# CS 30: Discrete Math in CS (Winter 2020): Lecture 23

Date: 19th February, 2020 (Wednesday)

Topic: Graphs: Proof of Hall's Theorem

*Disclaimer: These notes have not gone through scrutiny and in all probability contain errors. Please discuss in Piazza/email errors to deeparnab@dartmouth.edu*

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1. **Recap.**

   A graph $G = (V, E)$ is ***bipartite*** if there is partition of $V = L \cup R$ such that $L \cap R = \emptyset$ and for every edge $e = (u, v) \in R$, we have $|\{u, v\} \cap L| = |\{u, v\} \cap R| = 1$. That is, every edge has exactly one endpoint in $L$ and exactly one endpoint in $R$.

   A ***matching*** $M$ in a graph is a subset of edges $M \subseteq E$ such that for any $e, e' \in M$, $e \cap e' = \emptyset$. That is, $M$ is a collection of edges which do not share end points. A vertex $v \in V$ participates in the matching $M$ if there is an edge in $M$ which is incident to $v$. In a bipartite graph $G = (L \cup R, E)$, a matching $M \subseteq E$ is an $L$-matching if all vertices in $L$ participate in $M$.

2. **Hall's Theorem** Given any subset $S \subseteq L$, we $N_G(S)$ are the set of vertices in $R$ which neighbors of some vertex in $S$. Hall's Theorem says the following.

   **Theorem 1.** Let $G = (V, E)$ be a bipartite graph with $V = L \cup R$. Then, $G$ has an $L$-matching if and only if

   $$\text{For every subset } S \subseteq L,\ |N_G(S)| \geq |S| \qquad \text{(Hall's Condition)}$$

   *Proof.* Again, one direction is easy. That is, if $G = (L \cup R, E)$ has an $L$-matching, then we must have (Hall's Condition). Why? Suppose there exists an $L$-matching called $M$. Then for any $S \subseteq L$, consider the set $T = \{v \in R : \exists u \in S : (u, v) \in M\}$. That is, look at all the partners in $M$, of vertices in $S$. Clearly, $T \subseteq N_G(S)$, and thus, $|N_G(S)| \geq |T|$. And $|T| = |S|$ since every vertex in $S$ has a partner in $M$ ($M$ is an $L$-matching). So, $|N_G(S)| \geq |S|$.

   The interesting direction is the converse. Given that (Hall's Condition) holds, we need to prove that $G = (L \cup R, E)$ has an $L$-matching. We will prove by induction. In fact, I will show two proofs. One proof is by induction on the number of *edges* — this is the proof we almost did to completion in class (I will point the part we didn't finish). The second proof is by induction on the number of *vertices*. Both of them are deep proofs, in that it has layers. So hold tight!

   **<u>Proof Number 1.</u>**

   > Let $P(m)$ be the predicate which is true if any bipartite graphs $G = (L \cup R, E)$ with $|E| = m$ satisfying (Hall's Condition) has an $L$-matching.

   We need to show $\forall m \in \mathbb{N} : P(m)$ is true; we proceed to prove this by induction.

   **Base Case:** Is $P(1)$ true? Fix a bipartite graph $G = (L \cup R, E)$ with only one edge $(u, v)$ with $u \in L$ and $v \in R$. Does $G$ have an $L$-matching. The main observation is that in this case $L$ has to be the singleton set $\{u\}$. Why? If not, that is, if $L$ contained a vertex $w \neq u$, then $\deg_G(w) = 0$ since $(u, v)$ is the only edge in $G$. But then the set $S = \{w\}$ would *violate* (Hall's Condition). Thus, $L = \{u\}$, and in this case the matching $M = \{(u, v)\}$ is the desired $L$-matching.

**Inductive Case:** Fix a natural number $k$. We assume $P(1), P(2), \ldots, P(k)$ are all true. That is,

> Any bipartite graph $G' = (L' \cup R', F)$ with $|F| \leq k$ and which satisfies (Hall's Condition), has an $L'$ matching.

We wish to prove $P(k+1)$. To that end, we fix a bipartite graph $G = (L \cup R, E)$ which satisfies (Hall's Condition) and $|E| = k + 1$.

Let $(u, v)$ be an *arbitrary* edge in $G$. Consider the graph $G' = G - e$. Note, $G' = (L \cup R, E \setminus \{(u, v)\})$. See an illustration in Figure 1

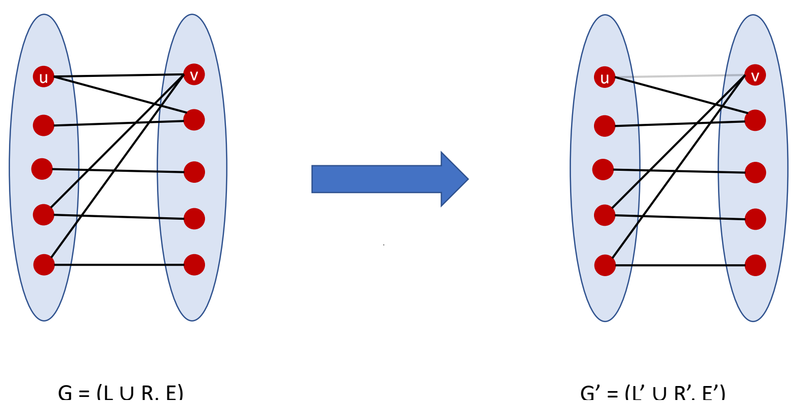



Figure 1: Illustration of going from $G$ to $G'$

*Case 1: $G'$ satisfies* (Hall's Condition). This is a nice accident to have. Why? Well, $|E(G')| = |E(G)| - 1 = k$. Thus, by the fact that $P(k)$ is true, we get that since $G'$ satisifies Hall's condition, $G'$ has an $L$-matching called $M'$. The vertex set didn't change — same $L$. And thus, the same $M'$ is also an $L$-matching in $G$. We are done. Therefore, the maximum fun is to be had in the next case, when $G'$ doesn't satisfy Hall's condition. This is where we are going to focus our energy now.

*Case 2: $G'$ **does not** satisfy* (Hall's Condition). What does this mean? It means there is some subset $S^* \subseteq L$ such that $|N_{G'}(S^*)| < |S^*|$. This is illustrated in Figure 2. We make a few crucial observations about this set $S^*$; so crucial, we are going to encapsulate them in a *lemma*. A lemma is an interesting statement to be proven on our journey in the proof of a theorem.

**Lemma 1.** Let $S^*$ be a subset of $L$ which satisfies $|N_{G'}(S^*)| < |S^*|$. Then,

(a) $u \in S^*$.

(b) For all $w \in S^* \setminus u$, we know $(w, v) \notin E(G)$.

(c) $|N_G(S^*)| = |S^*|$.

*Proof.* To prove this, consider any subset $S \subseteq L$ and let us ask ourselves how do the sets $N_G(S)$ and $N_{G'}(S)$ look like? Recall, $G' = G - (u, v)$. So, if these neighborhoods of $S$ are indeed different in these different graphs, they can only differ in $v$ and that also only if (i) $u \in S$, and (ii) no other vertex $w \in S \setminus u$ has an edge to $v$. If any of the above don't hold, then note that $N_G(S) = N_{G'}(S)$. Indeed, if $u \notin S$, the presence or absence of the edge $(u, v)$ doesn't play any role in determining the neighborhood of $S$. Furthermore, the only way $N_{G'}(S)$ can be now smaller is if $(u, v)$ was the *only* edge from vertices in $S$ to $v$; otherwise $v$ would be in $N_{G'}(S)$ as well.

Thus we get part (a) and (b). To see part (c), we note that the previous argument also implies $|N_{G'}(S)| \geq |N_G(S)| - 1$; if the neighborhood drops, it can only drop by this vertex $v$. Thus, if $|S^*| > |N_{G'}(S^*)|$ and $|S^*| \leq |N_G(S^*)|$, then the only way this can occur if (c) is true; the Hall's condition holds with equality on $S^*$. □

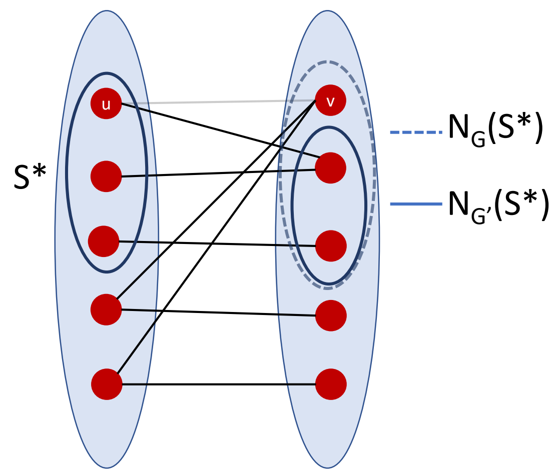



Figure 2: The set $S^*$ if $G'$ doesn't satisfy (Hall's Condition)

The fact that $|N_G(S^*)| = |S^*|$ implies that if $G$ had an $L$-matching (and we are trying to prove it does), then all the vertices of $S^*$ would have to match to some vertex in $N_G(S^*)$ and vice-versa. However, the vertex $v \in N_G(S^*)$ has only *one* neighbor $u$ in $S^*$ (by part(b)). Thus, if $G$ has an $L$-matching, the edge $(u, v)$ better be in it! And so, we should remove it, and then hope we can prove the remaining graph has an $(L - u)$-matching as well. Indeed, this motivates the following definition.

Let $H$ be the graph obtained by deleting both *vertices* $u$ and $v$, and all edges incident on either vertex, from $G$. That is, $H = G - \{u, v\}$. More precisely,

$$H = (L' \cup R', F) \quad \text{where} \quad L' = L - u, \quad R' = R - v, \quad F = E - (\partial_G(u) \cup \partial_G(v))$$

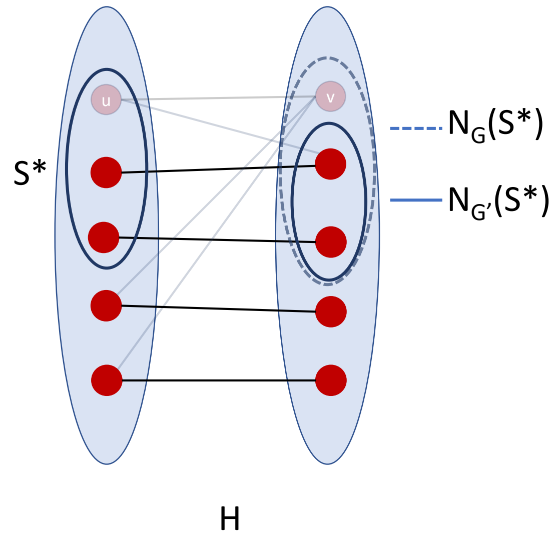



Figure 3: The graph $H$

This is illustrated in Figure 3.

Note that $|E(H)| \le |E(G)| - 1$ because at the very least the edge $(u, v)$ is deleted. However, $|E(H)|$ could be much less; let's call it the number $m$.

The next lemma is the crucial one. We assert that this graph $H$ satisfies (Hall's Condition).

**Lemma 2.** For every subset $T \subseteq L'$, we have $|N_H(T)| \ge |T|$.
That is, $H$ satisfies (Hall's Condition)

Before we prove Lemma 2, let us see that if we believe it, we are done. And then we can prove the claim. Indeed, if $H = (L' \cup R', F)$ satisfies (Hall's Condition) and because $|E(H)| = m \le k$, by the fact that $P(m)$ is true, we know $H$ has an $L'$-matching, call it $M'$. And then, we can use this matching to obtain an $L$-matching in $G$. How? Simply define $M = M' \cup \{(u, v)\}$. Since $u$ was the only vertex missing from $L'$ and since $v$ doesn't even appear in $R'$, the set $M$ is a matching and an $L$-matching at that. We would have established $P(k+1)$. Thus, the only thing that remains is to prove Lemma 2.

**Proof of Lemma 2.** We proceed by contradiction. Suppose not. Suppose there is a bad set $B \subseteq L'$ such that $|N_H(B)| < |B|$. Figure 4 (left side) is an illustration of such a set.

Once again, as in the proof of Lemma 1, let is investigate the sets $N_G(S)$ and $N_H(S)$ for *any* set $S$. The *only* difference between $N_G(S)$ and $N_H(S)$ at all can be the vertex $v$; that is the *only* vertex removed from our set $R$ to get $R'$. Thus,

$$|N_H(S)| = \begin{cases} |N_G(S)| - 1 & \text{if } v \in N_G(S) \\ |N_G(S)| & \text{otherwise} \end{cases} \tag{1}$$

Furthermore, since $|N_H(B)| < |B|$, that is, $|N_H(B)| \le |B| - 1$, *and* since $|N_G(B)| \ge |B|$ (for $G$ satisfies (Hall's Condition)), using (1) we can assert

$$|N_G(B)| = |B| \quad \text{and} \quad |N_H(B)| \ge |N_G(B)| - 1 \quad \text{implying} \quad v \in N_G(B) \tag{2}$$

Just to give an intuition of how the proof goes, let us assume ***not without loss of generality*** that $B$ was *disjoint* from $S^*$. This is what we did in class, and I believe it really tells what is going on. I am going to do the full proof after this.

The key thing is to look at the neighborhood of the set $B \cup S^*$ in the graph $G$. See Figure 4 (right side) for this. We see,

$$N_G(B \cup S^*) \ = \ N_G(B) \cup N_G(S^*) \quad \text{and} \quad v \in N_G(B) \cap N_G(S^*)$$

The latter follows from (2) and the fact that $u \in S^*$ and $(u, v) \in E(G)$. Therefore, by the baby inclusion exclusion, we get

$$|N_G(B \cup S^*)| = |N_G(B)| + |N_G(S^*)| - |N_G(B) \cap N_G(S^*)| \le |N_G(B)| + |N_G(S^*)| - 1 = |B| + |S^*| - 1$$

If $B$ and $S^*$ are disjoint, then $|B \cup S^*| = |B| + |S^*|$, and thus, we get $|N_G(B \cup S^*)| \le |B \cup S^*| - 1$. This *contradicts* (Hall's Condition). And thus our supposition that there was a bad set must be wrong. That is, $H$ satisfies (Hall's Condition) as well.

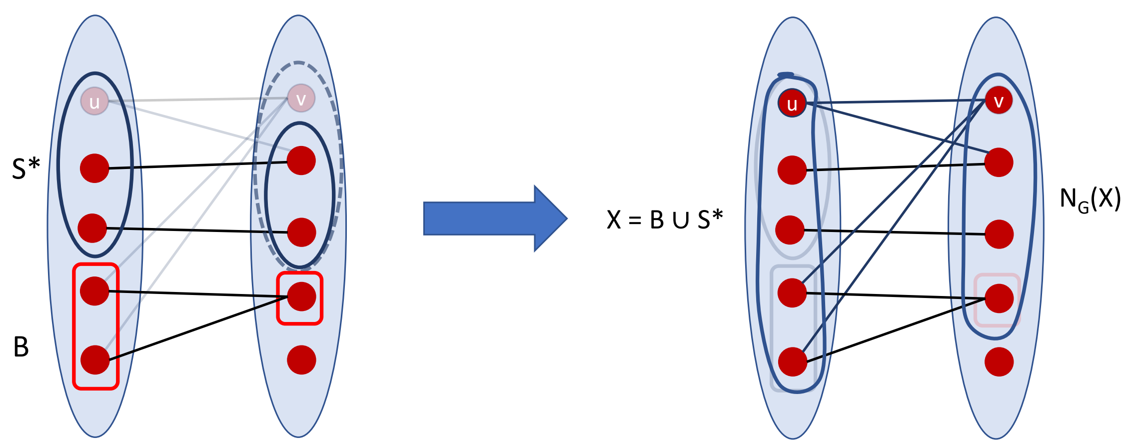



Figure 4: A bad set in $H$ violating (Hall's Condition), and how it implies a violation of (Hall's Condition) in $G$ if it were disjoint from $S^*$.

However, $B$ ***may not*** be disjoint from $S^*$. In that case, we need to a little more work. Now that you have come so far, stick with me a bit more.

We first partition $B$ into the part that is disjoint from $S^*$ and the part that is not. Let $A := B \setminus S^*$ and $C = B \cap S^*$. That is, $C$ is the set of common vertices between $C$ and $S^*$ and $A$ is disjoint from $S^*$. See Figure 5 for an illustration.

The following claim asserts that $A$ cannot have too many neighbors "outside" $N_G(S^*)$.

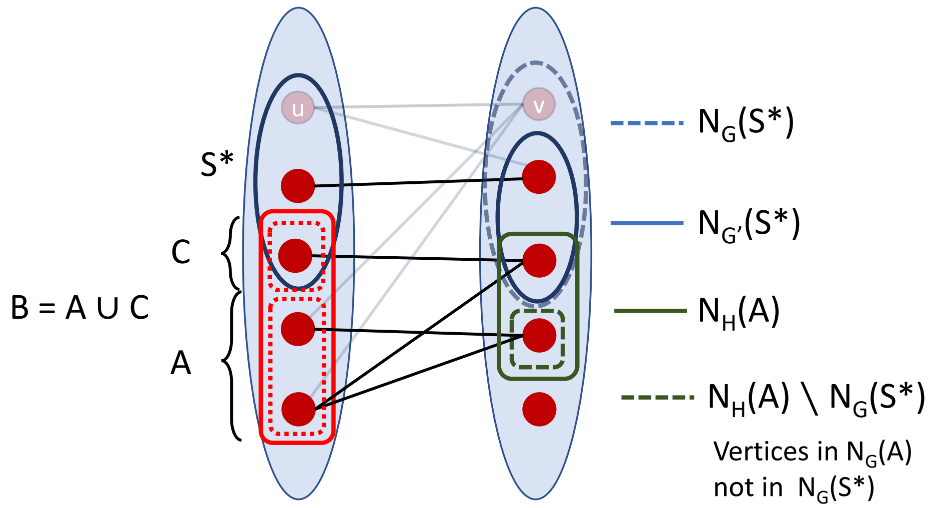



Figure 5: A bad set $B$ in $H$ violating (Hall's Condition) which intersects $S^*$.

**Claim 1.** $|N_H(A) \setminus N_G(S^*)| \le |A| - 1$

Why is this claim useful? Firstly, note: $N_H(A) \setminus N_G(S^*) = N_G(A) \setminus N_G(S^*)$ because the only vertex $N_G(A)$ and $N_H(A)$ can possibly differ in is $v$ which is already in $N_G(S^*)$. Therefore, the claim implies

$$|N_G(A) \setminus N_G(S^*)| \le |A| - 1 \tag{3}$$

Now consider the set $X = A \cup S^*$. We have

$$|N_G(A \cup S^*)| = |N_G(S^*)| + |N_G(A) \setminus N_G(S^*)|$$

that is, the neighbors of the set $A \cup S^*$ in $G$ are the neighbors of $S^*$ plus the number of neighbors $A$ has "outside" $S^*$. Plugging in part (c) of Lemma 1 and (3), we get

$$|N_G(A \cup S^*)| = |S^*| + |A| - 1 < |A \cup S^*|$$

where we used the fact that $A$ and $S^*$ are disjoint to get $|A| + |S^*| = |A \cup S^*|$. But this is a contradiction to the fact that $G$ satisfies (Hall's Condition). So our supposition is wrong, thus completing the proof of Lemma 2.

So all that remains (phew!) is

**Proof of Claim 1.** Suppose not; suppose for the sake of contradiction

$$|N_H(A) \setminus N_G(S^*)| \ge |A| \tag{4}$$

Recall the bad set $B = A \cup C$. Thus, the neighborhood $N_H(B)$ definitely contains all $N_H(C)$ (see Figure 5 for illustration), and also $N_H(A) \setminus N_G(S^*)$. And these two sets are disjoint. Therefore, the size of $N_H(B)$ is at least the sum of sizes of these two sets. In math,

$$|N_H(B)| \geq |N_H(C)| + |N_H(A) \setminus N_G(S^*)| \tag{5}$$

The final thing: what is $|N_H(C)|$? Here we are going to use part (b) of Lemma 1 (we never really used it so far). We know that for all vertices $w$ in $L'$ which are in $S^*$, there is no edge from $w$ to $v$ in $G$. Therefore, $v \notin N_G(C)$ for $C$ defined above. Thus, by (1), we get that $|N_H(C)| = |N_G(C)|$. And since $G$ satisfies (Hall's Condition), we get

$$|N_H(C)| = |N_G(C)| \geq |C| \tag{6}$$

Substituting (6) and (4) into (5), we get

$$|N_H(B)| \geq |A| + |C| = |B|$$

where the last equality follows since $B$ is the union of disjoint sets $A$ and $C$. But $B$ was a bad set with $|N_H(B)| < |B|$. Contradiction. Thus the claim must be true. □

□

**Proof Number** 2.

> Let $P(n)$ be the predicate which is true if any bipartite graphs $G = (L \cup R, E)$ with $|L| = n$ satisfying (Hall's Condition) has an $L$-matching.

We need to show $\forall n \in \mathbb{N} : P(n)$ is true; we proceed to prove this by induction.

**Base Case:** Is $P(1)$ true? Fix any graph $G = (L \cup R, E)$ with $|L| = 1$. Let $L = \{v\}$. (Hall's Condition) implies, $\deg_G(v) \geq 1$. So, there is some edge $(v, w)$ incident on $v$. $M = \{(v, w)\}$ is an $L$-matching. So, $P(1)$ is true.

**Inductive Case:** Fix a natural number $k$. We assume $P(1), P(2), \ldots, P(k)$ are all true. We wish to prove $P(k+1)$. To that end, we fix a bipartite graph $G = (L \cup R, E)$ which satisfies (Hall's Condition) and $|L| = k + 1$.

Let $u \in L$ be an arbitrary vertex. (Hall's Condition) implies $\deg(u) \geq 1$, thus there is at least one edge $(u, v) \in E$. Pick one such edge *arbitrarily*. Consider the graph $G' = G - \{u, v\}$. That is, we delete both **vertices** $u$ and $v$ (and not just the edge $(u, v)$). $G'$ is also a bipartite graph, with $G = (L' \cup R', E')$ where $L' = L - u$, $R' = R - v$ and $E' = E \setminus (N_G(u) \cup N_G(v))$. See Figure 6 for an illustration.

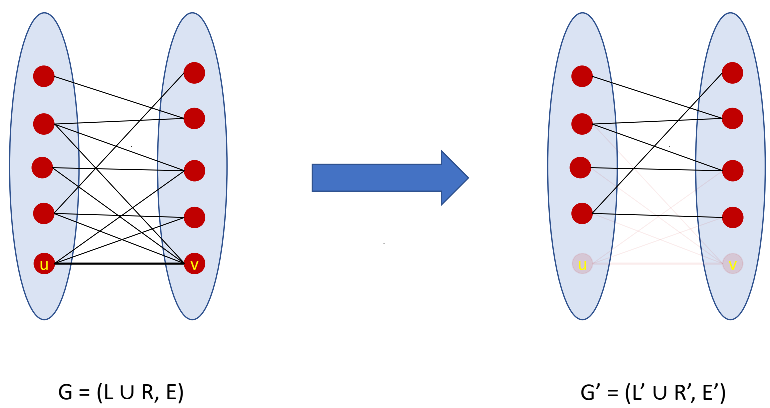



Figure 6: Deleting the vertices $u$ and $v$.

We now fork into two cases.

*Case 1: $G'$ satisfies* (Hall's Condition). This is the easy case. Since $|L'| = |L| - 1 = k$, and since by the induction hypothesis, $P(k)$ is true, we get that $G'$ has an $L'$-matching; let's call it $M'$. Then, $M := M' \cup (u, v)$ is the required $L$-matching in $G$. So in this case, we have proven $P(k+1)$.

*Case 2: $G'$ doesn't satisfy* (Hall's Condition). What does this mean? It means there is some subset $S \subseteq L'$, such that $|N_{G'}(S)| < |S|$. On the other hand, since $G$ did satisfy (Hall's Condition), we have $|N_G(S)| \geq |S|$. Finally, note that the only way $N_{G'}(S)$ and $N_G(S)$ can be different is that if $N_G(S)$ has the vertex $v$ in it. And in that case, $N_{G'}(S) = N_G(S) \setminus v$. See Figure 7 for an illustration.

Therefore, we have $v \in N_G(S)$ and furthermore, $|N_G(S)| = |S|$; if $|N_G(S)| > |S|$, then indeed, $|N_G(S)| \geq |S| + 1$ because the LHS is an integer, which in turn implies $|N_{G'}(S)| = |N_G(S)| - 1 \geq |S|$.

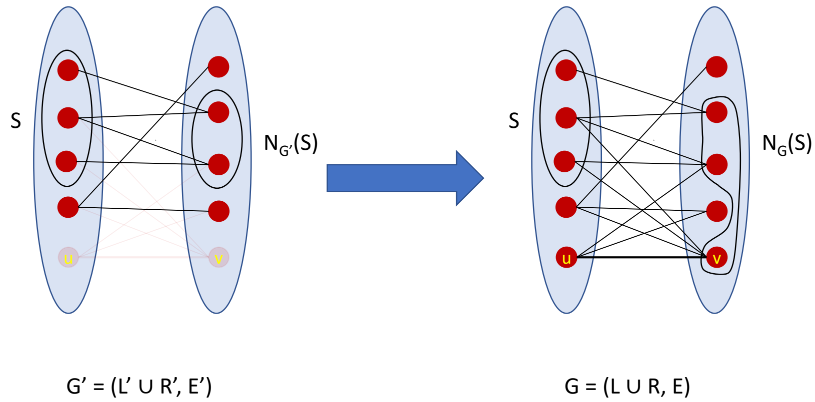



Figure 7: How to related $N_{G'}(S)$ and $N_G(S)$.

Now, we consider two different graphs. We consider $G_1 = G[S \cup N_G(S)]$ and $G_2 = G[(L \setminus S) \cup (R \setminus N_G(S))]$. Recall, the notion of induced subgraphs. See Figure 8 for an illustration.

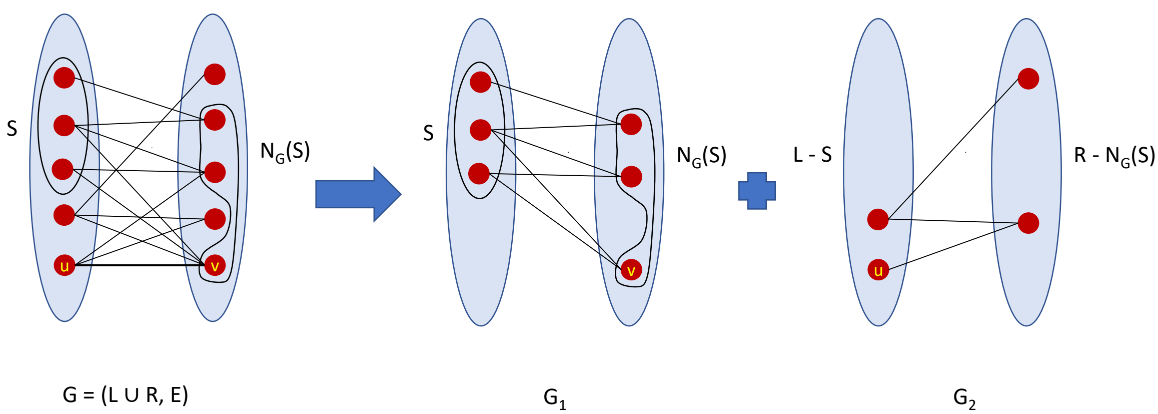



Figure 8: Breaking into two graphs.

**Claim 2.** Both $G_1$ and $G_2$ satisfy (Hall's Condition).

*Proof.* Let's first prove for $G_1$. Any subset $T \subseteq S$ has $N_G(T) \subseteq N_G(S)$. Thus, $N_{G_1}(T) = N_G(T)$ as well. Since $G$ satisfied (Hall's Condition), we get $|N_{G_1}(T)| = |N_G(T)| \geq |T|$. Thus, $G_1$ satisfies (Hall's Condition).

Moving on to $G_2$. Fix a subset $T \subseteq L \setminus S$. What is $N_{G_2}(T)$? Here is an useful observation:

$$N_{G_2}(T) = N_G(T) \setminus N_G(S) = N_G(S \cup T) \setminus N_G(S)$$

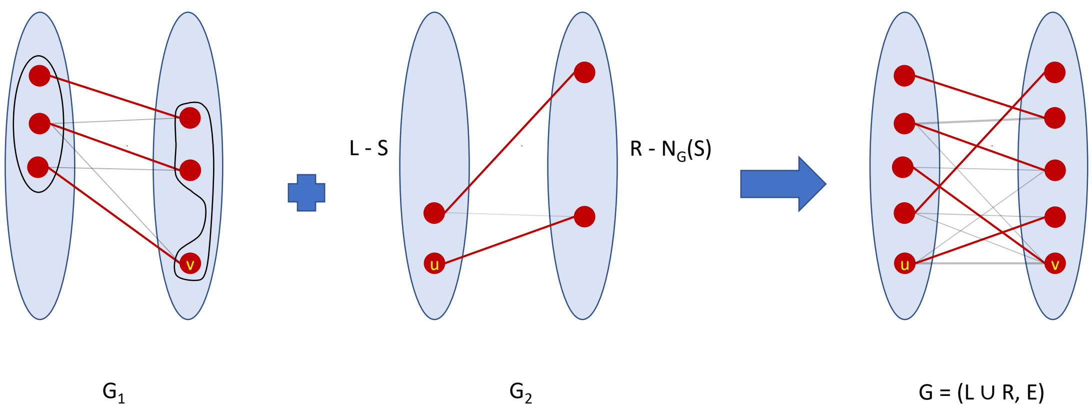
The first equality follows since the neighbors of $T$ in $G_2$ are precisely the neighbors of $T$ in $G$ which are not the neighbors of $S$ in $G$. The second equality is the clever part; it is noting that even if we look at neighbors of $S \cup T$ in $G$ and remove the neighbors of $S$, we still get the neighbors of $T$ in $G$ which are not in $N_G(S)$. Why is this useful? Because, $N_G(S) \subseteq N_G(S \cup T)$. Thus, we know that $|N_G(S \cup T) \setminus N_G(S)| = |N_G(S \cup T)| - |N_G(S)|$.

Putting all together, we get

$$|N_{G_2}(T)| = |N_G(S \cup T)| - |N_G(S)| \geq |S \cup T| - |S| = |T|$$

where the inequality follows since $|N_G(S \cup T)| \geq |S \cup T|$ by (Hall's Condition) and since $|N_G(S)| = |S|$, and the second equality follows since $S \cap T = \emptyset$. □

Since both $G_1$ and $G_2$ satisfy (Hall's Condition), and since both $|S|$ and $|L \setminus S|$ are $< |L|$, by the induction hypothesis, we get that $G_1$ has an $S$-matching called $M_1$ and $G_2$ has an $L \setminus S$-matching called $M_2$. Thus, $M_1 \cup M_2$ is the $L$-matching in $G$.




Done! □