be the time taken to divide an n-bit number by an m-bit number. Once again, time for us is the number of BIT-ADDS.

- Let's first understand the base cases. Line 5 and Line 7 take **0** time. Thus, we get $T(n, m) = 0$ if $n < m$. Note, we cannot say $n = m$ for x and y can both be m bits big and yet $x > y$.
- Line 8 and Line 10 also takes no BIT-ADDS. This is for the same reason as in MULT.
- The recursive call in Line 9 takes time at most $T(n-1, m)$. This is because x' has $n-1$ bits, and the definition of worst-case runtime.
- Now consider Line 12 and Line 14. Note that in both cases we are adding 1 to an even number (see Line 10), that is, a number whose last bit is 0. Thus, one needs only one BIT-ADD to increment an even number by one. Thus, these steps cost **2** BIT-ADDS.
- Line 15 is the "time-taking" step of *subtraction*. How does one subtract? As you can see in the supplement, subtraction is simply an addition[3] with "complementing", the time (or number of BIT-ADDS) is the same as to add. And thus, since both y and r have $\leq (m+1)$ bits (note $r \leq 2y$), this step takes $\mathbf{m} + \mathbf{1}$ time.

Therefore, we get the following recurrence for DIVIDE.

$$
\begin{aligned}
T(n, m) &= 0 \quad \text{if } n < m \\
T(n, m) &\leq T(n-1, m) + (m+3)
\end{aligned}
\tag{2}
$$

[3]If you have never seen this before, then I recommend going and reading this in the supplement.

Theorem 3. DIVIDE takes $T(n,m) \leq (m+3)\cdot(n-m+1)$ time (i.e. BIT-ADDs/elementary operations) to divide an n-bit number by an m-bit number where $n \geq m$.

Proof. Once again, this can be solved by the kitty method or "opening up the brackets" as follows:

$$\begin{aligned} T(n,m) &\leq T(n-1,m) + (m+3) \\ &\leq T(n-2,m) + (m+3) + (m+3) \\ &\vdots \\ &\leq T(m-1,m) + (m+3)\cdot(n-m+1) \end{aligned}$$

The proof completes by noting $T(m-1,m) = 0$. □

Corollary 1. If $n = m + c$, that is, x has only c more bits than y, then DIVIDE takes $\leq Cn$ time for some constant C dependent on c.