# CS 10: Problem solving via Object Oriented Programming

## Hierarchies 3: Balance

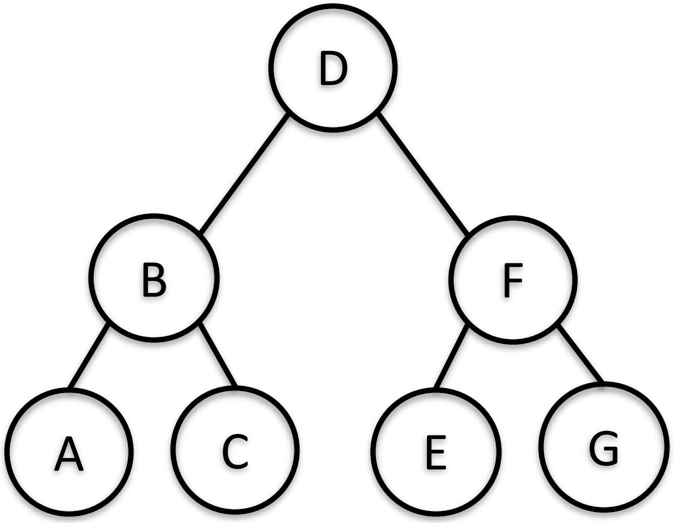
# Agenda

1. Balanced Binary Trees
2. 2-3-4 Trees
3. Red-Black Trees

**Key points:**

1. **BSTs keep data sorted in a tree structure**
2. **Each node in the tree has a Key and a Value**
3. **BSTs search by Key and return the matching Value**

# Review: Binary Search Trees (BSTS) are an ordered collection of Key/Value nodes




**Binary Search Tree property**

Let x be a node in a binary search tree s.t.:

- left.key < x.key
- right.key > x.key

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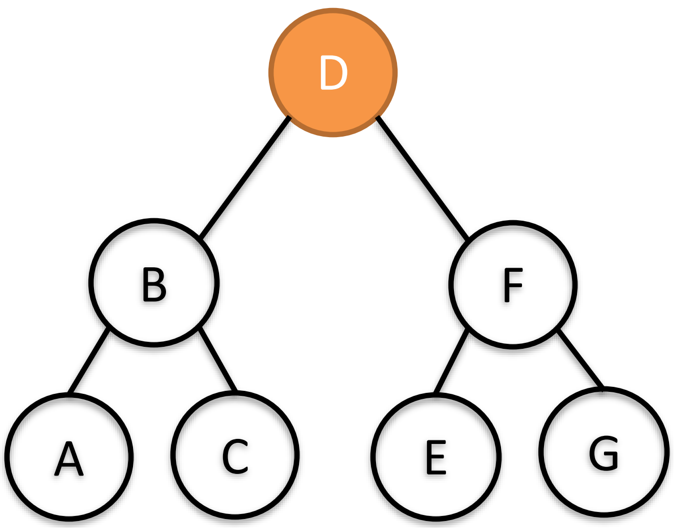


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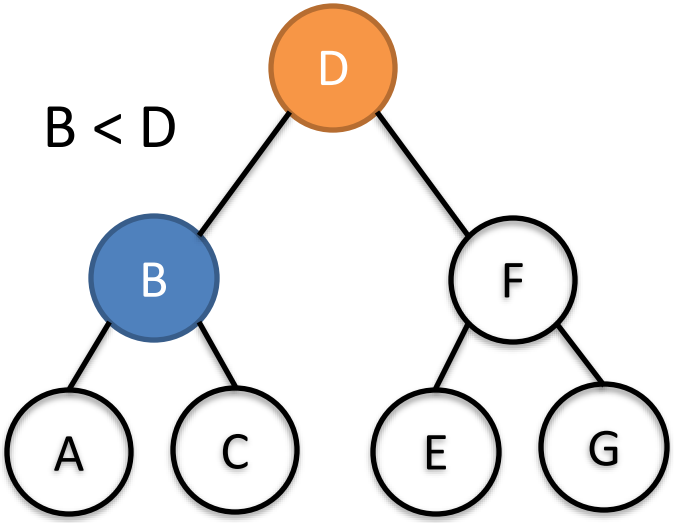


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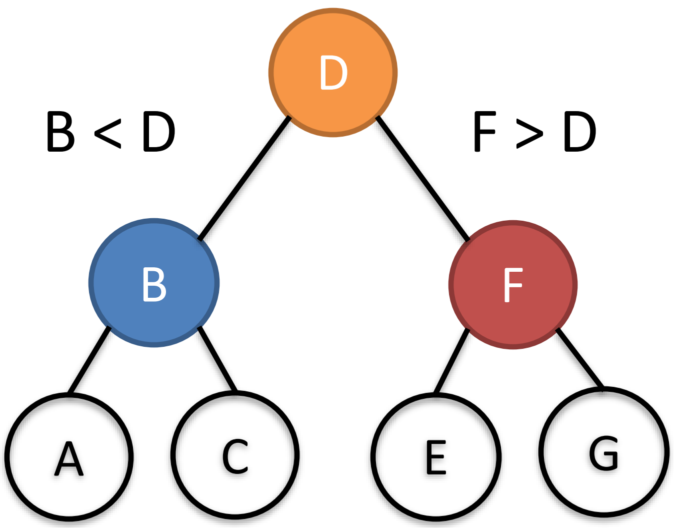



**Binary Search Tree property**

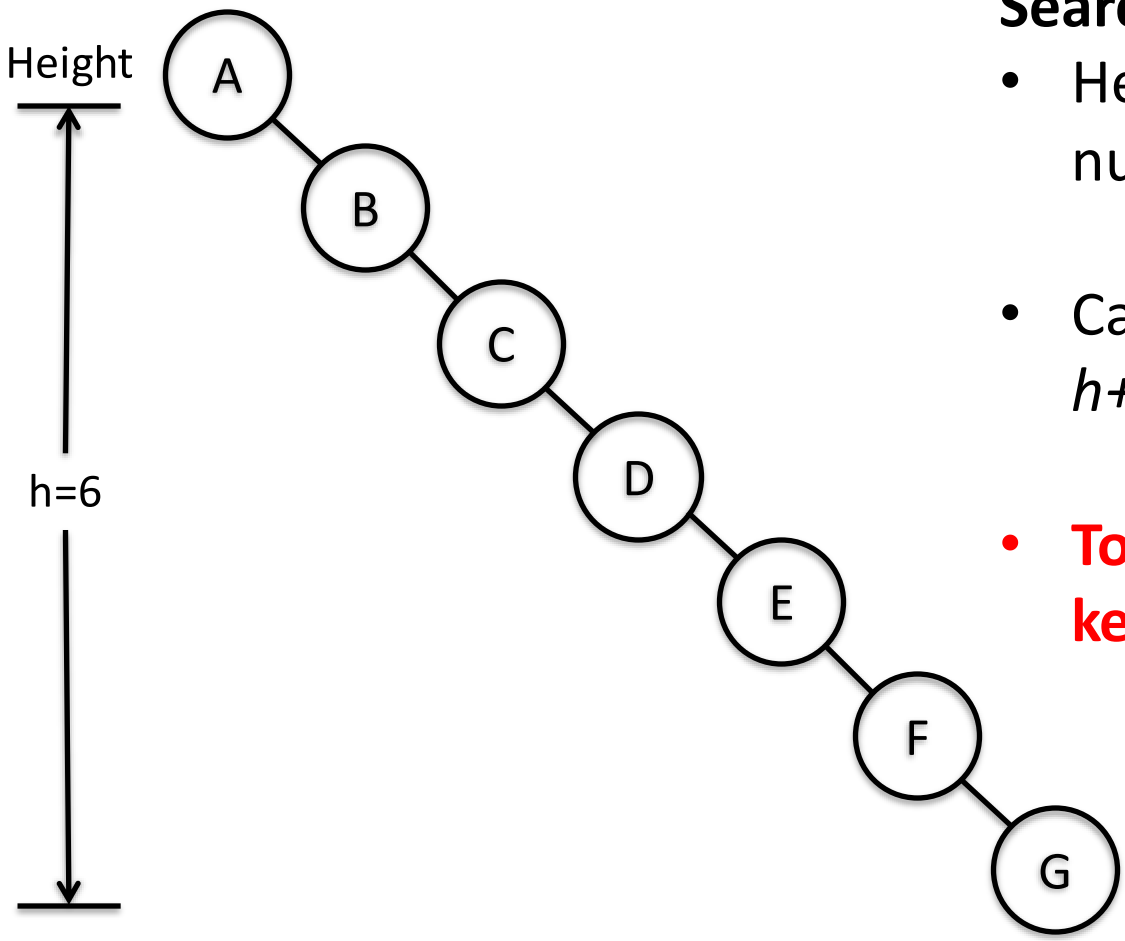
Let x be a node in a binary search tree s.t.:

- left.key < x.key
- right.key > x.key

**Remember, I'm showing the Keys for each node, but there is also a Value for each node that is not shown**

# BSTs do not have to be balanced! Can not make tight bound assumptions

**Find Key "G"**

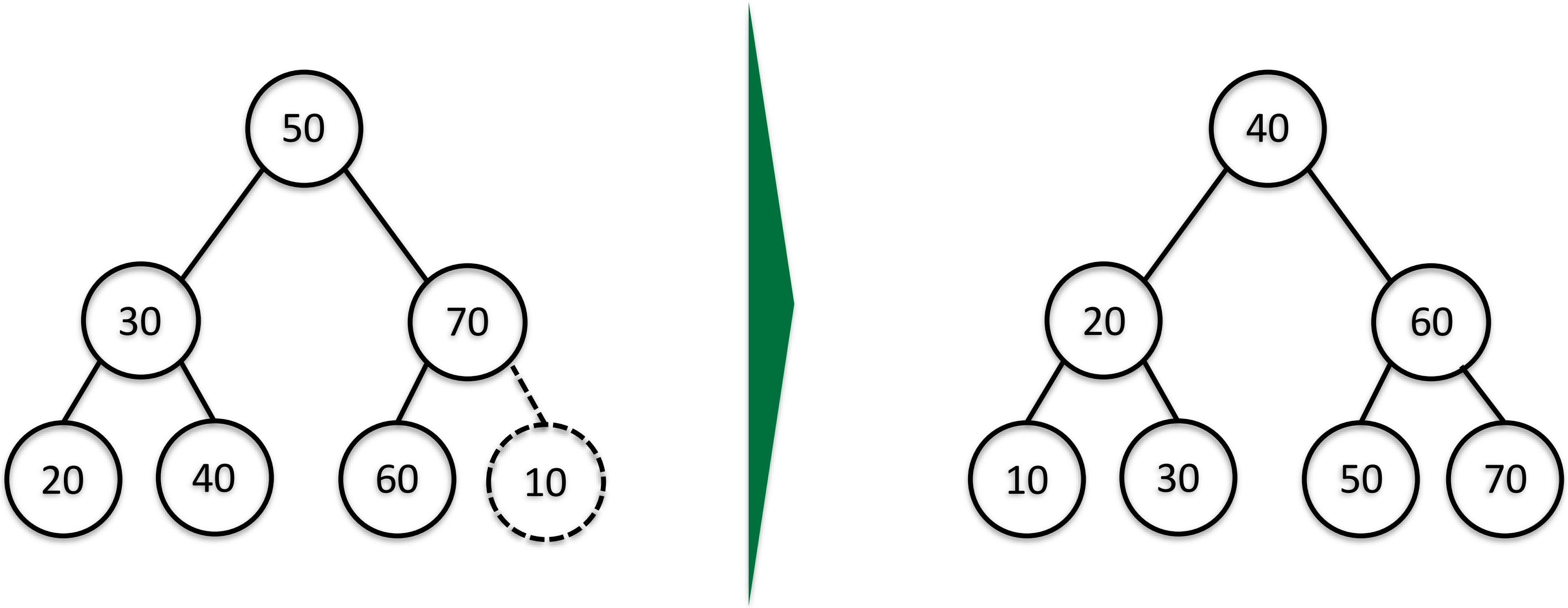



**Search process**

- Height *h = 6* (count number of edges to leaf)
- Can take no more than *h+1* checks, O(h)
- **Today we will see how to keep trees "balanced"**

# Could try to "fix up" tree to keep balance as nodes are added/removed

**Keeping balance is tricky**




Insert 10

"Fix up"

All nodes changed position
O(n) possible on many updates!
Need another way

# We consider two other options to keep “binary” trees “perfectly balanced”

1. Give up on “binary” – allow nodes to have multiple keys (2-3-4 trees)

2. Give up on “perfect” – keep tree “close” to perfectly balanced (Red-Black trees)

# Agenda

1. Balanced Binary Trees
2. 2-3-4 Trees
3. Red-Black Trees

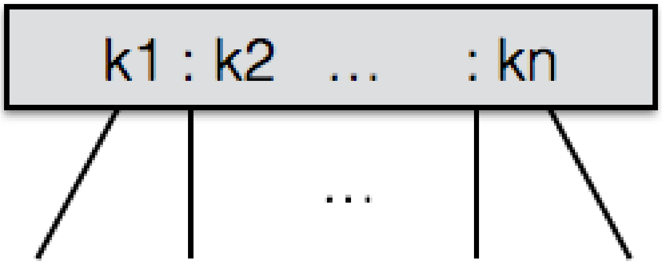
**Key points:**

1. **2-3-4 trees give up on binary**
2. **Nodes have 2, 3, or 4 children**
3. **All leaves at the same level**
4. **Height of 2-3-4 tree $O(\log_2 n)$**
5. **Ensures O(log n) performance**

# 2-3-4 trees (aka 2,4 trees) give up on binary but keep tree balanced

**Intuition:**

- Allow multiple keys to be stored at each node
- A node will have one more child than it has keys:
    - leftmost child — all keys less than the first key
    - next child — all keys between the first and second keys
    - … etc …
    - last child — all keys greater than the last key
- We will work with nodes that have 2, 3, or 4 children (nodes are named after number of children, not the number of keys)



# 2-3-4 trees maintain two properties: Size and Depth

**Size property**

Each node has either 2, 3, or 4 children (1, 2, or 3 keys per node)

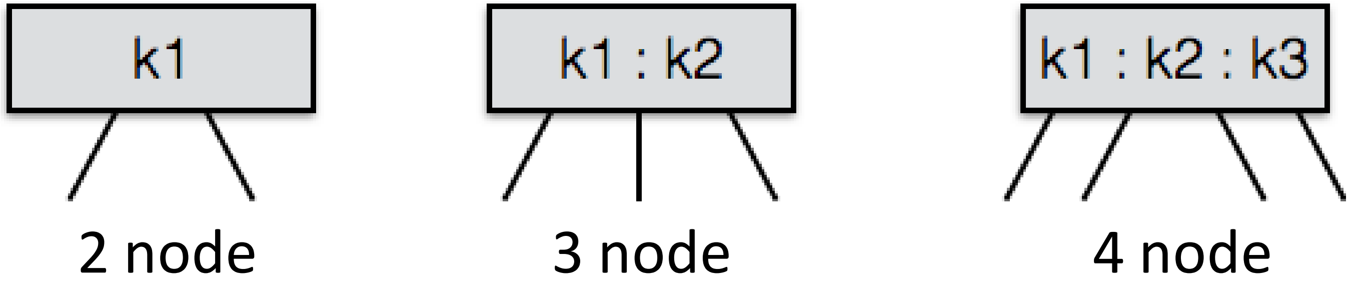
Each node type named after number of children, not keys

**Depth property**

All leaves of the tree (external nodes) are on the same level

**It can be shown that the height of the Tree is O($\log_2$ n) if these properties are maintained (see book)**

**Node types**



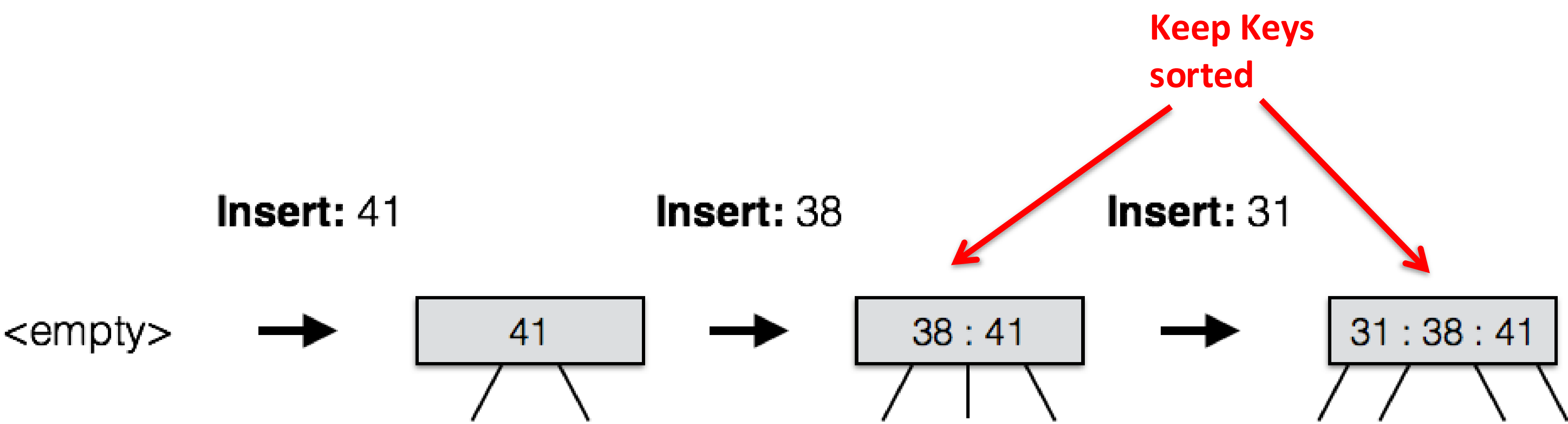
# Inserting into a 2-3-4 Tree must maintain Size and Depth properties

**Insertion:**

1. Begin by searching for Key in 2-3-4 Tree
2. If found, update Value
3. If not found, search terminates at a leaf
4. Do an insert at the leaf
5. Maintain the Size and Depth properties (next slides)

# Insert into the lowest node, but do not violate the size property

**Inserting into 2 or 3 node**




**Inserting into a 2 or 3 node:**

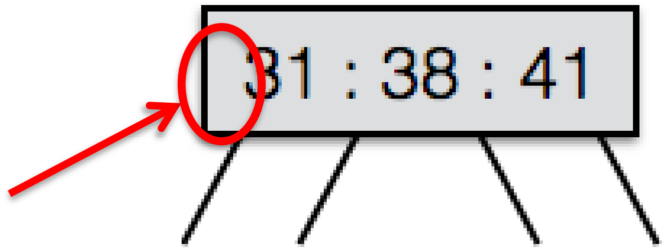
- Keep keys ordered inside each node
- Can insert key inside a *node* in O(1) because there are only three places where Key could go
- So, we can update a node in constant, O(1) time

# If insert would violate size rule, split 4 node into two 2 nodes, then insert new object

**Inserting into 4 node**

**Insert:** 12

Would go here

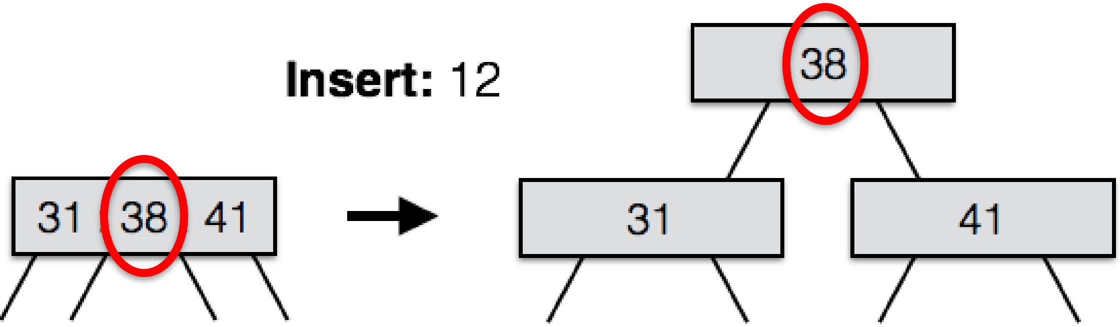



Insert would cause size violation for this node

Insert in a two step process

# If insert would violate size rule, split 4 node into two 2 nodes, then insert new object

**Inserting into 4 node, two step process**



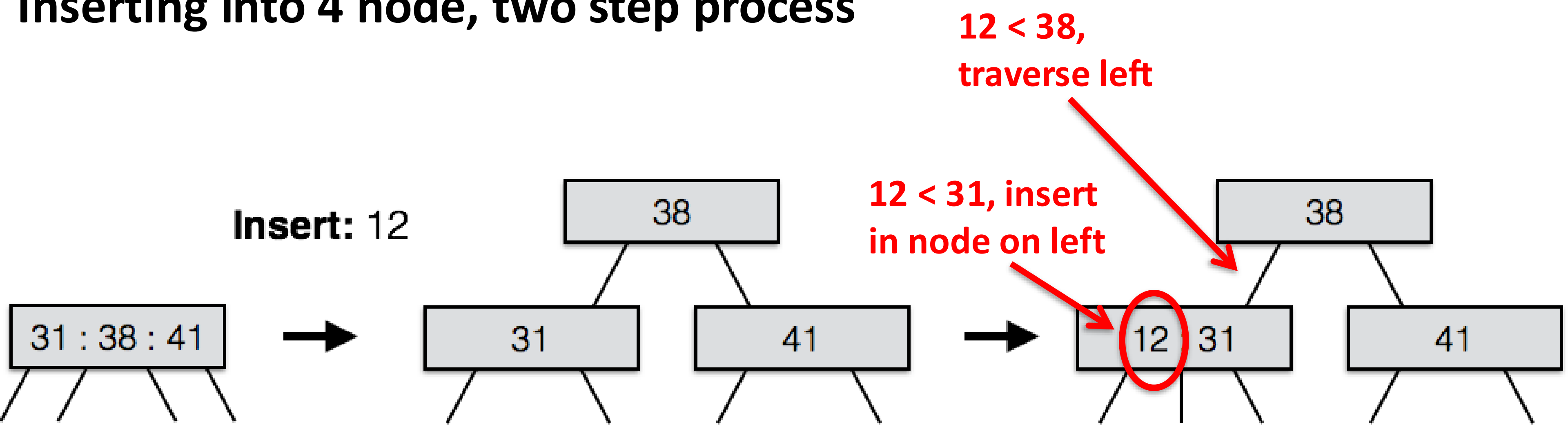

**Step 1: split/promote**

Promote middle key to higher level

- May become new root
- Parent may have to be split also!

# If insert would violate size rule, split 4 node into two 2 nodes, then insert new object

**Inserting into 4 node, two step process**




**Step 1: split/promote**

Promote middle key to higher level

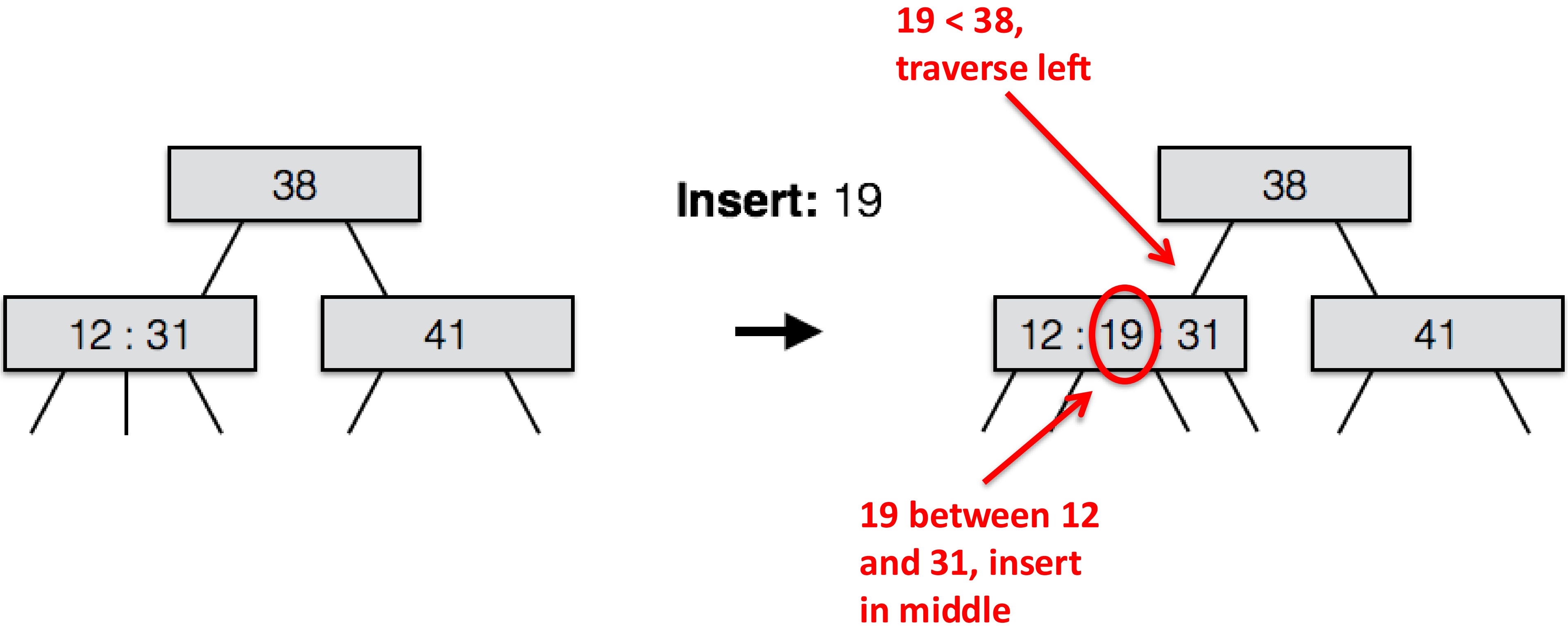
- May become new root
- Parent may have to be split also!

**Step 2: insert**

Insert 12 into appropriate node at lowest level

# Continue inserting until need to split nodes

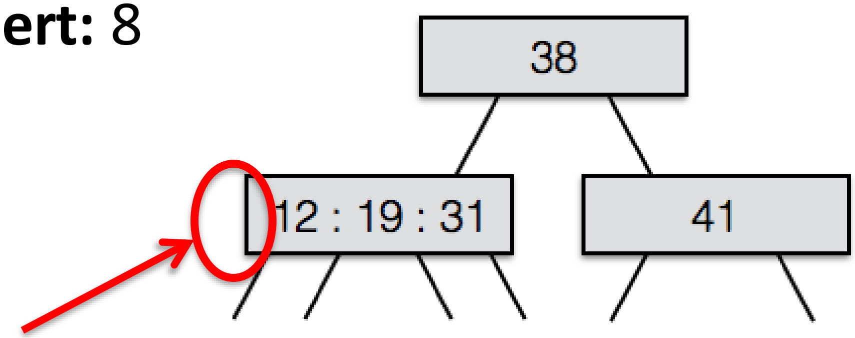
**Insert process**



# Promote middle key to higher level and insert new key into proper position

**Insert process**

**Insert:** 8

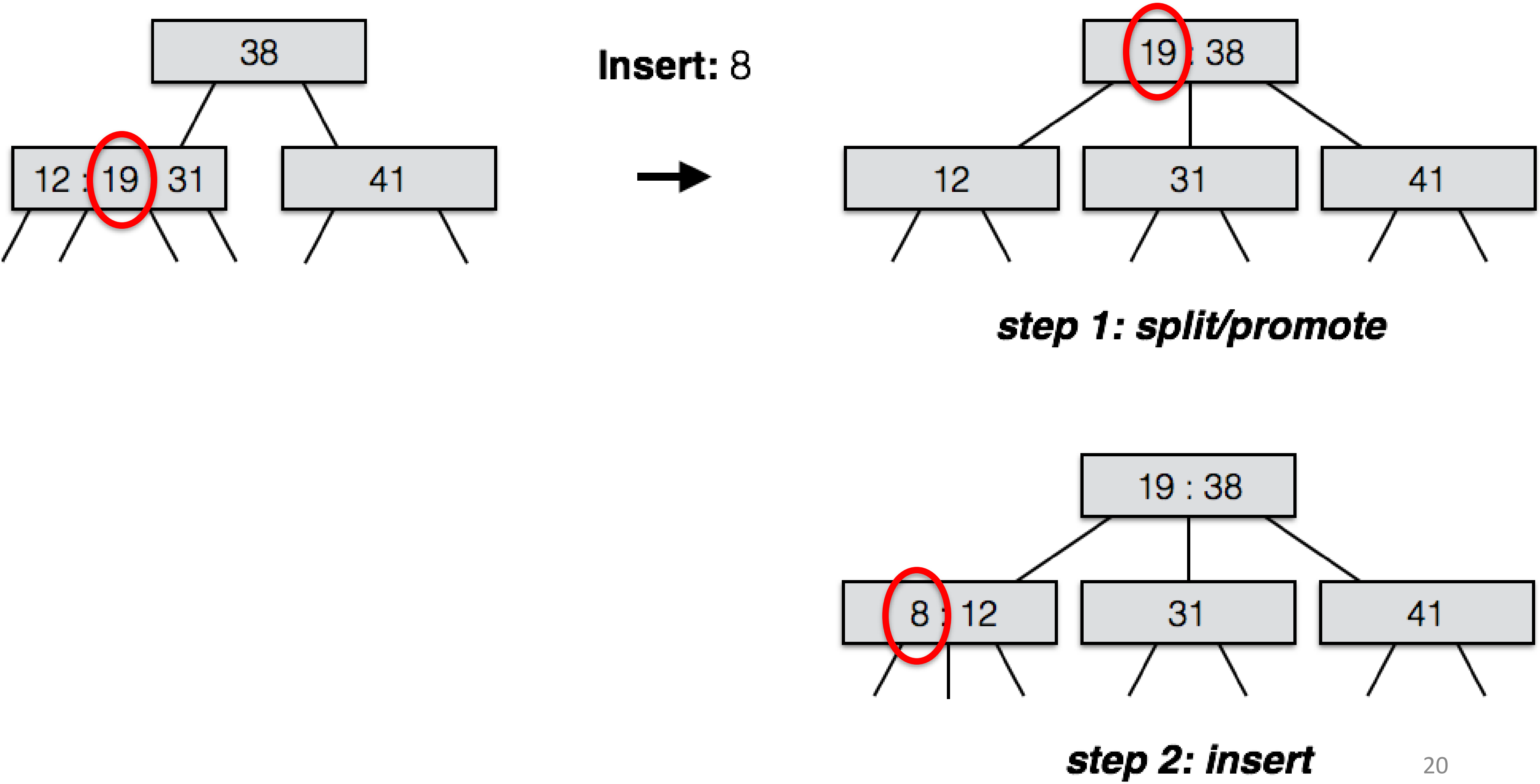



Would go here

Insert would cause size violation for this node

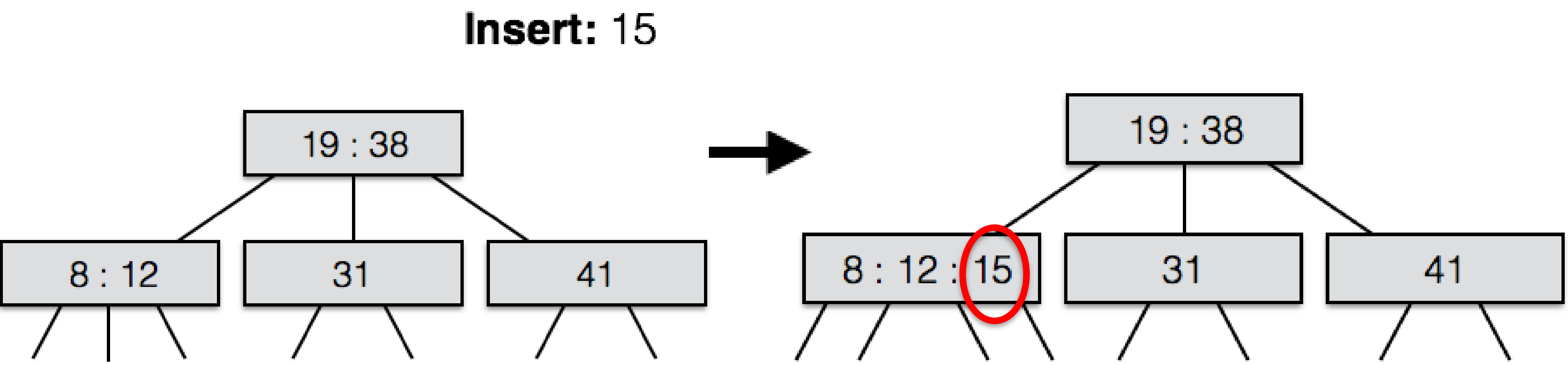
# Promote middle key to higher level and insert new key into proper position

**Insert process**



# Always insert new key in lowest level

**Insert process**



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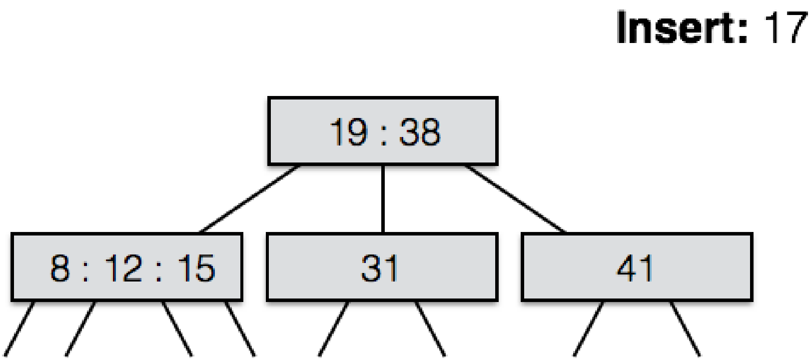
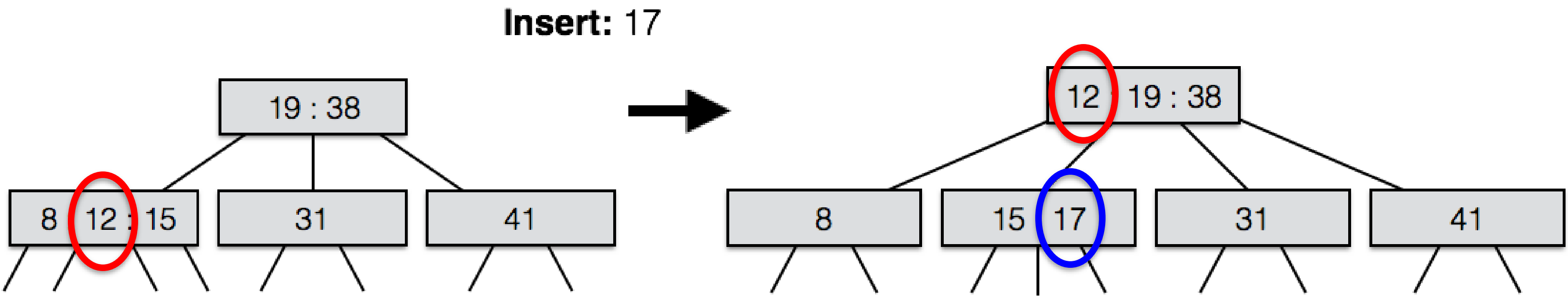



Step 1: Split and promote 12
Step 2: Insert 17

# Always insert new key in lowest level

**Insert process**

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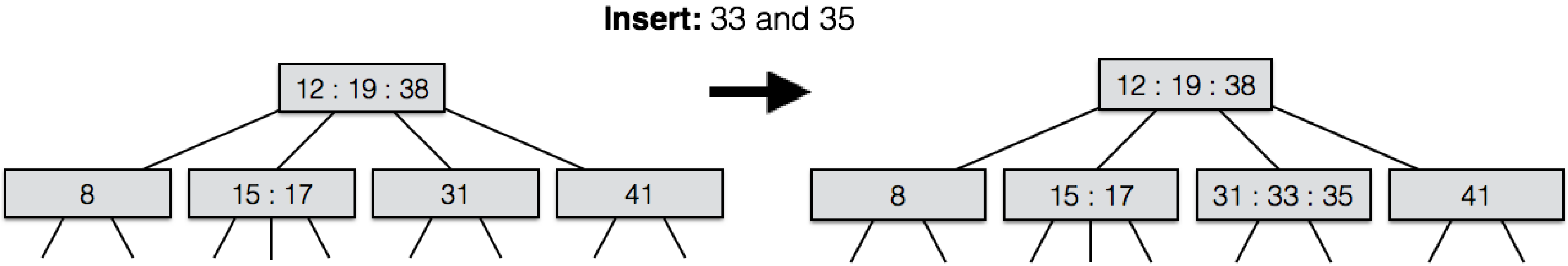
# Always insert new key in lowest level

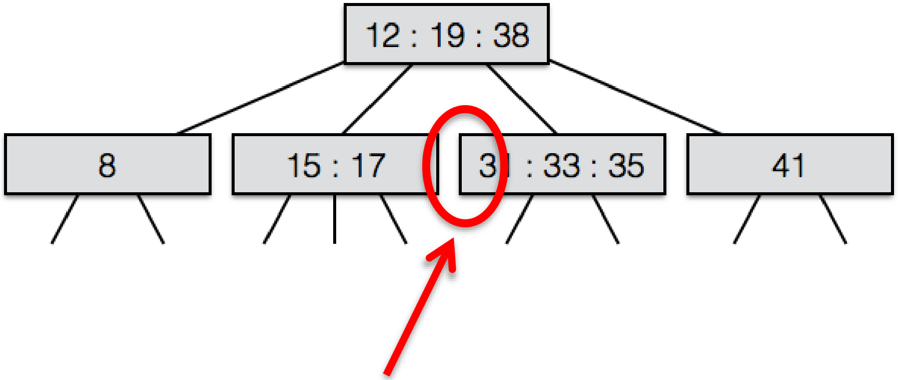
**Insert process**



# Might have to split multiple nodes to ensure parent size property is not violated

**Insert process**

**Insert:** 20




Would go here

Insert would cause size violation for this node

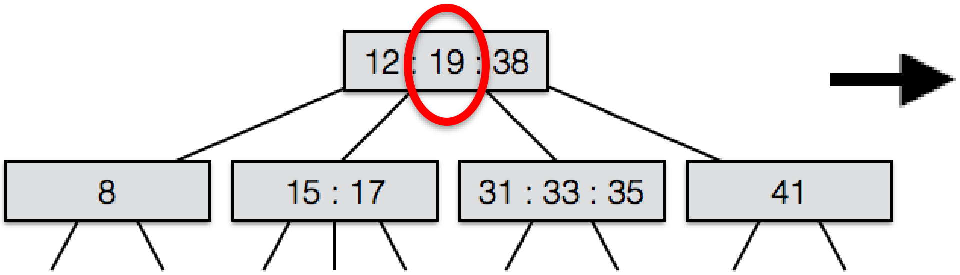
Promoting would cause parent size violation

Split parent first, then split child, then insert

Could bubble up all the way to the root

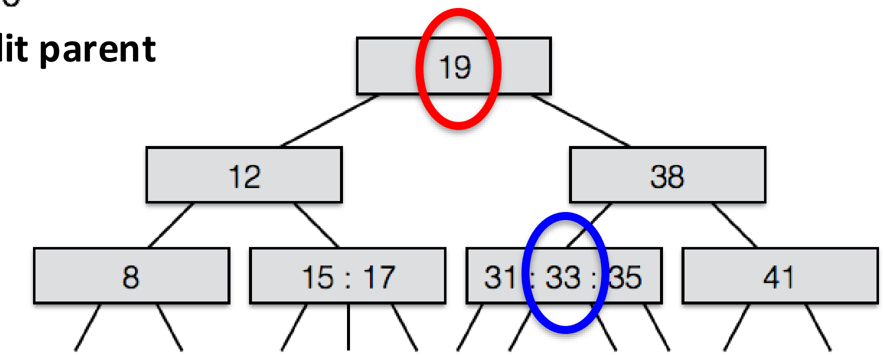
# Might have to split multiple nodes to ensure parent size property is not violated

**Insert process**

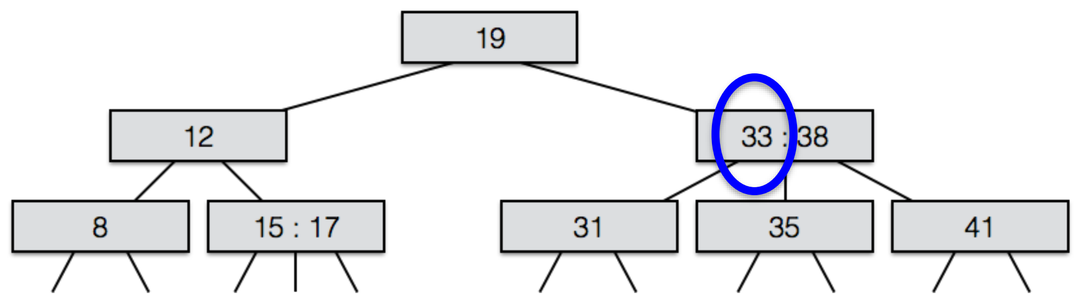



**Insert:** 20

**First split parent**

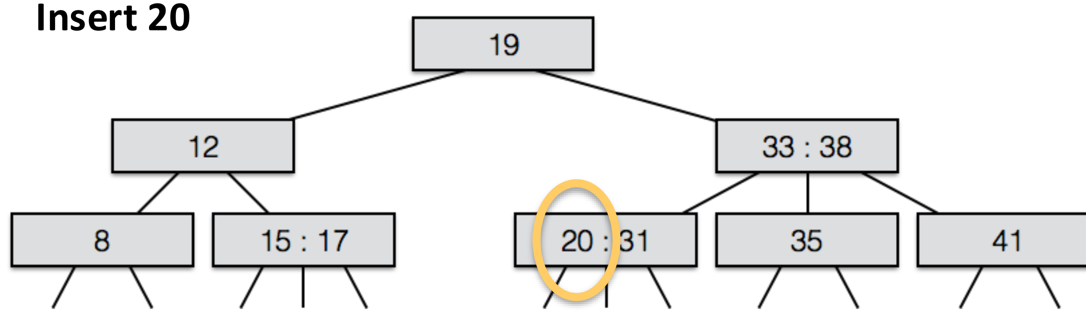



**Second split**




**Performance?**
**O(h) = O($Log_2$n)**

**Insert 20**



# 2-3-4 work, but are tricky to implement

- Need three different types of nodes
- Create new nodes as you need them, then copy information from old node to new node
- Can waste space if nodes have few keys
- Book has more info on insertion and deletion
- There are generally easier ways to implement as a Binary Tree

# Agenda

1. Balanced Binary Trees

2. 2-3-4 Trees

3. Red-Black Trees

**Key points:**

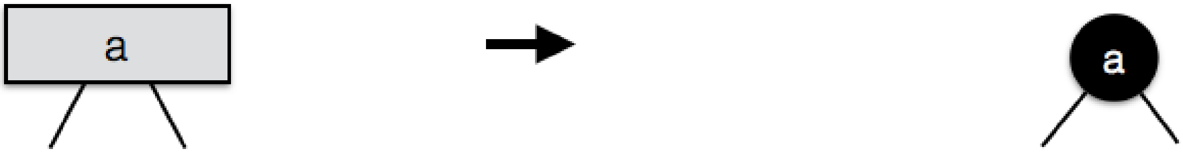
1. **Red-Black trees are binary trees**
2. **Maintain ”close enough” balance to ensure O(log n) performance**

# Red-Black trees are binary trees conceptually related to 2-3-4 trees

**Overview**

- Can think of each 2, 3, or 4 node as miniature binary tree
- "Color" each vertex so that we can tell which nodes belong together as part of a larger 2-3-4 tree node
- Paint node red if would be part of a 2-3-4 node with parent

***2-node***



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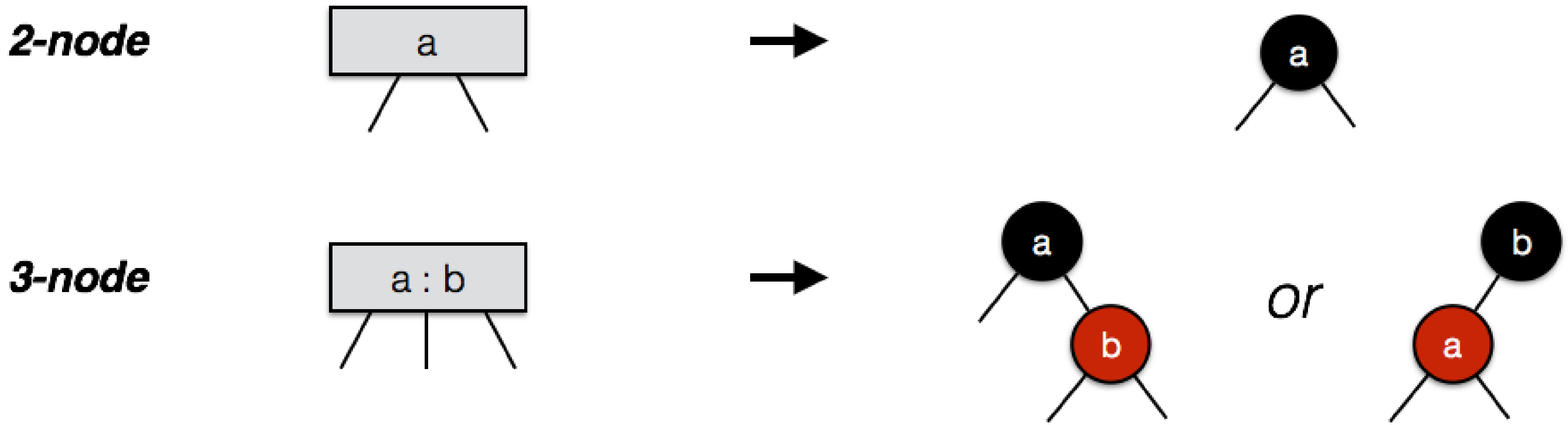


**Red node would be in the same node as black parent in a 2-3-4 Tree**

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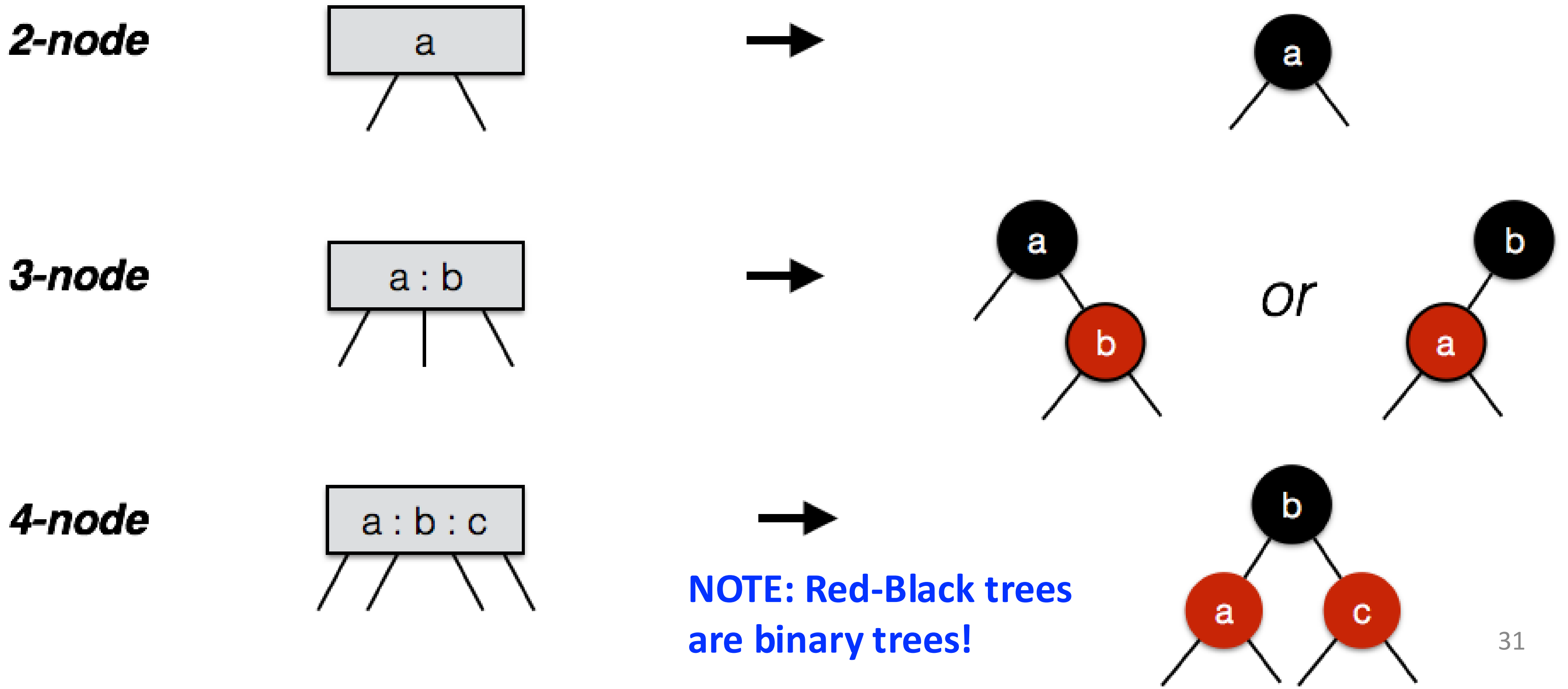
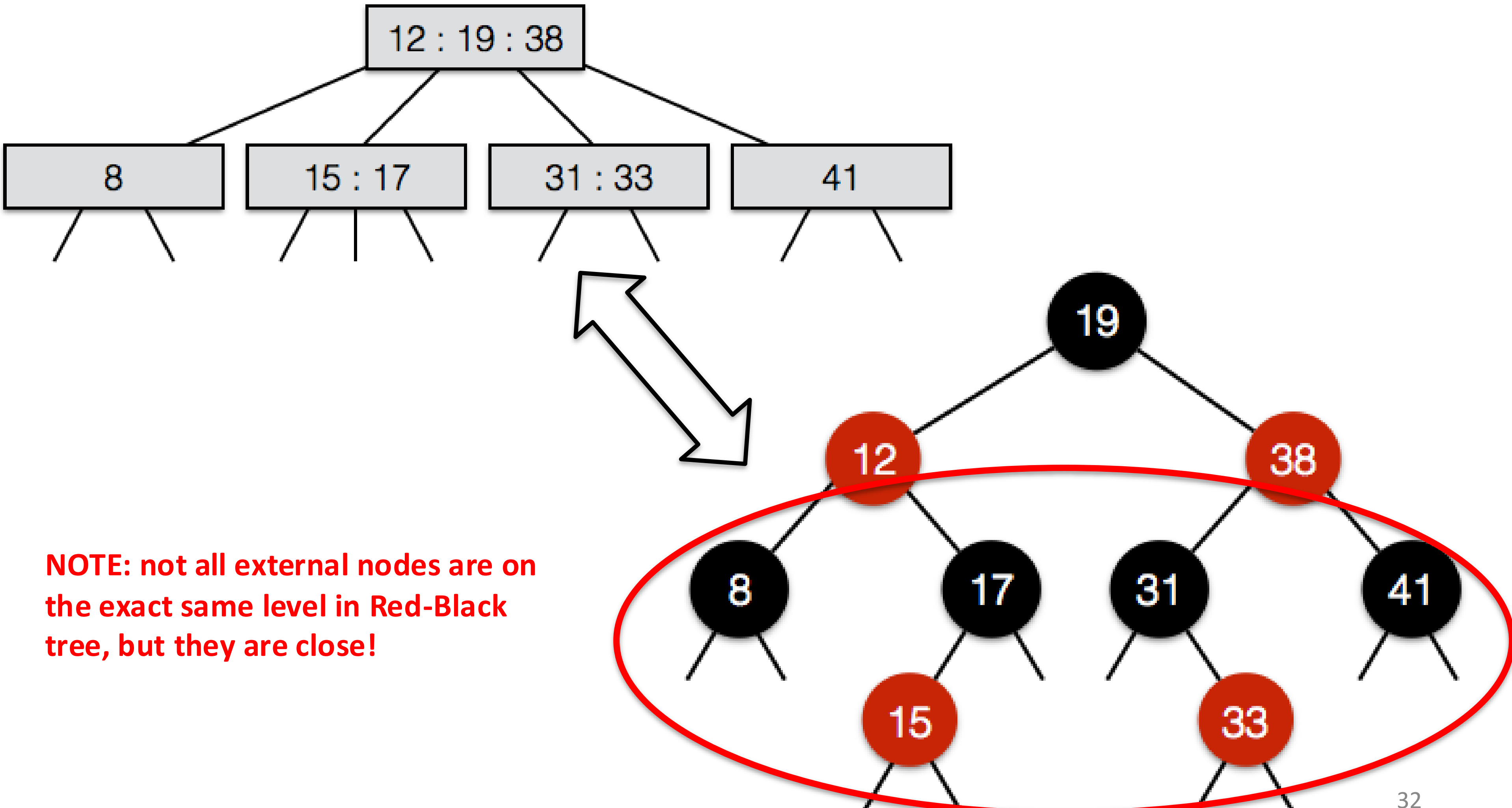



**NOTE: Red-Black trees are binary trees!**

# You can convert between 2-3-4 trees and Red-Black trees and vice versa

**Red-Black as related to 2-3-4 trees**




**NOTE: not all external nodes are on the exact same level in Red-Black tree, but they are close!**

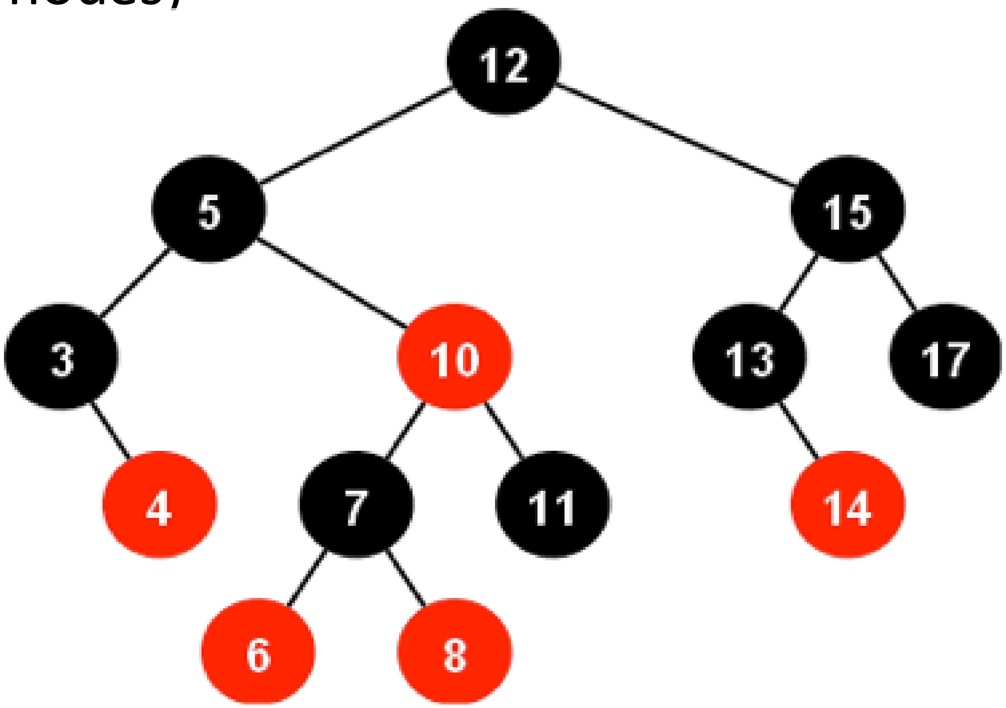
# Red-Black trees maintain four properties

**Red-Black trees properties**

1. Every nodes is either red or black
2. Root is always black, if operation changes it red, turn it black again
3. Children of a red node are black (no consecutive red nodes)
4. All external nodes have the same black depth (same number of black ancestor nodes)

**Black depth: 3**

**No node more than 3 black nodes away from root**



# Red-Black properties ensure depth of tree is $O(log_2 n)$, given *n* nodes in tree

## Informal justification

- Since every path from the root to a leaf has the same number of black nodes (by property 4), the ***shortest*** possible path would be one which has *no* red nodes in it
- Suppose $k$ is the number of black nodes along any path from the root to a leaf
- What is the ***longest*** possible path?
  - It would have alternating black and red nodes
  - Since there can't be two red nodes in a row (property 3) and root is black (property 2), the longest path given $k$ black nodes is $2k$ or $h \le 2k$, where $h$ is Tree height
- It can be shown that if *each* path from root to leaf has $k$ black nodes, there must be ***at least*** $2^k - 1$ nodes in the tree
- Since $h \le 2k$, then $k \ge h/2$, so there must be ***at least*** $2^{(h/2)} - 1$ nodes in the tree
- If there are $n$ nodes in the tree then:
  - $n \ge 2^{(h/2)} - 1$
  - Adding 1 to both sides gives: $n+1 \ge 2^{(h/2)}$
  - Taking the log (base 2) of both sides gives:
    - $\log_2(n+1) \ge h/2$
    - $2\log_2(n+1) \ge h$, which means $h$ is upper bound by $2\log_2(n+1) = O(\log_2 n)$

**Run time complexity of a search operation is O($h$) in a Binary Tree, which we just argued is O($\log_2 n$) in the worst case here**

# Searching a Red-Black Tree is O(log n)

- Red-Black tree *is* a Binary Tree with search time proportional to height

- Search time takes $O(\log_2 n)$ since $h$ is $O(\log_2 n)$

- Hard part is maintaining the tree with inserts and deletes

# Insertion into Red-Black trees must deal with several cases

**Four Red-Black Tree properties:**

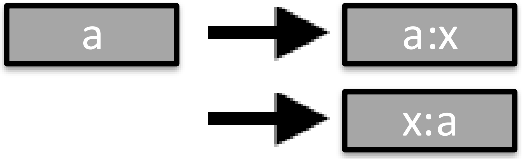
1. Every nodes is either red or black
2. Root is always black, if operation changes it red, turn it black again
3. Children of a red node are black (no consecutive red nodes)
4. All external nodes have the same black depth (same number of black ancestors)

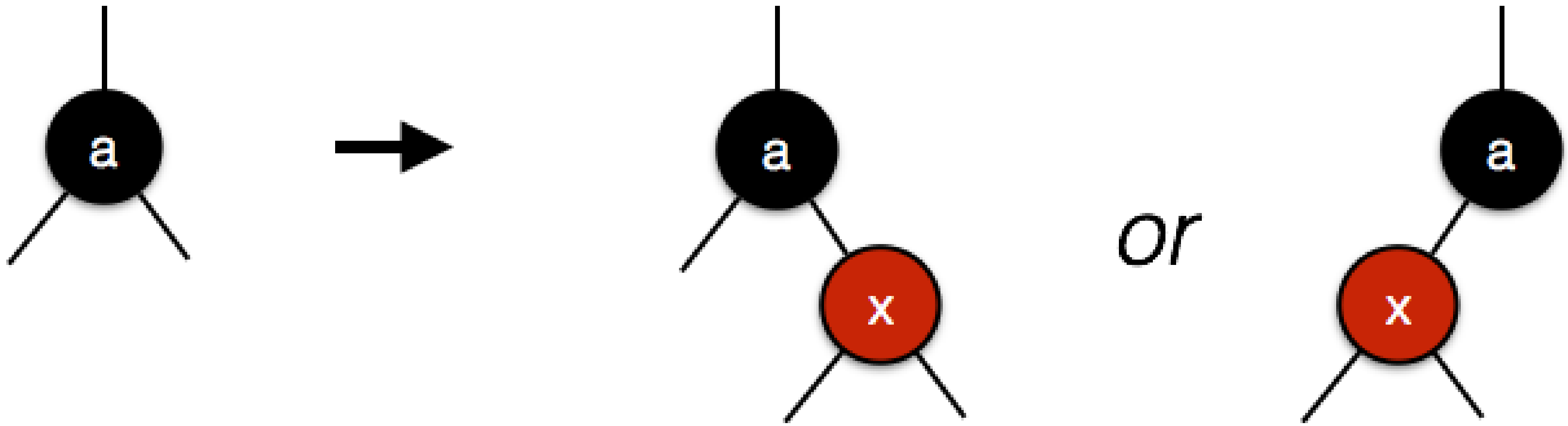
**Insert procedures**

- As with BSTs, find location in tree where new element goes and insert
- Color new node red – ensures rules 1, 2 and 4 are preserved
- Rule 3 might be violated (red node must have black children)
- Three cases can arise on insert (equivalent to 2, 3, or 4 node inserts)
- Inserting into a 2 or 4 node fairly straightforward
- 3 node is more complex

# Case 1: Insert into 2 node, no violation

**Insert into 2 node causes no violation**




Insert new node <x> as child of <a>

Color <x> red

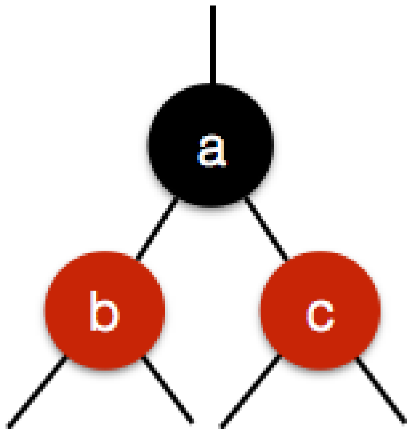
No violations

Each of these Trees are possible depending on the value of <x>

# Case 2: Insert into 4 node is a violation, resolve with "color flip"

**4 nodes are black with red children**

b:a:c




Insert new node <x> as child of <b> or <c> would cause two red nodes in a row

Violates rule 3

# Case 2: Insert into 4 node is a violation, resolve with “color flip”

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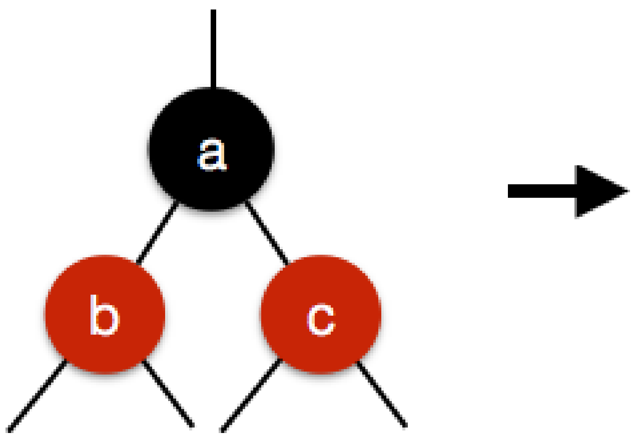


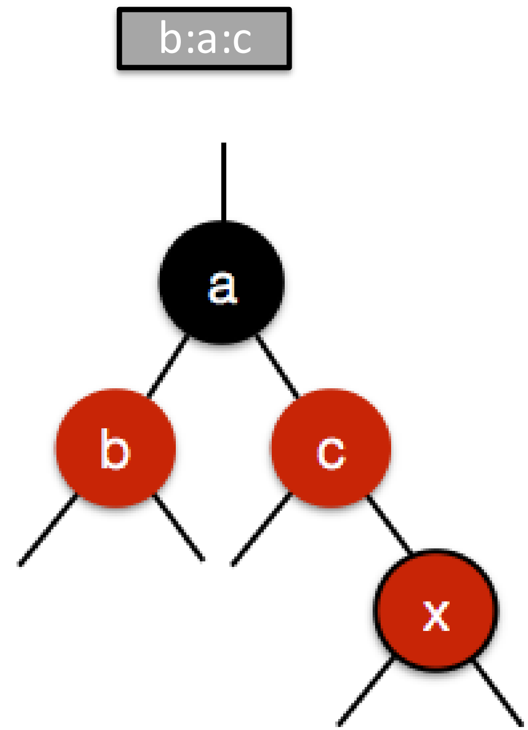


Insert new node <x> as child of <b> or <c> would cause two red nodes in a row

Violates rule 3

Must split node, promoting middle key

- Could promote <a> to parent, and unjoin <b> and <c> from <a>
- Amounts to a “color flip”

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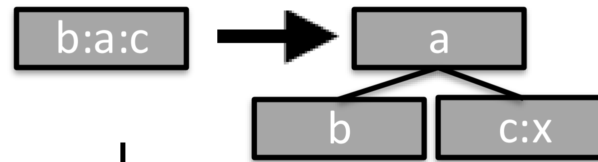


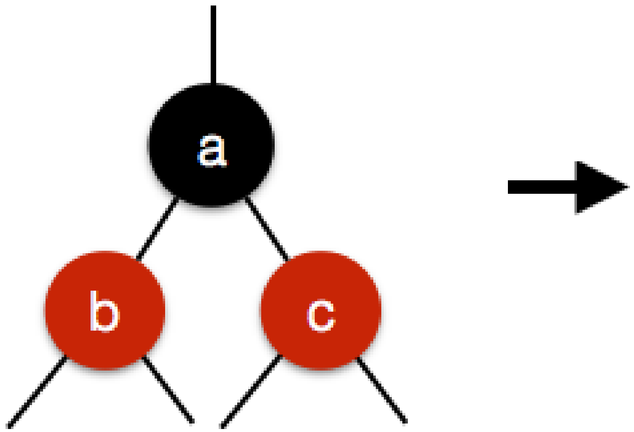


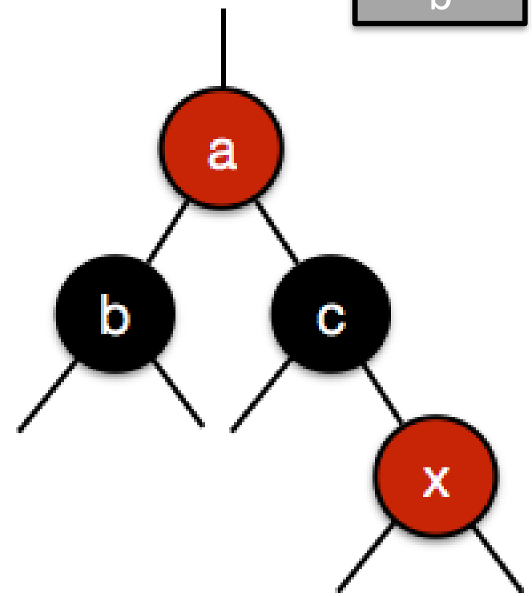


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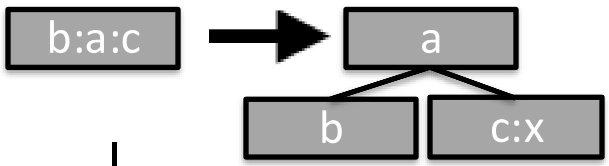
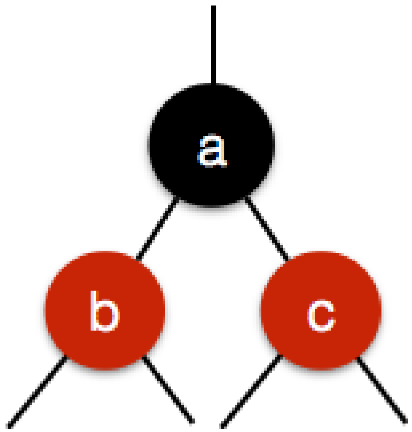
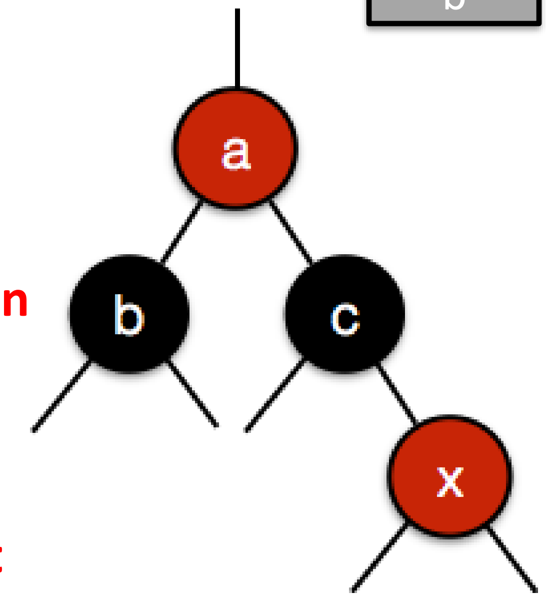



**Black depth not changed**

**Must check <a> doesn't violate parent two reds in a row**

**Might bubble up color flips to root**

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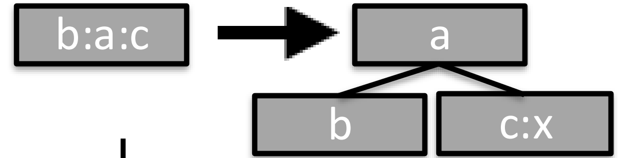


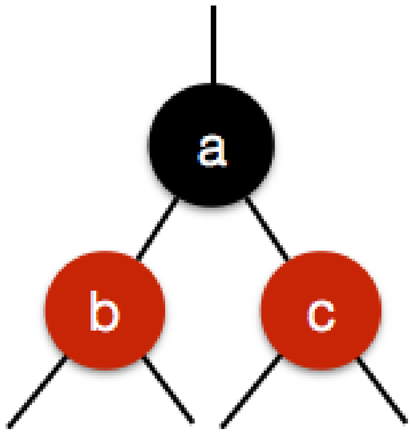


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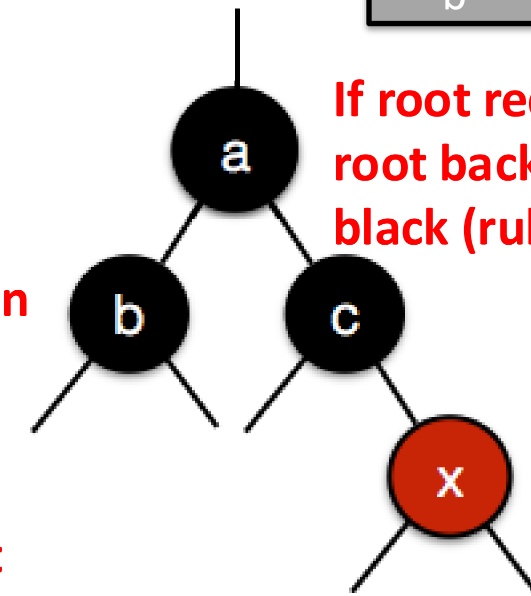



**If root red, flip root back to black (rule 2)**

Insert new node <x> as child of <b> or <c> would cause two red nodes in a row

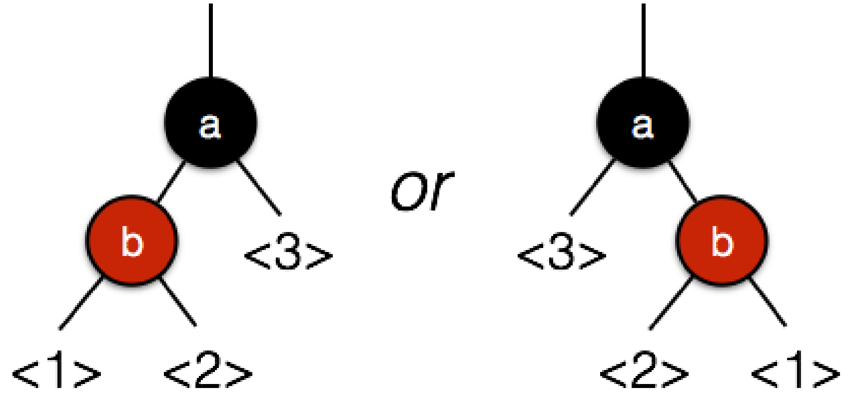
Violates rule 3

Must split node, promoting middle key

- Could promote <a> to parent, and unjoin <b> and <c> from <a>
- Amounts to a “color flip”

# Case 3: Insert into 3 node, might be violation

**3 nodes are black with one red child**




**With a 3 node there are three places where node could be added: <1>, <2>, or <3>**

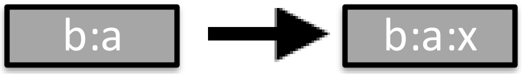
<3> is easy

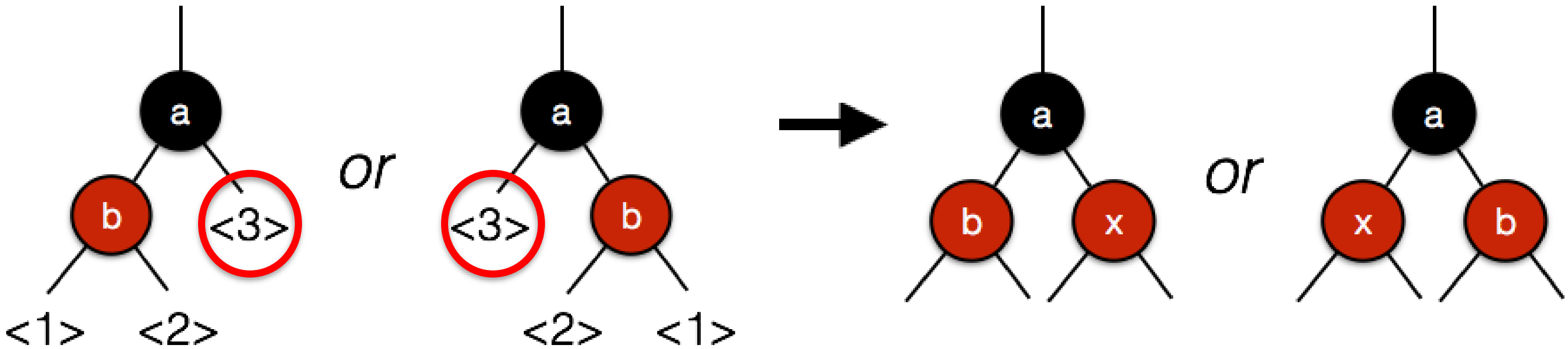
<1> involves a single rotation (2 reds in straight line)

<2> involves a double rotation (2 reds in zig-zag)

# Case 3: Inserting at position <3> is easy

**3 nodes are black with one red child**



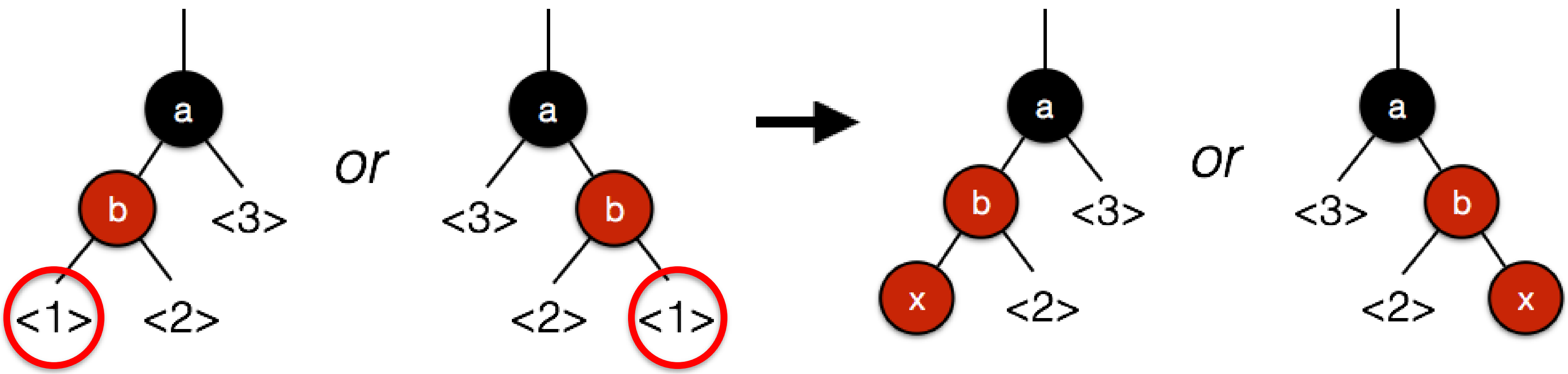

**Inserting into position <3> makes a 4 node**

- No problem if inserting at position <3>
- Makes a 4 node

# Case 3: Inserting at position <1> (two red in straight line) causes single rotation

**3 nodes are black with one red child**

b:a

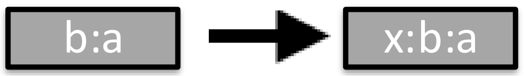


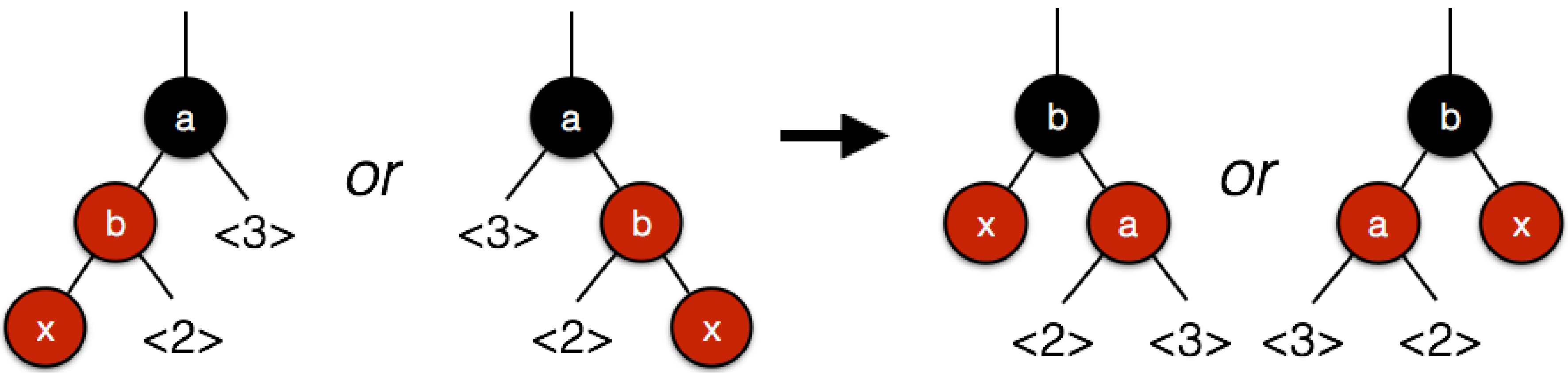

**Inserting at <1> do a single rotation**

- Violation of no two red nodes in a straight line
- Since x < b < a or x > b > a, could fix by rotating whole structure
- Lift <b> to root (color black), while dropping down <a> (color red) to be child of <b>

# Case 3: Inserting at position <1> (two red in straight line) causes single rotation

**3 nodes are black with one red child**



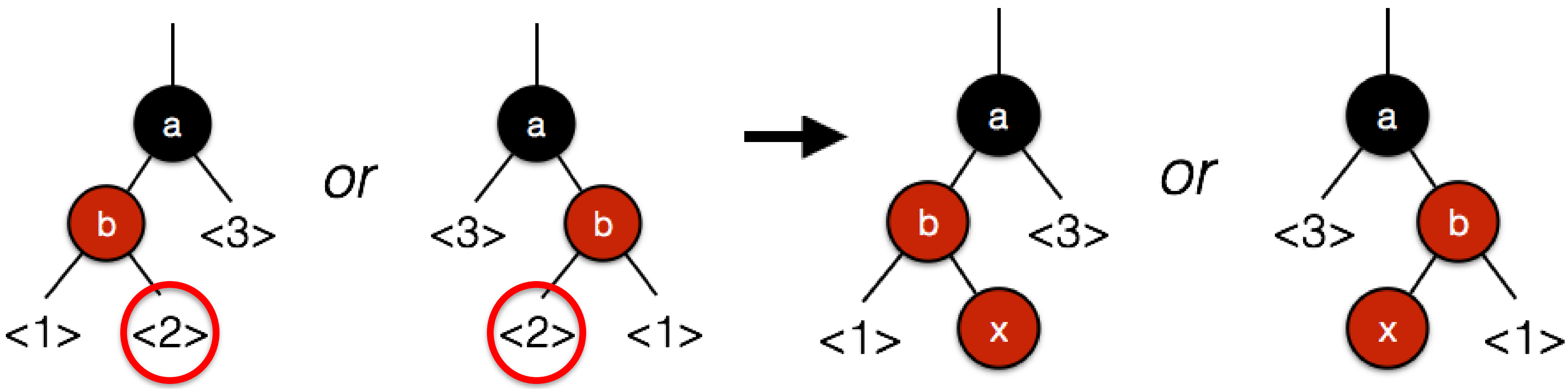

**Inserting at <1> do a single rotation**

- Violation of no two red nodes in a straight line
- Since x < b < a or x > b > a, could fix by rotating whole structure
- Lift <b> to root (color black), while dropping down <a> (color red) to be child of <b>
- Still maintains ordered property
- Called a ***single rotation***

# Case 3: Inserting at position <2> (two red in zig-zag) causes double rotation

**3 nodes are black with one red child**

b:a

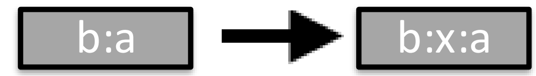


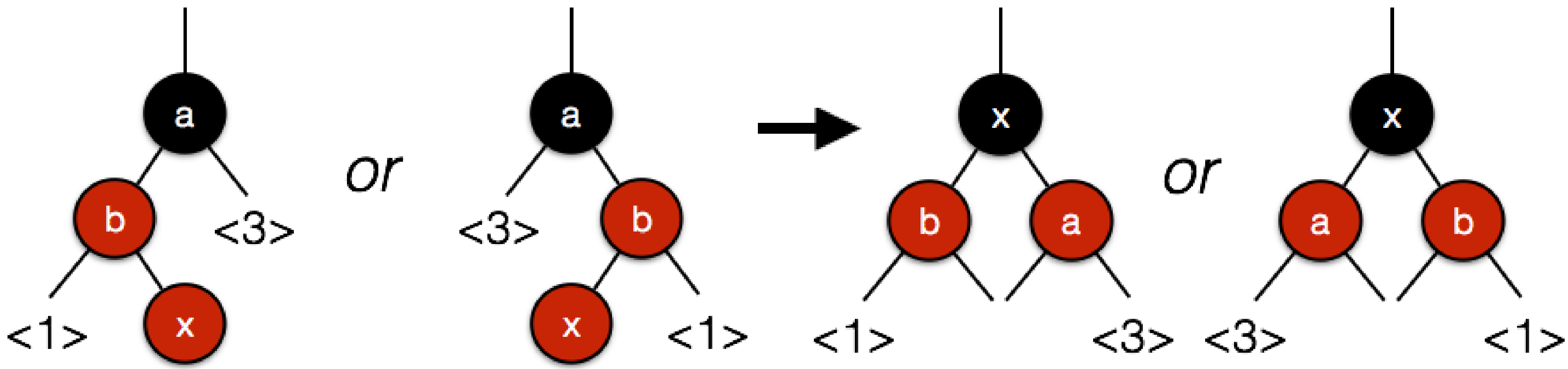

**Inserting at <2>, do double rotation**

- Two red nodes in zig-zag pattern
- Lift <x> to root (color black) and have <a> and <b> as children (colored red)
- Called a ***double rotation***

# Case 3: Inserting at position <2> (two red in zig-zag) causes double rotation

**3 nodes are black with one red child**



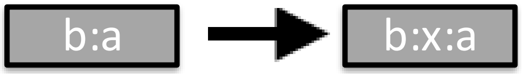



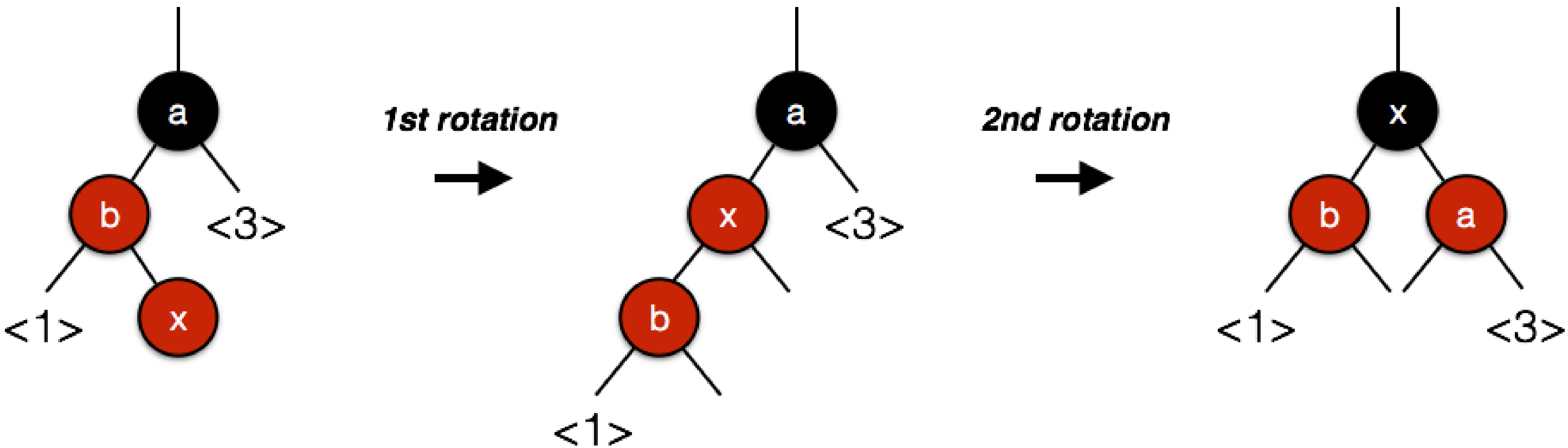

**Inserting at <2>, do double rotation**

- Two red nodes in zig-zag pattern
- Lift <x> to root (color black) and have <a> and <b> as children (colored red)
- Called a ***double rotation***

# Case 3: Inserting at position <2> (two red in zig-zag) causes double rotation

**3 nodes are black with one red child**




**Inserting at <2>, do double rotation**

- Two red nodes in zig-zag pattern
- Lift <x> to root (color black) and have <a> and <b> as children (colored red)
- Called a ***double rotation***
- Rotate once around <b>, then again around <x>

# Insert run time is O($\log_2 n$)

- Worse case we only have to fix colors along the path between new node and root, O($\log_2 n$) path length

- Each operation is constant time
  - It can be shown we only need to do *at most* one single-rotation or one double-rotation to fix the tree, O(1)
  - All other changes done with color flips, O(1)
  - But, might have to traverse up to root

**See textbook for details on delete operations**

- Leads to O($\log_2 n$) insert run-time complexity

# Summary

- Binary Search Trees performance suffers if they are unbalanced
- Two options to keep $O(\log_2 n)$ find, insert, and delete performance:
  1. **2-3-4 trees – give up on binary**
     - All leaves are at the same level, all paths the same length
     - Memory inefficient if nodes have small number of keys
     - Difficult to implement due to different node types
  2. **Red-Black trees – give up on perfectly balanced**
     - *Conceptually* think of 2-3-4 nodes as "mini trees"
     - Nodes colored to indicate they are conjoined with their parent
     - Use rotations and color flips to keep tree in approximate balance
     - Find, insert and delete take no more than $O(\log_2 n)$
     - All Map operations $O(\log_2 n)$ using Red-Black tree
     - Java uses for Red-Black Trees for TreeMap

# Key Points

1. BSTs keep data sorted in a tree structure
2. Each node in the tree has a Key and a Value
3. BSTs search by Key and return the matching Value
4. 2-3-4 trees give up on binary
5. Nodes have 2, 3, or 4 children
6. All leaves at the same level
7. Height of 2-3-4 tree $O(\log_2 n)$
8. Ensures O(log n) performance
9. Red-Black trees are binary trees
10. Maintain ”close enough” balance to ensure O(log n) performance