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Fourier Analysis of Correlated Monte Carlo Importance Sampling: Supplementary document

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Contents

1. Overview

In this supplemental, we provide all essential mathematical derivations† that are required to perform our analytical analysis in the main paper. Using Fourier series, we first derive the sampling-based integrator in Sec. 2, the expected error (bias) due to these sampling-based integrators in Sec. 3 and the corresponding variance in Sec. 4. We show that the third (covariance) term is a real entity (Sec. 4.1) and expand it to relate variance with a pair correlation function (Sec. 4.2). We also derive analytic expressions for the expected power spectra for both random Sec. 5 and jittered samples Sec. 6. We further expand these expected power spectra for any arbitrary PDF in the case of random samples. The expectation term $\langle \mathbf{S}_m \mathbf{S}_l^* \rangle$ is also derived analytically both for random and jittered samples. In each section, all important results are boxed for better exposition. Later in Sec. 8, we show that our Fourier series based variance formulation respects conventional (although impractical) cases of perfect importance sampling for which the resultant variance is zero. We conclude by showing some preliminary results on improving variance/convergence by simply shifting the strata boundaries over the sampling domain.

2. Sampling-based integrator

We expand the estimator $\mu_N = \int_0^L f(x)\, S(x)\, \mathrm{d}x$ (where $L \in \mathbb{R}^+$) using the Fourier series as follows:

$$\mu_N = \int_0^L \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{f}_m e^{i\frac{2\pi m x}{L}} \mathbf{S}_l e^{i\frac{2\pi l x}{L}}\, \mathrm{d}x \tag{1}$$

$$\mu_N = \int_0^L \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{f}_m \mathbf{S}_l e^{i\frac{2\pi(m+l)x}{L}}\, \mathrm{d}x \tag{2}$$

Substituting $q = m + l$, we obtain:

$$\mu_N = \int_0^L \sum_{q=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \mathbf{f}_{q-m} \mathbf{S}_m e^{i\frac{2\pi q x}{L}}\, \mathrm{d}x \tag{3}$$

The inner summation represents a convolution operation ($\otimes$) on a discrete domain:

$$\mu_N = \int_0^L \sum_{q=-\infty}^{\infty} \mathbf{f} \otimes \mathbf{S}_q e^{i\frac{2\pi q x}{L}}\, \mathrm{d}x \text{ where, } \mathbf{f} \otimes \mathbf{S}_q = \sum_{m=-\infty}^{\infty} \mathbf{f}_{q-m} \mathbf{S}_m \tag{4}$$

Since we integrate over a unit period ($L = 1$), the integral is non-zero only at the DC ($q = 0$) frequency. This allows us to rewrite the estimator as:

$$\mu_N = \int_0^1 \mathbf{f} \otimes \mathbf{S}_0 e^{i2\pi 0 x}\, \mathrm{d}x = \mathbf{f} \otimes \mathbf{S}_0 \int_0^1 1\, \mathrm{d}x = \mathbf{f} \otimes \mathbf{S}_0 \tag{5}$$

Using the convolution definition from (4) and the fact that with real functions we can replace $\mathbf{f}_{-m}$ with it's conjugate $\mathbf{f}_m^*$, we get:

$$\boxed{\mu_N = \sum_{m=-\infty}^{\infty} \mathbf{f}_m^* \mathbf{S}_m} \tag{6}$$

3. Bias for sampling-based integrator

Following the work by Subr and Kautz [SK13], we rederive the bias $\langle \Delta \rangle := I - \langle \mu_N \rangle$ term as a function of the Fourier series representation that respects the finite sampling domain. Using the definition of the MC estimator derived in appendix A.1 of the main paper, we write the bias

† The derivations are done for a non-zero PDF distributions. To consider PDFs that can go to zero, the same results can be obtained by combining the $p(x)$ term with $f(x)$ instead of using it as weights in the sampling function $S(x)$.

term in the following form (where $\mathbf{f}_0^* = \mathbf{f}_0 = I$):

$$\langle \Delta \rangle = \mathbf{f}_0^* - \sum_{m=-\infty}^{\infty} \mathbf{f}_m^* \langle \mathbf{S}_m \rangle , \tag{7}$$

$$\langle \Delta \rangle = \mathbf{f}_0^* - \left(\mathbf{f}_0^* \langle \mathbf{S}_0 \rangle + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{f}_m^* \langle \mathbf{S}_m \rangle \right) . \tag{8}$$

This can be rearranged into the following form:

$$\boxed{\langle \Delta \rangle = \mathbf{f}_0^* (1 - \langle \mathbf{S}_0 \rangle) - \left(\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{f}_m^* \langle \mathbf{S}_m \rangle \right)} , \tag{9}$$

Subr and Kautz mentioned that an unbiased estimator can be obtained if $\langle \mathbf{S}_m \rangle = \delta(m)$. In the next section, we show that this condition can be achieved if the sampling weights $\alpha(x) = 1/p(x)$,i.e. are equal to the inverse of the pdf from which the samples are derived.

3.1. Expectation of sampling Fourier coefficients

The Fourier transform of the sampling function $S(x) = \frac{1}{N}\sum_{k=1}^{N} \alpha_k \delta(x - \mathbf{x}_k)$ has the form:

$$\mathbf{S}_m = \int_0^1 \frac{1}{N} \sum_{k=1}^{N} \frac{e^{-i2\pi m x}}{p(x)} \delta(x - \mathbf{x}_k) \mathrm{d}x = \frac{1}{N} \sum_{k=1}^{N} \alpha_k e^{-i2\pi m \mathbf{x}_k} \tag{10}$$

For sampling weights $\alpha(x) = 1/p(x)$, the expected Fourier spectrum can be written as:

$$\langle \mathbf{S}_m \rangle = \left\langle \frac{1}{N} \sum_{k=1}^{N} \alpha_k e^{-i2\pi m \mathbf{x}_k} \right\rangle = \left\langle \frac{1}{N} \sum_{k=1}^{N} \frac{e^{-i2\pi m \mathbf{x}_k}}{p(\mathbf{x}_k)} \right\rangle \tag{11}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \left\langle \frac{e^{-i2\pi m \mathbf{x}_k}}{p(\mathbf{x}_k)} \right\rangle \tag{12}$$

$$= \frac{1}{N} \sum_{k=1}^{N} \int_0^1 \frac{e^{-i2\pi m x}}{p(x)} p(x) \mathrm{d}x \tag{13}$$

$$= \int_0^1 e^{-i2\pi m x} \mathrm{d}x = 0 \; \forall m \neq 0 \tag{14}$$

This show that the expectation of the sampling Fourier coefficients is non-zero only at the DC ($m = 0$) frequency as long as the samples are weighted by $\alpha(x) = 1/p(x)$.

4. Variance of sampling-based integrator

The variance due to Monte Carlo estimation can be written in the Fourier domain thanks to the early work by Durand [Dur11], Subr and Kautz [SNJ*14], and Pilleboue et al. [PSC*15]. However, these formulations does not properly consider the finite sampling domain. In this section, we derive step-by-step the variance formulation using Fourier series into a general form that covers both correlated and Importance sampling on top of uniform sample distributions. We can start by applying the variance operator to the Monte Carlo estimator:

$$\mathrm{Var}(\mu_N) = \mathrm{Var}\left(\sum_{m=-\infty}^{\infty} \mathbf{f}_m^* \mathbf{S}_m \right) \tag{15}$$

From first principles, variance applied to a sum of complex numbers:

$$\mathrm{Var}\left(\sum_{i=1}^{N} a_i X_i \right) = \sum_{i=1}^{N} ||a_i||^2 \mathrm{Var}(X_i) + \sum_{i\neq j}^{N} a_i {a_j}^* Cov\left(X_i, {X_j}^*\right) , \tag{16}$$

yields

$$\mathrm{Var}(\mu_N) = \sum_{m=-\infty}^{\infty} \mathbf{f}_m^* \mathbf{f}_m \mathrm{Var}(\mathbf{S}_m) + \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ l\neq m}}^{\infty} \mathbf{f}_m^* \mathbf{f}_m Cov\left(\mathbf{S}_m, \mathbf{S}_l^*\right) , \tag{17}$$

where we can further separate the DC component from the first summand. This gives:

$$\mathrm{Var}\left(\mu_N\right)=\mathbf{f}_0^*\mathbf{f}_0\mathrm{Var}\left(\mathbf{S}_0\right)+\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\mathbf{f}_m^*\mathbf{f}_m\mathrm{Var}\left(\mathbf{S}_m\right)+\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\l\neq m}}^{\infty}\mathbf{f}_m^*\mathbf{f}_l\,Cov\left(\mathbf{S}_m,\mathbf{S}_l^*\right). \tag{18}$$

Since $Cov\left(X,Y^*\right):=\left\langle XY^*\right\rangle-\left\langle X\right\rangle\left\langle Y^*\right\rangle$, the covariance term becomes $Cov\left(\mathbf{S}_m,\mathbf{S}_l^*\right)=\left\langle\mathbf{S}_m\mathbf{S}_l^*\right\rangle-\left\langle\mathbf{S}_m\right\rangle\left\langle\mathbf{S}_l^*\right\rangle$. Thanks to the Fourier series representation, since we are only considering integer frequencies, it is easy to show that the expectation of the Fourier coefficients for stochastically generated samples is given by $\left\langle\mathbf{f}_m\right\rangle=\delta(m)$ when samples are weighted by $\alpha(x)=1/p\left(x\right)$ (see supplemental Sec. 1.1). This is also the condition proposed by Subr and Kautz [SK13] to obtain an unbiased estimator. Since $m\neq l$, we get: $\left\langle\mathbf{S}_0\right\rangle\left\langle\mathbf{S}_l{}^*\right\rangle=\left\langle\mathbf{S}_m\right\rangle\left\langle\mathbf{S}_0{}^*\right\rangle=0$, which simplifes (18) in the form:

$$\mathrm{Var}\left(\mu_N\right)=\mathbf{f}_0^*\mathbf{f}_0\mathrm{Var}\left(\mathbf{S}_0\right)+\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\mathbf{f}_m^*\mathbf{f}_m\mathrm{Var}\left(\mathbf{S}_m\right)+\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\l\neq m}}^{\infty}\mathbf{f}_m^*\mathbf{f}_l\left\langle\mathbf{S}_m\mathbf{S}_l^*\right\rangle. \tag{19}$$

Using the fact that $\left\langle\mathbf{f}_m\right\rangle=\delta(m)$, we can further simplify $\mathrm{Var}\left(\mathbf{S}_m\right)=\left\langle\mathbf{S}_m^*\mathbf{S}_m\right\rangle-\left\langle\mathbf{S}_m\right\rangle^2=\left\langle\mathbf{S}_m^*\mathbf{S}_m\right\rangle$. Also, since $\mathbf{f}_0=I$ and is real, we can simplify (19) as follows:

$$\boxed{\mathrm{Var}\left(\mu_N\right)=I^2\mathrm{Var}\left(\mathbf{S}_0\right)+\sum_{\substack{m=-\infty\\m\neq 0}}^{\infty}\mathbf{f}_m^*\mathbf{f}_m\left\langle\mathbf{S}_m^*\mathbf{S}_m\right\rangle+\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\l\neq m}}^{\infty}\mathbf{f}_m^*\mathbf{f}_l\left\langle\mathbf{S}_m\mathbf{S}_l^*\right\rangle.} \tag{20}$$

This is the generalized variance formulation proposed in the main paper. Here, $\mathrm{Var}\left(\mathbf{S}_0\right)=\left\langle\mathbf{S}_0^*\mathbf{S}_0\right\rangle-\left\langle\mathbf{S}_0\right\rangle^2$ and since $\left\langle\mathbf{S}_0\right\rangle=1$, regardless of the sampling pattern, we get: $\mathrm{Var}\left(\mathbf{S}_0\right)=\left\langle\mathbf{S}_0^*\mathbf{S}_0\right\rangle-1$.

4.1. Third term is a real entity

In the above variance formulation (20), the first term contains only the DC component which is real, the second term contains the power spectra which is also real. However, it is not straightforward to see that the third term is also real. This is important to verify since these three terms together defines the variance for a sampling-base estimator which is a real entity. By using the linearity property of the expectation operator, we can rewrite the third term as follows:

$$\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\l\neq m}}^{\infty}\mathbf{f}_m^*\mathbf{f}_l\left\langle\mathbf{S}_m\mathbf{S}_l^*\right\rangle=\left\langle\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\m\neq l}}^{\infty}\mathbf{f}_m^*\mathbf{S}_m\mathbf{f}_l\mathbf{S}_l^*\right\rangle. \tag{21}$$

We can rewrite the double summation for all m & l frequencies by subtracting the diagonal elements ($m=l$):

$$\left\langle\sum_{m=-\infty}^{\infty}\sum_{\substack{l=-\infty\\m\neq l}}^{\infty}\mathbf{f}_m^*\mathbf{S}_m\mathbf{f}_l\mathbf{S}_l^*\right\rangle=\left\langle\sum_{m=-\infty}^{\infty}\mathbf{f}_m^*\mathbf{S}_m\sum_{l=-\infty}^{\infty}\mathbf{f}_l\mathbf{S}_l^*-\sum_{m=-\infty}^{\infty}\mathbf{f}_m^*\mathbf{f}_m\mathbf{S}_m^*\mathbf{S}_m\right\rangle \tag{22a}$$

$$=\left\langle\sum_{m=-\infty}^{\infty}\mathbf{f}_m^*\mathbf{S}_m\sum_{l=-\infty}^{\infty}\mathbf{f}_l\mathbf{S}_l^*\right\rangle-\left\langle\sum_{m=-\infty}^{\infty}\mathbf{f}_m^*\mathbf{f}_m\mathbf{S}_m^*\mathbf{S}_m\right\rangle. \tag{22b}$$

Both the entities in the expectation operators in (22b) are real quantities: first one is the complex inner product of the Fourier representation of the sampling-based estimator with itself and second is the power spectrum.

4.2. Relation to the PCF

The Fourier domain variance formulation in (20) can be explicitly expressed in terms of joint probability (second order correlations), that defines the *pair correlation function* [IPSS08]. We start by expanding the expectation operator $\left\langle\mathbf{S}_m\mathbf{S}_l^*\right\rangle$ in the third (covariance) term of the variance formulation in (20), given that the samples are derived from a pdf $p(x)$:

$$\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \frac{1}{N^2} \left\langle \frac{\sum_k e^{-i2\pi m x_k}}{p(x_k)} \frac{\sum_k e^{i2\pi l x_k}}{p(x_k)} \right\rangle \tag{23}$$

$$= \frac{1}{N^2} \left\langle \frac{\sum_k \sum_j e^{-i2\pi(m x_k - l x_j)}}{p(x_k)\, p(x_j)} \right\rangle \tag{24}$$

$$= \frac{1}{N^2} \left\langle \frac{\sum_k e^{-i2\pi(m-l)x_k}}{p(x_k)^2} + \frac{\sum_k \sum_{j\neq k} e^{-i2\pi(m x_k - l x_j)}}{p(x_k)\, p(x_j)} \right\rangle \tag{25}$$

$$= \frac{1}{N^2} \left\langle \frac{\sum_k e^{-i2\pi(m-l)x_k}}{p(x_k)^2} \right\rangle + \left\langle \frac{\sum_k \sum_{j\neq k} e^{-i2\pi(m x_k - l x_j)}}{p(x_k)\, p(x_j)} \right\rangle \tag{26}$$

$$= \frac{1}{N^2} \left[\sum_k \left\langle \frac{e^{-i2\pi(m-l)x_k}}{p(x_k)^2} \right\rangle + \sum_k \sum_{j\neq k} \left\langle \frac{e^{-i2\pi(m x_k - l x_j)}}{p(x_k)\, p(x_j)} \right\rangle \right] \tag{27}$$

$$= \frac{1}{N^2} \sum_k \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)^2} p(x)\, \mathrm{d}x + \frac{1}{N^2} \sum_k \sum_{j\neq k} \int_0^1 \int_0^1 \frac{e^{-i2\pi(mx-ly)}}{p(x)\, p(y)} \rho(x,y) \mathrm{d}x\mathrm{d}y \tag{28}$$

$$= \frac{1}{N^2} N \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)} \mathrm{d}x + \frac{N(N-1)}{N^2} \int_0^1 \int_0^1 \frac{e^{-i2\pi(mx-ly)}}{p(x)\, p(y)} \rho(x,y) \mathrm{d}x\mathrm{d}y\,. \tag{29}$$

That simplifies to:

$$\boxed{\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \frac{1}{N} \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)} \mathrm{d}x + \frac{N-1}{N} \int_0^1 \int_0^1 \frac{e^{-i2\pi(mx-ly)}}{p(x)\, p(y)} \rho(x,y) \mathrm{d}x\mathrm{d}y}\,. \tag{30}$$

Here, $\rho(x,y)$ represents the joint probability density function of two samples at locations x and y which is also related to the pair correlation function [OG12, IPSS08].

5. Random samples

For random i.i.d. samples $\rho(x,y) := p(x)\, p(y)$. This renders the second part containing the double integral in (30) equals to:

$$\frac{1}{N^2} \sum_k \sum_{j\neq k} \int_0^1 \int_0^1 \frac{e^{-i2\pi(mx-ly)}}{p(x)\, p(y)} \rho(x,y) \mathrm{d}x\mathrm{d}y = \frac{N(N-1)}{N^2} \int_0^1 \int_0^1 \frac{e^{-i2\pi(mx-ly)}}{p(x)\, p(y)} p(x)\, p(y)\, \mathrm{d}x\mathrm{d}y \tag{31}$$

$$= \frac{N-1}{N} \int_0^1 \int_0^1 e^{-i2\pi(mx-ly)} \mathrm{d}x\mathrm{d}y \tag{32}$$

$$= \frac{N-1}{N} \mathrm{Sinc}(\pi m) \mathrm{Sinc}(\pi l) e^{-i\pi(l-m)} \tag{33}$$

$$= 0\,, \tag{34}$$

Here we use the fact that $\mathrm{Sinc}(\pi m)\mathrm{Sinc}(\pi l) = 0$ when m, l are integer frequencies. This implies that, for random samples:

$$\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \frac{1}{N^2} \sum_k \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)} \mathrm{d}x = \frac{1}{N} \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)} \mathrm{d}x = \frac{1}{N} \int_0^1 \alpha(x) e^{-i2\pi(m-l)x} \mathrm{d}x \tag{35}$$

where, $\alpha(x) := 1/p(x)$. This renders the final term:

$$\boxed{\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \frac{1}{N} \mathbf{W}_{m-l}\,,} \tag{36}$$

where $\mathbf{W}_{m-l}$ is the Fourier transform of the weighting function $\alpha(x)$ at the $(m-l)$-th frequency. If the sampling pdf is constant: $\alpha(x) = c$ where $c \in \mathbb{R}$ is a constant, the third term goes to zero:

$$\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = c \frac{1}{N^2} \sum_k \int_0^1 e^{-i2\pi(m-l)x} \mathrm{d}x = c \frac{1}{N} \left[\mathrm{Sinc}(2\pi(l-m)) - i\, \mathrm{Sinc}(\pi(l-m)) \sin(\pi(l-m)) \right] = 0, \tag{37}$$

at all integer frequencies. Note that, this does not yield that $\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \left\langle \mathbf{S}_m \right\rangle \left\langle \mathbf{S}_l^* \right\rangle$ even for random i.i.d. samples.

5.1. Deriving analytic expression for the power spectrum

To analytically solve the variance formulation in (20), we need the analytic expression for the expected power spectrum for random samples. We start with the Fourier transform of $S(x)$ which can be written as:

$$\mathbf{S}_m = \frac{1}{N}\sum_{k=1}^{N}\frac{e^{-i2\pi m x_k}}{p(x_k)} = \frac{1}{N}\sum_{k=1}^{N}\left(\frac{\cos(2\pi m x_k)}{p(x_k)} + i\,\frac{\sin(2\pi m x_k)}{p(x_k)}\right) \tag{38}$$

The corresponding power spectrum $\mathbf{S}_m^*\mathbf{S}_m := |\mathbf{S}_m|^2$ can be derived as follows:

$$\mathbf{S}_m^*\mathbf{S}_m = \frac{1}{N^2}\left[\left(\sum_{k=1}^{N}\frac{\cos(2\pi m x_k)}{p(x_k)}\right)^2 + \left(\sum_{k=1}^{N}\frac{\sin(2\pi m x_k)}{p(x_k)}\right)^2\right] \tag{39}$$

where,

$$\left(\sum_{k=1}^{N}\frac{\cos(2\pi m x_k)}{p(x_k)}\right)^2 = \sum_{k=1}^{N}\left(\frac{\cos(2\pi m x_k)}{p(x_k)}\right)^2 + \sum_{k}^{N}\sum_{j\neq k}^{N}\frac{\cos(2\pi m x_k)\cos(2\pi m x_j)}{p(x_k)\,p(x_j)} \tag{40}$$

$$\left(\sum_{k=1}^{N}\frac{\sin(2\pi m x_k)}{p(x_k)}\right)^2 = \sum_{k=1}^{N}\left(\frac{\sin(2\pi m x_k)}{p(x_k)}\right)^2 + \sum_{k}^{N}\sum_{j\neq k}^{N}\frac{\sin(2\pi m x_k)\sin(2\pi m x_j)}{p(x_k)\,p(x_j)}\,. \tag{41}$$

Adding (40) and (41), we obtain:

$$\sum_{k=1}^{N}\frac{\cos(2\pi m x_k))^2 + \sin(2\pi m x_k))^2}{p(x_k)^2} + \sum_{k}^{N}\sum_{j\neq k}^{N}\frac{\cos(2\pi m x_k)\cos(2\pi m x_j) + \sin(2\pi m x_k)\sin(2\pi m x_j)}{p(x_k)\,p(x_j)}\,, \tag{42}$$

which gives the power spectrum of random samples in the form:

$$\mathbf{S}_m^*\mathbf{S}_m = \frac{1}{N^2}\left[\sum_{k=1}^{N}\frac{1}{p(x_k)^2} + \sum_{k}^{N}\sum_{j\neq k}^{N}\frac{\cos(2\pi m(x_k - x_j))}{p(x_k)\,p(x_j)}\right]. \tag{43}$$

We will now derive the expected power spectrum first for uniform Sec. 5.2 and then for non-uniform Sec. 5.3 random sampling distributions.

5.2. Expected power spectrum for uniform distribution

For samples generated from a constant pdf value $p(x) := 1$, the power spectrum equation (43) simplifies to the following:

$$\mathbf{S}_m^*\mathbf{S}_m = \frac{1}{N^2}\left[\sum_{k=1}^{N}1 + \sum_{k}^{N}\sum_{j\neq k}^{N}\left[\cos(2\pi m(x_k - x_j))\right]\right] \tag{44}$$

$$= \frac{1}{N^2}\left[N + \sum_{k}^{N}\sum_{j\neq k}^{N}\left[\cos(2\pi m(x_k - x_j))\right]\right]. \tag{45}$$

$$\left\langle \mathbf{S}_m^*\mathbf{S}_m\right\rangle = \frac{1}{N^2}\left[N + \sum_{k}^{N}\sum_{j\neq k}^{N}\left\langle\cos(2\pi m(x_k - x_j))\right\rangle\right] \tag{46}$$

$$= \frac{1}{N^2}\left[N + \sum_{k}^{N}\sum_{j\neq k}^{N}\int_0^1\int_0^1 \cos(2\pi m(x-y))\,\mathrm{d}x\,\mathrm{d}y\right] \tag{47}$$

$$= \frac{1}{N^2}\left[N + \sum_{k}^{N}\sum_{j\neq k}^{N}\mathrm{Sinc}(\pi m)^2\right] \tag{48}$$

$$= \frac{1}{N^2}\left[N + (N-1)N\,\mathrm{Sinc}(\pi m)^2\right]. \tag{49}$$

$$\boxed{\left\langle \mathbf{S}_m^*\mathbf{S}_m\right\rangle = \begin{cases} 1 & m = 0 \\ \frac{1}{N} + \frac{N-1}{N}\mathrm{Sinc}(\pi m)^2 & m \neq 0 \end{cases}} \tag{50}$$

Since in our case m frequencies are always integer values, the $\mathrm{Sinc}(\cdot)^2$ term goes to zero leaving the expected power spectrum equal to $1/N$. In d-dimensions, the above expressions can be easily derived in the form:

$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \begin{cases} 1 & m = 0 \\ \frac{1}{N} + \frac{N-1}{N} \prod_{j=1}^{d} \mathrm{Sinc}(\pi m_j)^2 & m \neq 0 \end{cases} \tag{51}$$

5.3. Expected power spectrum for non-uniform distribution

For random samples derived from an arbitrary non-constant PDF, we can obtain the expected power spectrum, starting from (43) as follows:

$$\begin{aligned} \langle \mathbf{S}_m^* \mathbf{S}_m \rangle &= \frac{1}{N^2} \left[\sum_{k=1}^{N} \left\langle \frac{1}{p(x_k)^2} \right\rangle + \sum_{k}^{N} \sum_{j \neq k}^{N} \left\langle \frac{\cos(2\pi m(x_k - x_j))}{p(x_k)\, p(x_j)} \right\rangle \right] & (52) \\ &= \frac{1}{N^2} \left[\sum_{k=1}^{N} \int_0^1 \frac{1}{p(x)^2} p(x)\, \mathrm{d}x + \sum_{k}^{N} \sum_{j \neq k}^{N} \int_0^1 \int_0^1 \frac{\cos(2\pi m(x-y))}{p(x)\, p(y)} p(x,y)\, \mathrm{d}x\, \mathrm{d}y \right] & (53) \\ &= \frac{1}{N^2} \left[N \int_0^1 \frac{1}{p(x)} \mathrm{d}x + (N-1) N \int_0^1 \int_0^1 \frac{\cos(2\pi m(x-y))}{p(x)\, p(y)} p(x,y)\, \mathrm{d}x\, \mathrm{d}y \right]. & (54) \end{aligned}$$

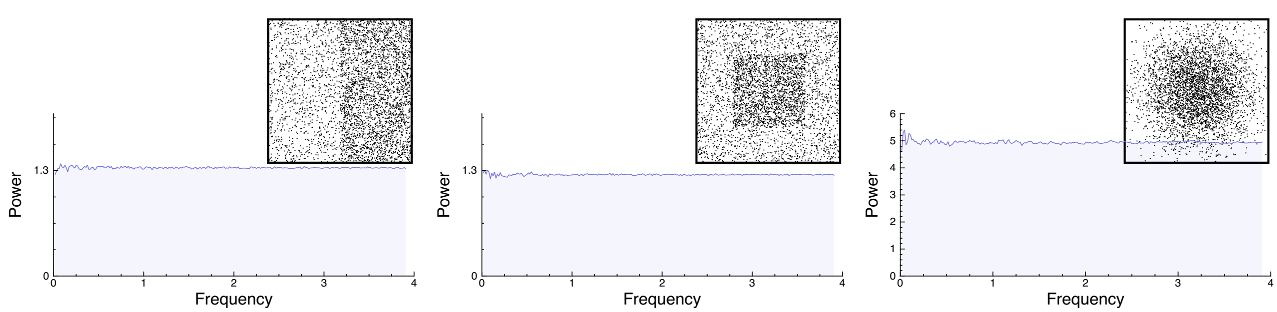
$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \frac{1}{N} \left[\int_0^1 \frac{1}{p(x)} \mathrm{d}x + \; + (N-1) \int_0^1 \int_0^1 \frac{\cos(2\pi m(x-y))}{p(x)\, p(y)} p(x,y)\, \mathrm{d}x\, \mathrm{d}y \right] \tag{55}$$

Since for random samples, $p(x,y) := p(x)\, p(y)$, we get:

$$\begin{aligned} \langle \mathbf{S}_m^* \mathbf{S}_m \rangle &= \frac{1}{N} \left[\int_0^1 \frac{1}{p(x)} \mathrm{d}x + (N-1) \int_0^1 \int_0^1 \frac{\cos(2\pi m(x-y))}{p(x)\, p(y)} p(x)\, p(y)\, \mathrm{d}x\, \mathrm{d}y \right] & (56) \\ &= \frac{1}{N} \left[\int_0^1 \frac{1}{p(x)} \mathrm{d}x + (N-1) \mathrm{Sinc}(\pi u)^2 \right]. & (57) \end{aligned}$$

$$\boxed{\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \begin{cases} \frac{\int_0^1 \alpha(x) \mathrm{d}x}{N} + \frac{(N-1)}{N} & m = 0 \\ \frac{\int_0^1 \alpha(x) \mathrm{d}x}{N} + \frac{N-1}{N} \mathrm{Sinc}(\pi u)^2 & m \neq 0 \end{cases},} \tag{58}$$

where $\alpha(x) := 1/p(x)$. This expression in (58) shows that there is an offset in the expected power spectrum of random samples generated for a given PDF (shown in Fig. 1). The DC term ($m = 0$) seems to depend on the integral of the weights. As the integral of the weights goes higher, the DC component diverges from 1 and, therefore, plays a role in the variance of the corresponding estimator.



(a) Step PDF (b) Box-Box PDF (c) Gaussian PDF

Figure 1: *Radially averaged expected power spectra for* ***random samples*** *derived from different probability distribution functions (PDFs). The corresponding expected power spectrum is computed by inversely weighting the samples with the corresponding PDFs. Sample distributions are shown as insets on top-right. The power depends on the integral $\int_0^1 \alpha(x)dx$ of the weighting function $\alpha(x)$. Since both the inverse of a step and a Box-Box PDF integrates to the same value ($\int_0^1 \alpha(x)dx = 1.3$), both a & b has the same power.*

5.4. Variance formulation for random samples

Using (58) and (36) in the variance formulation in (20) we can write the analytical expression for the variance of random samples from any given PDF in the following form:

$$\mathrm{Var}(\mu_N) = \mathbf{f}_0^* \mathbf{f}_0 \mathrm{Var}(\mathbf{S}_0) + \frac{\int_0^1 \alpha(x)\mathrm{d}x}{N} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \mathbf{f}_m^* \mathbf{f}_m + \frac{1}{N} \sum_{m=-\infty}^{\infty} \sum_{\substack{l=-\infty \\ l\neq m}}^{\infty} \mathbf{f}_m^* \mathbf{f}_l \, \mathbf{W}_{m-l}\,. \tag{59}$$

6. Jittered samples

Unlike independent random sampling, jittered sampling has correlations within samples. In jittered sampling, one sample is generated within each stratum randomly. However, samples across the strata are correlated to each other. Following the work by [Ö16], we first write down the probability density function $p(x)$ of jittered samples followed by the joint probability density function $\rho(\mathbf{y},\mathbf{z})$ between any two samples $\mathbf{y}$ & $\mathbf{z}$ within the two different strata.

6.1. Joint probability density function

We first represent the *probability* of a sample $\mathbf{y} \in [j/N, (j+1)/N)$ to be in a j-th stratum by $\lambda(\mathbf{y}) := p(\mathbf{y})\mathrm{d}\mathbf{y}$ where $p(\mathbf{y}) := N$ represent its probability density function. Similarly, $p(\mathbf{y},\mathbf{z}) := \rho(\mathbf{y},\mathbf{z})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z}$ represents the joint probability of having points at locations $\mathbf{y}$ and $\mathbf{z}$ at the same time, where $\rho(\mathbf{y},\mathbf{z})$ represent its joint probability density function. The joint probability of having a point in $\mathrm{d}\mathbf{y}$ around $\mathbf{y} \in T_i$ stratum and in $\mathrm{d}\mathbf{z}$ around $\mathbf{z} \in T_j$ stratum, is given by:

$$\lambda(\mathbf{y},\mathbf{z}) = \begin{cases} 0 & \text{when } \mathbf{y} \text{ \& } \mathbf{z} \text{ belongs to the same stratum} \\ p(\mathbf{y})\,p(\mathbf{z})\,\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} & \text{otherwise} \end{cases} \tag{60}$$

Here $i,j \in [0,...,N-1)$. This shows that if we consider two strata, then the joint probability density function here would be $\rho(\mathbf{y},\mathbf{z}) = p(\mathbf{y})\,p(\mathbf{z}) = N^2$.

6.2. Deriving expected Fourier spectrum in 1D

Next, we derive the expected Fourier spectrum $\langle \mathbf{S}_m \rangle$ of jittered samples for unit constant PDF. We start with a definition of $\mathbf{S}_m$:

$$\mathbf{S}_m = \frac{1}{N}\sum_{k=0}^{N-1} e^{-i2\pi m \mathbf{x}_k} = \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{S}_m^1\,, \tag{61}$$

where $\mathbf{S}_m^1$ represents the Fourier transform of one sample. The corresponding expected power spectrum can be written in the following form using the linearity property of the expectation operator:

$$\langle \mathbf{S}_m \rangle = \frac{1}{N}\sum_{k=0}^{N-1} \left\langle \mathbf{S}_m^1 \right\rangle. \tag{62}$$

The above notation says that the expected Fourier spectrum of a given point set (N samples) is actually equal to the sum over the expected Fourier spectra of each sample $\mathbf{x}_k$, divided by the number of samples N. In 1D, we have one sample in each stratum with a stratum length $= 1/N$ in which a sample is generated with a constant probability density ($p(\mathbf{x}) = N$). We consider one such stratum (let's call it k-th stratum) within a range $\mathbf{x} \in [\frac{k}{N}, \frac{k+1}{N}]$, where $k \in [0,...,N-1]$. We can compute the expected Fourier transform of one such sample in a given stratum as follows:

$$\left\langle \mathbf{S}_m^1 \right\rangle = \int_{\frac{k}{N}}^{(k+1)/N} e^{-i2\pi m \mathbf{x}} p(\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{63}$$

By expanding the complex exponential into the sin and cos terms, we get:

$$\begin{aligned}
\left\langle \mathbf{S}_m^1 \right\rangle &= N \int_{\frac{k}{N}}^{(k+1)/N} (\cos 2\pi m\mathbf{x} - i \sin 2\pi m\mathbf{x})\,\mathrm{d}\mathbf{x} && (64)\\
&= N\left[\int_{\frac{k}{N}}^{(k+1)/N} \cos 2\pi m\mathbf{x}\,\mathrm{d}\mathbf{x} - i\int_{\frac{k}{N}}^{(k+1)/N} \sin 2\pi m\mathbf{x}\,\mathrm{d}\mathbf{x}\right] && (65)\\
&= N\frac{2\sin\frac{\pi m}{N}}{2\pi m} e^{-i\pi m(2k+1)/N} && (66)\\
&= \mathrm{Sinc}\left(\frac{\pi m}{N}\right) e^{-i\pi m(2k+1)/N}. && (67)
\end{aligned}$$

By plugging the above expression back in (62), we get:

$$\langle \mathbf{S}_m \rangle = \frac{1}{N} \mathrm{Sinc}\left(\frac{\pi m}{N}\right) \sum_{k=0}^{N-1} e^{-i2\pi mk/N} . \tag{68}$$

6.3. Deriving expected power spectrum in 1D

To compute the expected power spectrum for a unit constant PDF, we start from (61) and expand the power spectrum which is the product of $\mathbf{S}_m$ with it's conjugate:

$$\begin{aligned}
\mathbf{S}_m^* \mathbf{S}_m &= \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} && (69)\\
&= \frac{1}{N^2} \left[\sum_{k=0}^{N-1} 1 + \sum_{k=0}^{N-1} \sum_{\substack{j=0\\ j\neq k}}^{N-1} e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} \right] && (70)\\
&= \frac{1}{N} + \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\substack{j=0\\ j\neq k}}^{N-1} e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} . && (71)
\end{aligned}$$

We are mainly interested in the expected power spectrum which can be derived as follows:

$$\begin{aligned}
\left\langle \mathbf{S}_m^* \mathbf{S}_m \right\rangle &= \left\langle \frac{1}{N} + \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\substack{j=0\\ j\neq k}}^{N-1} e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} \right\rangle && (72)\\
&= \frac{1}{N} + \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{\substack{j=0\\ j\neq k}}^{N-1} \left\langle e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} \right\rangle . && (73)
\end{aligned}$$

To simplify the derivation we can first compute the expectation for any given j-th and k-th strata within the double summation as follows:

$$\begin{aligned}
\left\langle e^{-i2\pi m(\mathbf{x}_j - \mathbf{x}_k)} \right\rangle &= \int_{\frac{k}{N}}^{\frac{k+1}{N}} \int_{\frac{j}{N}}^{\frac{j+1}{N}} e^{-i2\pi m(\mathbf{y}-\mathbf{z})} \rho(\mathbf{y},\mathbf{z}) \mathrm{d}\mathbf{y} \mathrm{d}\mathbf{z} && (74)\\
&= N^2 \frac{2 \sin\left(\frac{\pi u}{N}\right)^2}{2\pi^2 m^2} \left[e^{i2\pi m(j-k)} \right] . && (75)
\end{aligned}$$

Plugging it back in (73), we get:

$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \frac{1}{N} + \frac{1}{N^2} \left[\sum_{k=0}^{N-1} \sum_{\substack{j=0 \\ j \neq k}}^{N-1} N^2 \frac{2\sin\left(\frac{\pi m}{N}\right)^2}{2\pi^2 m^2} e^{i2\pi m(j-k)} \right] \tag{76}$$

$$= \frac{1}{N} + \frac{1}{N^2} N^2 \frac{2\sin\left(\frac{\pi m}{N}\right)^2}{2\pi^2 m^2} \left[\sum_{k=0}^{N-1} \sum_{\substack{j=0 \\ j \neq k}}^{N-1} e^{i2\pi m(j-k)} \right] \tag{77}$$

$$= \frac{1}{N} + \frac{1}{N^2} N^2 \frac{2\sin\left(\frac{\pi m}{N}\right)^2}{2\pi^2 m^2} \left[\sum_{k=0}^{N-1} \sum_{j=0}^{N-1} e^{i2\pi m(j-k)} - \sum_{k=0}^{N-1} e^{i2\pi m(0)} \right] \tag{78}$$

$$= \frac{1}{N} + \frac{1}{N^2} N^2 \frac{2\sin\left(\frac{\pi m}{N}\right)^2}{2\pi^2 m^2} \left[\frac{\sin(\pi m)^2}{\sin\left(\frac{\pi m}{N}\right)^2} - N \right] \tag{79}$$

$$= \frac{1}{N} + \frac{1}{N^2} \left[N^2 \frac{\sin(\pi m)^2}{\pi^2 m^2} - N^3 \frac{\sin\left(\frac{\pi m}{N}\right)^2}{\pi^2 m^2} \right] \tag{80}$$

$$= \frac{1}{N} + \frac{1}{N^2} \left[N^2 \frac{\sin(\pi m)^2}{\pi^2 m^2} - N \frac{\sin\left(\frac{\pi m}{N}\right)^2}{\left(\frac{\pi m}{N}\right)^2} \right] \tag{81}$$

$$= \frac{1}{N} + \frac{1}{N^2} \left[N^2 \mathrm{Sinc}(\pi m)^2 - N \mathrm{Sinc}\left(\frac{\pi m}{N}\right)^2 \right] \tag{82}$$

$$= \frac{1}{N} + \frac{1}{N} \left[N \mathrm{Sinc}(\pi m)^2 - \mathrm{Sinc}\left(\frac{\pi m}{N}\right)^2 \right] \tag{83}$$

This can be simplified in the following form:

$$\boxed{\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \frac{1}{N} \left(1 - \mathrm{Sinc}\left(\frac{\pi m}{N}\right)^2 \right) + \mathrm{Sinc}(\pi m)^2.} \tag{84}$$

For integer frequencies, this gives us:

$$\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \begin{cases} 1 & m = 0 \\ \frac{1}{N} \left[1 - \mathrm{Sinc}\left(\frac{\pi m}{N}\right)^2 \right] & m \neq 0 \end{cases} \tag{85}$$

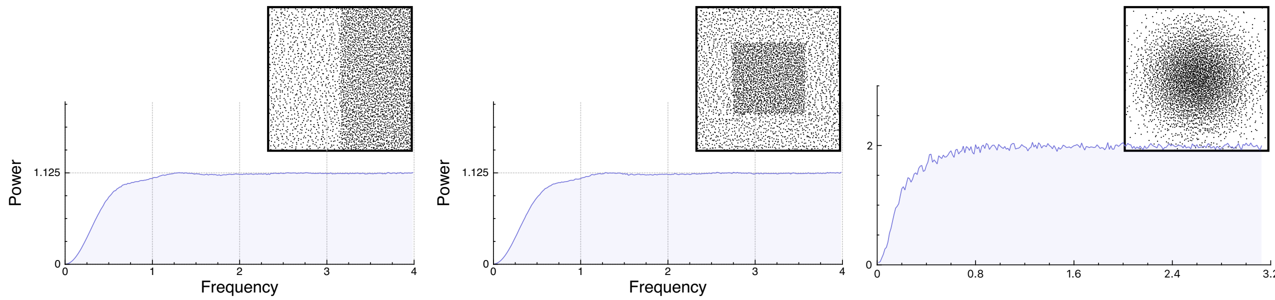
Here we use the following: for $m := 0$, we have $\mathrm{Sinc}(\pi m) = 1$ and $\mathrm{Sinc}\left(\frac{\pi m}{N}\right) = 1$ which renders (84): $\frac{1}{N}(1-0)+1 = 1$. For $m \neq 0$, $\mathrm{Sinc}(\pi m)^2 = 0$ in (84) for all interger non-zero frequencies. In multi-dimensional case, it is straight forward to show that the corresponding expressions from (84) becomes:

$$\boxed{\langle \mathbf{S}_m^* \mathbf{S}_m \rangle = \frac{1}{N} \left(1 - \prod_{j=1}^{d} \mathrm{Sinc}\left(\frac{\pi m_j}{\sqrt[d]{N}}\right)^2 \right) + \prod_{j=1}^{d} \mathrm{Sinc}\left(\pi m_j\right)^2,} \tag{86}$$

where d represents the dimension. The expression is valid for uniform sample distribution with constant PDF. For non-constant PDFs, the expected power spectra are shown in Fig. 2.

6.4. Variance formulation: Covariance term for uniform 1D jittered samples

To predict variance due to jittered samples, we need the covariance term in the variance formulation (20). We compute this term analytically in 1D and then extend it to the shifted strata version that we visualize in Fig. 2 of the main paper. We start with the definition of the Fourier



(a) Step PDF (b) Box-Box PDF (c) Gaussian PDF

Figure 2: *Radially averaged expected power spectra for **jittered samples** derived from different probability distribution functions (PDFs). The corresponding expected power spectrum is computed by inversely weighting the samples with the corresponding PDFs. Sample distributions are shown as insets on top-right. The power depends on the integral* $\int_0^1 \alpha(x)dx$ *of the weighting function* $\alpha(x)$*. This explains why a & b converges to the same value* ($= 1.125$)*. For the Gaussian PDF in (c), the radial average converges to a value* $\int_0^1 \alpha(x)dx = 2$*.*

transform: $\mathbf{S}_m = \frac{1}{N}\sum_{k=0}^{N-1} e^{-i2\pi m \mathbf{x}_k}$ and write the expectation term within the third (covariance) term as:

$$\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = \frac{1}{N^2}\left\langle \sum_{k=0}^{N-1}\sum_{j=0}^{N-1} e^{-i2\pi(mx_j - lx_k)} \right\rangle \tag{87}$$

$$= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\left\langle e^{-i2\pi(mx_j - lx_k)} \right\rangle \tag{88}$$

$$= \frac{1}{N^2}\left[\sum_{k=0}^{N-1}\left\langle e^{-i2\pi(m-l)x_k}\right\rangle + \sum_{k=0}^{N-1}\sum_{\substack{j=0\\ j\neq k}}^{N-1}\left\langle e^{-i2\pi(mx_j - lx_k)}\right\rangle\right]. \tag{89}$$

We first solve the summand with a single summation in the above equation:

$$\sum_{k=0}^{N-1}\left\langle e^{-i2\pi(m-l)x_k}\right\rangle = \sum_{k=0}^{N-1}\int_{\frac{k}{N}}^{\frac{k+1}{N}} e^{-i2\pi(m-l)\mathbf{x}} p(\mathbf{x})\, d\mathbf{x} \tag{90}$$

$$= \sum_{k=0}^{N-1}\int_{\frac{k}{N}}^{\frac{k+1}{N}} e^{-i2\pi(m-l)\mathbf{x}} N d\mathbf{x} \tag{91}$$

$$= N\sum_{k=0}^{N-1}\int_{\frac{k}{N}}^{\frac{k+1}{N}} e^{-i2\pi(m-l)\mathbf{x}} d\mathbf{x} \tag{92}$$

$$= N\sum_{k=0}^{N-1}\left[\frac{\sin\frac{\pi(m-l)}{N}}{\pi(m-l)} e^{-i\frac{\pi(m-l)(2k+1)}{N}}\right] \tag{93}$$

$$= N\frac{\sin\frac{\pi(m-l)}{N}}{\pi(m-l)}\left[\sum_{k=0}^{N-1} e^{-i\frac{\pi(m-l)(2k+1)}{N}}\right] \tag{94}$$

$$= N\frac{\sin\frac{\pi(m-l)}{N}}{\pi(m-l)}\left[\frac{\sin(2\pi(m-l))}{2\sin(\frac{\pi(m-l)}{N})} - i\frac{\sin(\pi(m-l))^2}{\sin(\frac{\pi(m-l)}{N})}\right] \tag{95}$$

$$= N\left[\frac{\sin(2\pi(m-l))}{2\pi(m-l)} - i\frac{\sin(\pi(m-l))^2}{\pi(m-l)}\right] \tag{96}$$

$$= N\left[\mathrm{Sinc}(2\pi(m-l)) - i\sin(\pi(m-l)\mathrm{Sinc}(\pi(m-l)))\right]. \tag{97}$$

Since we are only interested in the integer frequencies with $m \neq l$, the above term becomes zero and has no contribution in the covariance term. Now for the summand with a double summation with $j \neq k$, we first compute the expectation over the exponential part before expanding the double summations:

$$\left\langle e^{-i2\pi(mx_j - lx_k)}\right\rangle = \int_{\frac{k}{N}}^{\frac{k+1}{N}}\int_{\frac{j}{N}}^{\frac{j+1}{N}} e^{-i2\pi(m\mathbf{y}-l\mathbf{z})}\rho(\mathbf{y},\mathbf{z})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \tag{98}$$

$$= \int_{\frac{k}{N}}^{\frac{k+1}{N}}\int_{\frac{j}{N}}^{\frac{j+1}{N}} e^{-i2\pi(m\mathbf{y}-l\mathbf{z})}N^2\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \tag{99}$$

$$= N^2\int_{\frac{k}{N}}^{\frac{k+1}{N}}\int_{\frac{j}{N}}^{\frac{j+1}{N}} e^{-i2\pi(m\mathbf{y}-l\mathbf{z})}\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{z} \tag{100}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}e^{-i\pi\frac{(2k+1)l-(2j+1)m}{N}} \tag{101}$$

Now, we compute the double summation as follows:

$$\sum_{k=0}^{N-1}\sum_{\substack{j=0\\ j\neq k}}^{N-1}\left\langle e^{-i2\pi(mx_j - lx_k)}\right\rangle = \sum_{k=0}^{N-1}\sum_{\substack{j=0\\ j\neq k}}^{N-1} N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}e^{-i\pi\frac{(2k+1)l-(2j+1)m}{N}} \tag{102}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\sum_{k=0}^{N-1}\sum_{\substack{j=0\\ j\neq k}}^{N-1} e^{-i\pi\frac{(2k+1)l-(2j+1)m}{N}} \tag{103}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left[\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} e^{-i\pi\frac{(2k+1)l-(2j+1)m}{N}} - \sum_{k=0}^{N-1} e^{-i\frac{\pi((l-m)+2k(l-m))}{N}}\right] \tag{104}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left[\frac{\sin\pi m\sin\pi l}{\sin\frac{\pi m}{N}\sin\frac{\pi l}{N}}e^{-i\pi(m-l)} - \sum_{k=0}^{N-1} e^{-i\frac{\pi((l-m)(2k+1))}{N}}\right] \tag{105}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left[\frac{\sin\pi m\sin\pi l}{\sin\frac{\pi m}{N}\sin\frac{\pi l}{N}}e^{-i\pi(m-l)} - \left[\frac{2\sin\pi(m-l)\cos\pi(m-l)}{2\sin\frac{\pi(m-l)}{N}} - i\frac{\sin\pi(m-l)^2}{\sin\frac{\pi(m-l)}{N}}\right]\right] \tag{106}$$

$$= N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left[\frac{\sin\pi m\sin\pi l}{\sin\frac{\pi m}{N}\sin\frac{\pi l}{N}}e^{-i\pi(m-l)} - \frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{N}}e^{-i\pi(m-l)}\right] \tag{107}$$

$$= e^{-i\pi(m-l)}\left[N^2\mathrm{Sinc}(\pi m)\mathrm{Sinc}(\pi l) - N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left(\frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{N}}\right)\right] \tag{108}$$

Here $\mathrm{Sinc}(\pi m)\mathrm{Sinc}(\pi l) = 0$, since both m and l are integers with $m \neq l$. This gives:

$$\sum_{k=0}^{N-1}\sum_{\substack{j=0\\ j\neq k}}^{N-1}\left\langle e^{-i2\pi(mx_j - lx_k)}\right\rangle = -e^{-i\pi(m-l)}\left[N^2\frac{\sin\frac{\pi l}{N}\sin\frac{\pi m}{N}}{\pi^2 ml}\left(\frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{N}}\right)\right] \tag{109}$$

$$= -e^{-i\pi(m-l)}\left[\mathrm{Sinc}\left(\frac{\pi l}{N}\right)\mathrm{Sinc}\left(\frac{\pi m}{N}\right)\frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{N}}\right]. \tag{110}$$

As a result, the final covariance term can be written as:

$$\left\langle \mathbf{S}_m\mathbf{S}_l^*\right\rangle = -\frac{e^{-i\pi(m-l)}}{N^2}\left[\mathrm{Sinc}\left(\frac{\pi l}{N}\right)\mathrm{Sinc}\left(\frac{\pi m}{N}\right)\frac{\sin\pi(m-l)}{\sin\frac{\pi(m-l)}{N}}\right]. \tag{111}$$

This can further be simplified in the following form:

$$\boxed{\left\langle \mathbf{S}_m\mathbf{S}_l^*\right\rangle = -\frac{e^{-i\pi(m-l)}}{N}\left[\mathrm{Sinc}\left(\frac{\pi l}{N}\right)\mathrm{Sinc}\left(\frac{\pi m}{N}\right)\frac{\mathrm{Sinc}(\pi(m-l))}{\mathrm{Sinc}\left(\frac{\pi(m-l)}{N}\right)}\right].} \tag{112}$$

which gives $\langle \mathbf{S}_m\mathbf{S}_l^*\rangle$ within the third (covariance) term of the variance formulation (20) for 1D jittered samples.

Covariance term for shifted 1D jittered samples When the 1D jittered sampling grid is shifted by a constant $a \in \mathbb{R}^d$, the corresponding equation (112) becomes:

$$\boxed{\left\langle \mathbf{S}_m \mathbf{S}_l^* \right\rangle = -\frac{e^{-i\pi(2a+1)(m-l)}}{N} \left[\mathrm{Sinc}\left(\frac{\pi l}{N}\right) \mathrm{Sinc}\left(\frac{\pi m}{N}\right) \frac{\mathrm{Sinc}(\pi(m-l))}{\mathrm{Sinc}\left(\frac{\pi(m-l)}{N}\right)} \right].} \tag{113}$$

Here we use the Fourier shift theorem which states that a translation in the primal becomes a modulation with a complex exponential (i.e. a shift in the phase).

7. Analytic test functions

In this section, we derive analytic expressions for the Fourier coefficients of few simple functions that later helps derive corresponding variance analytically for analysis purposes.

7.1. Step 1D function

Fourier spectrum for Step 1D function We can symbolically obtain the analytic expressions for $\mathbf{f}_m$ for a step 1D function defined as:

$$f(x) := \begin{cases} 0.5 & -0.5 < x < a \\ 1 & a \leq x < 0.5 \\ 0 & \text{otherwise}, \end{cases} \tag{114}$$

for $-0.5 < a < 0.5$, in the following form:

$$\mathbf{f}_m = \mathrm{Sinc}(2\pi m) - \frac{a}{2}\mathrm{Sinc}(2a\pi m) - i\frac{1}{2}\left[\mathrm{Sinc}(\pi m)\sin \pi m + (1+a)\mathrm{Sinc}(\pi m(1+a))\sin(\pi m(1-a))\right]. \tag{115}$$

Since we are considering only integer frequencies, the above expression can be simplified in the following form:

$$\mathbf{f}_m = \mathrm{Sinc}(2\pi m) - \frac{a}{2}\mathrm{Sinc}(2a\pi m) - i\frac{1}{2}(1+a)\mathrm{Sinc}(\pi m(1+a))\sin(\pi m(1-a)). \tag{116}$$

From the above expression, the power spectrum can be easily derived using:

$$\mathbf{f}_m^*\mathbf{f}_m = \mathbf{f}_m{\mathbf{f}_m}^* \tag{117}$$

$$= \frac{1}{2}\mathrm{Sinc}(\pi m)^2 + \frac{(a-1)^2}{4}\mathrm{Sinc}(\pi m(a-1))^2 - \frac{1}{4}(2a-1)\mathrm{Sinc}(\pi m(2a-1)\mathrm{Sinc}(\pi m)). \tag{118}$$

Covariance Term for Step1D function Similarly, we can obtain the product $\mathbf{f}_m^*\mathbf{f}_l$ by first solving for intermediate terms as:

$$A := -\frac{1}{2}\sin(\pi m)\mathrm{Sinc}(\pi m) - \frac{a+1}{2}\mathrm{Sinc}(\pi m(a+1))\sin(\pi m(1-a)) + i\left[\mathrm{Sinc}(2\pi m) + \frac{a}{2}\mathrm{Sinc}(2\pi m a)\right] \tag{119}$$

$$B := -\frac{1}{2}\sin(\pi l)\mathrm{Sinc}(\pi l) - \frac{a+1}{2}\mathrm{Sinc}(\pi l(a+1))\sin(\pi l(1-a)) - i\left[\mathrm{Sinc}(2\pi l) + \frac{a}{2}\mathrm{Sinc}(2\pi l a)\right]. \tag{120}$$

Since we are considering only the integer frequencies, we can simplify above expressions to:

$$A = -\frac{a+1}{2}\mathrm{Sinc}(\pi m(a+1))\sin(\pi m(1-a)) + i\left[\mathrm{Sinc}(2\pi m) + \frac{a}{2}\mathrm{Sinc}(2\pi m a)\right] \tag{121}$$

$$B = -\frac{a+1}{2}\mathrm{Sinc}(\pi l(a+1))\sin(\pi l(1-a)) - i\left[\mathrm{Sinc}(2\pi l) + \frac{a}{2}\mathrm{Sinc}(2\pi l a)\right]. \tag{122}$$

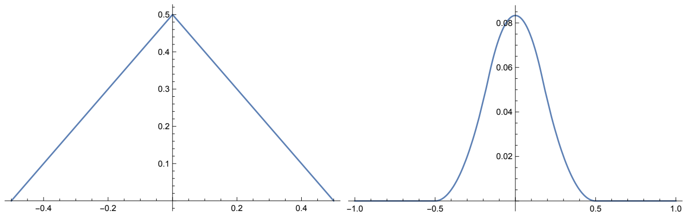
Finally, the covariance term for the step1D integrand can be written as:

$$\mathbf{f}_m^*\mathbf{f}_l . = A \times B \tag{123}$$

7.2. Tent 1D function

: This is a simple convolution of two Box functions:

$$f(x) := \frac{1}{2}(1 - |2x|) \ \text{ with } -0.5 \leq x < 0.5, \tag{124}$$



(a) Tent1D function (b) Box3Convolution1D function

Figure 3: *Visualizing integrands which are used in the motivation example in Fig.1 of the main paper.*

whose Fourier transform is simply a multiplication of the Fourier transforms of the Box functions. The resultant power spectrum can be obtained in the following form:

$$\mathbf{f}_m^*\mathbf{f}_m = \mathbf{f}_m\mathbf{f}_m{}^* = \frac{1}{16}\mathrm{Sinc}\left(\frac{\pi m}{2}\right)^4 \text{ where, } \mathbf{f}_m = \frac{1}{4}\mathrm{Sinc}\left(\frac{\pi m}{2}\right)^2. \tag{125}$$

7.3. Box3Convolution 1D function

: This is a simple convolution of three Box functions:

$$f(x) := \begin{cases} \frac{1}{8}(1+2x)^2 & -\frac{1}{2} < x \leq -\frac{1}{6} \\ \frac{1}{12} - x^2 & -\frac{1}{6} < x \leq \frac{1}{6} \\ \frac{1}{8}(1+2x)^2 & \frac{1}{6} < x < \frac{1}{2} \end{cases} \tag{126}$$

whose Fourier transform is simply a multiplication of the Fourier transforms of three Box functions. The resultant power spectrum can be obtained in the following form:

$$\mathbf{f}_m^*\mathbf{f}_m = \mathbf{f}_m\mathbf{f}_m{}^* = \frac{1}{729}\mathrm{Sinc}\left(\frac{\pi m}{3}\right)^6 \text{ where, } \mathbf{f}_m = \frac{1}{27}\mathrm{Sinc}\left(\frac{\pi m}{3}\right)^3. \tag{127}$$

8. Zero variance for perfect importance sampling

Here, we analytically validate our variance formulation by showing that for perfect importance sampling the corresponding variance is zero. For random samples, we have derived the expected power spectrum in (58) which is of the form:

$$\boxed{\langle \mathcal{P}_S(m) \rangle = \begin{cases} \frac{\int_x \alpha(x)\mathrm{d}x}{N} + \frac{(N-1)}{N} & m = 0 \\ \frac{\int_x \alpha(x)\mathrm{d}x}{N} + \frac{(N-1)\mathrm{Sinc}(\pi u)^2}{N} & m \neq 0, \end{cases}} \tag{128}$$

where $\alpha(x) := 1/p(x)$ and $x \in [0,1)$. We now analytically derive zero variance for two simple 1D functions: *BoxBox*1*D* and a step 1D in the next subsections.

8.1. For Box-Box 1D PDF

For $x \in [-0.5, 0.5)$:

$$f(x) := \begin{cases} B & -0.25 < x < 0.25 \\ 0 & x > 0.5 \,\|\, x < -0.5 \\ A & \text{otherwise} \end{cases} \tag{129}$$

$$p(x) := \frac{f(x)}{I_f} \tag{130}$$

$$x = \begin{cases} \dfrac{\xi I_f}{A} - 0.5 & 0 \geq \xi < \dfrac{0.25A}{I} \\ \dfrac{\xi I_f - 0.25(A+B)}{B} & \dfrac{0.25A}{I} \geq \xi < \dfrac{0.25A + 0.5B}{I} \\ \dfrac{\xi I_f - 0.5B}{A} & \dfrac{0.25A + 0.5B}{I} \geq \xi < 1 \end{cases} \tag{131}$$

where $I_f := \int f(x)dx = 0.5(A+B)$. For $A = 0.5, B = 1.0$:

$$\mathbf{f}_m = 0.25\mathrm{Sinc}\left(\frac{m\pi}{2}\right) + 0.5\mathrm{Sinc}(m\pi) \tag{132}$$

$$\mathbf{f}_0 = 0.75 \tag{133}$$

$$\mathbf{f}_0^*\mathbf{f}_0 = ||\mathbf{f}_0||^2 = 0.5625 \tag{134}$$

$$\mathbf{f}_m^*\mathbf{f}_m = \frac{\left(\sin(\frac{m\pi}{2}) + \sin(m\pi)\right)^2}{4m^2\pi^2}\,. \tag{135}$$

For random samples:

$$\boxed{\left\langle \mathbf{S}_m\mathbf{S}_l^* \right\rangle = \frac{1}{N^2}\sum_k \int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)}\mathrm{d}x = \frac{1}{N}\int_0^1 \frac{e^{-i2\pi(m-l)x}}{p(x)}\mathrm{d}x = \frac{1}{N}\mathbf{W}_{m-l}\,.} \tag{136}$$

where, $\alpha(x) := 1/p(x)$. Now, the integral of weighting function is:

$$\int_{-0.5}^{0.5} \alpha(x)dx = 1.125 \tag{137}$$

$$\mathrm{Var}(\mathbf{S}_0) = \left\langle \mathbf{S}_0^2 \right\rangle - \langle \mathbf{S}_0 \rangle^2 = \frac{1}{N}\int \alpha(x)\mathrm{d}x + \frac{N-1}{N} - 1 = \frac{0.125}{N} \tag{138}$$

$$\left\langle \mathbf{S}_m^*\mathbf{S}_m \right\rangle = \frac{1.125}{N} \quad \forall\, m \neq 0 \tag{139}$$

$$\left\langle \mathbf{S}_0\mathbf{S}_m{}^* \right\rangle = \frac{1}{N}\mathcal{F}_w(m)^* \tag{140}$$

$$\mathbf{W}_u = \frac{-0.238732\sin(1.5708u) + 0.477465\sin(\pi u)}{u} \tag{141}$$

When we compute the variance terms in (20) for a given $f(x)$ using perfect importance random sampling with pdf $p(x)$:

$$\mathbf{f}_0^*\mathbf{f}_0\mathrm{Var}(\mathbf{S}_0) = \frac{0.0703125}{N} \tag{142}$$

$$2\sum_{m=1}^{\infty} \mathbf{f}_m^*\mathbf{f}_m \left\langle \mathbf{S}_m^*\mathbf{S}_m \right\rangle = \frac{0.0703125}{N} \tag{143}$$

$$2\sum_{m=1}^{\infty}\sum_{\substack{l=1\\ l\neq m}}^{\infty} Re\left(\mathbf{f}_m^*\mathbf{f}_l\left\langle \mathbf{S}_m\mathbf{S}_l^* \right\rangle\right) = \frac{3\left(\sin(\frac{l\pi}{2}) + \sin(l\pi)\right)\left(2\sin((l-m)\pi) + \sin(\frac{(m-l)\pi}{2})\right)\left(\sin(\frac{m\pi}{2}) + \sin(m\pi)\right)}{16Nl(l-m)\pi^3 m} \tag{144}$$

$$= 0 \tag{145}$$

The third which is divided into real components can be symbolically solved to get:

$$4\sum_{m=1}^{\infty} Re\left(\mathbf{f}_0\mathbf{f}_m\left\langle \mathbf{S}_0\mathbf{S}_m{}^* \right\rangle\right) = \frac{-0.140625}{N}\,. \tag{146}$$

All these terms add up to form the variance $\mathrm{Var}(I_N) = 0$.

8.2. For Step 1D PDF

$$f(x) := \begin{cases} A & -0.5 < x < 0 \\ B & 0 \leq x < 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{147}$$

$$x := \begin{cases} \dfrac{\xi(A+B)}{2A} - 0.5 & 0 \leq \xi < \dfrac{A}{A+B} \\ \dfrac{\xi(A+B)-A}{2B} & \dfrac{A}{A+B} \leq \xi < 1 \end{cases} \tag{148}$$

$$\mathbf{f}_m = 0.75\mathrm{Sinc}(m\pi) + i\frac{\cos(m\pi) - 1}{4m\pi} \tag{149}$$

$$\mathbf{f}_0 = 0.75 \tag{150}$$

$$\mathbf{f}_0^*\mathbf{f}_0 = 0.5625 \tag{151}$$

$$\mathbf{f}_m^*\mathbf{f}_m = -\frac{2\cos(2m\pi) + \cos(m\pi) - 3}{8m^2\pi^2} \tag{152}$$

$\alpha(x) := 1/p(x)$ where $p(x) := I_f/f(x)$

$$\int_{-0.5}^{0.5} \alpha(x)dx = 1.125 \tag{153}$$

$$\mathrm{Var}(\mathbf{S}_0) = \left\langle \mathbf{S}_0^2 \right\rangle - < \mathbf{S}_0 >^2 = \frac{1}{N}\int \alpha(x)\mathrm{d}x + \frac{N-1}{N} - 1 = \frac{0.125}{N} \tag{154}$$

$$\left\langle \mathbf{S}_m^*\mathbf{S}_m \right\rangle = \frac{1.125}{N} \quad \forall\, m \neq 0 \tag{155}$$

$$\left\langle \mathbf{S}_0\mathbf{S}_m{}^* \right\rangle = \frac{1}{N}\mathcal{F}_w(m)^* \tag{156}$$

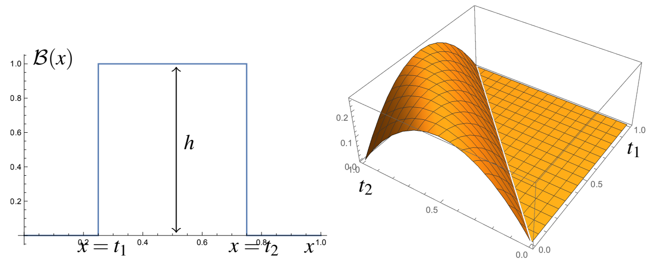
$$\mathcal{F}_w(m) = \frac{0.358099\sin(\pi m)}{m} - i\frac{0.119366 - 0.119366\cos(\pi m)}{m} \tag{157}$$

again compute the variance terms for this tent function using perfect importance random sampling with pdf $p(x)$:

$$\mathbf{f}_0^*\mathbf{f}_0\mathrm{Var}(\mathbf{S}_0) = \frac{0.0703125}{N} \tag{158}$$

$$2\sum_{m=1}^{\infty}\mathbf{f}_m^*\mathbf{f}_m \left\langle \mathbf{S}_m^*\mathbf{S}_m \right\rangle = \frac{0.0703125}{N} \tag{159}$$

and the third term is $= \frac{-0.140625}{N}$. All these terms add up to form the variance $\mathrm{Var}(I_N) = 0$.



(a) Pulse

(b) Variance of Pulse with $t_1 < t_2$

Figure 4: *Variance for a pulse $\mathcal{B}(x)$ function having C_0 discontinuities is shown to depend on the locations of these C_0 discontinuities.*

9. C_0 Discontinuities

9.1. Pulse discontinuities

Following a similar procedure from the main paper, we can obtain the variance of a pulse function $\mathcal{B}(x) = H(x-t_1) - H(x-t_2)$, in the form: $\mathrm{Var}(\mathcal{B}(x)) = h^2(t_2-t_1)(1-t_2+t_1)$, where, $0 \le t_1 < t_2 \le 1$ and h is the height of the pulse Fig. 4(a). For jittered sampling, this is similar (after appropriate scaling) to the case when both the edges fall within the same stratum leading to the variance in the form: $\mathrm{Var}(\mu_N) = h^2 N(t_2-t_1)(1-N(t_2-t_1))$. As shown in Fig. 4(b), this equation also has a parabolic form suggesting that if the discontinuities can be shifted near the boundaries of the domain (or stratrum) then the variance can be reduced significantly.

If both the edges of the pulse falls in separate strata (say j-th and k-th strata) then the variance of the MC estimator would simply be the sum of the variance of these two strata (since the function is constant for all other strata) and can be easily derived in the form: $\mathrm{Var}(\mu_N) = (Nt_1 - j)(j+1-Nt_1)h^2 + (Nt_2-k)(k+1-Nt_2)h^2$.

More evaluations regarding the impact of shifting strata boundaries on the variance and convergence rate is shown in the following figures.

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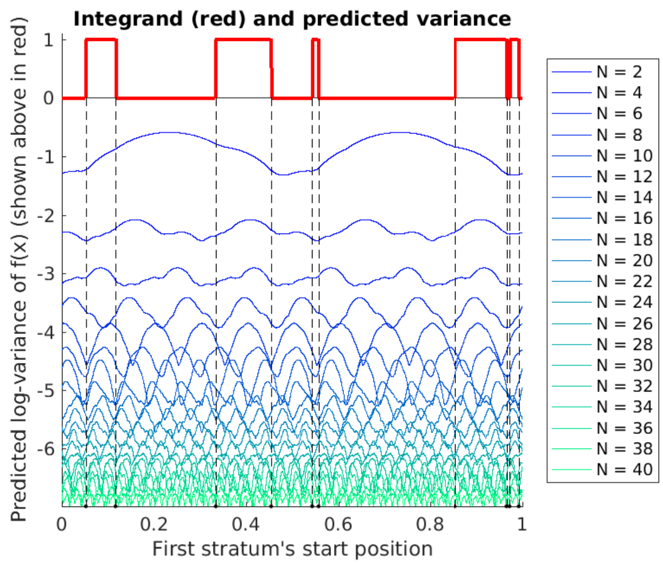

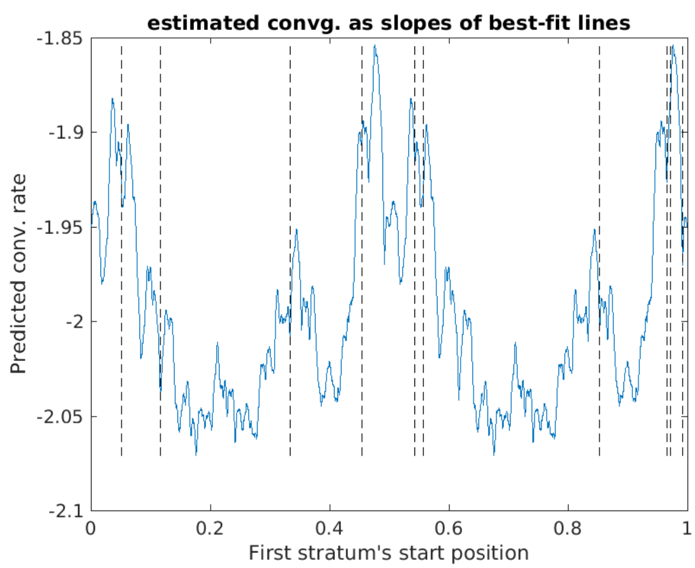

Figure 5: *Variance as a function of the location of the discontinuities.*

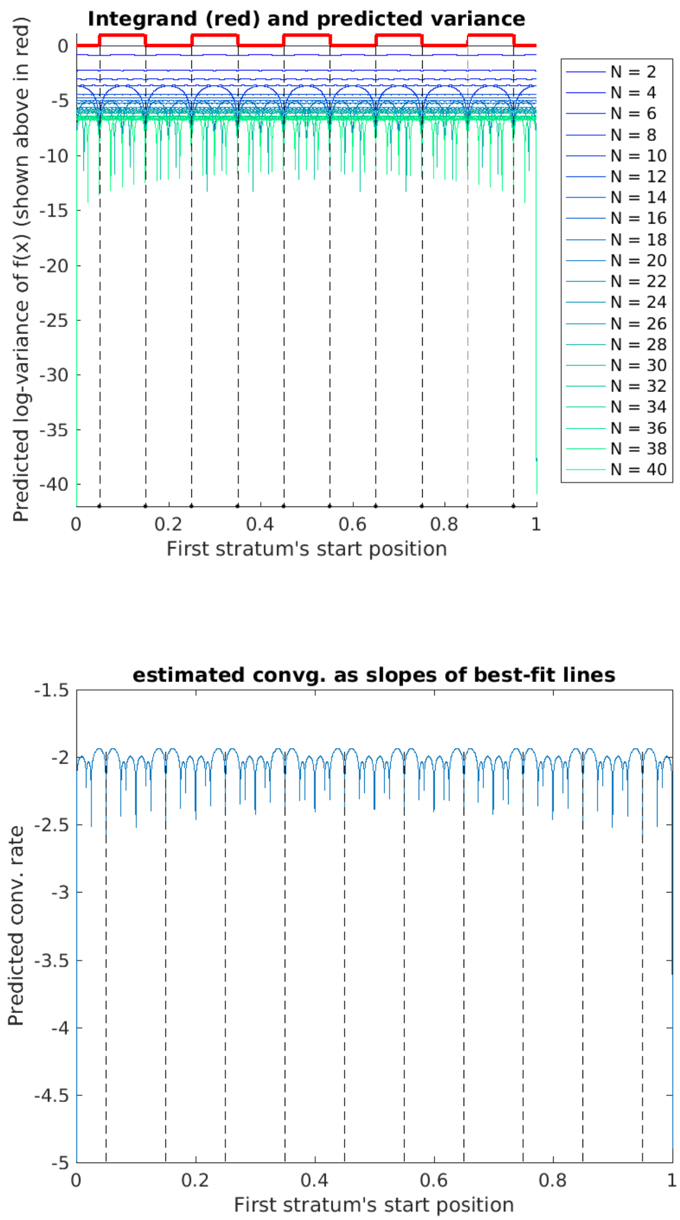

Figure 6: *Variance as a function of the location of the discontinuities.*

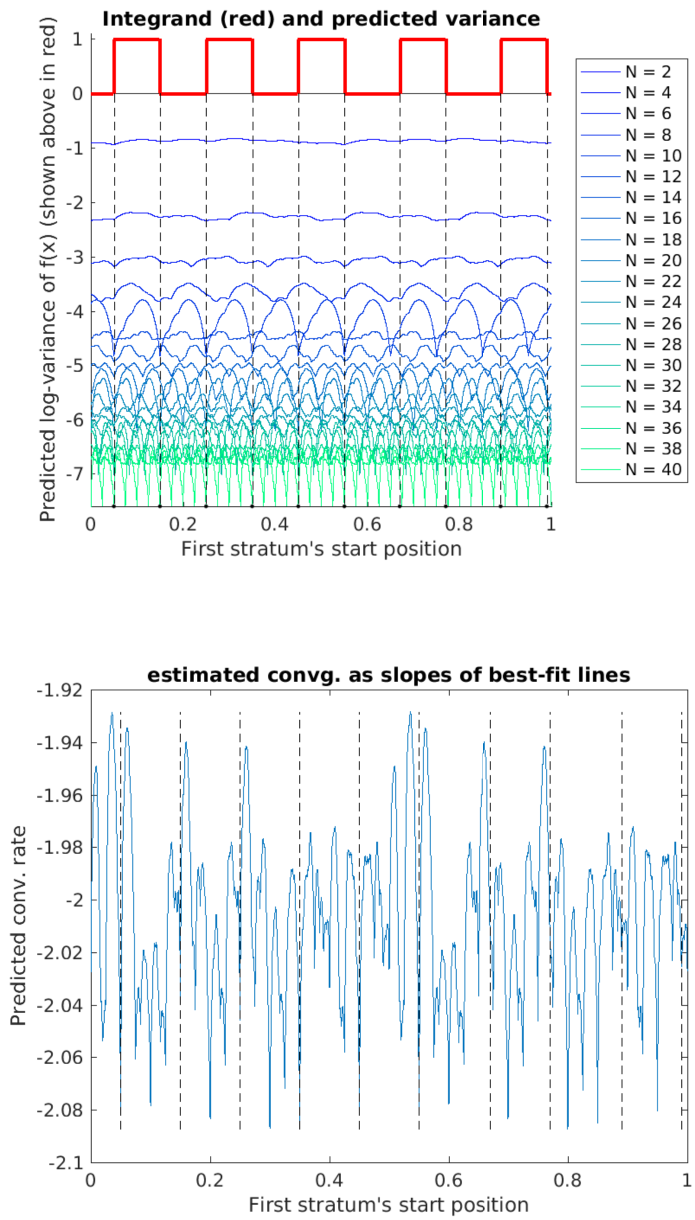

Figure 7: *Variance as a function of the location of the discontinuities.*