A radiative transfer framework for non-exponential media: Supplemental document

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1 NON-EXPONENTIAL TRANSPORT IN ARBITRARY PARTICLE DISTRIBUTIONS

The derivation in Section 2 relies on the classical radiative transport theory to define the realizations of the random process over which the macroscopic problem is averaged. This may appear to limit the generality, but in this section we start from a weaker set of assumptions to show the same fundamental relationships between transmittance and free-flight PDF for light originating from the medium and from outside the medium still hold.

We consider a realization of the medium to be a binary volume in which transport is described simply by a visibility function $V_\mu(\mathbf{x},\mathbf{y})$ which is equal to 1 when the line segment from $\mathbf{x}$ and $\mathbf{y}$ is unoccluded and 0 when it is occluded. This allows for any arrangements of particles of any shape and orientation [Jakob et al. 2010], with no limits on correlation between particles. Visibility is reciprocal: $V_\mu(\mathbf{x},\mathbf{y}) = V_\mu(\mathbf{y},\mathbf{x})$; and it is monotonic: for a point $\mathbf{z}$ on the line between $\mathbf{x}$ and $\mathbf{y}$, $V_\mu(\mathbf{x},\mathbf{y}) = V_\mu(\mathbf{x},\mathbf{z}) \wedge V_\mu(\mathbf{z},\mathbf{y})$; in particular, $V_\mu(\mathbf{x},\mathbf{y}) \implies V_\mu(\mathbf{x},\mathbf{z})$ for all $\mathbf{z}$ between $\mathbf{x}$ and $\mathbf{y}$.

The expected value of V_μ over the ensemble of random volumes is the unconditional probability of visibility from one point to another, called the *simple transmittance* or just transmittance, and denoted ff:

$$\mathrm{ff}(\mathbf{x},\mathbf{y}) = \Pr\{V_\mu(\mathbf{x},\mathbf{y}) = 1\} = \mathrm{E}\{V_\mu(\mathbf{x},\mathbf{y})\}.$$

ff inherits reciprocity and monotonicity from V_μ: $\mathrm{ff}(\mathbf{x},\mathbf{y}) = \mathrm{ff}(\mathbf{y},\mathbf{x})$ and $\mathrm{ff}(\mathbf{x},\mathbf{z}) \geq \mathrm{ff}(\mathbf{x},\mathbf{y})$ for all $\mathbf{z}$ between $\mathbf{x}$ and $\mathbf{y}$. In order to exclude some awkward cases we assume $\mathrm{ff}(\mathbf{x},\mathbf{x}) = 1$ and that ff is continuous. (This excludes infinitely thin opaque surfaces, which we consider separately in section 2, and also media that have a non-negligible packing rate.) Transmittance is appropriate for computing how much light makes it from a surface source to a surface detector through the medium: the surfaces are the same in all realizations.

To make later definitions simple we define a restriction of ff to a ray:

$$\mathrm{ff}_{\mathbf{x},\omega}(s) = \mathrm{ff}(\mathbf{x},\mathbf{x}+s\omega) \tag{54}$$

The cross-section, or attenuation coefficient, is the probability per unit length of hitting the medium. In terms of $\mathrm{ff}_{\mathbf{x},\omega}$, it is

$$\sigma(\mathbf{x},\omega) = \lim_{s\to 0} \frac{\Pr\{\neg V_\mu(\mathbf{x},\mathbf{x}+s\omega)\}}{s} \tag{55}$$

$$= \lim_{s\to 0} \frac{1-\mathrm{ff}_{\mathbf{x},\omega}(s)}{s} \tag{56}$$

$$= -\mathrm{ff}'_{\mathbf{x},\omega}(0) \tag{57}$$

To simplify notation we write $\mathrm{ff}'(\mathbf{x},\omega)$ for $\mathrm{ff}'_{\mathbf{x},\omega}(0)$.

The free-flight distribution $\mathrm{fp}(\mathbf{x},\mathbf{y})$ is a probability density along rays that describes the distribution of the first intersection with the volume for light starting at $\mathbf{x}$ and heading towards $\mathbf{y}$. First intersecting at $\mathbf{y}$ involves two events happening together: $\mathbf{y}$ is visible from $\mathbf{x}$, and a point infinitesimally beyond $\mathbf{y}$ is not visible from $\mathbf{x}$. This lets us compute the probability, as a density along the line joining $\mathbf{x}$ and $\mathbf{y}$:

$$\mathrm{fp}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \frac{\Pr\{V_\mu(\mathbf{x},\mathbf{y}) \wedge \neg V_\mu(\mathbf{x},\mathbf{y}+s\overrightarrow{\mathbf{xy}})\}}{s} \tag{58}$$

$$= \lim_{s\to 0} \frac{\mathrm{E}\{V_\mu(\mathbf{x},\mathbf{y})(1-V_\mu(\mathbf{x},\mathbf{y}+s\overrightarrow{\mathbf{xy}}))\}}{s} \tag{59}$$

$$= \lim_{s\to 0} \frac{\mathrm{E}\{V_\mu(\mathbf{x},\mathbf{y})-V_\mu(\mathbf{x},\mathbf{y}+s\overrightarrow{\mathbf{xy}})\}}{s} \tag{60}$$

$$= \lim_{s\to 0} \frac{\mathrm{ff}(\mathbf{x},\mathbf{y})-\mathrm{ff}(\mathbf{x},\mathbf{y}+s\overrightarrow{\mathbf{xy}})}{s} \tag{61}$$

$$= -\mathrm{ff}'_{\mathbf{x},\overrightarrow{\mathbf{xy}}}(|\mathbf{y}-\mathbf{x}|) \tag{62}$$

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This says that the probability per unit length of stopping at $\mathbf{y}$ is equal to the rate of decrease of transmittance along the ray from $\mathbf{x}$ towards $\mathbf{y}$. This derivative of $\mathrm{ff}_{\mathbf{x},\overrightarrow{\mathbf{xy}}}(s)$ with respect to s can also be seen as a directional partial derivative of $\mathrm{ff}(\mathbf{x},\mathbf{y})$ with respect to $\mathbf{y}$.

We define the notation

$$\mathrm{ff}'(\mathbf{x},\mathbf{y}) = \mathrm{ff}'_{\mathbf{x},\overrightarrow{\mathbf{xy}}}(|\mathbf{x}-\mathbf{y}|)$$

to simplify writing these derivatives along the line joining two segments. Note that even though $\mathrm{ff}(\mathbf{x},\mathbf{y}) = \mathrm{ff}(\mathbf{y},\mathbf{x})$, in general $\mathrm{ff}'(\mathbf{x},\mathbf{y}) \neq \mathrm{ff}'(\mathbf{y},\mathbf{x})$.

Light that is emitted by or scattered by the volume itself experiences a different transmittance and a different free flight distribution. The *volume-source transmittance* $\mathrm{pf}(\mathbf{x},\mathbf{y})$ is the fraction of light emitted by the volume at $\mathbf{x}$ towards $\mathbf{y}$ that reaches $\mathbf{y}$. In a random medium the volume-source transmittance from $\mathbf{x}$ to $\mathbf{y}$ is the probability that $\mathbf{y}$ is visible from $\mathbf{x}$ *conditioned on the presence of a particle at* $\mathbf{x}$. Similarly, the *volume-source free flight distribution* $\mathrm{pp}(\mathbf{x},\mathbf{y})$ is the length distribution of first intersections with the volume, also conditioned on an intersection at $\mathbf{x}$.

Because of the assumption of low packing rate, we can't condition literally on a particle *at* $\mathbf{x}$; rather we look for an intersection in an infinitesimal interval beyond $\mathbf{x}$, just as when defining σ.

We can calculate pf using the probability of visibility from $\mathbf{x}$ to $\mathbf{y}$ conditioned on having an intersection in a short interval before $\mathbf{x}$, in the limit as the collision interval shrinks. Using the familiar rule for conditional probability $\Pr\{P|Q\} = \Pr\{P \wedge Q\}/\Pr\{Q\}$,

$$\mathrm{pf}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \Pr\{V_\mu(\mathbf{x},\mathbf{y}) \mid \neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{x})\} \tag{63}$$

$$\mathrm{pf}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \frac{\Pr\{V_\mu(\mathbf{x},\mathbf{y}) \wedge \neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{x})\}}{\Pr\{\neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{x})\}} \tag{64}$$

$$\mathrm{pf}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \frac{\Pr\{V_\mu(\mathbf{x},\mathbf{y}) \wedge \neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{y})\}}{\Pr\{\neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{x})\}} \tag{65}$$

$$\mathrm{pf}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \frac{(\mathrm{ff}(\mathbf{x},\mathbf{y}) - \mathrm{ff}(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{y}))/s}{\Pr\{\neg V_\mu(\mathbf{x}+s\overrightarrow{\mathbf{yx}},\mathbf{x})\}/s} \tag{66}$$

$$\mathrm{pf}(\mathbf{x},\mathbf{y}) = \frac{\lim_{s\to 0}(\mathrm{ff}(\mathbf{y},\mathbf{x}) - \mathrm{ff}(\mathbf{y},\mathbf{x}+s\overrightarrow{\mathbf{yx}}))/s}{\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}})} \tag{67}$$

$$= \frac{-\mathrm{ff}'(\mathbf{y},\mathbf{x})}{\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}})} \tag{68}$$

Since $\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}})$ is the limiting value of $-\mathrm{ff}'(\mathbf{y},\mathbf{x})$ as $\mathbf{y}$ approaches $\mathbf{x}$, this normalization makes pf into a proper transmittance that starts at 1 and decreases to 0 with distance.

Since they are both derivatives of ff, it is clear that fp and pf are related by the reciprocity relation:

$$\mathrm{pf}(\mathbf{x},\mathbf{y})\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}}) = -\mathrm{ff}'(\mathbf{y},\mathbf{x}) = \mathrm{fp}(\mathbf{y},\mathbf{x})$$

Finally, we can calculate $\mathrm{pp}(\mathbf{x},\mathbf{y})$ from pf in the same way we derived fp from ff: the derivative along the ray of transmittance gives the corresponding free-flight PDF:

$$\mathrm{pp}(\mathbf{x},\mathbf{y}) = \lim_{s\to 0} \frac{\mathrm{pf}(\mathbf{x},\mathbf{y}) - \mathrm{pf}(\mathbf{x},\mathbf{y}+s\overrightarrow{\mathbf{xy}})}{s} \tag{69}$$

$$= -\frac{\mathrm{pf}'_{\mathbf{x},\overrightarrow{\mathbf{xy}}}(|\mathbf{y}-\mathbf{x}|)}{\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}})} \tag{70}$$

$$= \frac{\mathrm{ff}''(\mathbf{x},\mathbf{y})}{\sigma(\mathbf{x},\overrightarrow{\mathbf{yx}})} \tag{71}$$

Since pf is already the derivative of ff along the ray with respect to $\mathbf{x}$, pp is a mixed second derivative of ff with respect to both $\mathbf{x}$ and $\mathbf{y}$, denoted $\mathrm{ff}''(\mathbf{x},\mathbf{y})$ on the last line above.

The conclusion of this section is that the relationships between ff, fp, pf, and pp, which were established in §3 using an ensemble average over random heterogeneous classical media, also hold in the more general case of arbitrary assemblies of correlated particles.

2 PROOF OF TRANSMITTANCE/FREE-FLIGHT PDF RELATIONSHIP

We can relate ff and fp as follows:

$$\begin{aligned}
\mathrm{ff}(\mathbf{x},\mathbf{x}_t) &= \left\langle \mathrm{Tr}_\mu(\mathbf{x},\mathbf{x}_t) \right\rangle && (72)\\
&= \left\langle \int_t^\infty p_\mu(\mathbf{x},\mathbf{x}_s)\,\mathrm{d}s \right\rangle \quad \text{from Equation (3)} && (73)\\
&= \int_t^\infty \left\langle p_\mu(\mathbf{x},\mathbf{x}_s) \right\rangle \mathrm{d}s \quad \text{from linearity of } \langle\cdot\rangle && (74)\\
&= \int_t^\infty \mathrm{fp}(\mathbf{x},\mathbf{x}_s)\,\mathrm{d}s \quad \text{from Equation (15b).} && (75)
\end{aligned}$$

The proof for pf and pp follows analogously.

3 SIMPLIFICATION OF $\mathrm{pp}(\mathbf{x},\mathbf{y})/\sigma(\mathbf{y})$

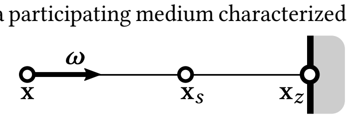
We simplify as follows:

$$\frac{\mathrm{pp}(\mathbf{x},\mathbf{y})}{\sigma(\mathbf{y})} = \frac{\left\langle \rho_\mu(\mathbf{x})\mathrm{Tr}_\mu(\mathbf{x},\mathbf{y})\sigma_\mu(\mathbf{y}) \right\rangle}{\sigma(\mathbf{y})} = \left\langle \rho_\mu(\mathbf{x})\mathrm{Tr}_\mu(\mathbf{x},\mathbf{y})\rho_\mu(\mathbf{y}) \right\rangle \tag{76}$$

We can interpret this as an ensemble averaged transmittance, conditioned on both $\mathbf{x}$ and $\mathbf{y}$ coinciding with a scatterer.

4 NAIVE NON-EXPONENTIALITY IS NOT ENERGY CONSERVING

We show a simple proof to demonstrate that simply replacing the exponential transmittance ($e^{-\tau}$) in the classical path integral with a non-exponential function $\mathrm{Tr}(\tau)$ must violate energy conservation. We consider a scene filled with a participating medium characterized by its extinction coefficient $\sigma(\mathbf{x})$ and albedo $\alpha(\mathbf{x})$, with the derived scattering coefficient $\sigma_s(\mathbf{x}) = \alpha(\mathbf{x})\sigma(\mathbf{x})$. We consider a pencil beam of light starting at $\mathbf{x}$ traveling in direction $\boldsymbol{\omega}$ that intersects a surface at point $\mathbf{x}_z$.



The radiance emitted at the beam origin is $L_0(\mathbf{x},\boldsymbol{\omega})$, and the radiance received by points on the beam decreases with distance as a result of extinction by the medium. The emitted radiance must then be distributed between two terms: the amount of light that scatters in the medium, and the remaining fraction that reaches the surface. Assuming a unit albedo, the sum of both terms must equal

$L_0(\mathbf{x}, \boldsymbol{\omega})$—otherwise, energy is either lost or gained along the way. This results in the following constraint:

$$L_0(\mathbf{x}, \boldsymbol{\omega}) = L_0(\mathbf{x}, \boldsymbol{\omega})\mathrm{Tr}(\tau(\mathbf{x}, \mathbf{x}_z)) + \int_0^z L_0(\mathbf{x}, \boldsymbol{\omega})\mathrm{Tr}(\tau(\mathbf{x}, \mathbf{x}_s))\sigma_s(\mathbf{x}_s)\,\mathrm{d}s, \tag{77}$$

The above equation can be simplified by dividing out $L_0(\mathbf{x}, \boldsymbol{\omega})$:

$$1 = \mathrm{Tr}(\tau(\mathbf{x}, \mathbf{x}_z)) + \int_0^z \mathrm{Tr}(\tau(\mathbf{x}, \mathbf{x}_s))\sigma_s(\mathbf{x}_s)\,\mathrm{d}s. \tag{78}$$

Using the fact that $\mathrm{d}\tau/\mathrm{d}s = \sigma(\mathbf{x}_s)$, and $\sigma = \sigma_s$ when $\alpha = 1$, we can perform a change of variable from $\mathrm{d}s$ to $\mathrm{d}\tau$ to obtain

$$1 = \mathrm{Tr}(\tau(\mathbf{x}, \mathbf{x}_z)) + \int_0^{\tau(\mathbf{x}, \mathbf{x}_z)} \mathrm{Tr}(\tau)\,\mathrm{d}\tau\,. \tag{79}$$

Any transmittance function Tr that does not satisfy the above constraint violates energy conservation and must inevitably lead to energy loss or gain when inserted into the classical path integral.

Equation (79) is an ordinary differential equation of the form $1 = f'(x) + f(x) - f(0)$. The only solutions that satisfy it are expressed by $\mathrm{Tr}(\tau) = c \cdot \mathrm{e}^{-\tau}$. In other words, *only an exponential transmittance* satisfies energy conservation in the classical path integral. Violation of energy conservation is not only a practical problem, but also means that the underlying process is non-physical. This means that the naive solution of substituting a non-exponential function for the transmittance in the classical path integral cannot work, and a more principled approach is needed.

5 VOLUMETRIC EMITTERS AND SENSORS

In the main paper, we have assumed that emission and measurements only occur on surfaces and therefore do not depend on the realization. This was done merely for conciseness, and we show in this section how to derive the general case of volumetric emitters and sensors. Our final path throughput is identical to the one of the main paper, with minor changes to the path integral.

We define a volumetric emitter (sensor) such that the emitted radiance (importance) is proportional to the local density of the medium. This means that the emission term becomes the product $L_e(\mathbf{x}_0, \mathbf{x}_1)\Sigma_\mu(\mathbf{x}_0)$, and similarly for importance. This results in a new path integral

$$I_\mu = \int_{\mathcal{P}} L_e(\mathbf{x}_0, \mathbf{x}_1)\Sigma(\mathbf{x}_0) g_\mu(\overline{\mathbf{x}}) W(\mathbf{x}_{k-1}, \mathbf{x}_k)\,\mathrm{d}\overline{\mathbf{x}} \tag{80}$$

$$g_\mu(\overline{\mathbf{x}}) = \frac{\Sigma_\mu(\mathbf{x}_0)}{\Sigma(\mathbf{x}_0)} \left[\prod_{i=1}^{k-1} f_\mu(\mathbf{x}_i)\Sigma_\mu(\mathbf{x}_i)\right] \left[\prod_{i=0}^{k-1} \mathrm{Tr}_\mu(\mathbf{x}_i, \mathbf{x}_{i+1}) G(\mathbf{x}_i, \mathbf{x}_{i+1})\right] \Sigma_\mu(\mathbf{x}_k) \tag{81}$$

where we have performed the transformation $\Sigma_\mu(\mathbf{x}_0) = \Sigma(\mathbf{x}_0) \cdot \Sigma_\mu(\mathbf{x}_0)/\Sigma(\mathbf{x}_0)$ and moved one of the Σ into the path integral and all other Σ terms directly into the path throughput.

Following the derivations of the main paper, we assume that phase function and albedo are independent of the medium and rearrange the dependent terms:

$$\langle I_\mu \rangle = \int_{\mathcal{P}} L_e(\mathbf{x}_0, \mathbf{x}_1)\Sigma(\mathbf{x}_0) \left\langle g_\mu(\overline{\mathbf{x}}) \right\rangle W(\mathbf{x}_{k-1}, \mathbf{x}_k)\,\mathrm{d}\overline{\mathbf{x}} \tag{82}$$

$$\left\langle g_\mu(\overline{\mathbf{x}}) \right\rangle \approx \prod_{i=1}^{k-1} f(\mathbf{x}_i) \prod_{i=0}^{k-1} G(\mathbf{x}_i, \mathbf{x}_{i+1}) \left\langle \frac{\Sigma_\mu(\mathbf{x}_0)}{\Sigma(\mathbf{x}_0)} \prod_{i=0}^{k-1} \mathrm{Tr}_\mu(\mathbf{x}_i, \mathbf{x}_{i+1})\Sigma_\mu(\mathbf{x}_{i+1}) \right\rangle. \tag{83}$$

Finally, we assume decorrelated path segments and obtain the approximation

$$\left\langle \frac{\Sigma_\mu(\mathbf{x}_0)}{\Sigma(\mathbf{x}_0)} \prod_{i=0}^{k-1} \mathrm{Tr}_\mu(\mathbf{x}_i, \mathbf{x}_{i+1})\Sigma_\mu(\mathbf{x}_{i+1}) \right\rangle \approx \prod_{i=0}^{k-1} \left\langle T_\mu(\mathbf{x}_i, \mathbf{x}_{i+1}) \right\rangle. \tag{84}$$

We claim in this equation that decorrelating path segments in this more general derivation results in the same right-hand side as in the main paper. There are two cases to consider: For the last segment on the path $(\mathbf{x}_{k-1}, \mathbf{x}_k)$, we have already introduced a $\Sigma(\mathbf{x}_k)$ term in the main paper. This was allowed because this term is 1 if $\mathbf{x}_k$ is on a surface, which we assumed in the main paper. In this more general derivation, $\Sigma(\mathbf{x}_k)$ was present from the start; it selects between the transmittance and the free-flight PDF on the camera segment. This selection is already present in the transport kernel from the main paper, and the term for the camera segment does not change.

For the first segment on the path $(\mathbf{x}_0, \mathbf{x}_1)$, we are presented with an additional $\Sigma_\mu(\mathbf{x}_0)/\Sigma(\mathbf{x}_0)$ term in the more general derivation. However, notice that this term is 1 if $\mathbf{x}_0$ is on a surface, and $\rho_\mu(\mathbf{x}_0)$ if $\mathbf{x}_0$ is in the medium. In other words, a volume emitter simply uses a correlated ensemble average, compared to the uncorrelated ensemble average of a surface emitter. However, the transport kernel introduced in the main paper already does that.

In other words, the path throughput presented in the paper does not need to change if we wish to support volumetric emitters or sensors. The only change that is required is to add a $\Sigma(\mathbf{x}_0)$ term to the path integral as in Equation (80).

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