Divide and Conquer: Merge Sort + Solving Recurrences[1]

In this and a few coming lectures, we look at the *divide-and-conquer* paradigm for algorithm design. This is applicable when any instance of a problem can be broken into smaller instances, such that the solutions of the smaller instances can be combined to get solution to the original instance. One usually has a "naive" but "slow" method to solve the problem at hand, and the D&C methodology (usually) "wins" if the combining can be done faster than the naive running time.

1 Merge Sort

SORTING AN ARRAY
Input: An array $A[1:n]$ of integers.
Output: A sorted (non-decreasing) permutation $B[1:n]$ of $A[1:n]$.
Size: The number n of entries in A.

Remark: *It is fair to ask why the size of the problem above doesn't include the number of bits required to encode $A[j]$'s and x. The short answer is that it is a modeling choice. If the numbers $A[j]$'s are "small", that is, they are at most some polynomial in n and thus fit in $O(\log n)$ sized registers, then we assume that simple operations such as adding, multiplying, dividing, comparing, reading, writing take $\Theta(1)$ time. The reason being that for numbers that fit in word/registers indeed these operations are fast compared to the other operations of the algorithm.*

The "naive" algorithm for sorting is one which repeatedly takes the minimum element and puts it in the front; this algorithm takes $O(n^2)$ time. We now show how to do better using divide-and-conquer. This algorithm is *merge-sort*. You have perhaps seen this algorithm before (in CS 10 or CS 1), and the idea nicely captures the Divide-and-Conquer strategy.

First we notice if $n = 1$, then we return the same array; this is the base case. Next, given an input $A[1:n]$ which needs to be sorted, we wish to divide into two smaller subproblems. We start with a natural way: we divide $A[1:n]$ into two halves $A[1:n/2]$ and $A[n/2+1:n]$. Next, we recursively apply the same algorithm to these halves to obtain sorted versions B_1 and B_2. Note that the final answer that we need is a sorted version of $B_1 \cup B_2$. So in the *combine/conquer* step we do exactly this.

At this point we need to figure out a "win": why is sorting $B_1 \cup B_2$ any easier than sorting $A[1:n]$ to begin with. The fact we exploit is that these B_1 and B_2 are individually sorted. We can use this to sort $B_1 \cup B_2$ way faster than the $\Theta(n^2)$ naive algorithm for $A[1:n]$. This is the non-trivial part of the algorithm, and once we get a "win" here over the naive algorithm, we will see that we get a win over all.

Combine Step. Let us then recall the Combine procedure of MERGESORT

[1] *Lecture notes by Deeparnab Chakrabarty. Last modified : 19th Mar, 2022*
These have not gone through scrutiny and may contain errors. If you find any, or have any other comments, please email me at deeparnab@dartmouth.edu. Highly appreciated!

COMBINE
Input: Two ***sorted*** arrays $P[1:p]$ and $Q[1:q]$.
Output: Sorted array $R[1:r]$ of elements of $P \cup Q$, with $r = p + q$.
Size: $p + q$.

This is an iterative algorithm which keeps three pointers i, j, k all set to 1. At each step we compare $P[i]$ and $Q[j]$, and $R[k]$ is set to whichever is smaller. That particular pointer and k are incremented. The process stops when either i reaches $p + 1$ or j reaches $q + 1$, in which case the rest of the other array is appended to R. This takes $O(p + q)$ time as each step takes $O(1)$ time, in each step either i or j increments, and so the algorithm terminates in $(p + q)$ steps. A formal pseudocode is given below both of the above combine step and the merge sort.

```
procedure COMBINE(P[1 : p], Q[1 : q]):
    ▷ P and Q are sorted; outputs R a sorted array of elements in P and Q.
    i = j = k ← 1.
    while i < p + 1 and j < q + 1 do:
        if (P[i] ≤ Q[j]) then:
            R[k] ← P[i]
            i ← i + 1
        else:
            R[k] ← Q[j]
            j ← j + 1
      k ← k + 1
    if i > p then:
        Append rest Q[j : q] to R
    else:
        Append rest of P[i : p] to R
    return R.
```

Theorem 1. If P and Q are sorted, then COMBINE(P, Q) returns a sorted array of the elements in P and Q.

Proof. We first show that R is sorted. The reason is that P and Q are sorted and thus elements added to R are increasing. Formally, in an iteration k of the while loop, an element is added to R as $R[k]$. This element is either an element $P[i]$ or an element $Q[j]$, for some i and j, whichever is smaller. Let us assume it was $P[i]$; the other case can be analogously argued. The previous element added to R (in the previous loop), that is, $R[k-1]$ was either $P[i-1]$ (in which case $i-1$ was incremented to i) or it was $Q[j-1]$. If the former, then $R[k-1] \leq R[k]$ because $P[i-1] \leq P[i]$ since P is sorted. If the latter, and this is crucial, then $Q[j-1]$ must have been compared with $P[i]$ since i didn't increment in the $(k-1)$th loop. And thus, $Q[j-1] \leq P[i] = R[k]$. That is, even in this case $R[k-1] \leq R[k]$. Thus in each step, what is added to R is increasing implying R is sorted. Secondly, all elements of P and Q are visited in this order. And thus, R is a sorted order of all elements in P and Q. □

Theorem 2. COMBINE(P, Q) takes $O(p+q)$ time.

Proof. Here is a general principle: when analyzing the running-time of an algorithm with a *while loop*, one needs to figure out a measure/quantity/potential which (a) either monotonically strictly increases or strictly decreases in each iteration of the while loop, (b) starts of at a known value, and (c) one can argue termination of the while loop if the value ever reaches a different quantity. Almost *all* while loop running times are measured that way.

For the COMBINE, what is this quantity? Well, we see that in the while loop, either i increments or j increments. Therefore, the quantity of interest is $(i+j)$. This quantity starts off at 2. It always increases by 1. And finally note that if it ever reaches $(p+q+2)$, then either $i \geq p+1$ or $j \geq q+1$ (for otherwise $i < p+1, j < q+1$ implying $i+j < p+q+2$.) That is, the while loop terminates. This shows the while loop cannot have more than $(p+q)$ iterations. Since each iteration takes $O(1)$ time, we get the desired running time for COMBINE. □

Armed with the above, the complete divide-and-conquer algorithm for sorting is given as:

```
1: procedure MERGESORT(A[1 : n]):
2:     ▷ Returns sorted order of A[1 : n]
3:     if n = 1 then:
4:         return A[1 : n]. ▷ Singleton Array
5:     m ← ⌊n/2⌋
6:     B_1 ←MERGESORT(A[1 : m])
7:     B_2 ←MERGESORT(A[m + 1 : n])
8:     return COMBINE(B_1, B_2)
```

Theorem 3. MERGESORT takes $O(n \log n)$ time.

Proof. Let $T(n)$ be the worst case running time of MERGESORT on arrays of size n. Since MERGESORT is a recursive algorithm, just as we did in Lecture 3 with MULT and DIVIDE, we try figuring out line by line the times taken. Let's begin.

First, we see that when $n = 1$, only Line 3 and Line 4 run. The time taken is $O(1)$. Thus, we get

$$T(1) = O(1) \tag{1}$$

Again, this is because *any* (finite) time algorithm on a constant sized input takes $O(1)$ time.

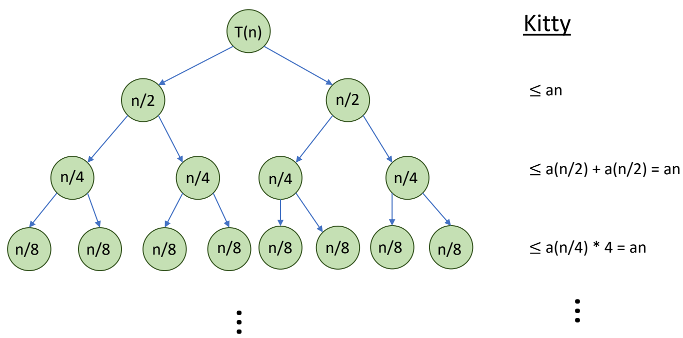
For larger n, the code proceeds to Lines 6 to 8. Line 6 is a recursive call. Since it is on an array of size m, by definition of $T()$, this takes *at most* $T(m)$ time. Similarly, Line 7 takes at most $T(n-m)$ time. And, Theorem 2 tells us that Line 8 takes $O(n)$ time. Finally, we note that since $m = \lfloor n/2 \rfloor$, we get that $n - m = \lceil n/2 \rceil$. Therefore, putting all together, we get the recurrence.

$$T(n) \leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n), \quad \forall n > 1 \tag{2}$$

To get to the big picture, we get rid of the floors and ceilings. In the supplement, you can see why this is kosher[2]. Furthermore, we also replace the $O(n)$ term by $\leq a \cdot n$ for some constant a, to get

$$T(n) \leq 2T(n/2) + a \cdot n, \quad \forall n > 1 \tag{3}$$

We will apply the "kitty method" or "opening up the brackets" method to solve the recurrence inequality given by (1) and (3).



$$\begin{aligned}
T(n) &\leq 2T(n/2) + an \\
&\leq 2\left(2T(n/4) + an/2\right) + an \\
&= 4T(n/4) + 2an \\
&\leq 4\left(2T(n/8) + an/4\right) + 2an \\
&= 8T(n/8) + 3an \\
&\vdots \\
&\leq 2^k T(n/2^k) + kan
\end{aligned}$$

Setting k such that $n/2^k \leq 1$ gives us $T(n) = O(n \log n)$. □

1.1 The Master Theorem

The following theorem is a useful hammer to solve many recurrences.

Theorem 4. Consider the following recurrence:

$$T(n) \leq a \cdot T(\lceil n/b \rceil) + O(n^d)$$

[2] if you are a little worried about this, then (a) good, and (b) note that for large x, $\lceil x \rceil$, $\lfloor x \rfloor$ are really $x\pm$ some "lower order" term, and so since we are talking using the Big-Oh notation, it shouldn't matter. This is ***not*** a rigorous argument — it is not meant to completely convince you, but it should give you a hint to why this may be true.

where a, b, d are non-negative reals. Then, the solution to the above is given by

$$T(n) = \begin{cases} O(n^d) & \text{if} \quad a < b^d \\ O(n^d \log n) & \text{if} \quad a = b^d \\ O(n^{\log_b a}) & \text{if} \quad a > b^d \end{cases}$$

The proof is similar in spirit to that of Theorem 3 and you are recommended trying it. Instead of splitting into two, each "ball" splits into a different balls each of size n/b (ignoring floors & ceilings), but it puts $\Theta(n^d)$ in the kitty. If you write the expression as above (it's a little more complicated than the one we say before), then you will get a *geometric series* whose base is exactly a/b^d. Thus if $a < b^d$ (that is, the base < 1), then the geometric sum is small, and the total cost is bounded by the first deposit in the kitty. If $a = b^d$ (that is, if the base $= 1$) then we make around the same amount of deposits in the kitty, and we do it $\Theta(\log n)$ times. Finally, if $a > b^d$, then the geometric series is bounded by the other end, and the "number of small balls" (which is this bizarrish term $n^{\log_b a}$) is what dominates. All this is perhaps too high-level to be useful – see the supplement for the proof.