

CS 30: Discrete Math in CS (Winter 2020): Lecture 1

Date: 6th January, 2020 (Monday)

Topic: Introduction, Sets

Disclaimer: These notes have not gone through scrutiny and in all probability contain errors.

Please discuss in Piazza/email errors to deeparnab@dartmouth.edu

1 Course Information

- **What?** Computer Science is mostly about *discrete* objects: bits, numbers, files, packets, memes, likes, etc. This is unlike in the other sciences where the objects of study – electricity, reagents, cytoplasmic fluid, prices, etc – are objects in *continuous* motion. Unsurprisingly, the Math one needs to learn is different. Introducing this Math is the main object of this course.
- **Organization.** We spend the first two weeks on learning some jargon, and understanding the concept of *proofs*. Subsequently, we spend roughly a week on *five* modules: combinatorics, probability, graphs, numbers, and infinities. Over the course, you will (a) learn the basics of these modules, and (b) keep learning how to read and write proofs.
- **Why?** (a) You will be learning concepts which you will see in all other (even non-mathematical) CS courses. (b) Proofs will help you reason about how you code, design, and more generally, think. (c) Fun.
- **Logistics.** Weekly assignments (3 problems) due every Wednesday at 11:59pm. Drills (simpler 1 or 2 problems) after almost every class, due in 36 hours. Two midterms, one final – closed book, in class.
- **Help.**

To many of you, this kind of Math may be new, and the Dartmouth quarter system does not much give breathing space to imbibe it in. Therefore, I cannot stress this enough:

“There is help available whenever you need it.”

Here are some ways: I hold an office hour four days a week. The TAs also will hold ample office hours. We will respond to Piazza queries promptly. If you cannot make my office hours, *email me* and we will figure out a time to meet.

Here is what *you* should do: Attend every lecture physically and mentally (I know the class time is early – plan ahead). Ask questions in class. Follow up on the lecture notes. Ask questions on Piazza. Do the drill. Do **not** leave the weekly assignments till the last minute — indeed, on the day it is out, read the three problems and spend 10 minutes on each of them to see if they make sense. If not, seek help. Come to office hours to clarify concepts, and try out ideas.

This course is tough. I won’t argue this. But if you try and follow the above steps, you may just end up enjoying the class!

2 Sets

1. **Definition.** A set is an *unordered* collection of *distinct* objects. These objects are called **elements** of the set. These elements could be *anything*, for instance, the element of a set could be a number, could be a string, could be tuples of numbers, and in fact can be other sets!
2. **Roster Notation.** A set can be described by explicitly writing down the elements, such as

$$S = \{1, 3, 5, 7, 9\} \quad \text{or} \quad T = \{\text{apple, banana, volcano, 100}\} \quad \text{or} \quad W = \{S, T\}$$

This is called the **roster notation**. Note that the *elements* of the set W are the sets S and T .

Remark: In the above example, $S \in W$ and $3 \in S$, but $3 \notin W$. When figuring out if an element is in a set, we don't "keep opening" the sets inside.


3. **Set Builder Notation.** A set can also be described *implicitly* by stating some rule which the elements follow. For example,

$$S = \{n : n \text{ is a positive odd integer less than } 10\} \quad \text{or} \quad V = \{x^2 : x \text{ is an integer and } 1 \leq x \leq 5\}$$

This is called the **set-builder notation**.

The sets S described in the above two examples correspond to the same set. The set V , written explicitly in the roster notation, is $V = \{1, 4, 9, 16, 25\}$.

Remark: Caution: Unless otherwise explicitly mentioned, duplicate items are removed from a set. For example, consider the set $A = \{x^2 : -2 \leq x \leq 2\}$ in the set-builder notation. In the roster notation, this set is $\{0, 1, 4\}$ and **not** $\{4, 1, 0, 1, 4\}$.

4. **\in operator.** If x is an element of a set S , we denote it as $x \in S$. If x is not an element of S , we denote it as $x \notin S$.
5. **Cardinality of a set.** The **cardinality** of a set S is denoted as $|S|$ is the number of elements in the set. For example if $A = \{\text{apple, banana, avocado}\}$, then $|A| = 3$. 

Exercise: What is $|A|$ when $A = \{x^2 : -3 \leq x \leq 3\}$?

If the set S has only finitely many elements, then $|S|$ is a finite number, and S is called a **finite** set.

$|S|$ could be ∞ in which case the set is called an infinite set.


6. **Famous examples of Infinite Sets.** \mathbb{N} , the set of all natural numbers; \mathbb{Z} , the set of all integers; \mathbb{Q} , the set of all rational numbers, \mathbb{R} , the set of all real numbers; and \mathbb{P} , the set of all computer programs written in Python. We will visit infinite sets in the very end of this course.
7. **Empty Set.** There is only one set which contains no elements and that set is called the **empty set** or sometimes the **null set**. It is denoted as \emptyset or $\{\}$.

8. **Subsets and Supersets.** A **subset** P of a set S is another set such that every element of P is an element of S . In that case, the notation used is $P \subset S$ or $P \subseteq S$. Note that $S \subseteq S$ as well, that is, a set is always a subset of itself. In case P is a subset and not equal to S , it is called a **proper subset**. It is denoted as $P \subsetneq S$.

For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then $B \subsetneq A$.

Remark: The empty set \emptyset is a subset of all sets. This is a convention.

If $A \subset B$, then B is called a **superset** of A . This is denoted as $B \supset A$.

9. **Power Set.** Given any set S , the **power set** $\mathcal{P}(S)$ is the set of *all* subsets of the set S . It is a set of sets. Note by the above convention, for any set S , the empty set $\emptyset \subseteq S$ and therefore, $\emptyset \in \mathcal{P}(S)$. 

Exercise: Write down all subsets of the sets $S = \{1, 2\}$, $T = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4\}$. Do you see a pattern in the number of subsets?

10. **Showing two sets equal.** Often, we will need to show two sets, A and B , described differently, are in fact the same set. The way to do is to prove $A \subseteq B$ and $B \subseteq A$. That is, every element of A is an element of B , and vice-versa.

11. Set Operations.

- **Union.** Given two sets A and B , the **union** $A \cup B$ is the set containing all elements which are either in A , or in B , or both. For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \text{ then } A \cup B = \{1, 2, 3, 4, 7, 9, 10\}$$


- **Intersection.** Given two sets A and B , the **intersection** $A \cap B$ is the set containing all elements which are in *both* in A and in B . For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \text{ then } A \cap B = \{4, 7, 10\}$$

Two sets A and B are called **disjoint** if $A \cap B = \emptyset$.

Remark: If A and B are two disjoint finite sets, then $|A \cup B| = |A| + |B|$.

Here is a proof (this was asked in the class today). Let $A = \{a_1, a_2, \dots, a_k\}$ where $k = |A|$. Similarly, let $B = \{b_1, b_2, \dots, b_\ell\}$ where $\ell = |B|$. Note, k and ℓ could be same, could be different. However, what we know for sure is $a_i \neq b_j$ for any indices i and j . This is because A and B are disjoint.

Now, $A \cup B = \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_\ell\}$. Thus, by inspection now, $|A \cup B| = k + \ell = |A| + |B|$. 

Exercise: True or False: If A and B are disjoint sets, and $C \subset A$, then are C and B disjoint?

- **Difference.** Given two sets A and B , the **set difference** $A \setminus B$ are all the elements in A which are *not in* B and $B \setminus A$ are the elements in B which are not in A . For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \text{ then } A \setminus B = \{1, 3\} \text{ and } B \setminus A = \{2, 9\}$$



Exercise: Can $A \setminus B = B \setminus A$ for any two sets A and B ?

Remark: A couple of useful observations:

- (a) A and $B \setminus A$ are disjoint since $B \setminus A$ doesn't contain elements of A .
- (b) In particular, this implies $(A \cap B)$ and $B \setminus A$ are disjoint since $A \cap B \subseteq A$.
- (c) $A \cup (B \setminus A) = A \cup B$. This is because every element of $A \cup B$ is either in A , and if not in A , must be in $B \setminus A$.
- (d) $(A \cap B) \cup (B \setminus A) = B$. This is because every element of B is either in A (in which case it is in $A \cap B$) or in $B \setminus A$.

- **Cartesian Product.** (We did not cover this in class.)

Given any two sets A and B , the **Cartesian product** $A \times B$ is another set whose elements are *tuples* (that is, ordered pairs) whose first entry comes from A and the second entry comes from B . Therefore, in the set-builder notation

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

For example, if $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

Remark: Note that $A \times B$ is in general not equal to $B \times A$. In particular, in the above example, the elements of $B \times A$ are $\{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$. The element $(a, 1)$ is not the same as $(1, a)$ for the order matters. A tuple is not a set.



Exercise: Can you figure out the cardinality of $|A \times B|$ in terms of $|A|$ and $|B|$?

12. Baby Inclusion Exclusion

Theorem 1. For any two finite sets A and B , we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Since $A \cup B = A \cup (B \setminus A)$ and since A and $B \setminus A$ are disjoint, we get

$$|A \cup B| = |A| + |B \setminus A| \quad (1)$$

Since $B = (A \cap B) \cup (B \setminus A)$ and since $(A \cap B)$ and $B \setminus A$ are disjoint, we get

$$|B| = |A \cap B| + |B \setminus A| \quad (2)$$

Subtracting (2) from (1), we get

$$|A \cup B| - |B| = |A| - |A \cap B|$$

The theorem follows by taking $|B|$ to the other side. \square

13. The Distributive Property

Theorem 2. For any three sets A, B, C , we have

- (a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
- (b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Proof. We prove (a) and leave (b) as an exercise.

To show equality of two sets, we need to show two things. For every x in the LHS set, we need to show it lies in the RHS set. And vice-versa.

Pick any $x \in (A \cup B) \cap C$. Therefore, $x \in C$ and $x \in A$ or $x \in B$. If $x \in A$, then since $x \in C$, we have $x \in A \cap C$, and therefore x is in the RHS set. If $x \in B$, then a similar argument shows $x \in B \cap C$ and therefore x is in the RHS set.

Now the vice-versa. Pick any $x \in (A \cap C) \cup (B \cap C)$. x is either in $A \cap C$ or in $B \cap C$. Suppose $x \in A \cap C$. Then, $x \in A$ which implies $x \in A \cup B$, and therefore, since $x \in C$, we have $x \in (A \cup B) \cap C$. The other possibility, that is if $x \in B \cap C$ also symmetrically implies $x \in (A \cup B) \cap C$. \square