Sets[1]

1 Basics

- **Definition.** A set is an *unordered* collection of *distinct* objects. These objects are called *elements* of the set. These elements could be *anything*, for instance, the element of a set could be a number, could be a string, could be tuples of numbers, and in fact can be other sets!

- **Roster Notation.** A set can be described by explicitly writing down the elements, such as

  $$S = \{1, 3, 5, 7, 9\} \quad \text{or} \quad T = \{\text{apple, banana, volcano}, 100\} \quad \text{or} \quad W = \{S, T\}$$

  This is called the *roster notation*. Note that the *elements* of the set W are the sets S and T.

- **The $\in$ and $\notin$ notation.** We use the notation "element" $\in$ "set" to indicate that the "element" is in the "set". We use $\notin$ to denote that the element is not in the set. In the above example, $3 \in S$ and apple $\in T$ and $S \in W$. But be wary : $3 \notin W$. When figuring out if an element is in a set, we don't "keep opening" the sets inside.

- **Set Builder Notation.** A set can also be described *implicitly* by stating some rule which the elements follow. For example,

  $$S = \{n : n \text{ is a positive odd integer less than } 10\} \quad \text{or} \quad V = \{x^2 : x \text{ is an integer and } 1 \le x \le 5\}$$

  This is called the *set-builder notation.*

  The sets S described in the above two examples correspond to the same set. The set V, written explicitly in the roster notation, is $V = \{1, 4, 9, 16, 25\}$.

  > **Remark:** *Caution: Unless otherwise explicitly mentioned, duplicate items are removed from a set. For example, consider the set $A = \{x^2 : -2 \le x \le 2\}$ in the set-builder notation. In the roster notation, this set is $\{0, 1, 4\}$ and* **not** $\{4, 1, 0, 1, 4\}$.

- **Cardinality of a set.** The *cardinality* of a set S is denoted as $|S|$ is the number of elements in the set. For example if $A = \{\text{apple, banana, avocado}\}$, then $|A| = 3$.

  > **Exercise:** *What is $|A|$ when $A = \{x^2 : -3 \le x \le 3, \;\; x \in \mathbb{Z}\}$?*

  If the set S has only finitely many elements, then $|S|$ is a finite number, and S is called a *finite* set. $|S|$ could be ∞ in which case the set is called an infinite set.

- **Famous examples of Infinite Sets.** $\mathbb{N}$, the set of all natural numbers; $\mathbb{Z}$, the set of all integers; $\mathbb{Q}$, the set of all rational numbers, $\mathbb{R}$, the set of all real numbers; and $\mathbb{P}$, the set of all computer programs written in Python. This course will mostly talk about finite sets. We will visit infinite sets (perhaps) in the very end of this course.

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 16th Jun, 2024*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

- **Empty Set.** There is only one set which contains no elements and that set is called the *empty set* or sometimes the *null set*. It is denoted as $\emptyset$ or $\{\}$.

- **Subsets and Supersets.** A *subset* P of a set S is another set such that every element of P is an element of S. In that case, the notation used is $P \subset S$ or $P \subseteq S$. Note that $S \subseteq S$ as well, that is, a set is always a subset of itself. In case P is a subset and not equal to S, it is called a *proper subset*. It is denoted as $P \subsetneq S$.

  For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2\}$, then $B \subsetneq A$.

  > **Remark:** *The empty set $\emptyset$ is a subset of* all *sets. This is a convention.*

  If $A \subset B$, then B is called a *superset* of A. This is denoted as $B \supset A$.

- **Power Set.** Given any set S, the *power set* $\mathcal{P}(S)$ is the set of *all* subsets of the set S. It is a set of sets. Note by the above convention, for any set S, the empty set $\emptyset \subseteq S$ and therefore, $\emptyset \in \mathcal{P}(S)$.

  > **Exercise:** *Write down* all *subsets of the sets $S = \{1, 2\}$, $T = \{1, 2, 3\}$ and $U = \{1, 2, 3, 4\}$. Do you see a pattern in the number of subsets?*

2 Set Operations.

- **Union.** Given two sets A and B, the *union* $A \cup B$ is the set containing all elements which are either in A, or in B, or both. For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \cup B = \{1, 2, 3, 4, 7, 9, 10\}$$

- **Intersection.** Given two sets A and B, the *intersection* $A \cap B$ is the set containing all elements which are in *both* in A and in B. For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \cap B = \{4, 7, 10\}$$

  Two sets A and B are called *disjoint* if $A \cap B = \emptyset$.

  > **Remark:** *By definition, $\cup$ and $\cap$ are commutative: that is $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*

- **Difference.** Given two sets A and B, the *set difference* $A \setminus B$ are all the elements in A which are *not in* B and $B \setminus A$ are the elements in B which are not in A. For example, if

$$A = \{1, 3, 4, 7, 10\} \text{ and } B = \{2, 4, 7, 9, 10\}, \quad \text{then } A \setminus B = \{1, 3\} \quad \text{and} \quad B \setminus A = \{2, 9\}$$

  > **Exercise:** *Is $\setminus$ operator commutative? Can it ever be that $A \setminus B = B \setminus A$ for any two sets A and B?*

- **Cartesian Product.**

  Given any two sets A and B, the *Cartesian product* $A \times B$ is another set whose elements are *tuples* (that is, ordered pairs) whose first entry comes from A and the second entry comes from B. Therefore, in the set-builder notation

  $$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

  For example, if $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then

  $$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

  Remark: *Note that $A \times B$ is in general not equal to $B \times A$. In particular, in the above example, the elements of $B \times A$ are $\{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$. The element $(a, 1)$ is not the same as $(1, a)$ for the order matters. A tuple is* **not** *a set.*

  Exercise: *Can you figure out the cardinality of $|A \times B|$ in terms of $|A|$ and $|B|$?*

- **Associativity.** We can apply $\cup$s and $\cap$s with three or more sets. For instance, given three sets A, B, C, the set $A \cup B \cup C$ is the set $(A \cup B) \cup C$. The associativity property means that it's the same as $A \cup (B \cup C)$ and along with commutativity, it's the same as $(C \cup A) \cup B$. The order of which sets are union-ed first doesn't matter. Similar to addition. Indeed, $A \cup B \cup C$ is the set of all elements which are either in A or in B or in C.

  The same is true for $\cap$-operator as well. The set $A \cap B \cap C$ is the set of elements which are common to all three sets.

  Remark: *The set-difference operator, however, is* **not** *associative. So, if you see the expression $A \setminus B \setminus C$, then you would have to ask for clarification: is it $A \setminus (B \setminus C)$ or is it $(A \setminus B) \setminus C$. These are two different sets. In particular, the first set can contain elements in $A \cap C$ which aren't in B, while the latter doesn't. The same also holds for the normal minus sign: $(5 - 3) - 2$ and $5 - (3 - 2)$ are different numbers. However, unlike arithmetic, there is no agreed upon convention of which one $A \setminus B \setminus C$ means.*

- **Distributive Property** When we have both operators, union and intersection, then the order in which we take them matters. Therefore, the expression $A \cup B \cap C$ is *ill-defined*. This is because, in general, $(A \cup B) \cap C$ and $A \cup (B \cap C)$ are two different sets. This is similar to the relation between $+$ and $\times$: $(2 + 3) \times 4$ and $2 + (3 \times 4)$ are different. And unlike arithmetic symbols, there is no accepted convention like PEDMAS/BODMAS which clarifies which takes precedence. So, please be careful and put parentheses.

  Exercise: *Is there any relation between $(A \cup B) \cap C$ and $A \cup (B \cap C)$? Is one subset of the other?*

  Just like $+$ and $\times$, the $\cup$ and $\cap$ satisfies the following distributive property.

Theorem 1. For any three sets A, B, C, we have

(a) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.
(b) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

Proof. We prove (a) and leave (b) as an exercise.

To show equality of two sets, we need to show two things. For every x in the LHS set, we need to show it lies in the RHS set. And vice-versa.

Pick any $x \in (A \cup B) \cap C$. Therefore, $x \in C$ and $x \in A$ or $x \in B$. If $x \in A$, then since $x \in C$, we have $x \in A \cap C$, and therefore x is in the RHS set. If $x \in B$, then a similar argument shows $x \in B \cap C$ and therefore x is in the RHS set.

Now the vice-versa. Pick any $x \in (A \cap C) \cup (B \cap C)$. x is either in $A \cap C$ or in $B \cap C$. Suppose $x \in A \cap C$. Then, $x \in A$ which implies $x \in A \cup B$, and therefore, since $x \in C$, we have $x \in (A \cup B) \cap C$. The other possibility, that is if $x \in B \cap C$ also symmetrically implies $x \in (A \cup B) \cap C$. □

Exercise: *True or False: If A and B are disjoint sets, and $C \subset A$, then are C and B disjoint?*

Remark: *Some useful observations:*

a. *A and $B \setminus A$ are* **disjoint** *since $B \setminus A$ doesn't contain elements of A.*
b. *In particular, this implies $(A \cap B)$ and $B \setminus A$ are disjoint since $A \cap B \subseteq A$.*
c. *$A \cup (B \setminus A) = A \cup B$. This is because every element of $A \cup B$ is either in A, and if not in A, must be in $B \setminus A$.*
d. *$(A \cap B) \cup (B \setminus A) = B$. This is because every element of B is either in A (in which case it is in $A \cap B$) or in $B \setminus A$.*

3 Baby Inclusion-Exclusion

- We now meet the first non-trivial (but simple) statement in the course. It is the "baby" inclusion-exclusion identity/equation/formula. It is "baby" because we will meet the grown-up version later in the course. But the baby is strong enough for many things.

- Before we go to the inclusion-exclusion, we start with a simpler but key claim.

  Claim 1. If A and B are two disjoint finite sets, then $|A \cup B| = |A| + |B|$.

  Proof. Since A and B are finite, they have well-defined cardinalities which are non-negative integers. Let $|A| = k$ and let $|B| = \ell$; note that these can be 0.

  We are now going to *name* the elements of our sets. This will be very helpful in our reasoning. Indeed naming objects is a key thing to learn in this course. There is fantastic power in this simple sounding step. And so, to this end, let $A = \{a_1, a_2, \ldots, a_k\}$ and let $B = \{b_1, b_2, \ldots, b_\ell\}$. Note that if either k

or ℓ or both are 0, then the corresponding set would be $\{\}$ that is, the empty set $\emptyset$. So this notation is well defined.

Now for the key observation : since A and B are disjoint, we know that $a_i \neq b_j$ for any indices i and j. Therefore, $A \cup B = \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_\ell\}$ since it must contain all items of A and B. Thus, by inspection now, $|A \cup B| = k + \ell = |A| + |B|$. □

Remark: *We can extend claim 1 to multiple sets. A definition first. A collection of n sets, $A_1, A_2, \ldots, A_n$, are called* **pairwise disjoint** *if $A_i \cap A_j = \emptyset$ for all $i \neq j$. That is, any two different sets are disjoint. Then, one can extend the claim above with a similar proof to the following generalization. We give a different argument below which contains the main idea behind the "proof by induction" technique.*

Claim 2. If $A_1, A_2, \ldots, A_n$ are pairwise disjoint finite sets, then $|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|$.

Proof. (Sketch) Let me show why the $n = 3$ case holds. The idea is to think of $A_1 \cup A_2$ as one set and apply the claim above. Since $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$, we have $(A_1 \cup A_2) \cap A_3 = \emptyset$. So,

$$|(A_1 \cup A_2) \cup A_3| = |A_1 \cup A_2| + |A_3|$$

And then since $A_1 \cap A_2 = \emptyset$, we get $|A_1 \cup A_2| = |A_1| + |A_2|$ which if we substitute above, we get $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3|$. Do you now see how to generalize for a general n? □

- Now we are ready for stating and proving the baby inclusion-exclusion theorem.

Theorem 2 (Baby Inclusion-Exclusion)**.** For any two finite sets A and B, we have

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof. Since $A \cup B = A \cup (B \setminus A)$ and since A and $B \setminus A$ are disjoint, we get

$$|A \cup B| = |A| + |B \setminus A| \tag{1}$$

Since $B = (A \cap B) \cup (B \setminus A)$ and since $(A \cap B)$ and $B \setminus A)$ are disjoint, we get

$$|B| = |A \cap B| + |B \setminus A| \tag{2}$$

Subtracting (2) from (1), we get

$$|A \cup B| - |B| = |A| - |A \cap B|$$

The theorem follows by taking $|B|$ to the other side. □

Answers to exercises

- Note that $A = \{9, 4, 1, 0\}$, and thus the answer is 4. Although -3 and $+3$ are distinct, their squares are not, and in the set A they are counted only once.
- The set of subsets of S are $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and there are **four** of them. The set of subsets of T are

  $$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

  and there are **eight** of them. I will let you write all the subsets of U. Do you see the pattern now?
- True. If A and B are disjoint, then no element of A is present in B. Since C is a subset of A, no element of C is present in B either. Conversely, no element of B is present in A (since they are disjoint), and thus no element of B can be present in C either.

  In general, $C \subseteq A$ implies $C \cap B \subseteq A \cap B$. If the second set is $\emptyset$, then $C \cap B$ has to be $\emptyset$ since that is the **only** subset of an empty set. Thus, C and B are disjoint too.
- In general, $A \setminus B \neq B \setminus A$. However, if $A = B$, then both $A \setminus B$ and $B \setminus A$ are $\emptyset$. Is that the only possibility?
- By the distributive property, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. Since $A \cap C \subseteq A$, we get that $(A \cup B) \cap C \subseteq A \cup (B \cap C)$
- It is simply $|A \times B| = |A| \cdot |B|$, the product of the two cardinalities. In the "combinatorics" module, this will be called the "product principle". Do you see why this is true? For each of the $|A|$ choices of the "first entry" in the tuple of $A \times B$, there are precisely $|B|$ choices for the "second entry".