CS 30: Discrete Math in CS (Winter 2019): Lecture 1 Supplement

Date: 3rd January, 2019 (X-hour)

Topic: Change of Variables, Nested Summations

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.
Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

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This material was not covered in class but supplements the stuff that was covered. Think of this material as what I would've covered if I had a semester instead of a quarter.

- **Nested or Double Summations.** In the class notes, we saw a *single* summation – there was a sequence indexed by a single variable and the summation summed these numbers from some index to another. We could evaluate it using one for-loop. But just as we have nested for-loops, we have nested summations.

  Suppose we have a sequence/matrix/2D array of numbers $f_{i,j}$ where i ranges from say 1 to n and j ranges from 1 to m. One can then write the double summation

  $$\sum_{i=p}^{q}\sum_{j=u}^{v} f_{i,j} = \sum_{i=p}^{q}\left(\sum_{j=u}^{v} f_{i,j}\right)$$

  That is, we first do the summation inside, and the summation outside. Note that the inner sum is computing for a fixed i (fixed row) the sum over the various j's (columns). For instance, if $p = 1$ and $q = n$, and $u = 1$ and $v = m$, then the ith inner sum corresponds to the sum of entries in row i. The outer sum is then computing the sum of the row-sums (which is the sum of every entry). Let's take a concrete example: say $n = 2$ and $m = 3$, and consider

  $$\sum_{i=1}^{2}\sum_{j=1}^{3} f_{i,j} = (f_{1,1} + f_{1,2} + f_{1,3}) \quad + \quad (f_{2,1} + f_{2,2} + f_{2,3})$$

  Again, in code, this would look like

  $\sum_{i=1}^{n}\sum_{j=1}^{m} f_{i,j}$ $:=$

  ```
  sum = 0.
  for i = p to q do:
      for j = u to v do:
          sum = sum + f_{i,j}.
  return sum.
  ```

  Remark: *Above we have implicitly assumed $p \le q$ and $u \le v$. As in the case of single summation, if $p > q$, then the corresponding for-loop would decrement instead of incrementing.*

  Exercise: *Evaluate the following double sum $\sum_{i=3}^{5}\sum_{j=2}^{7}\left(i^2 + j\right)$. You can use a computer if you like.*

- **Changing the Order in a Double Summation.** A very useful trick to know is of *switching summations*. This is akin to changing the order of for-loops. It is quite a powerful technique as it allows one to establish non-trivial facts such as the following one.

Theorem 1. For any positive integer n, we have

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{j}\right) = n$$

Before, we prove let us first point to the subtlety that, unlike before, j goes from i to n, and not 1 to n. So if we think of the double summation as code, in the second for-loop the indexing of j begins from i instead of 1. More precisely, we get

$\sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{j}\right)$ $:=$

```
sum = 0.
for i = 1 to n do:
    for j = i to m do:
        sum = sum + (1/j).
return sum.
```

Exercise: *Before reading further, code this up and check the statement of the theorem for the first 100 integers.*

Note that there is something slightly amazing going on here. In the LHS, we are adding a bunch of fractions of the form $1/j$, and in the RHS we have an integer! The "mystery" is resolved in the proof which illustrates the change of summation trick.

Proof. In the summation in the LHS, instead of summing over the i's and then summing over the j's, we do it vice-versa. But we *need to be careful*. When we reverse, we need to respect the fact that for any i, the j's add up from i to n. When we reverse, for any fixed j, the i's will add up from 1 to j. That is,

$$S = \sum_{i=1}^{n}\sum_{j=i}^{n}\left(\frac{1}{j}\right) = \sum_{j=1}^{n}\sum_{i=1}^{j}\left(\frac{1}{j}\right)$$

Let's pause and be sure we are doing correctly. Let's look at all (i, j) pairs that appear in the sum on the left. They are $(1, 1), (1, 2), \ldots, (1, n), (2, 2), (2, 3), \ldots, (2, n), \ldots$. Thus, the second number (j) goes from 1 to n, but the left number can go from 1 to j. This is the same logic one uses if you have nested for-loops and you want to swap them (did you do this in CS 1?)

Why is this useful? Well now we can take the $(1/j)$ out of the second summation to get

$$S = \sum_{j=1}^{n}\frac{1}{j}\cdot\left(\sum_{i=1}^{j}1\right) = \sum_{j=1}^{n}\frac{1}{j}\cdot j$$

The j's cancel out! And thus, we have that $S = \sum_{j=1}^{n} 1 = n$. $\square$

Exercise: *Given two sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$, prove that*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot b_j) = \left(\sum_{i=1}^{n} a_i \right) \cdot \left(\sum_{j=1}^{m} b_j \right)$$

As a corollary, figure out the value of

$$S := \sum_{i=1}^{100} \sum_{j=1}^{99} (-1)^i \cdot j^3$$