CS31 (Algorithms), Spring 2020 : Lecture 16 Supp

Date:

Topic: Graph Algorithms 6 Supp: Proof of Hall's Theorem

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
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1 Hall's Theorem via Max-Flow-Min-Cut

We can also derive a theorem you may have seen in previous courses: Hall's Theorem. A matching in a graph G is *perfect* if all vertices participate as endpoints in the matching. This theorem states a necessary and sufficient condition for a bipartite graph G to have a perfect matching. A definition: given any subset $S \subseteq L$, we define $\Gamma S := \{r \in R : \exists \ell \in S, (\ell, r) \in E\}$ to be the set of neighbors of S.

Theorem 1 (Hall's Theorem)**.** A bipartite graph $G = (L \cup R, E)$ with $|L| = |R|$ has a perfect matching if and only if for all $S \subseteq L$, $|\Gamma S| \geq |S|$.

Proof. We consider the network $\mathcal{N}$ defined above with one extra change: for all $e \in G$, we set $u(e) = \infty$. Note that the maximum flow doesn't change since the total capacity incoming into any $\ell \in L$ is 1, and also the total capacity out going from any $r \in R$ is also 1. Thus, the infinite capacity in the "middle" doesn't help in sending more flow. We see that it makes our arguments easier.

Now, we know that G has a perfect matching if and only if the maximum s, t flow in $\mathcal{N}$ is of value $|L|$. Using the max-flow-min-cut theorem, we get G has a perfect matching iff the minimum s, t cut in $\mathcal{N}$ is of value $|L|$.

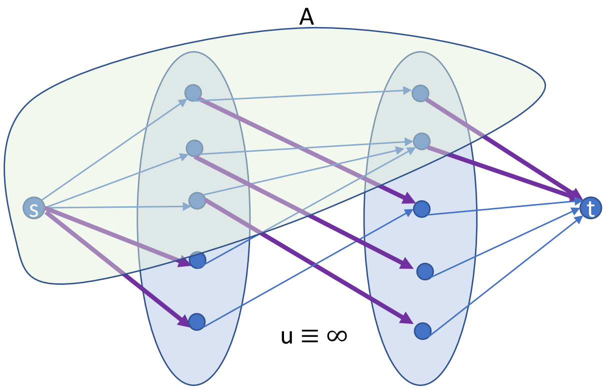
Let us consider an s, t cut in $\mathcal{N}$. Let A be the subset inducing the cut; $A = s \cup S \cup T$ where $S \subseteq L$ and $T \subseteq R$. Note, $t \notin A$. The capacity of this cut is as follows.

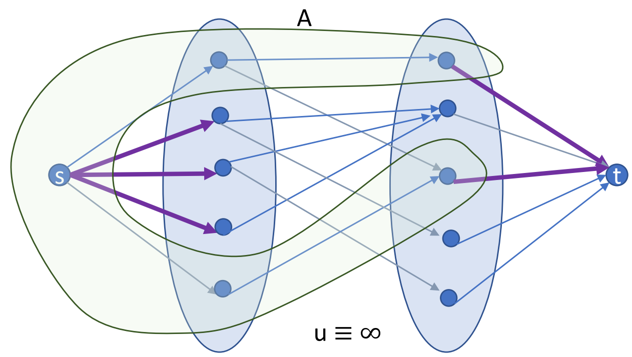
$$u(\partial^+ A) = \begin{cases} \infty & \text{if } \Gamma S \nsubseteq T \\ (|L| - |S|) + |T| & \text{otherwise} \end{cases}$$

To see this, note that if $\Gamma S \nsubseteq T$, then there is an edge $(\ell, r) \in \partial^+ A$ of capacity ∞ (this is why we defined it so). If $\Gamma S \subset T$, then the only edges in $\partial^+ A$ are of the form (s, ℓ) for $\ell \notin S$ and (r, t) for $r \in T$.

Now, if A were the minimum s, t cut, then observe that we would pick $T = \Gamma S$ (we would like to pick T as minimal as possible). Therefore, the value of the minimum s, t cut is precisely $\min_{S \subseteq L}(|L| - |S| + |\Gamma S|) = |L| + \min_{S \subseteq L} (|\Gamma S| - |S|)$.

See the figure below for an illustration.





Putting everything together, we get G has a perfect matching if and only if

$$|L| + \min_{S \subseteq L} (|\Gamma S| - |S|) = |L|$$

or, in other words, for every $S \subseteq L$, we have $|\Gamma S| \geq |S|$. □