



Harmonic Caching for Walk on Spheres

Zihong Zhou

Dartmouth College

Eugene d'Eon

NVIDIA

Rohan Sawhney

NVIDIA

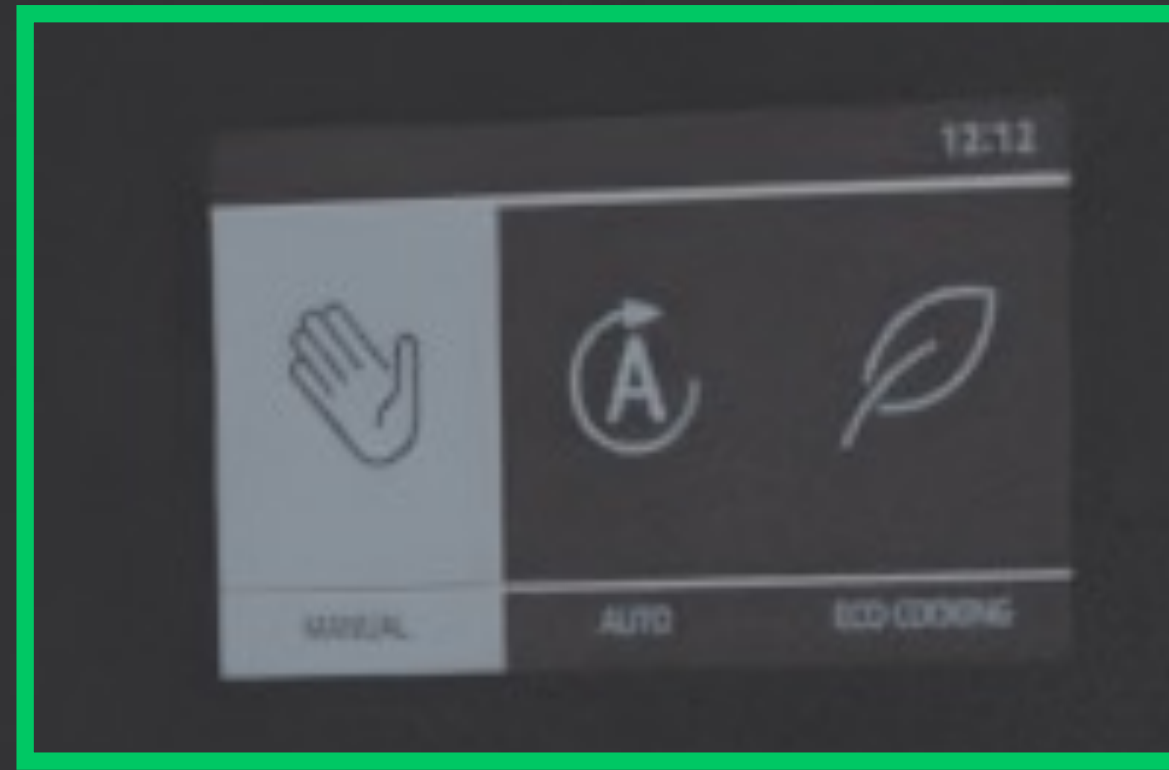
Wojciech Jarosz

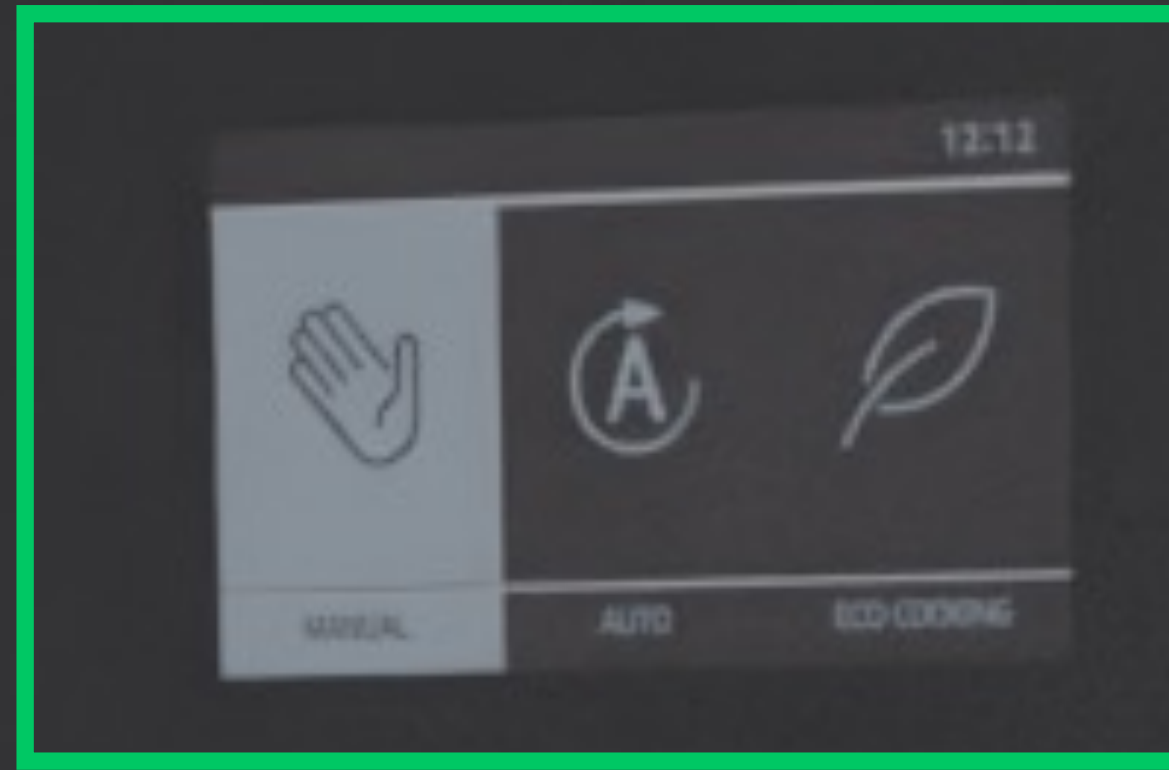
Dartmouth College

Rendering

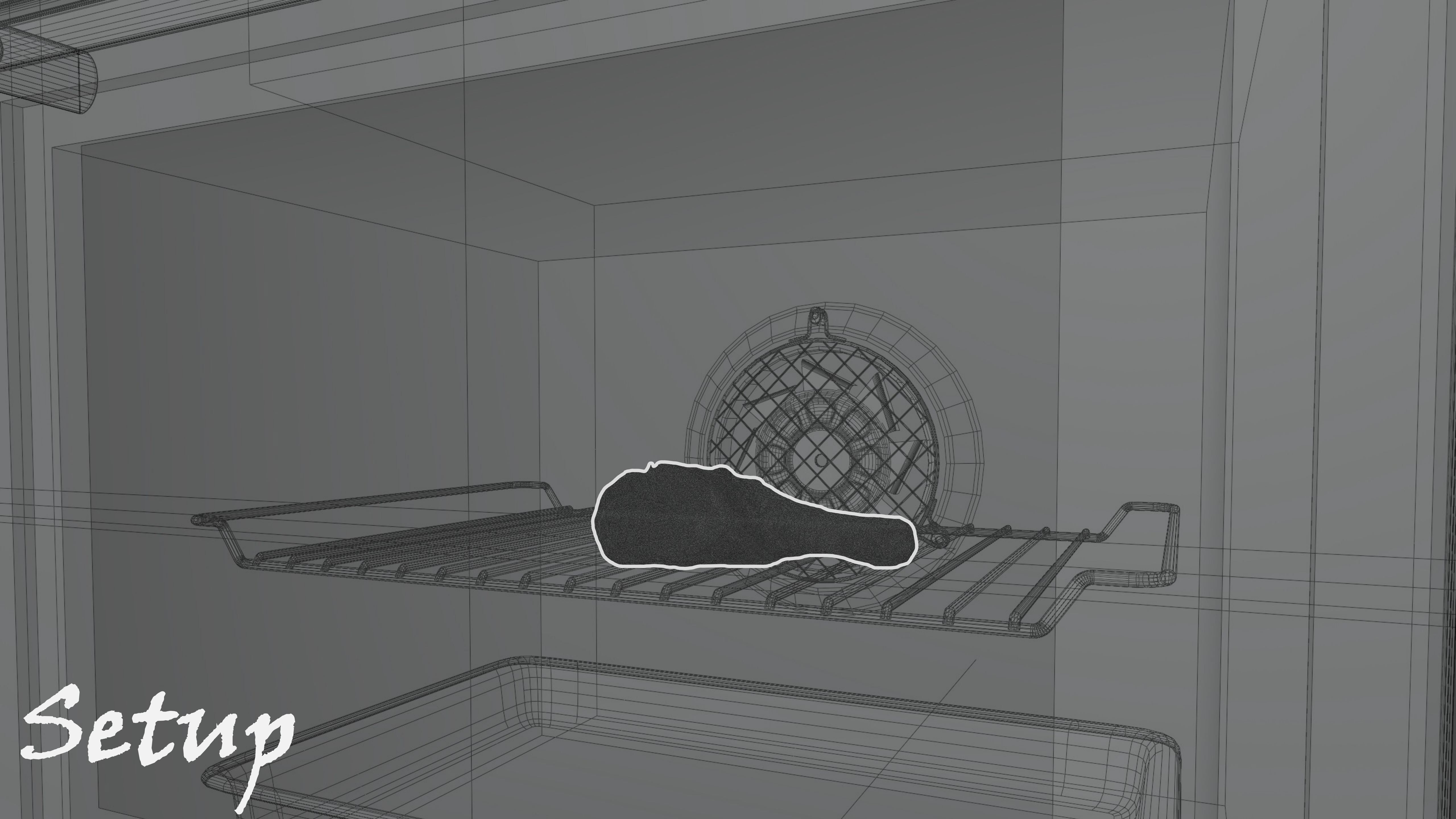


Rendering

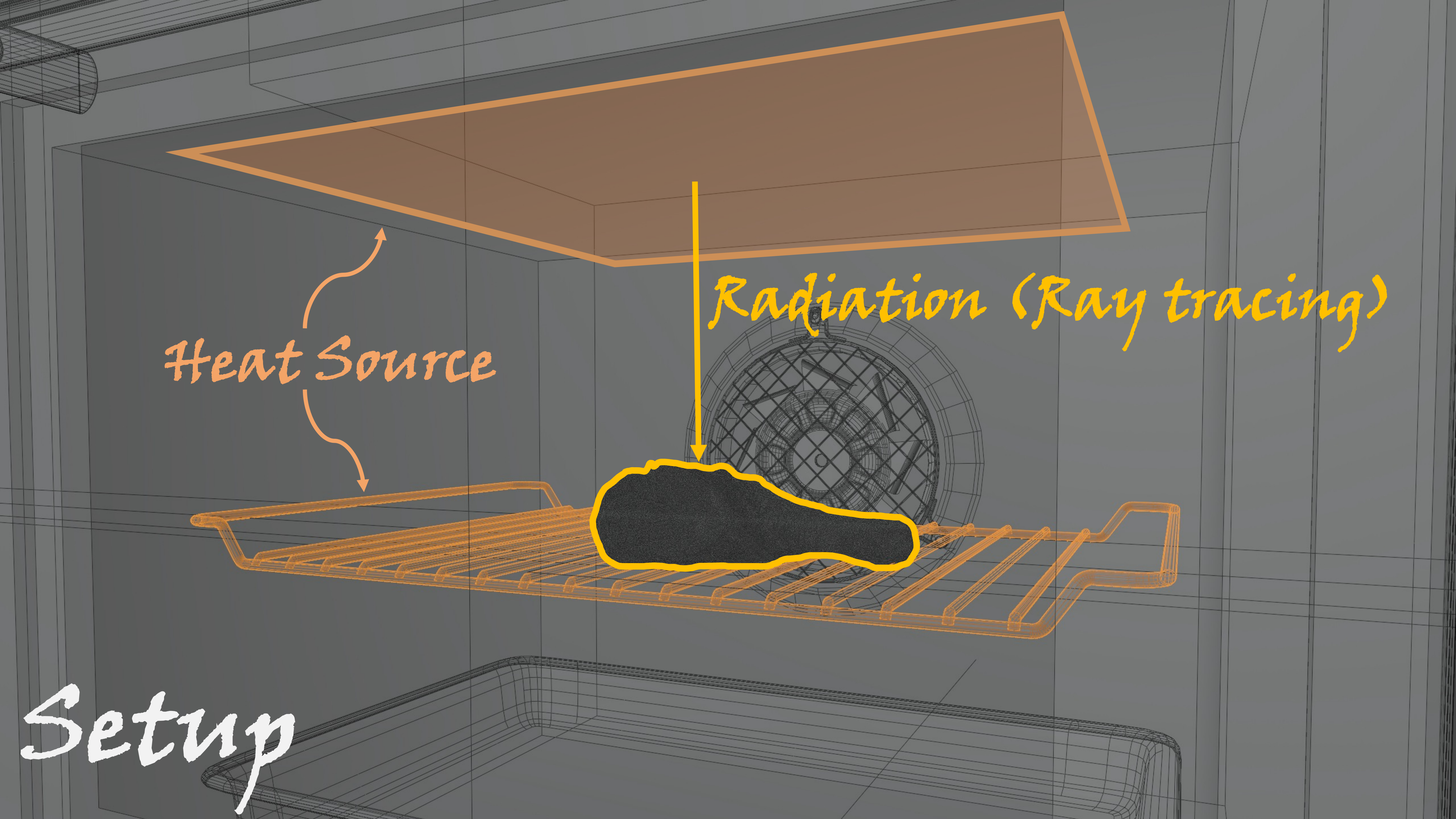




Heat Simulation?



Setup

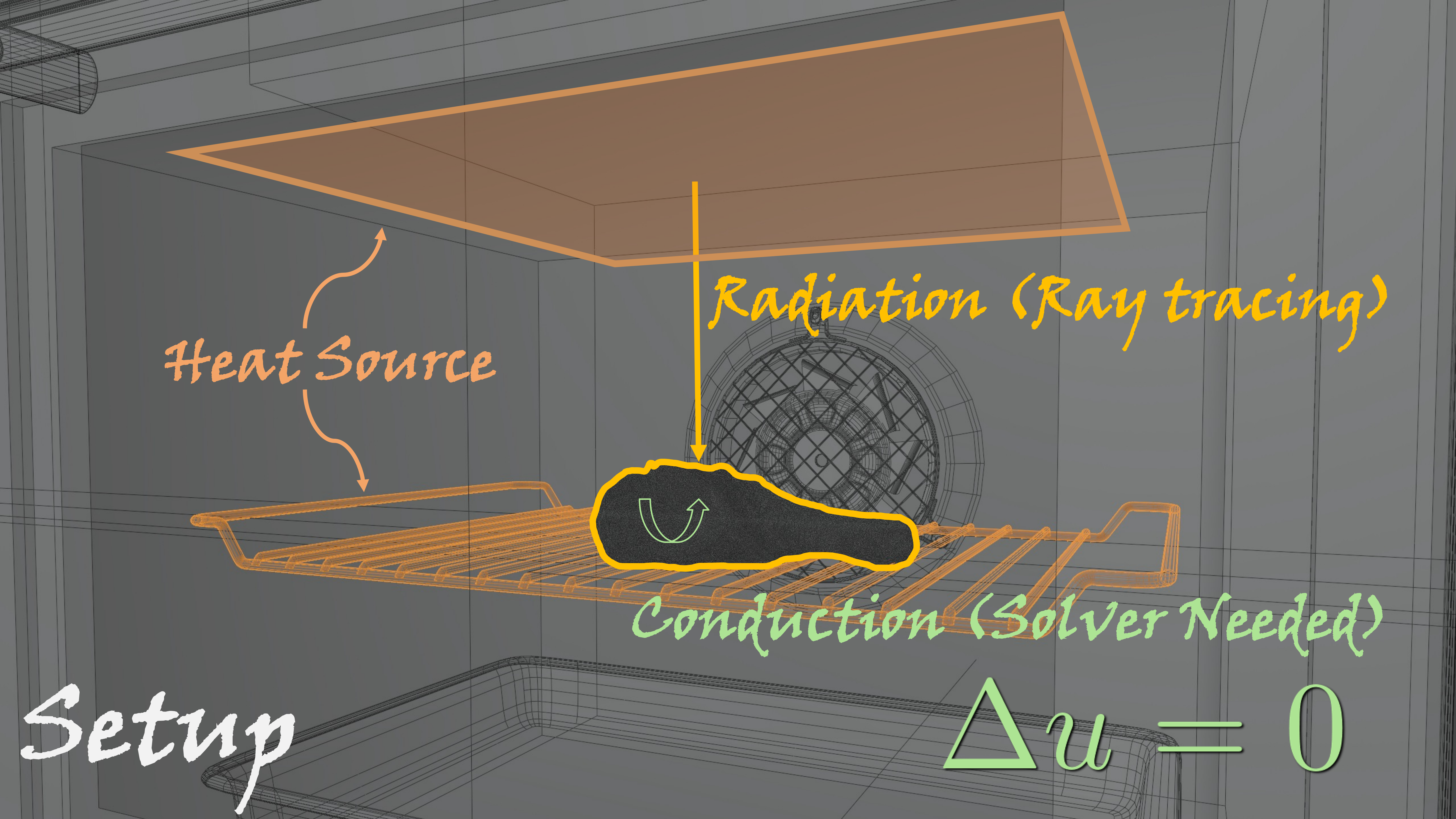


A 3D wireframe schematic of a mechanical setup. A brown rectangular plate is positioned above a black, irregularly shaped object. The black object sits on a yellow wireframe tray. A yellow arrow points from the text 'Radiation (Ray tracing)' to the black object. A yellow bracket labeled 'Heat Source' encompasses the black object and the tray. A circular fan-like structure is visible in the background. The entire scene is set against a gray grid background.

Heat Source

Radiation (Ray tracing)

Setup



Heat Source

Radiation (Ray tracing)

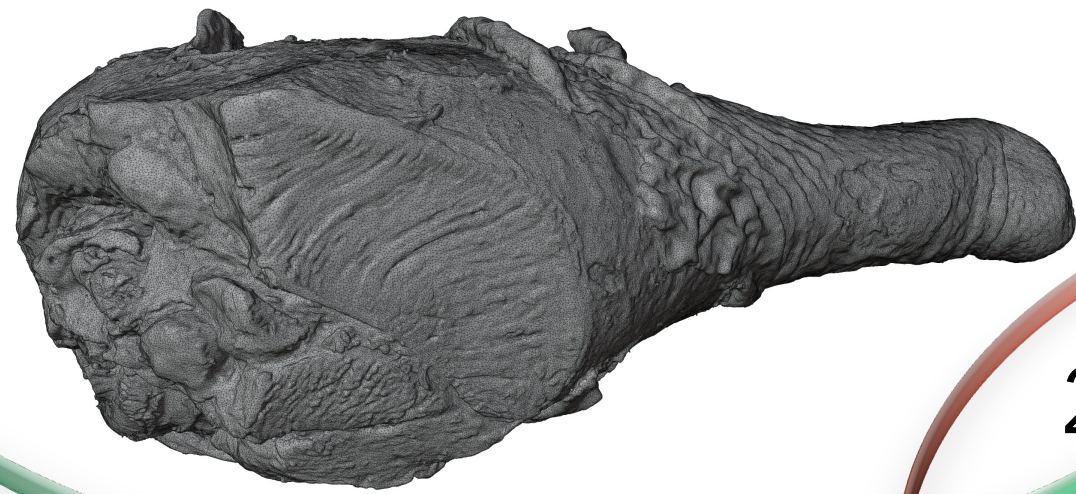
Conduction (Solver Needed)

Setup

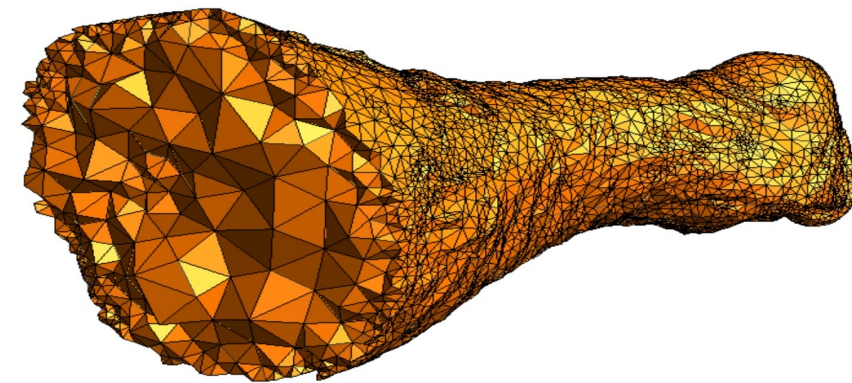
$$\Delta u = 0$$

Grid-based solvers require meshing

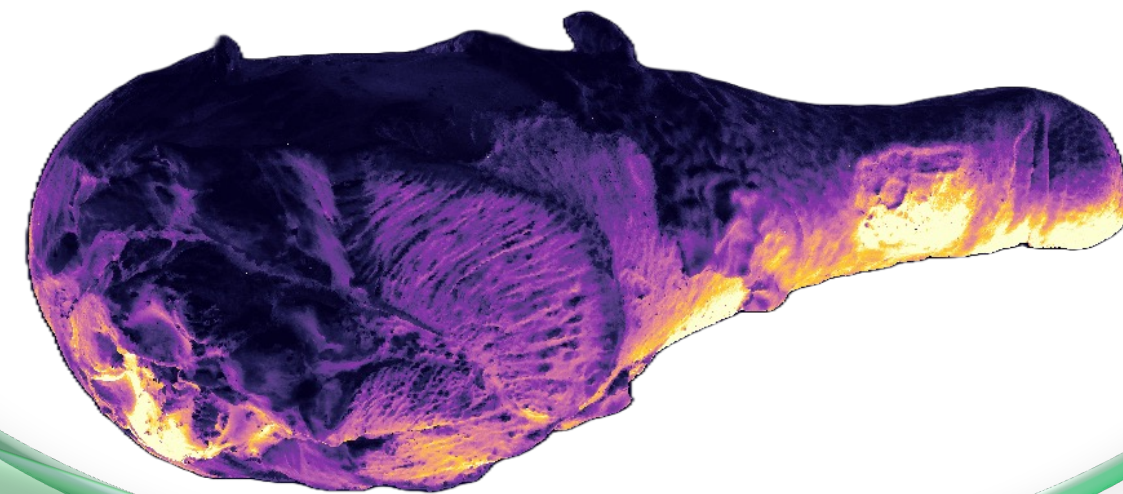
1. Modeling boundary mesh



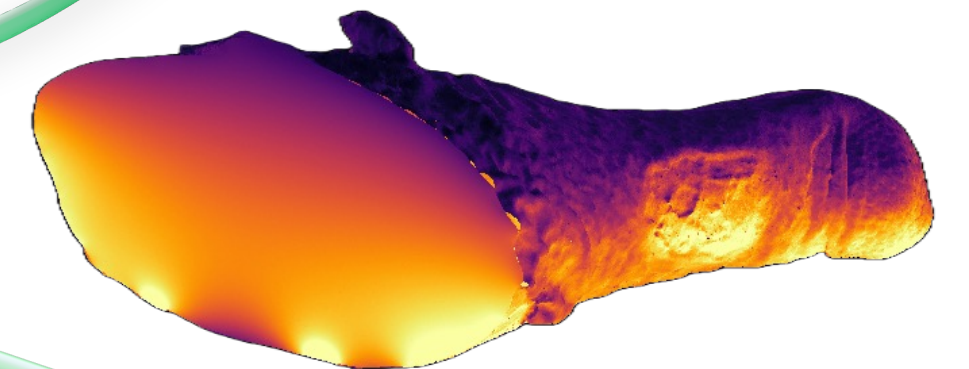
2. Meshing volumetric domain



3. Assembling boundary conditions



4. FEM solving



FEM Pipeline

Meshing can be challenging

Input boundary mesh



★★★★☆ 4.19 from 22 votes

Grilled Turkey Legs (Caveman Pops)

Grilled Turkey Legs are a fun take on Thanksgiving dinner! These Caveman Pops are moist and flavorful with a simple brine and so easy to cook on the grill!

🕒 Prep Time: 10 mins Cook Time: 1 hr
Brining: 4 hrs **Total Time: 1 hr 10 mins**

👤 Author: Lalaine Manalo

🍴 Course: Appetizer, Main Entree

<https://www.onionringsandthings.com/grilled-turkey-legs/>

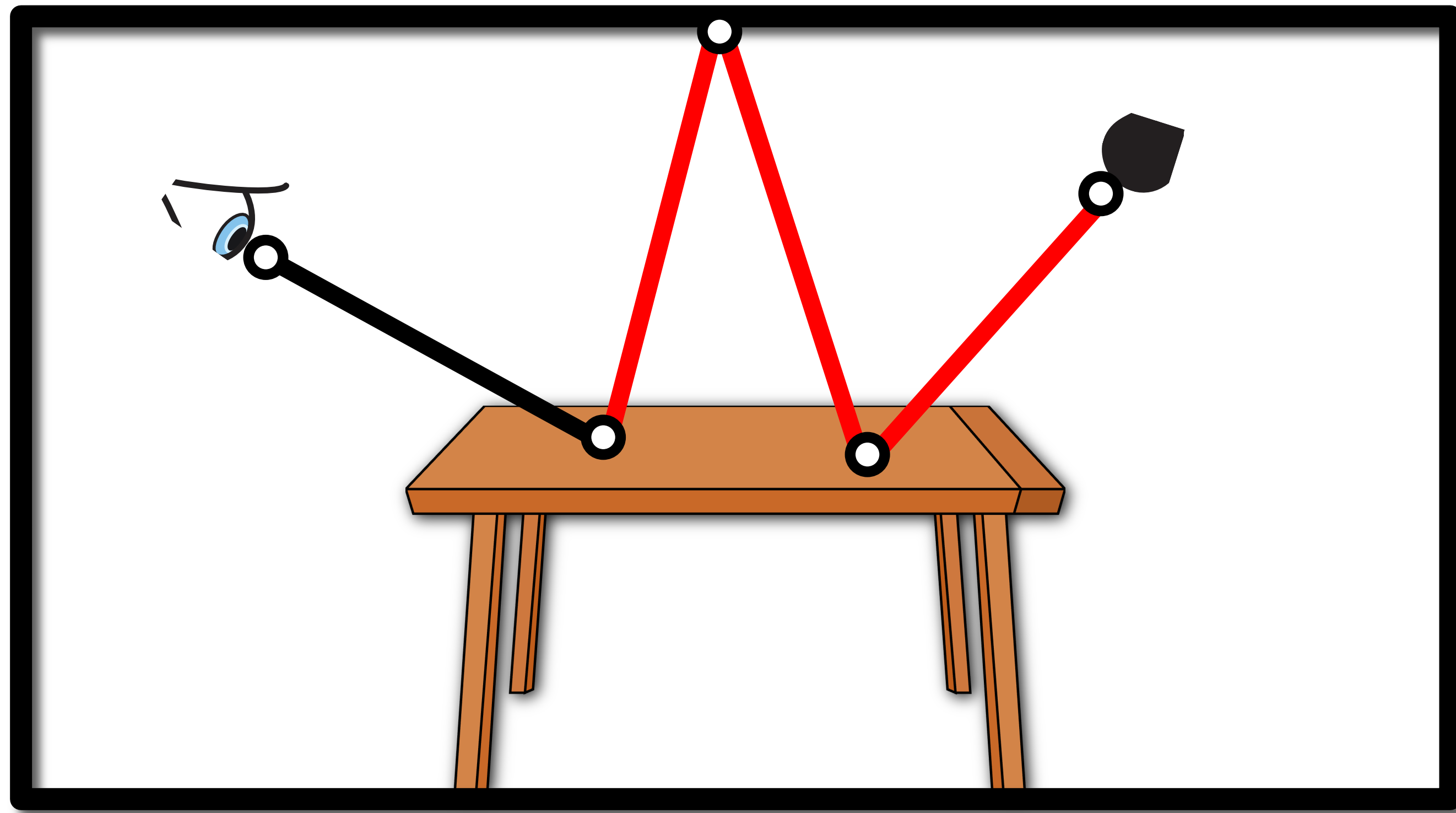


Tetrahedral mesh boundary

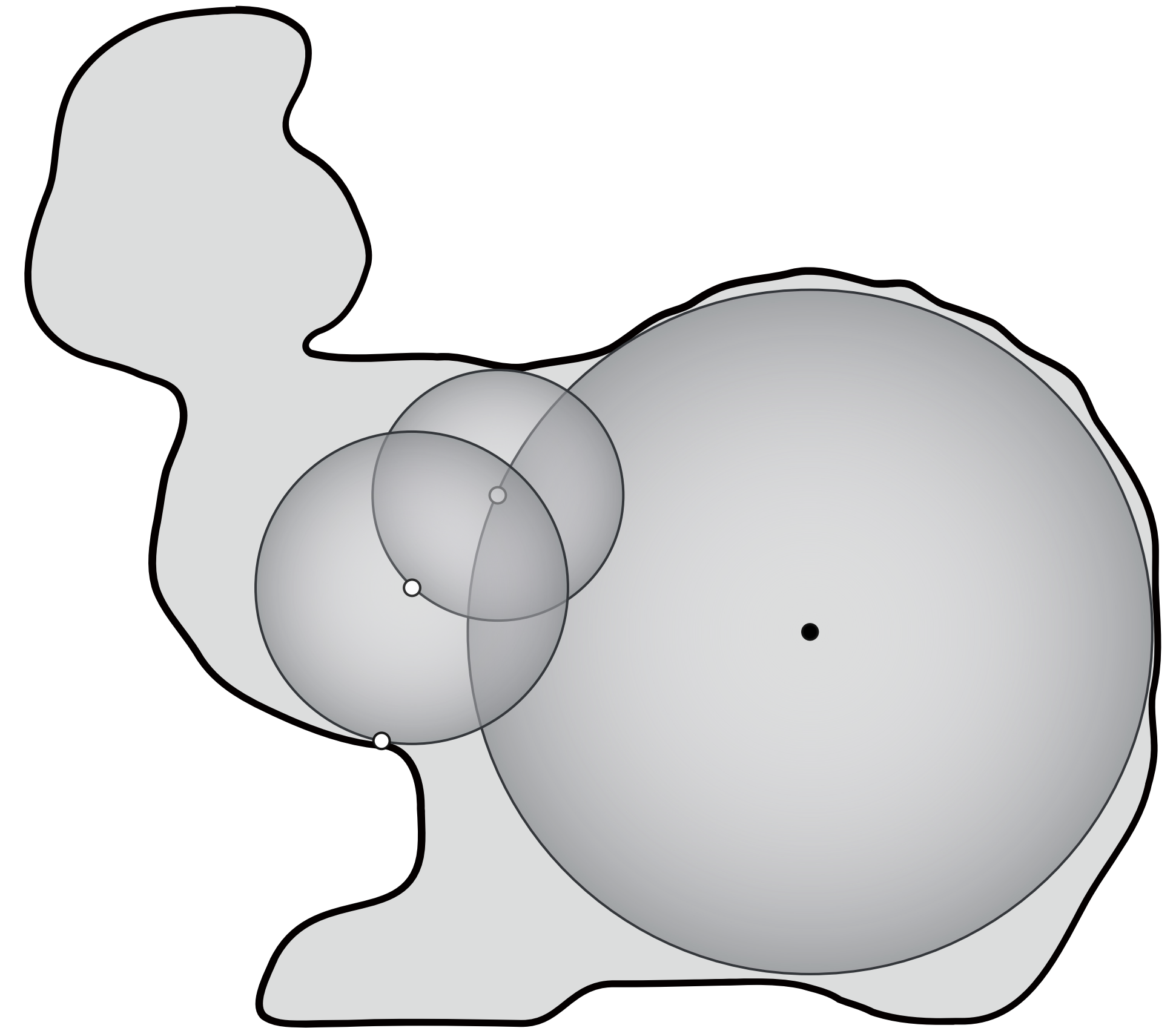


[Fast Tetrahedral Meshing in the Wild]

Path tracing and grid free PDE solvers



Path tracing

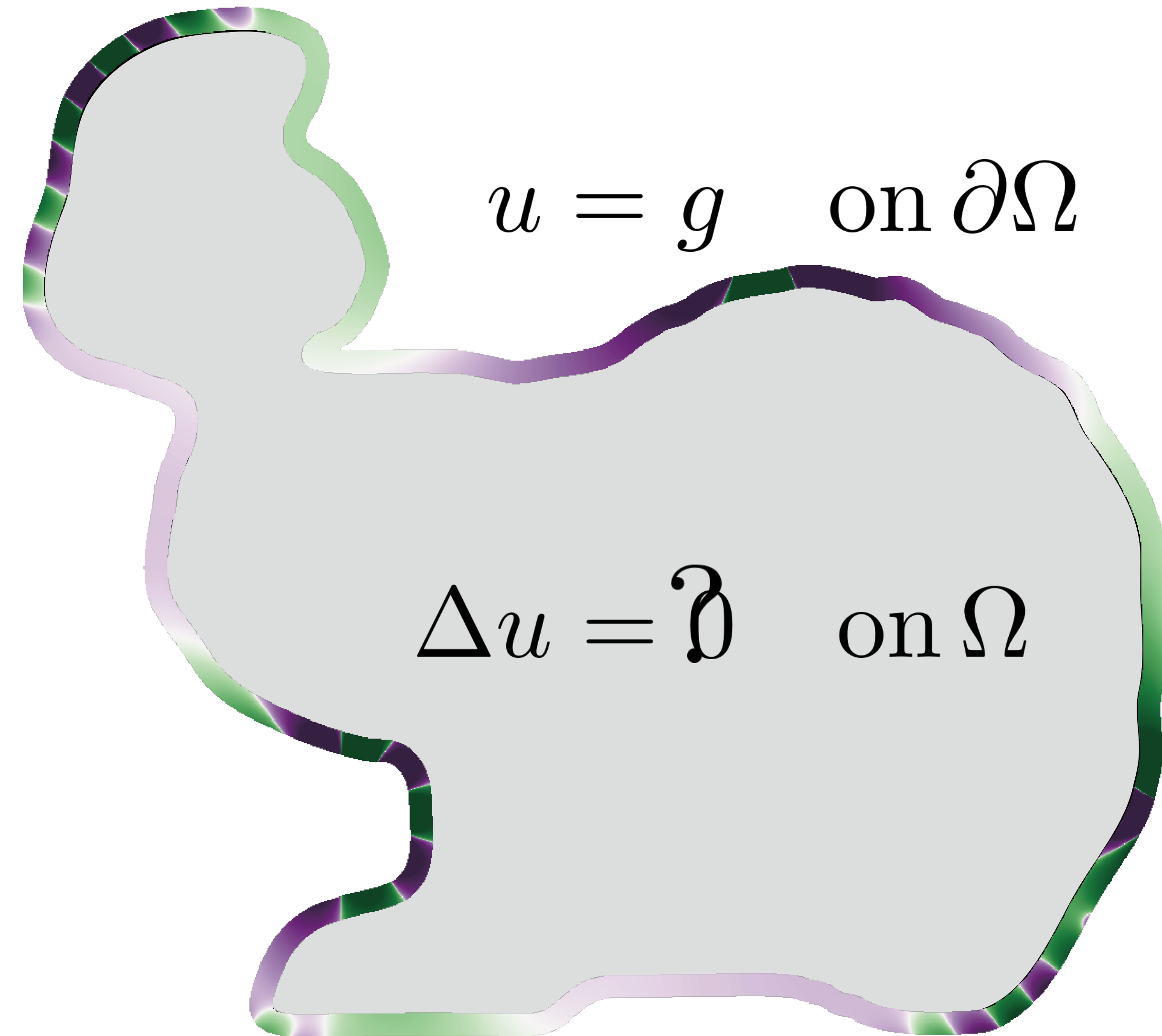


Walk on Spheres



Background

Dirichlet problem



Dirichlet problem

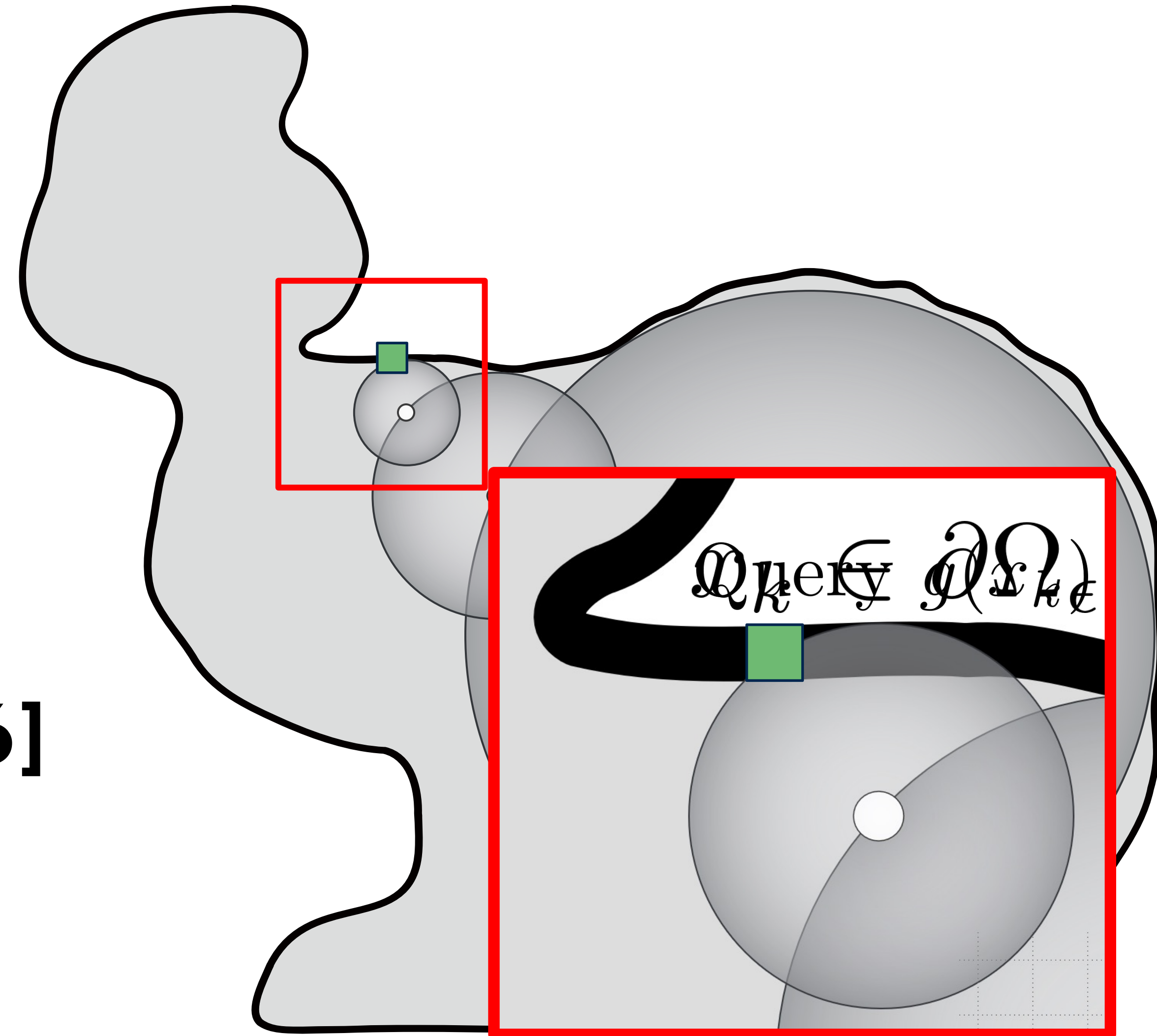
$$\begin{aligned}\Delta u &= 0 && \text{on } \Omega \\ u &= g && \text{on } \partial\Omega\end{aligned}$$

Mean Value Property

$$u(x) = \frac{1}{|\partial B(x)|} \int_{\partial B(x)} u(z) \, dz$$

Walk on Spheres [Muller 1956]

$$\hat{u}(x_k) := \begin{cases} g(x_k), & x_k \in \partial\Omega_\epsilon, \\ \hat{u}(x_{k+1}), & \text{otherwise.} \end{cases}$$



Dirichlet problem

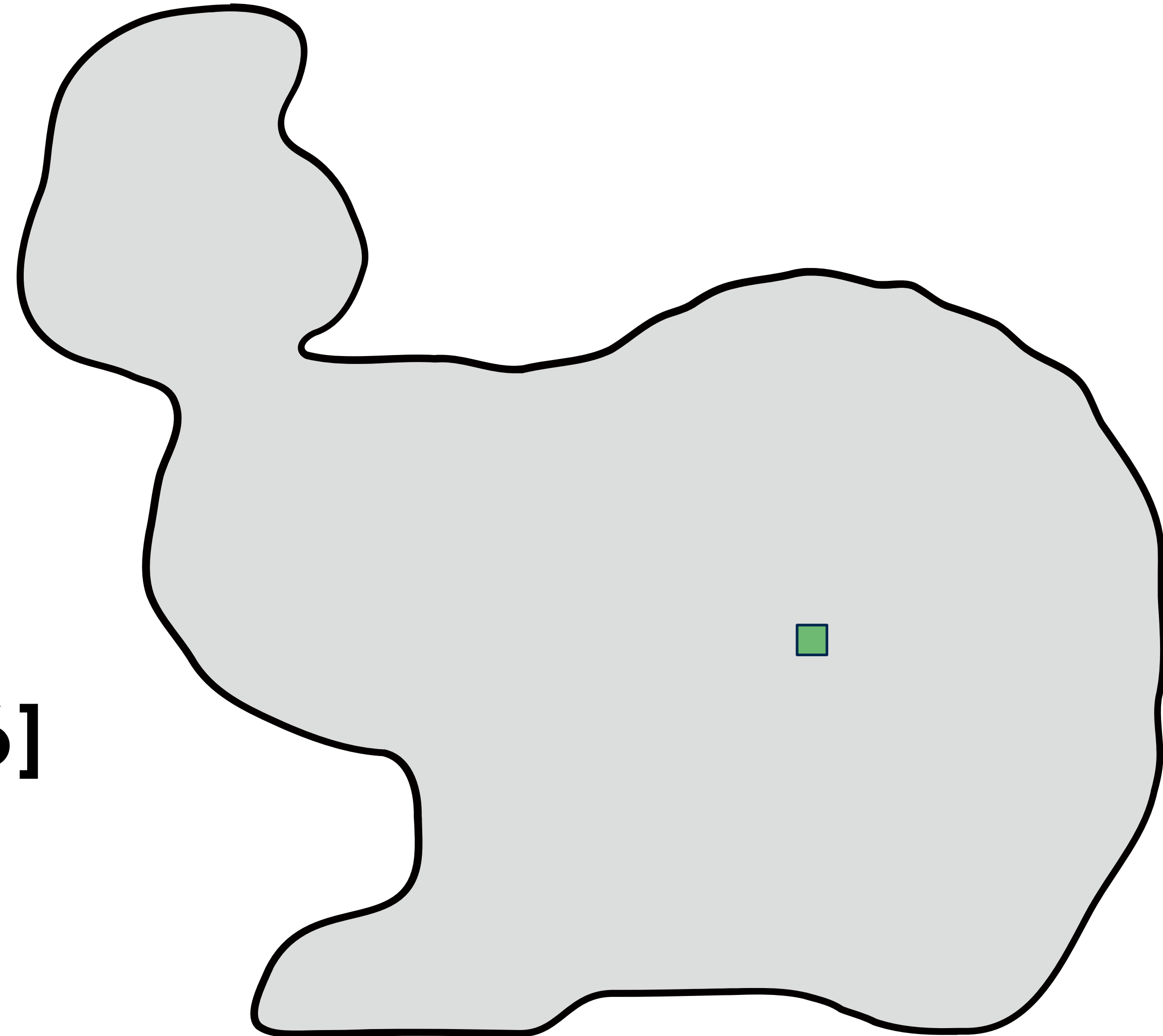
$$\begin{aligned}\Delta u &= 0 && \text{on } \Omega \\ u &= g && \text{on } \partial\Omega\end{aligned}$$

Mean Value Property

$$u(x) = \frac{1}{|\partial B(x)|} \int_{\partial B(x)} u(z) \, dz$$

Walk on Spheres [Muller 1956]

$$\hat{u}(x_k) := \begin{cases} g(x_k), & x_k \in \partial\Omega_\epsilon, \\ \hat{u}(x_{k+1}), & \text{otherwise.} \end{cases}$$



Dirichlet problem

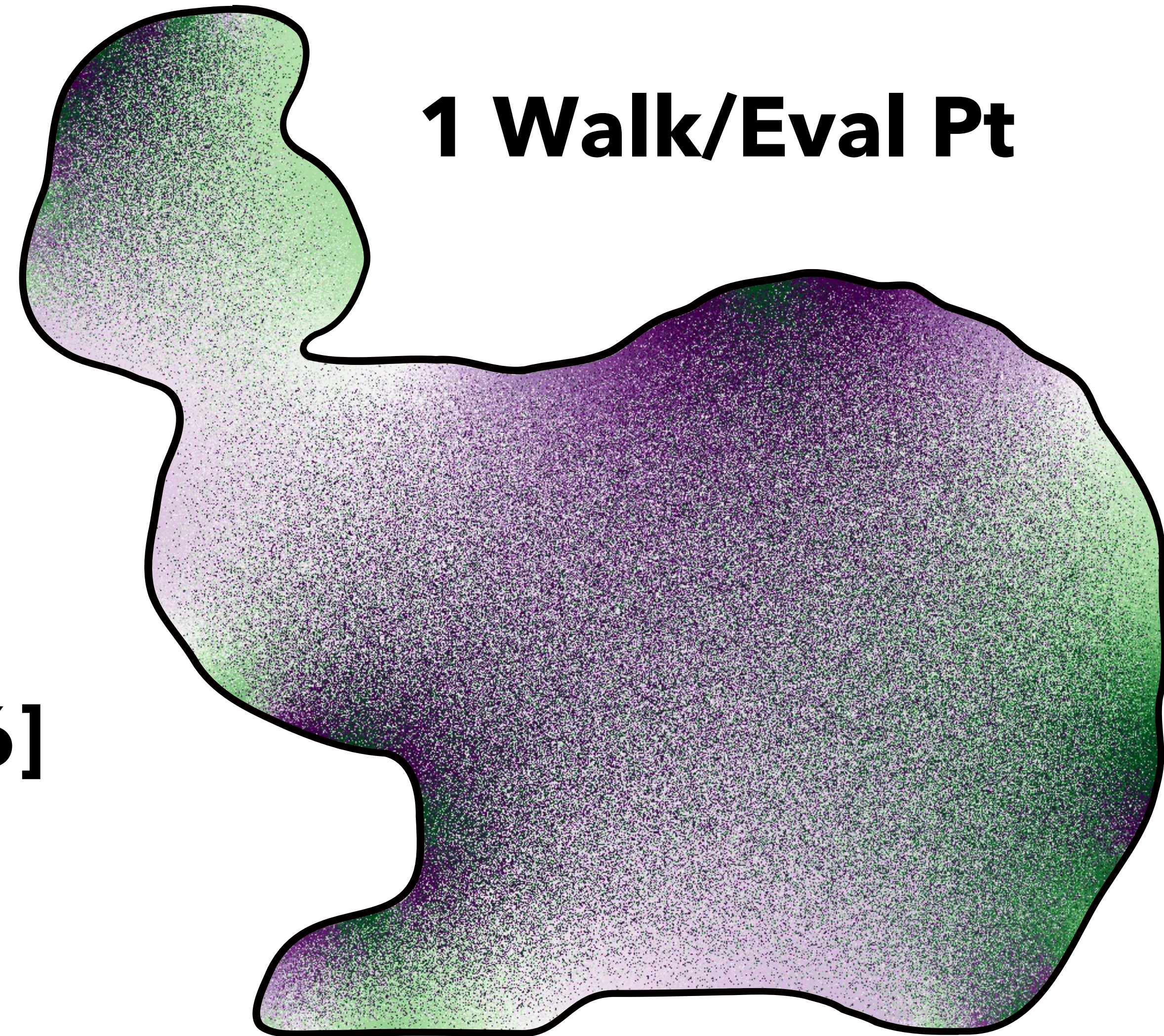
$$\begin{aligned}\Delta u &= 0 && \text{on } \Omega \\ u &= g && \text{on } \partial\Omega\end{aligned}$$

Mean Value Property

$$u(x) = \frac{1}{|\partial B(x)|} \int_{\partial B(x)} u(z) \, dz$$

Walk on Spheres [Muller 1956]

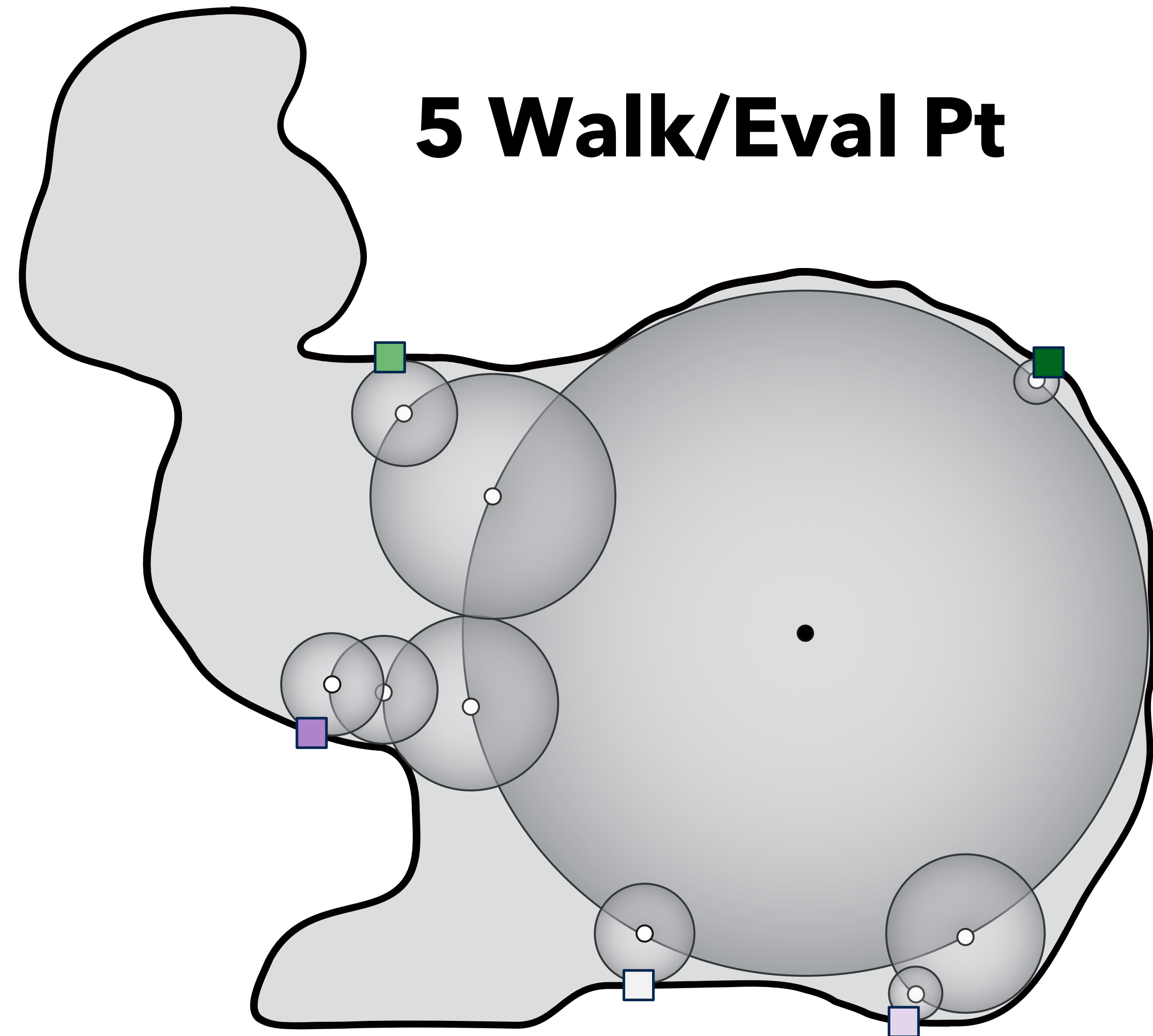
$$\hat{u}(x_k) := \begin{cases} g(x_k), & x_k \in \partial\Omega_\epsilon, \\ \hat{u}(x_{k+1}), & \text{otherwise.} \end{cases}$$



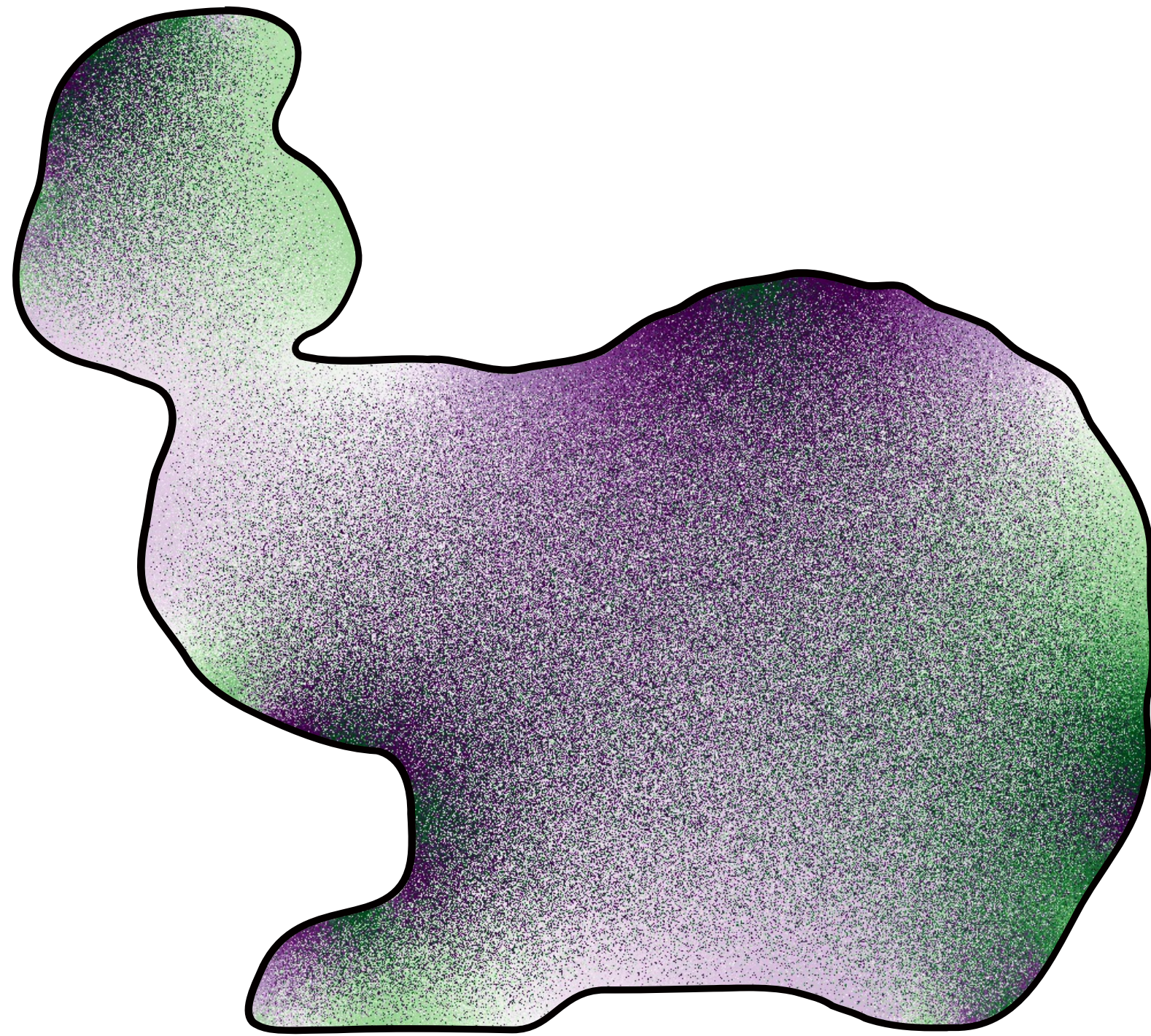
Run more walks to reduce variance

Maybe do more walks?

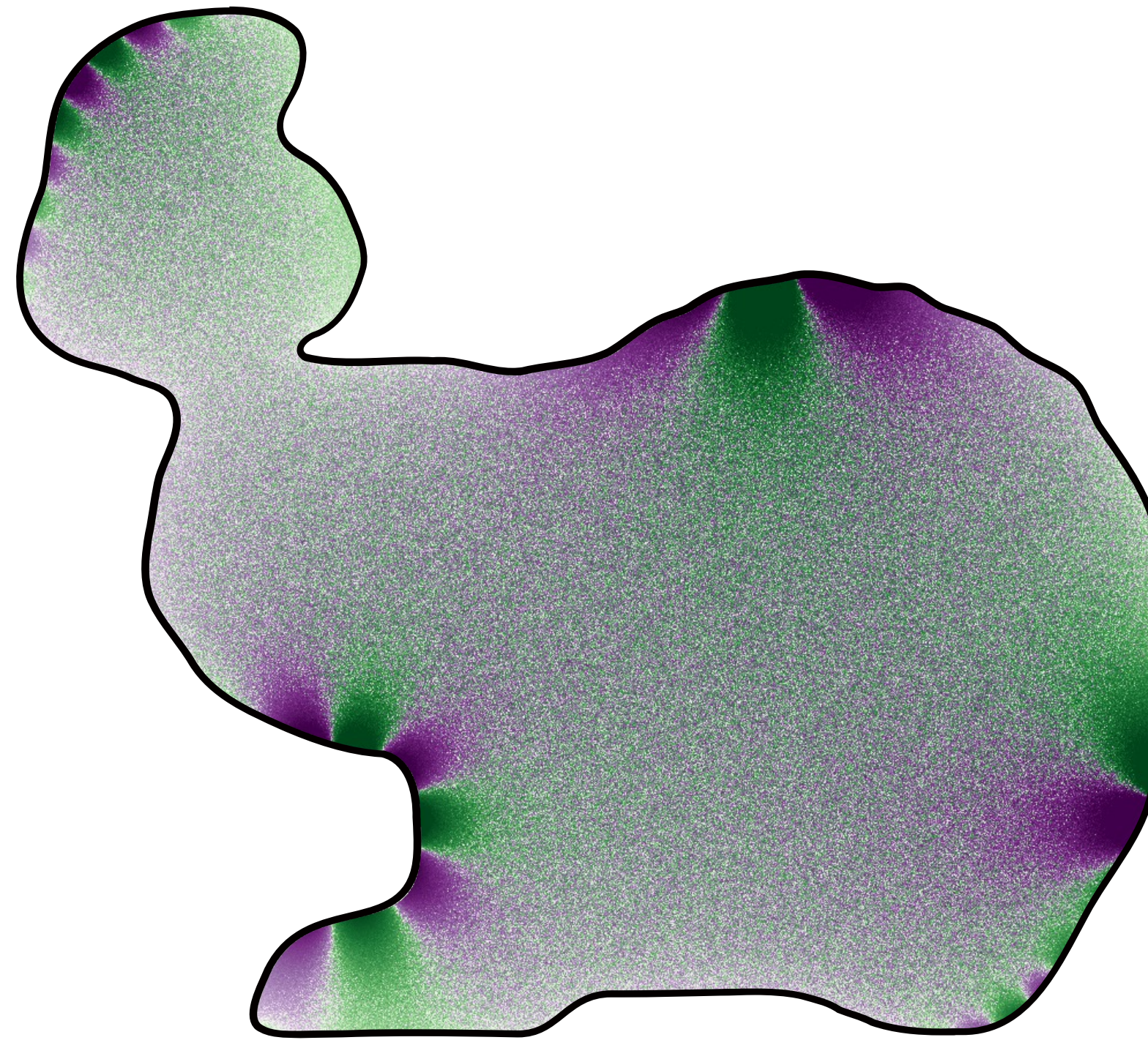
$$u(x) \approx \frac{\text{Green} + \text{Purple} + \text{Light Blue} + \text{Light Purple} + \text{Dark Green}}{5}$$



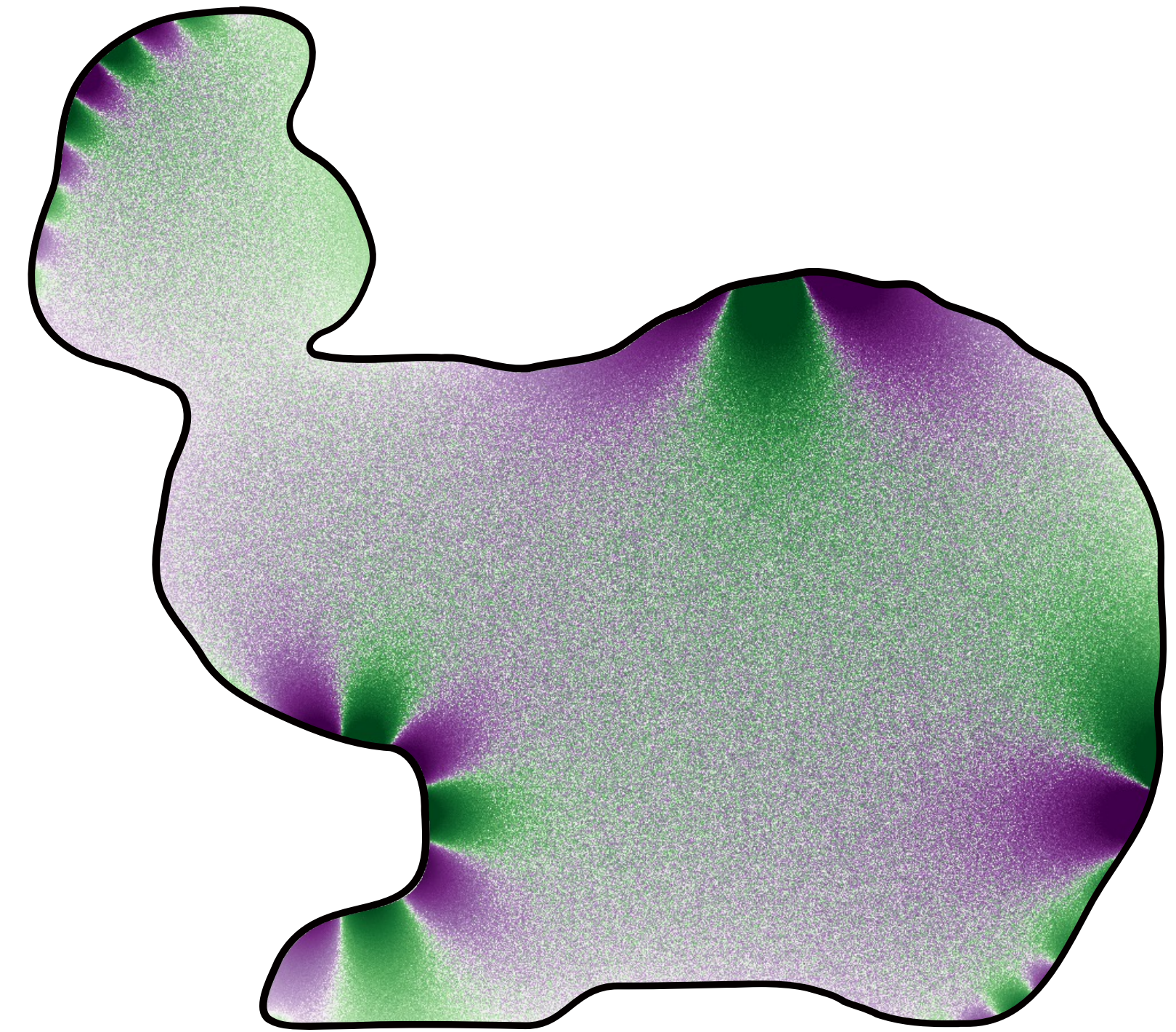
WoS produces noisy estimations



1 Walk/Eval Pt
2s



512 Walk/Eval Pt
180s

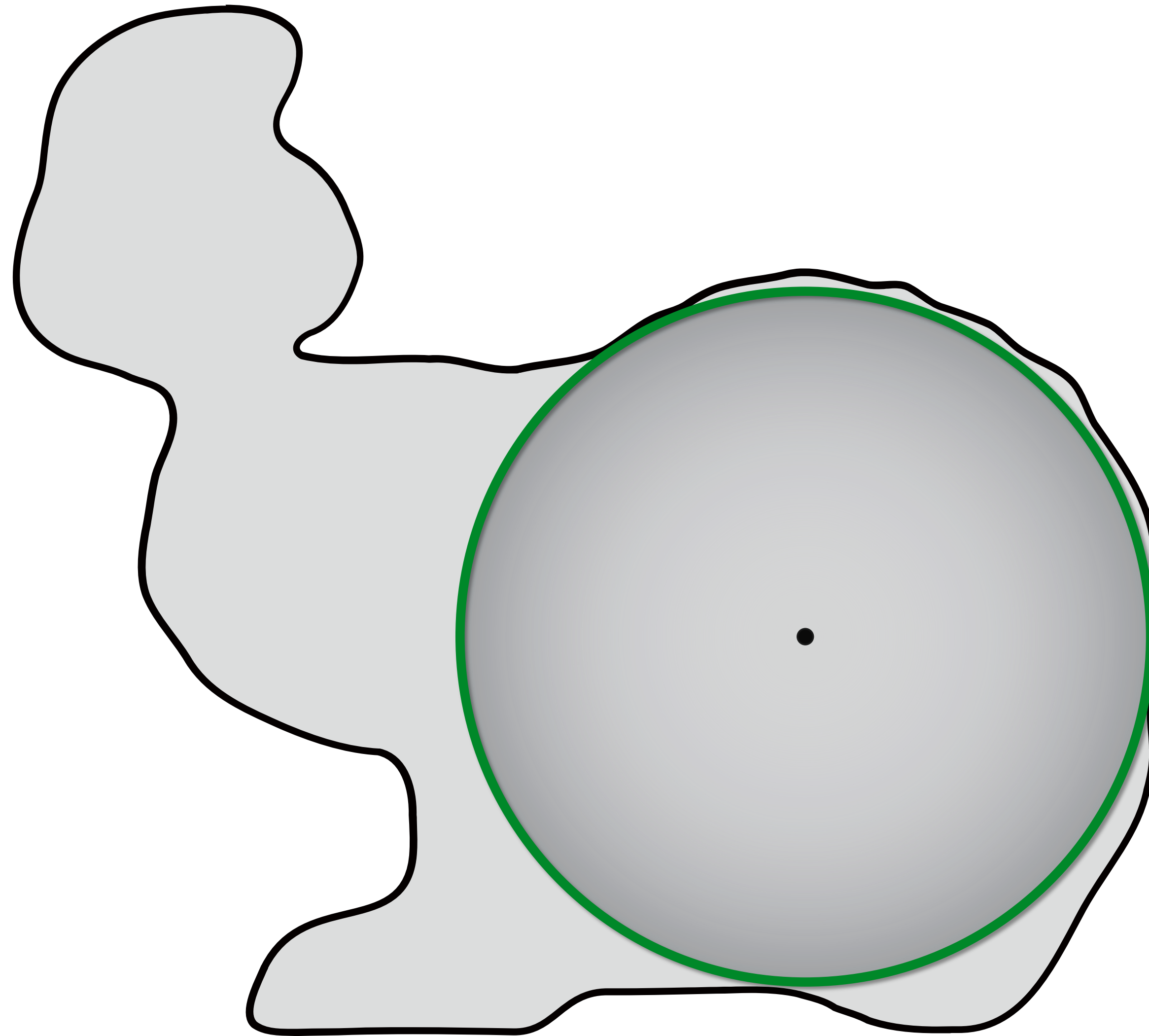


1024 Walk/Eval Pt
360s

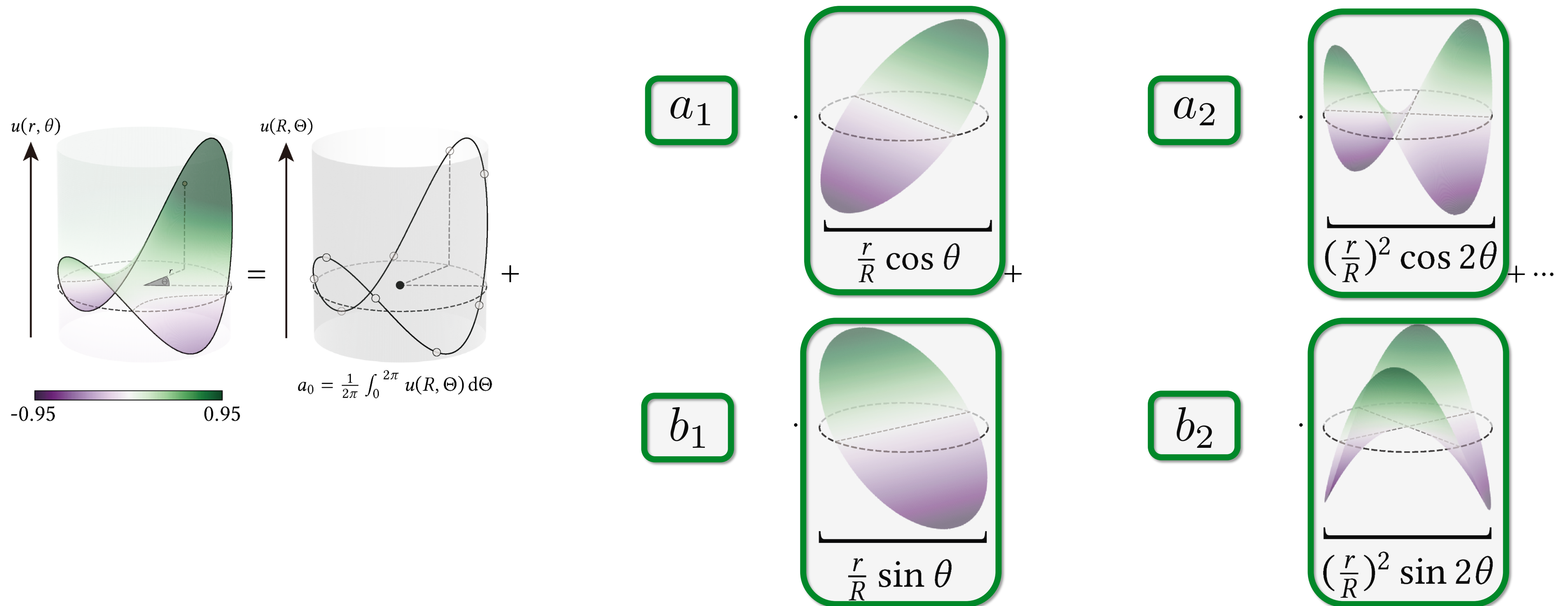


Method

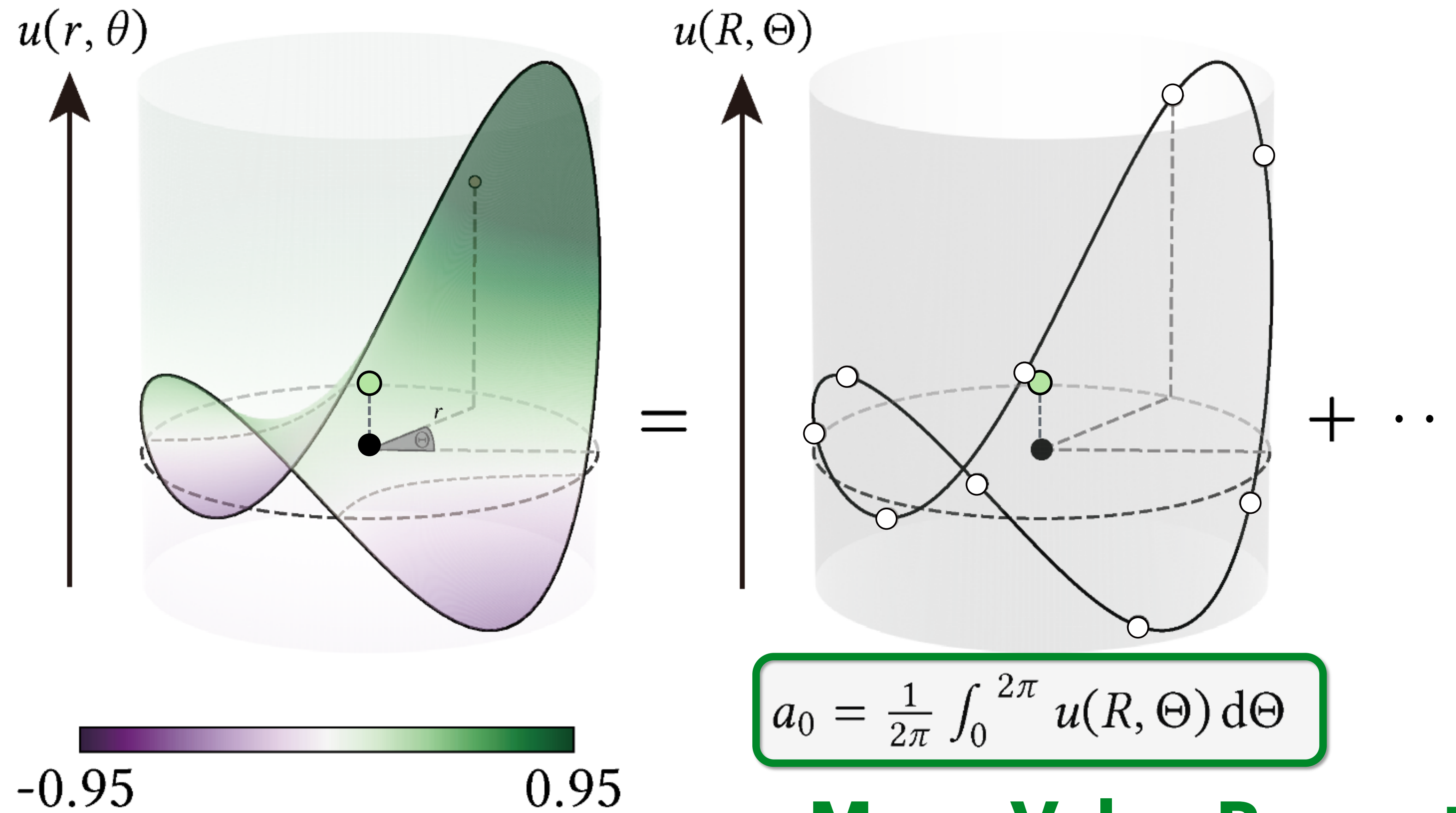
Harmonic series expansion [Booth 1981, 1982]



Harmonic series expansion [Booth 1981, 1982]



Harmonic series expansion [Booth 1981, 1982]



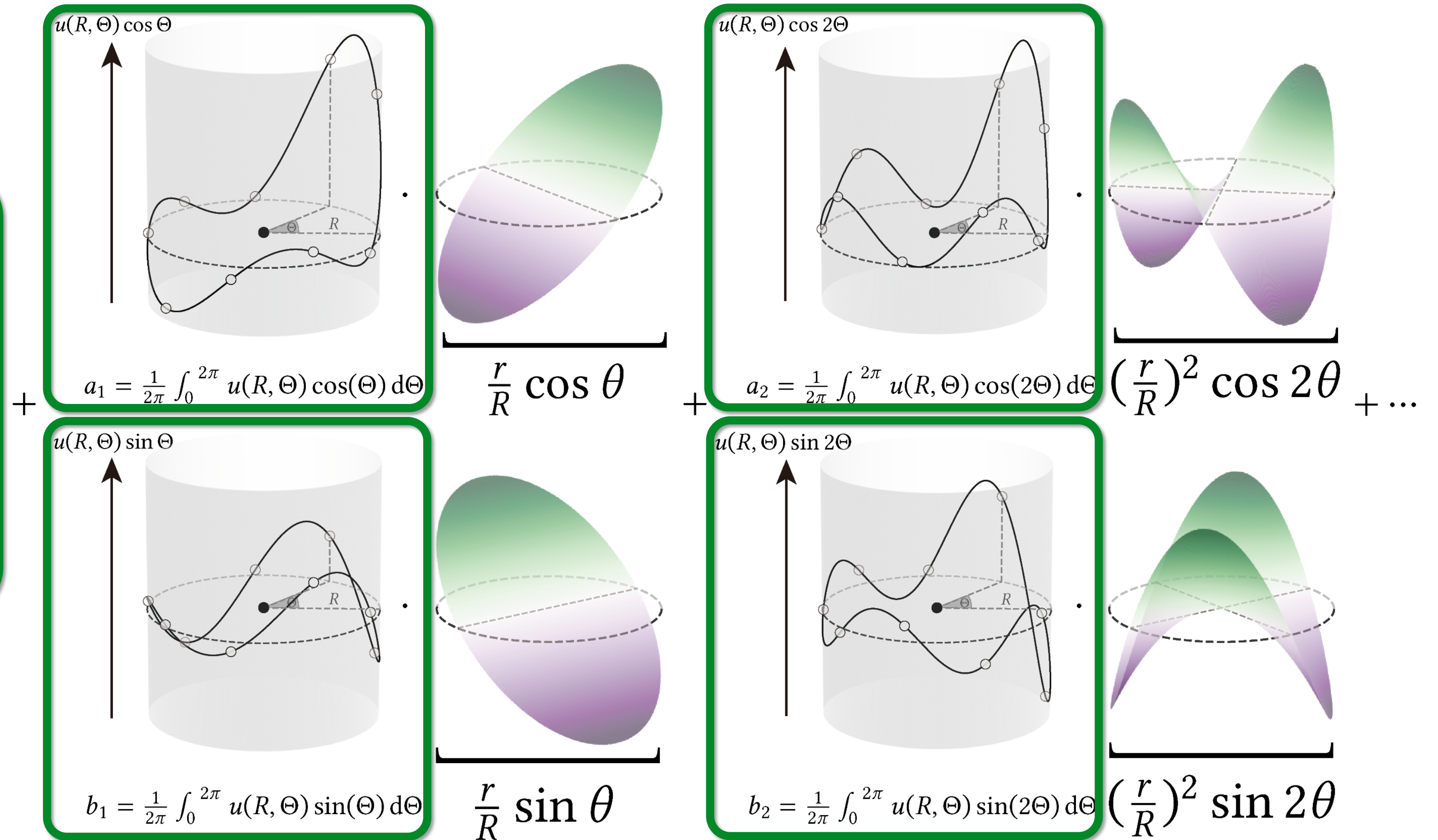
Mean Value Property

Harmonic series expansion [Booth 1981, 1982]

$$\hat{a}_l = \frac{1}{N} \sum_{i=1}^N u(R, \Theta_i) \cos l\Theta_i$$

$$\hat{b}_l = \frac{1}{N} \sum_{i=1}^N u(R, \Theta_i) \sin l\Theta_i$$

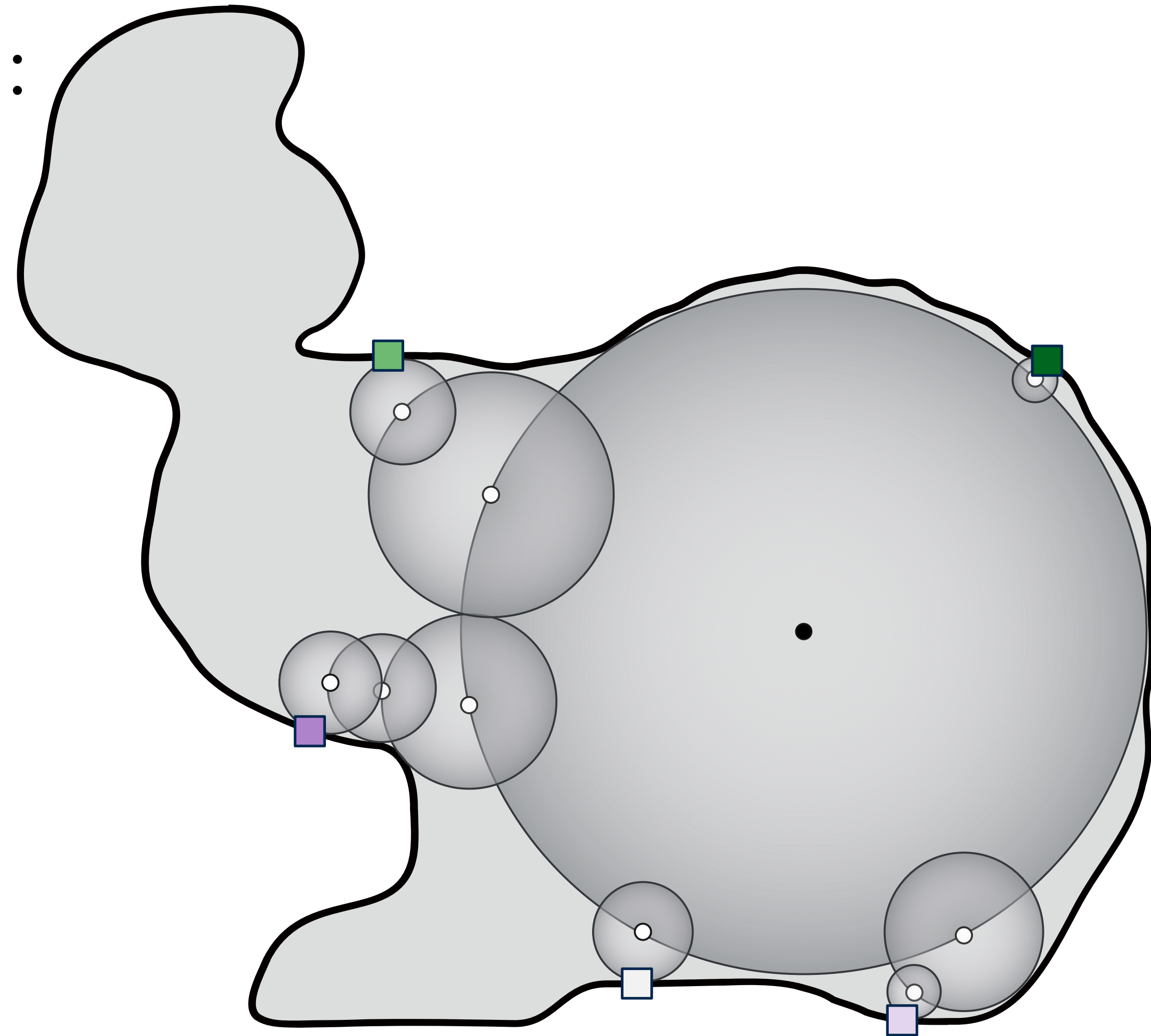
$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} u(R, \Theta) d\Theta$$



Noise free reconstruction after WoS

- WoS returns single estimation:

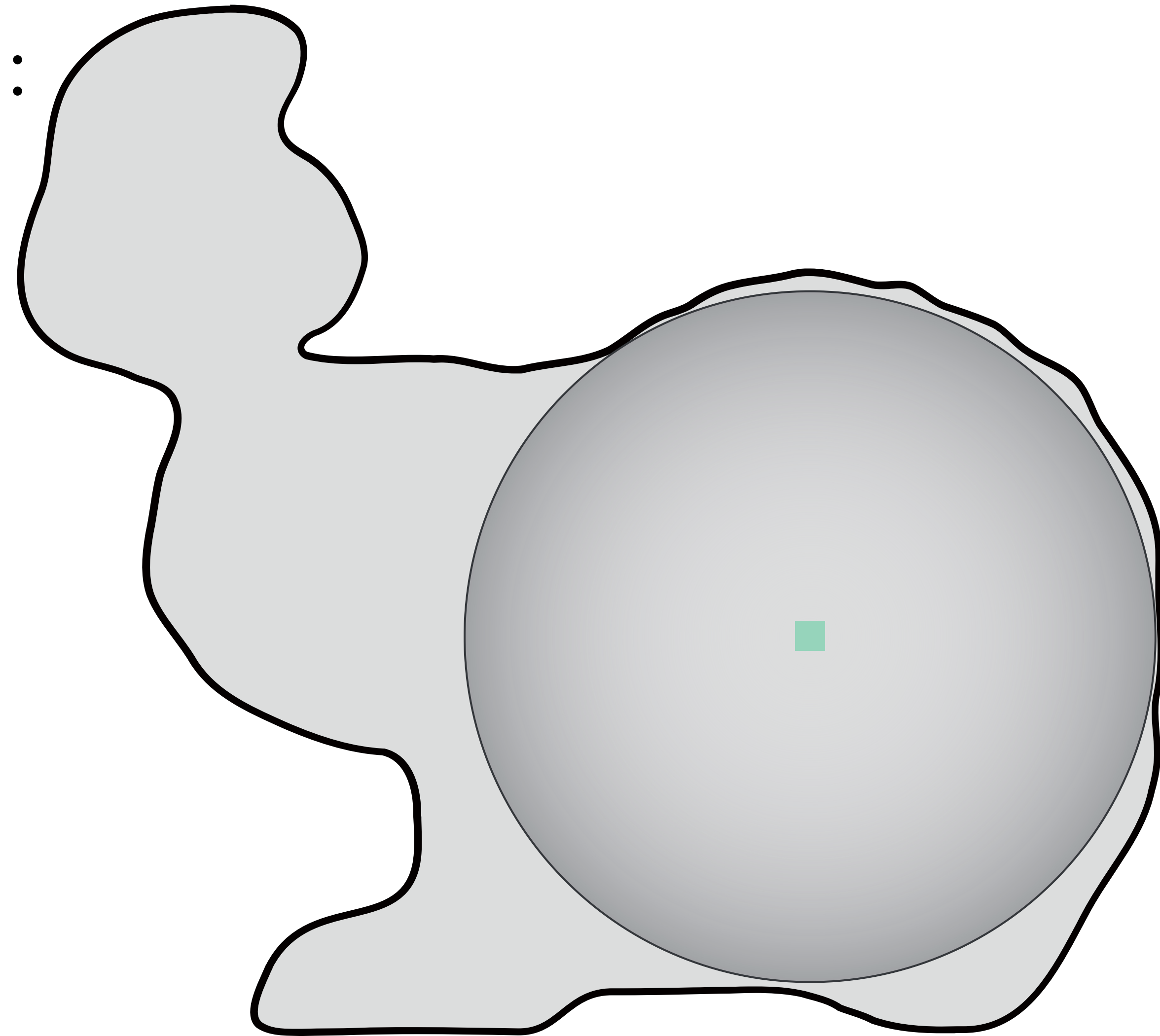
$$u(x) \approx \frac{\text{green square} + \text{purple square} + \dots + \text{light purple square} + \text{dark green square}}{N}$$



Noise free reconstruction after WoS

- WoS returns single estimation:

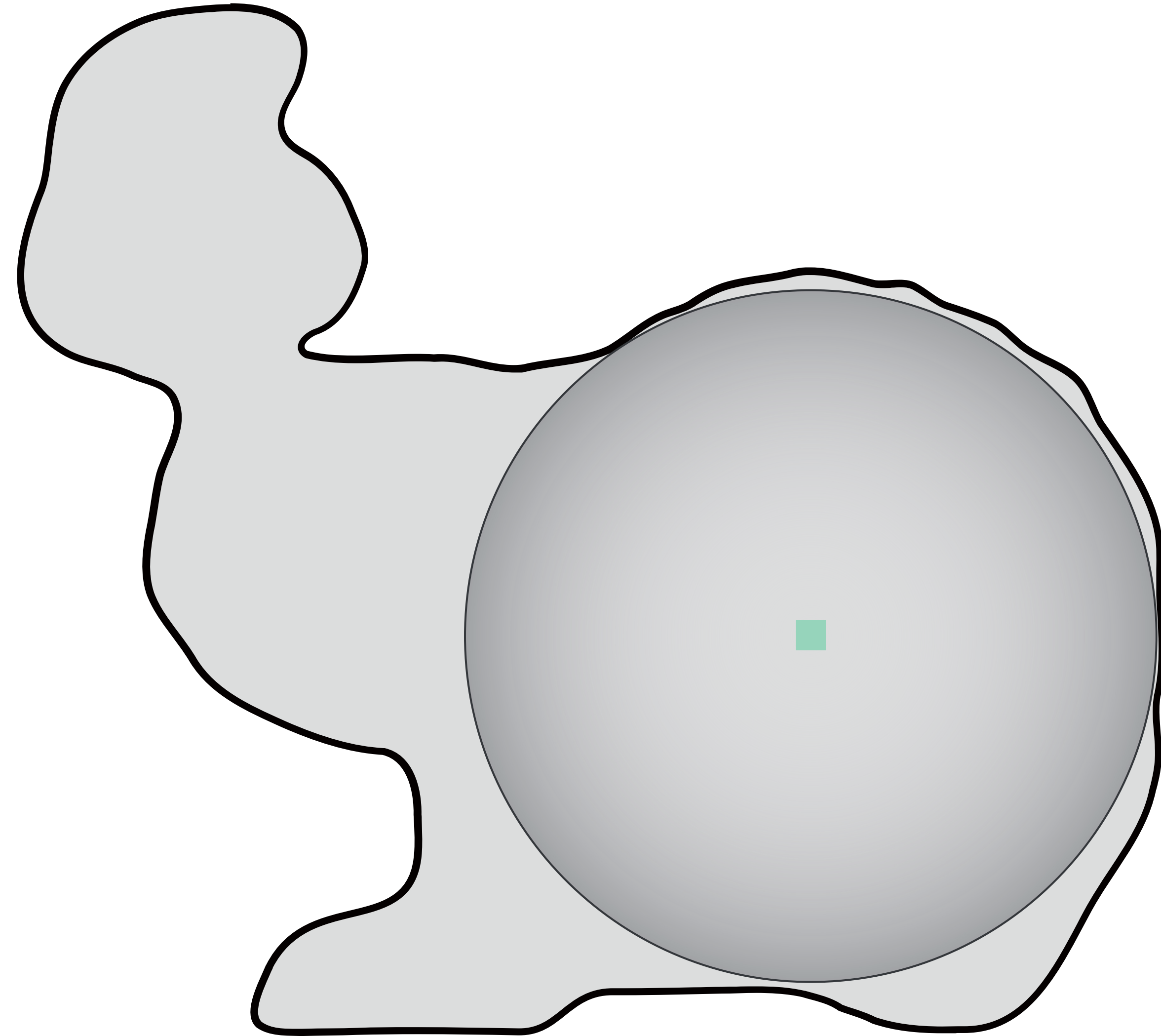
$$u(x) \approx \frac{\text{green square} + \text{purple square} + \dots + \text{light purple square} + \text{dark green square}}{N}$$



Noise free reconstruction after WoS

- 0th order coefficient from N walks:

$$\hat{a}_0 = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i)$$



Noise free reconstruction after WoS

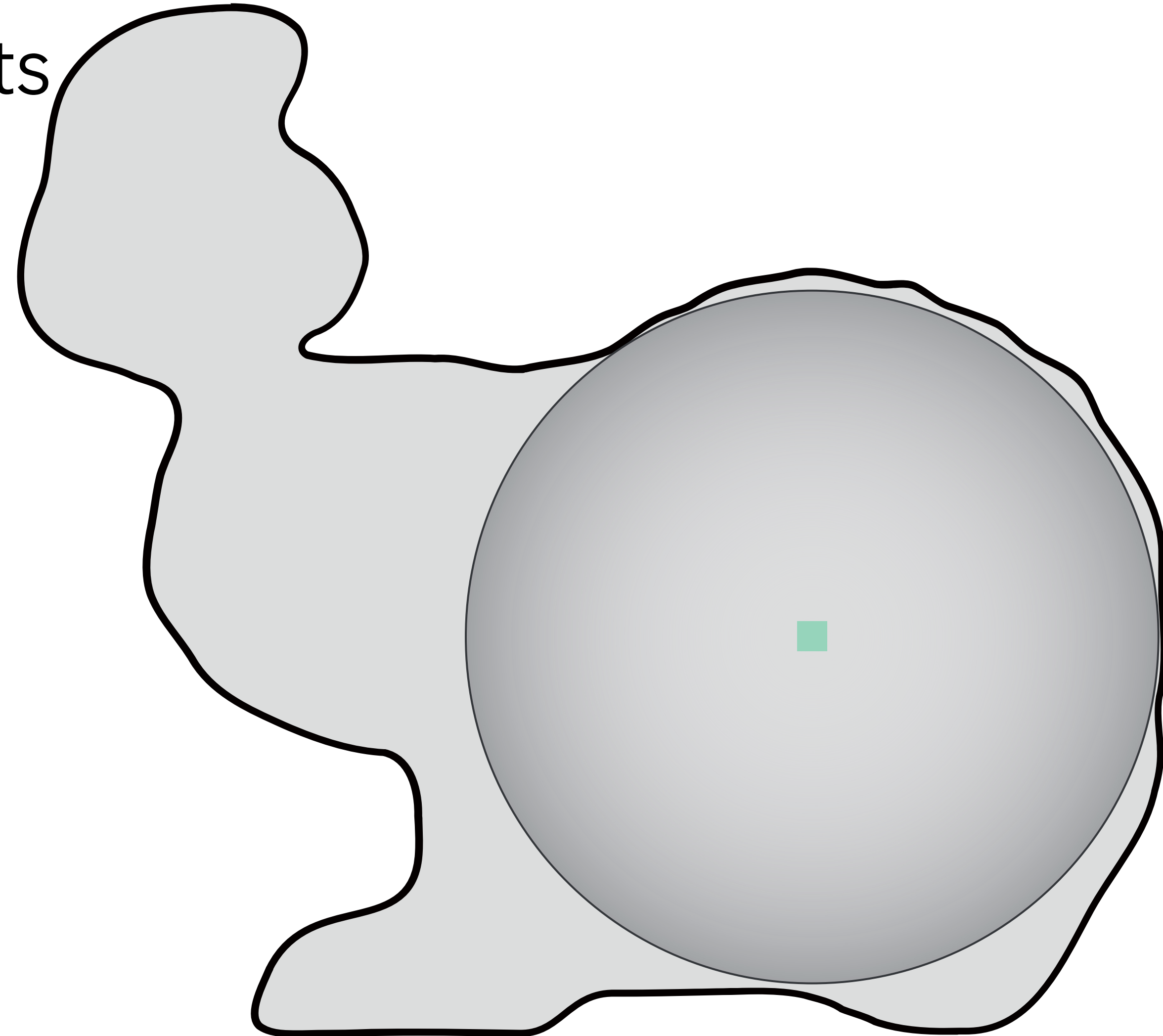
- 0th, 1st, 2nd, ... order coefficients from the same N walks:

$$\hat{a}_0 = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i)$$

⋮

$$\hat{a}_L = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i) \cos(L\Theta_i)$$

$$\hat{b}_L = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i) \sin(L\Theta_i)$$

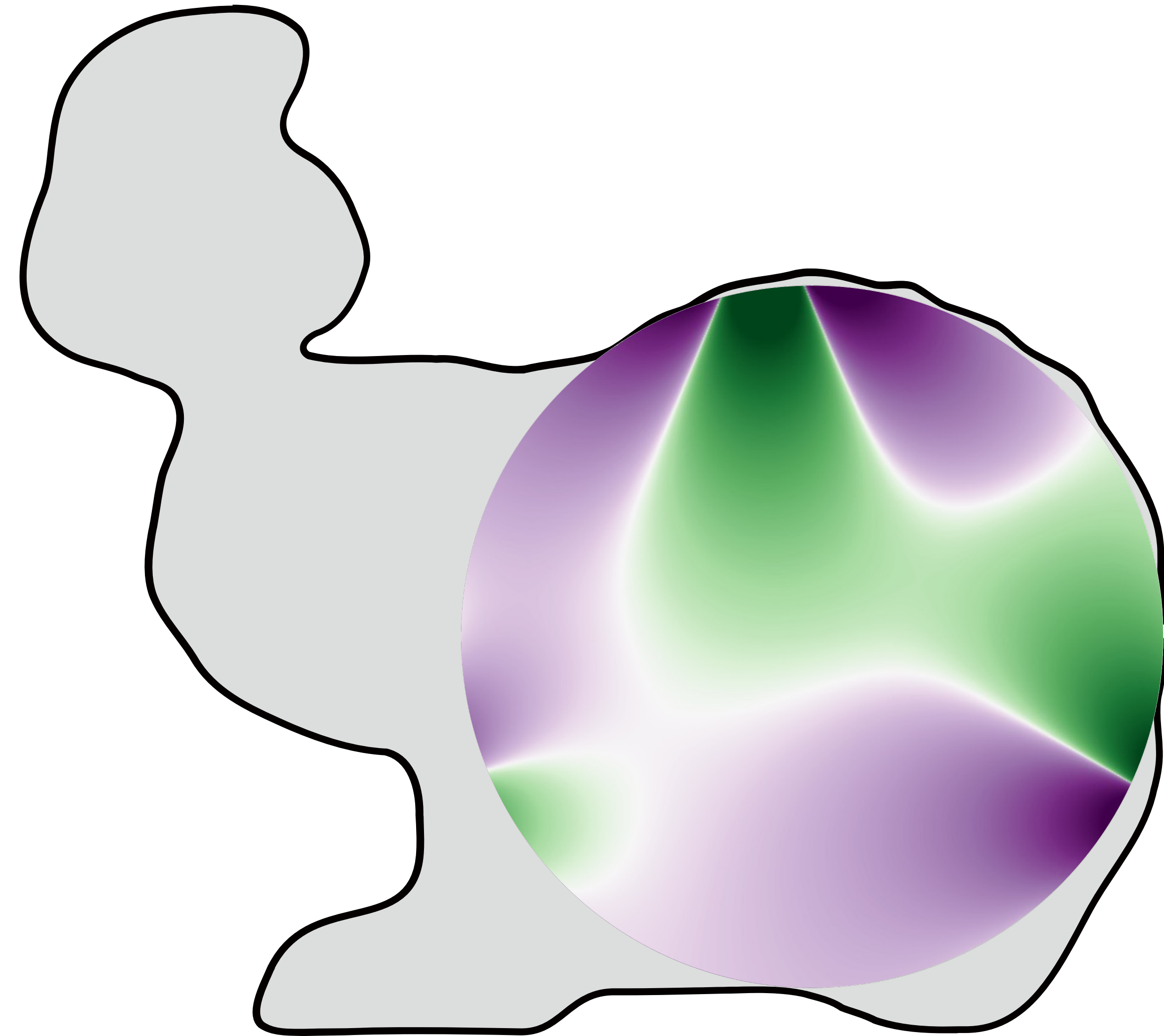


Noise free reconstruction after WoS

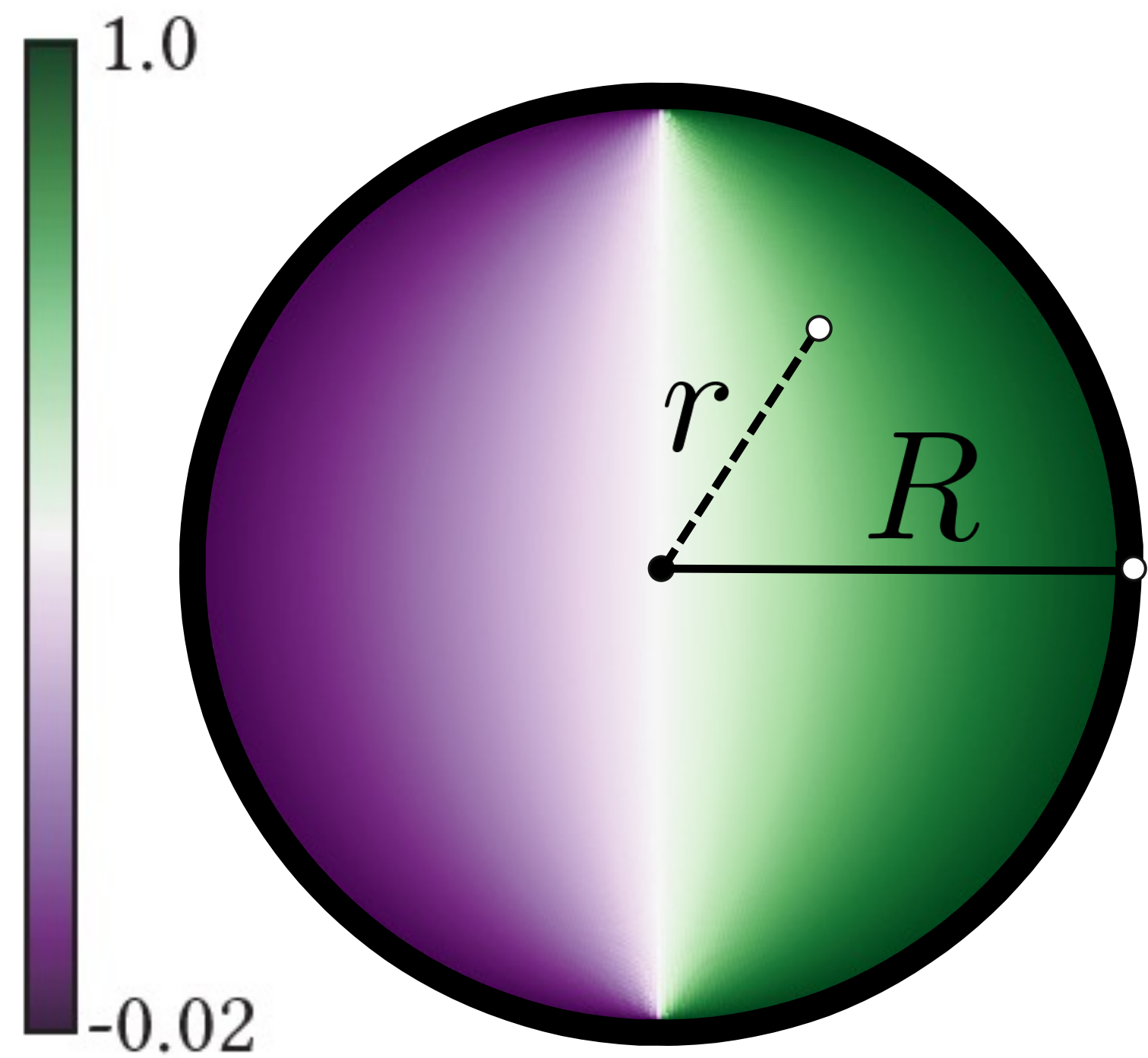
- Reconstruct solutions within the entire disk by:

$$u(r, \theta) \approx a_0 + 2 \sum_{l=1}^L \left(\frac{r}{R}\right)^l (\hat{a}_l \cos l\theta + \hat{b}_l \sin l\theta)$$

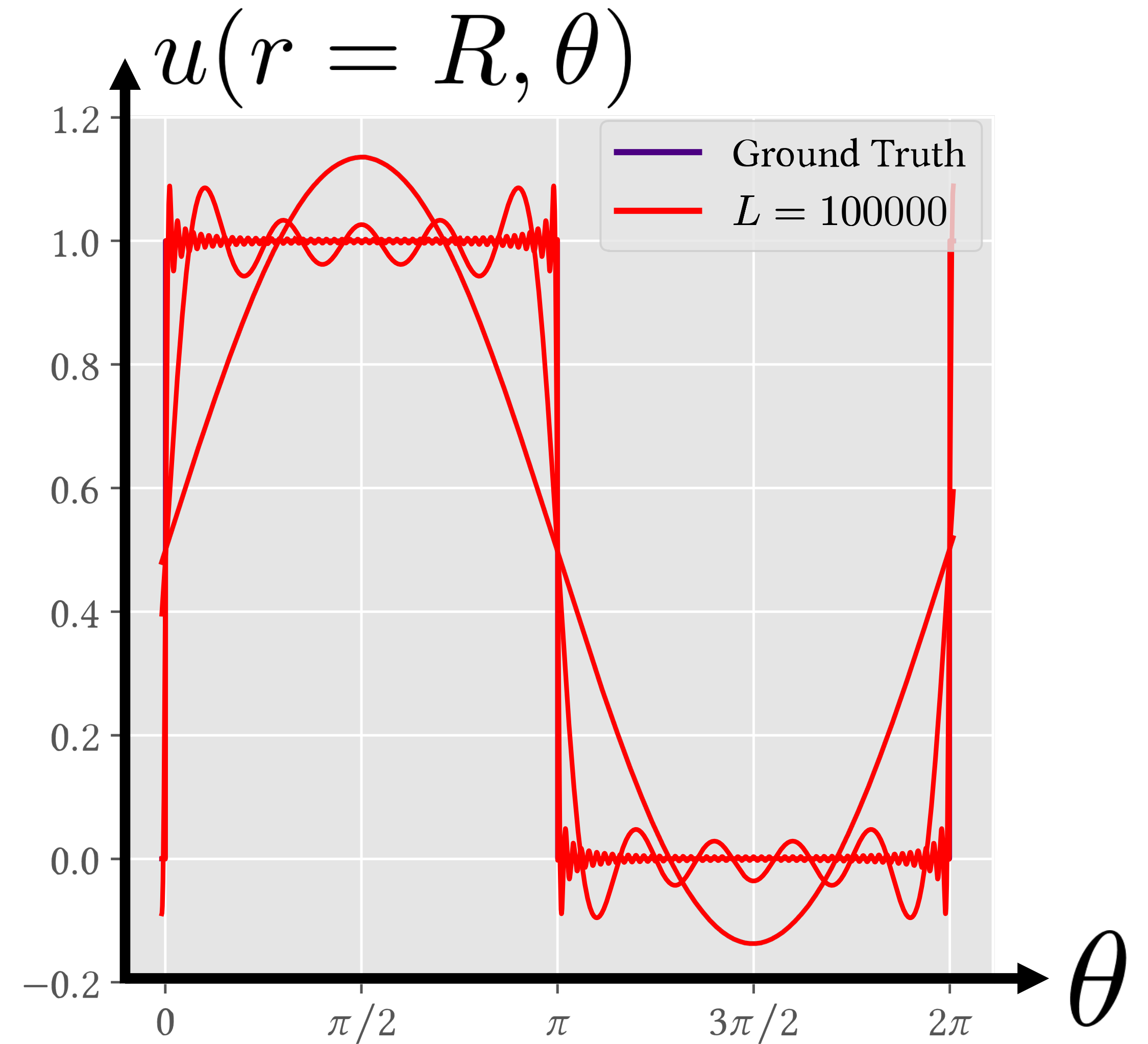
- In theory, we need infinite number of terms...
- In practice, $L = 10$ is enough even in the worst scenario.



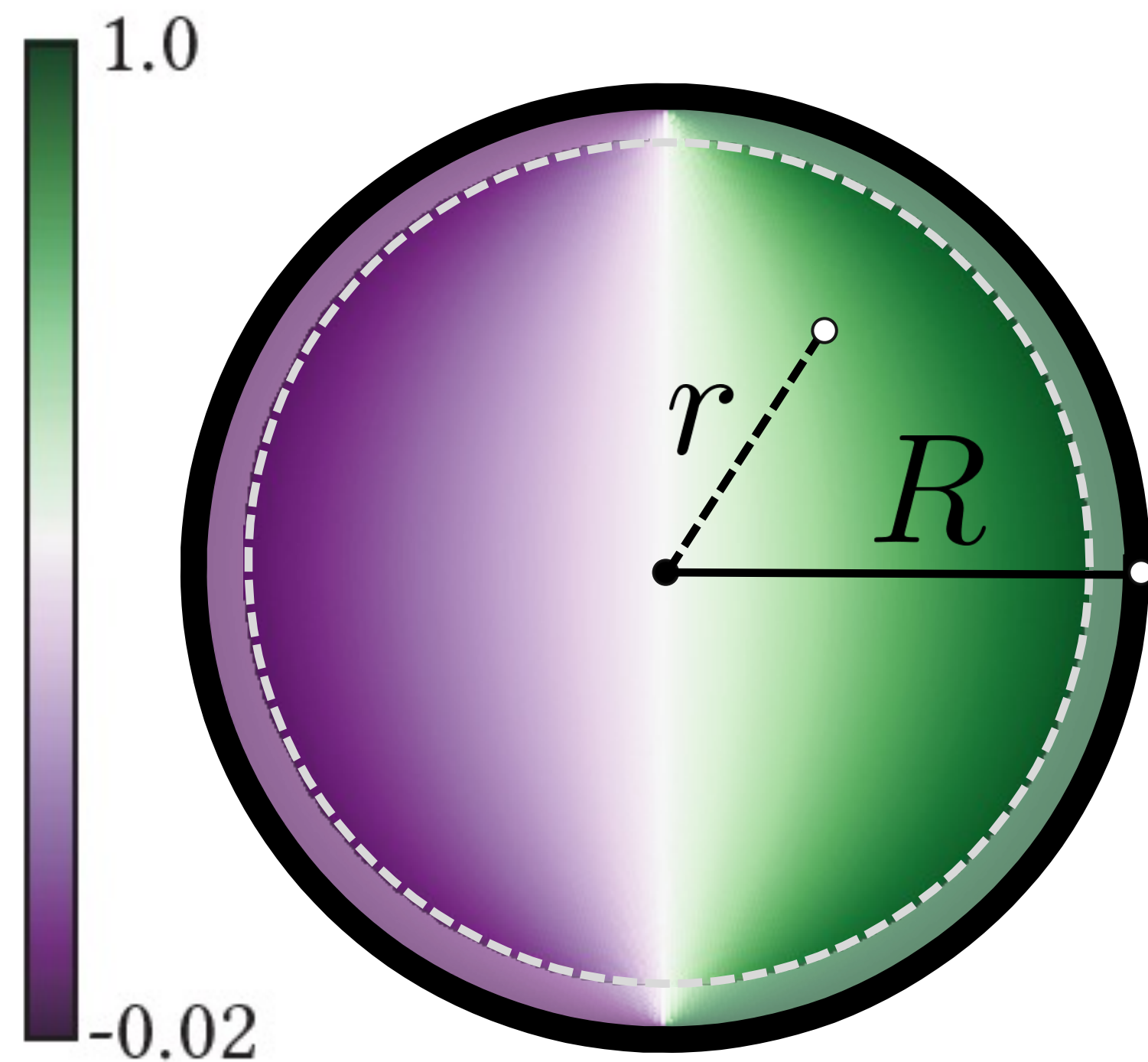
How to choose L ?



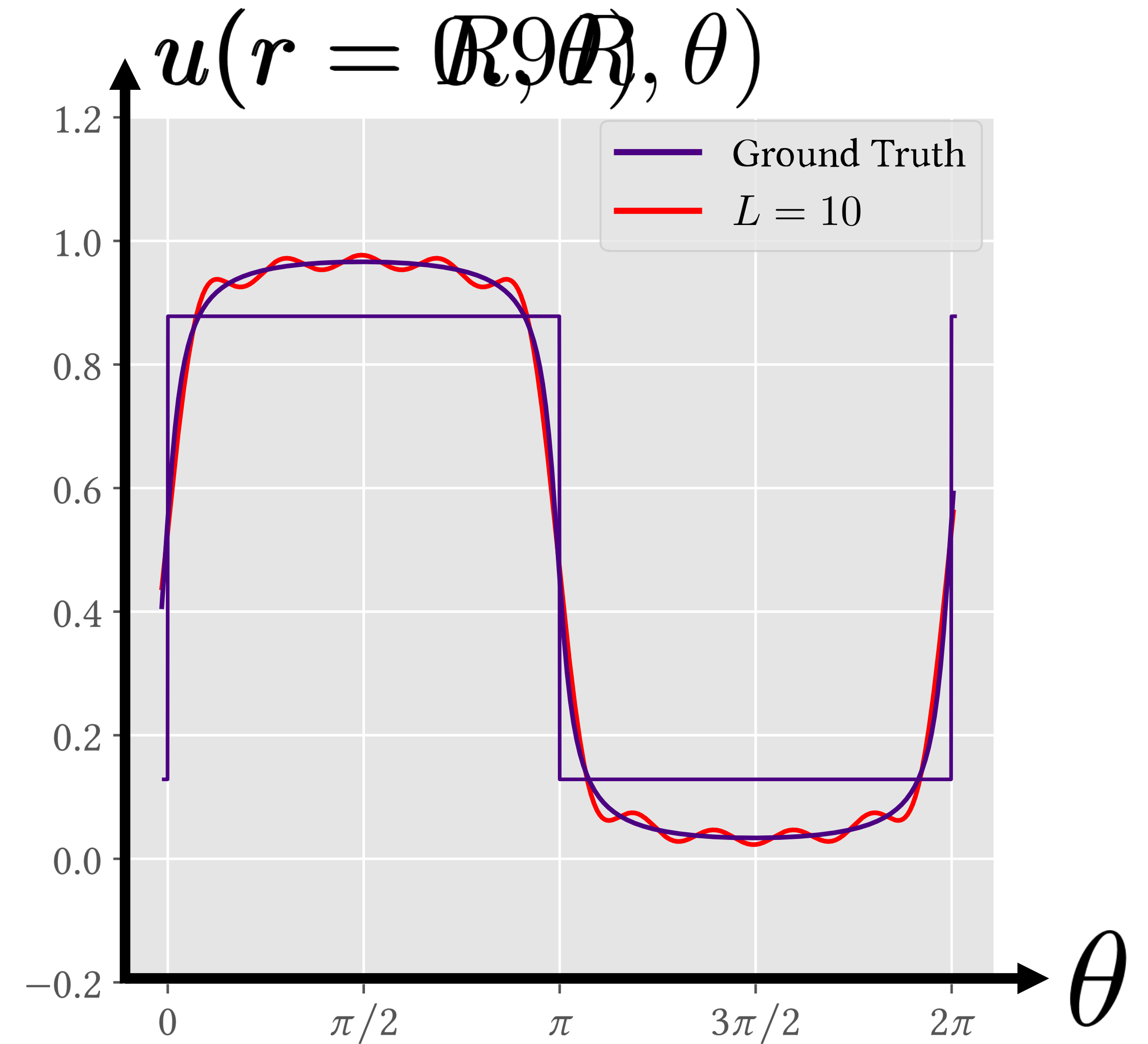
Ground Truth in the Disk



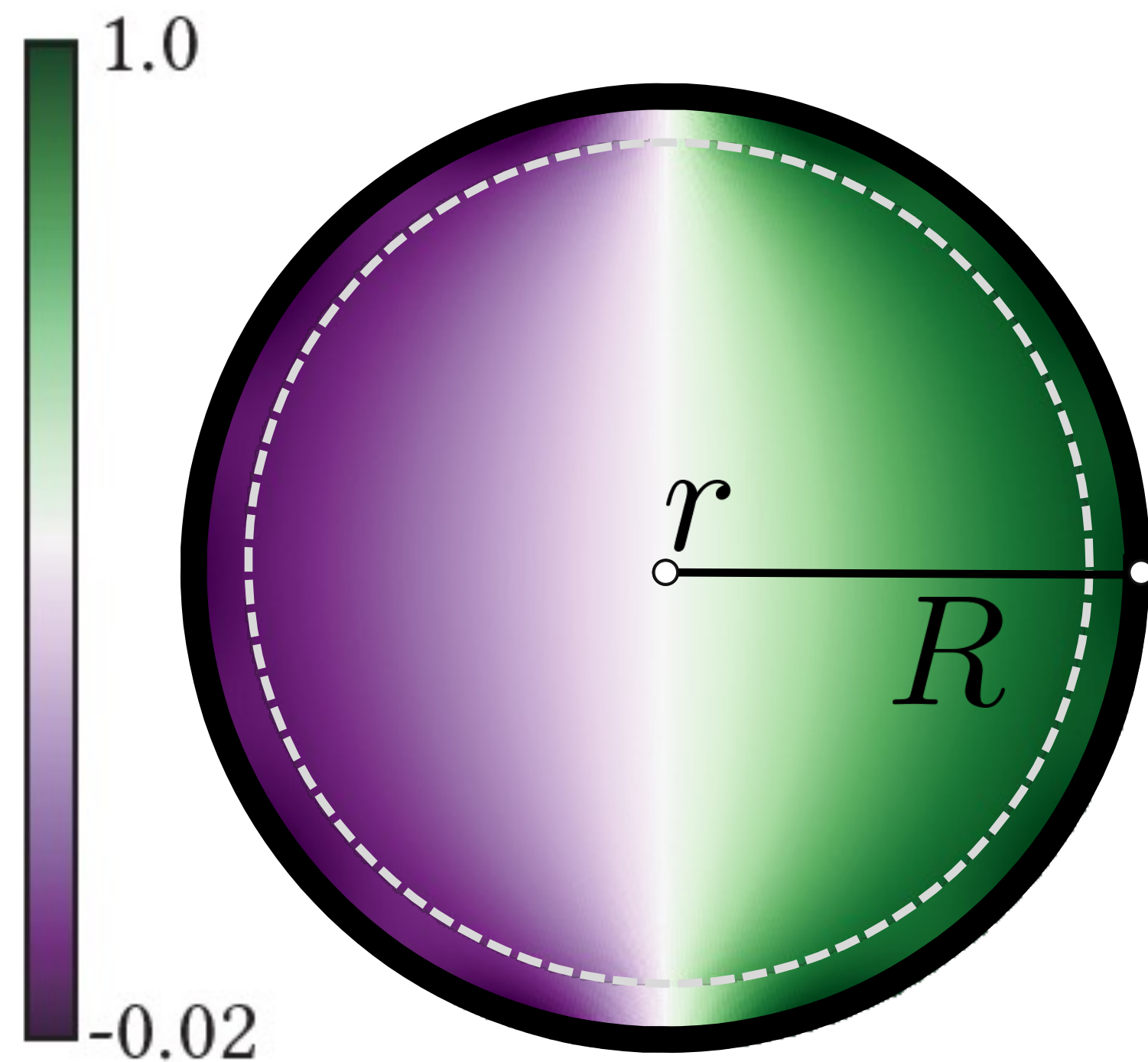
How to choose L ?



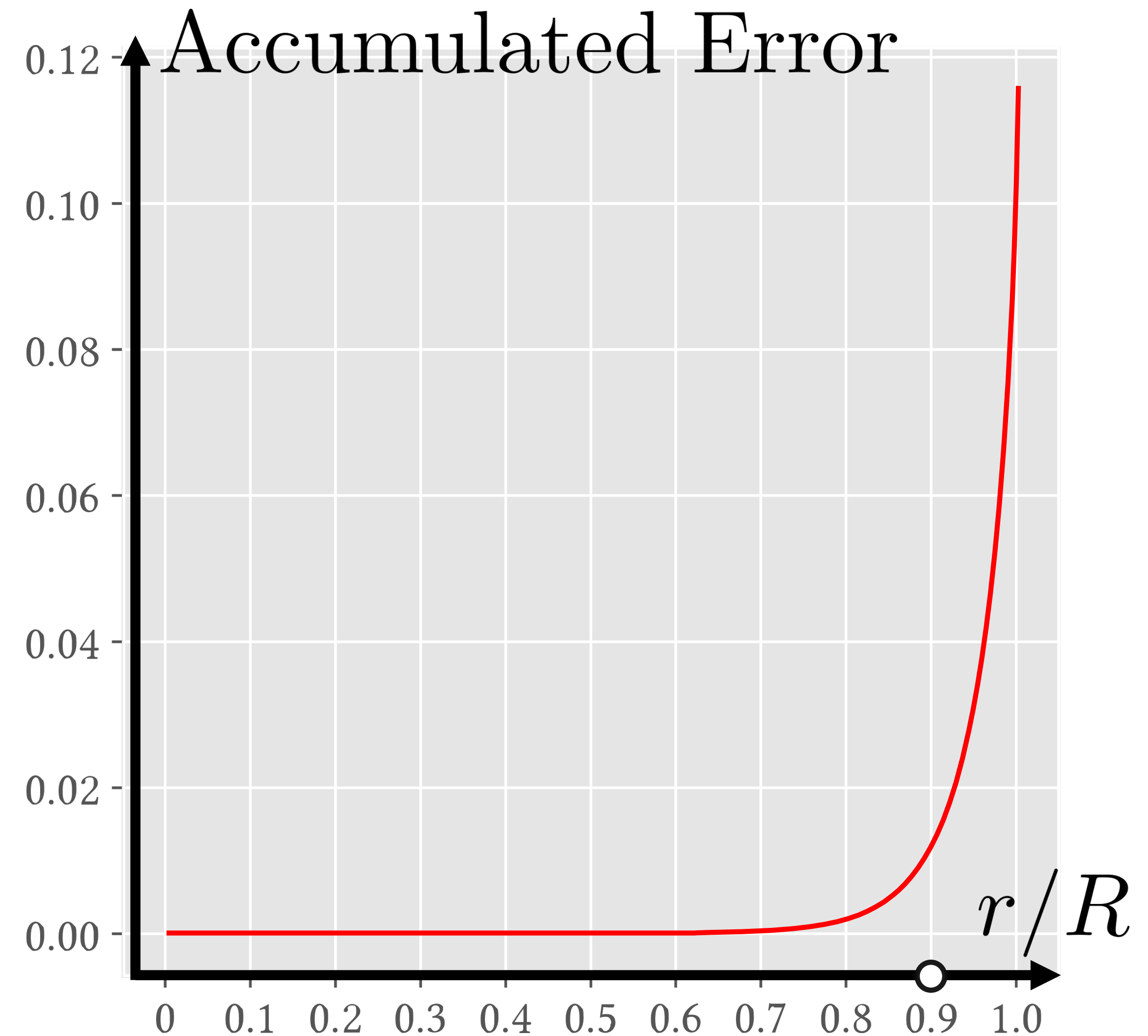
Ground Truth in the Disk



Let's choose $L = 10$ even in the worst scenario



Reconstruction in the Disk

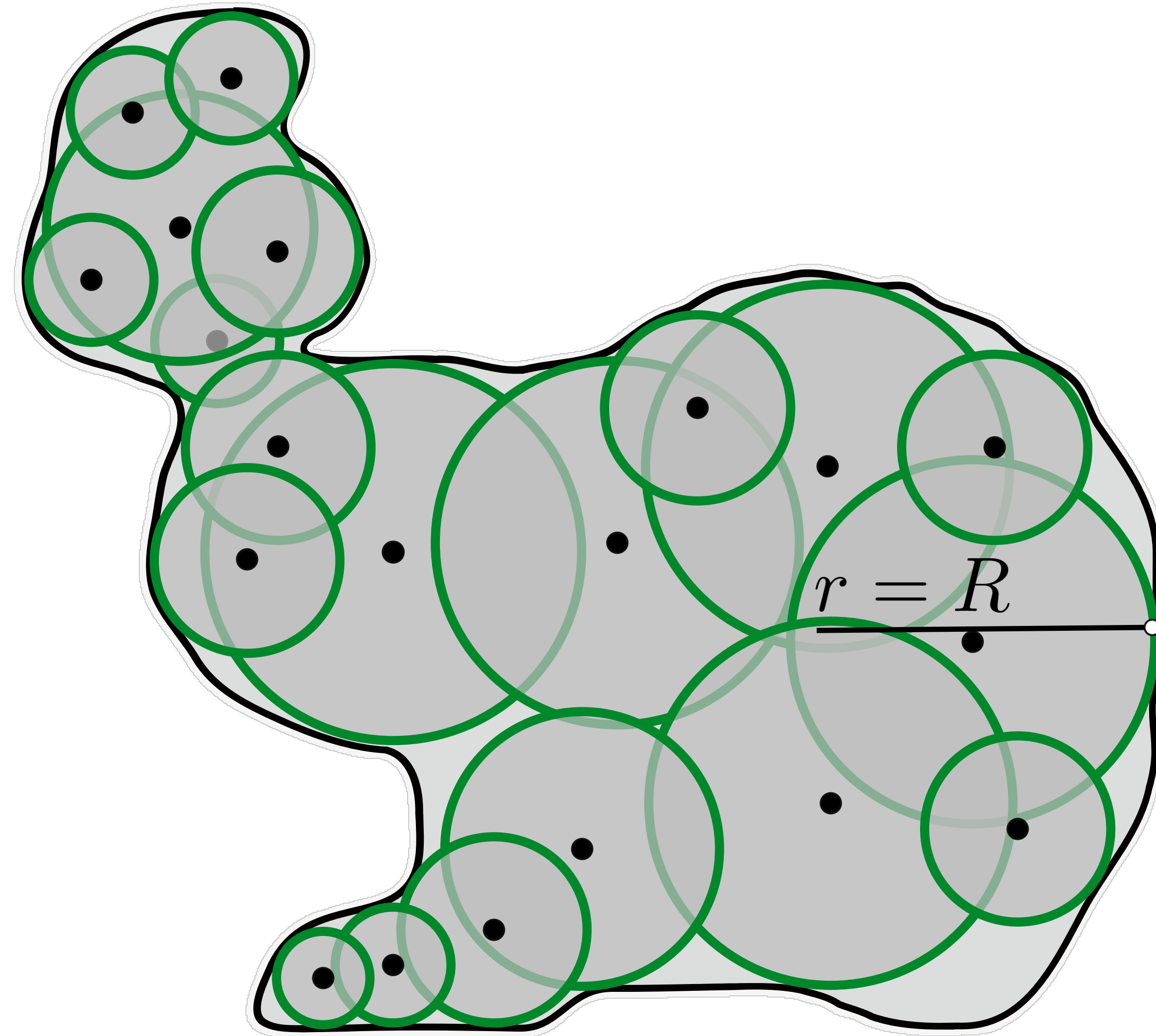


Noise free reconstruction after WoS

- Reconstruct solutions within the ~~entire disk~~ **entire disk**: maximum disk:

$$u(r, \theta) \approx a_0 + 2 \sum_{l=1}^{\infty} \left(\frac{r}{R} \right)^l (\hat{a}_l \cos l\theta + \hat{b}_l \sin l\theta)$$

- How about non-spherical regions of interest?
- ✓ Cover by more disks!





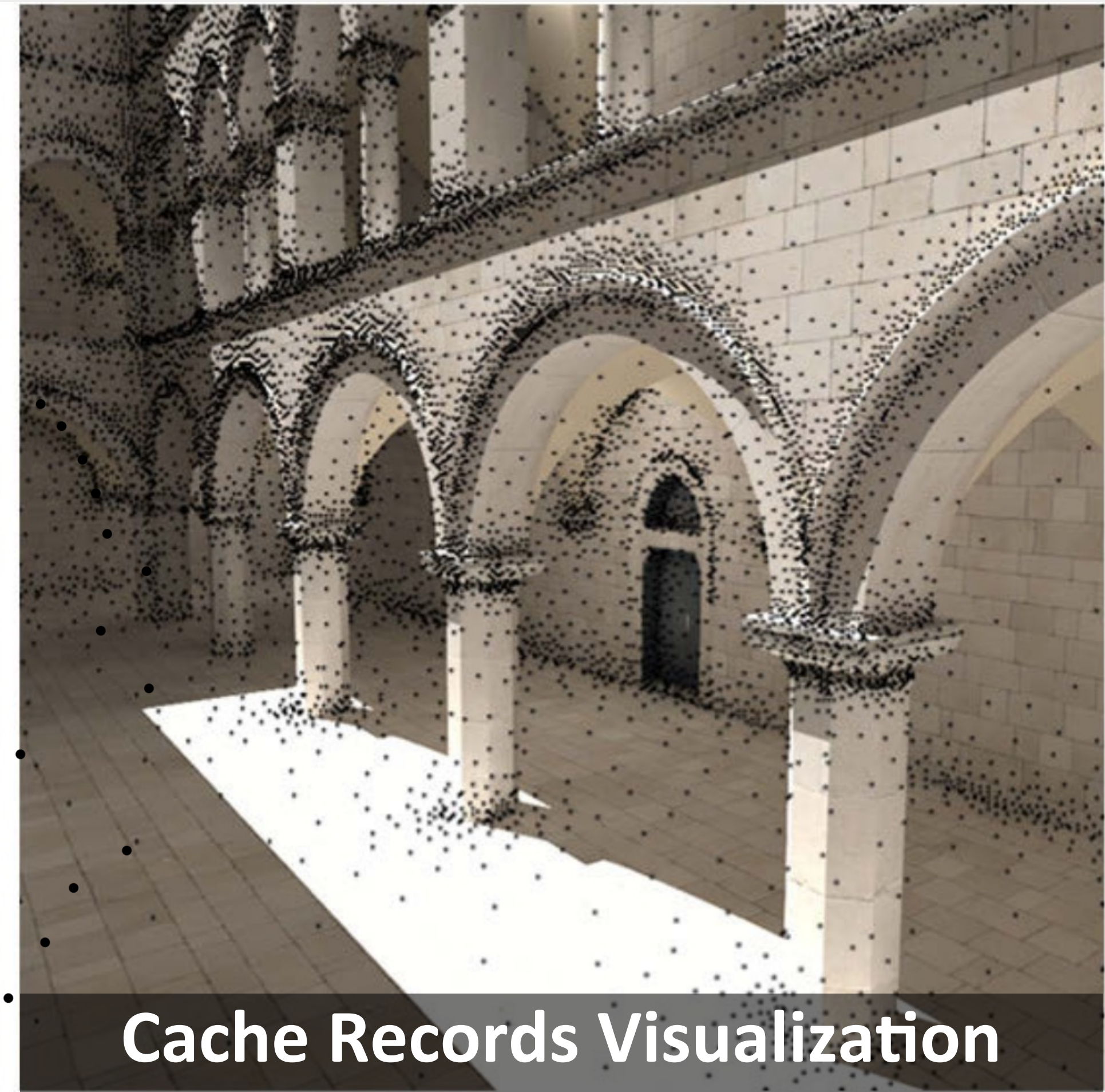
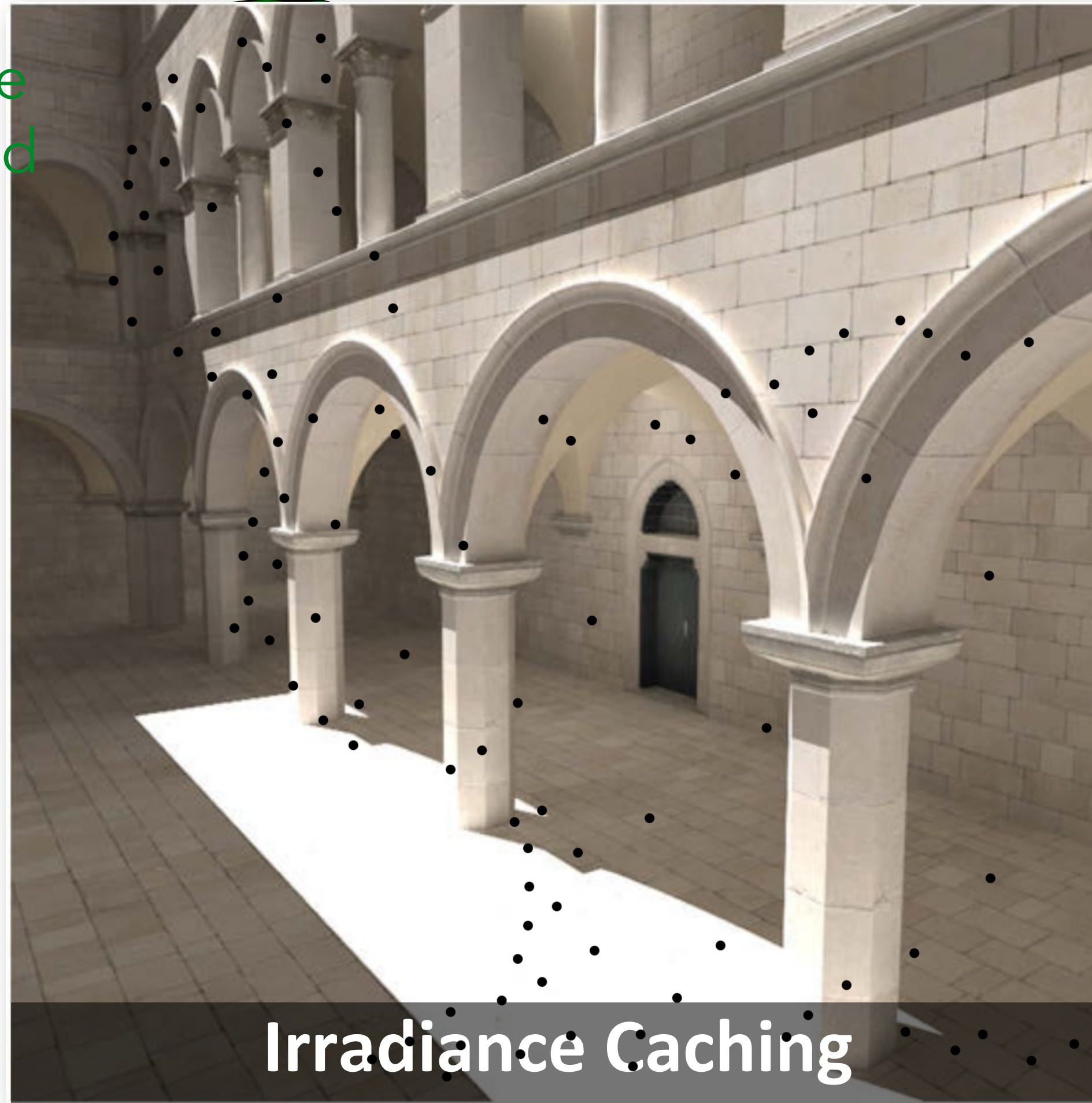
Harmonic Caching

Cache records population

Lazy evaluation style population [Ward et al. 1988]

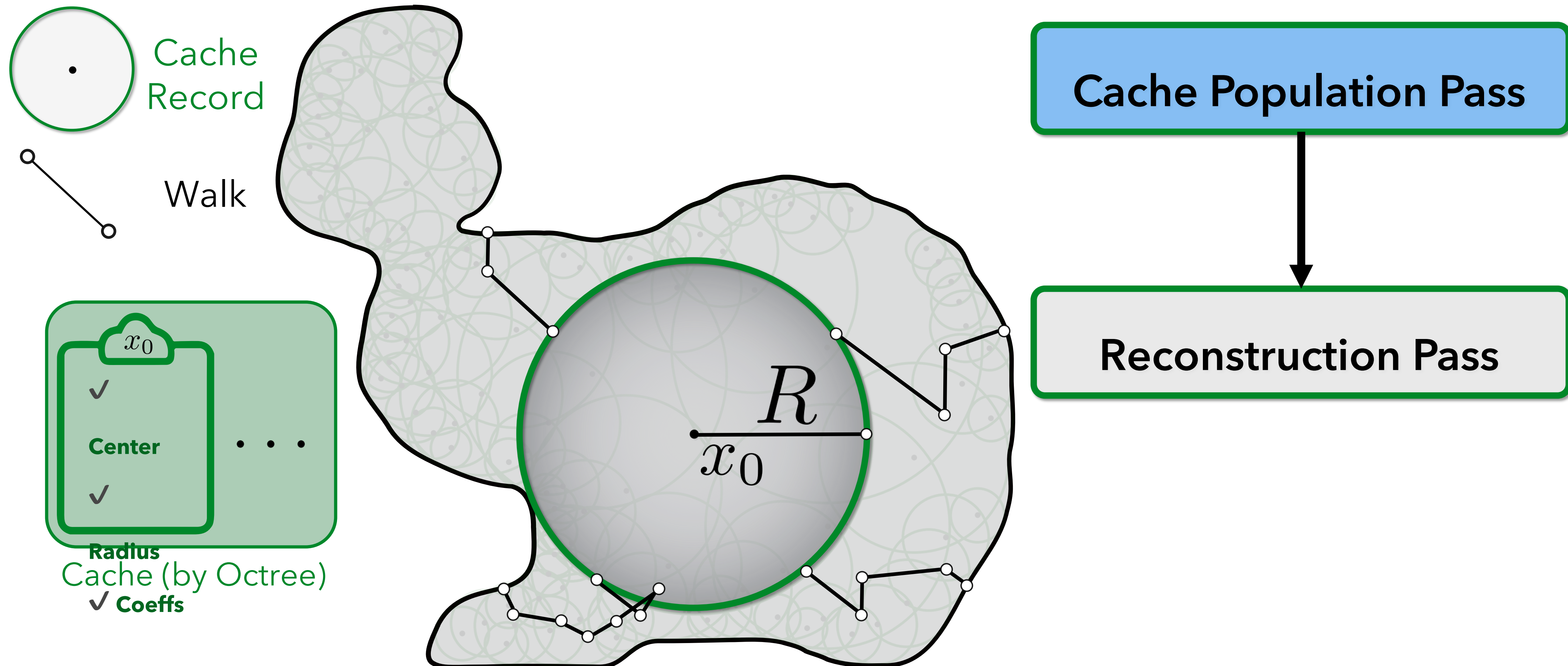


[Practical Global Illumination with Irradiance Caching]



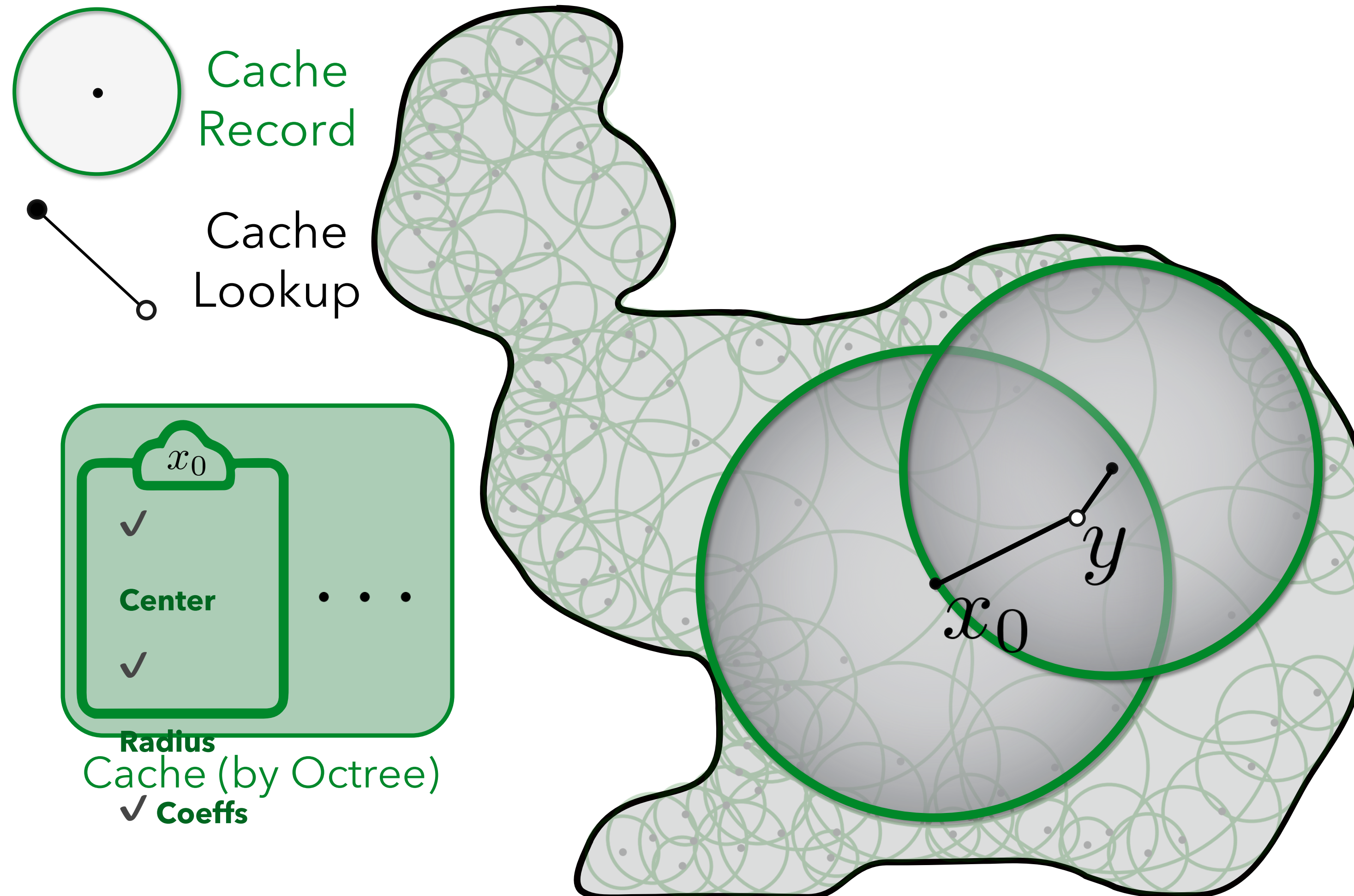
Cache records population

Lazy evaluation style population [Ward 2008]



Reconstruction by series expansion

(Weighted) sum of series expansions from cache balls

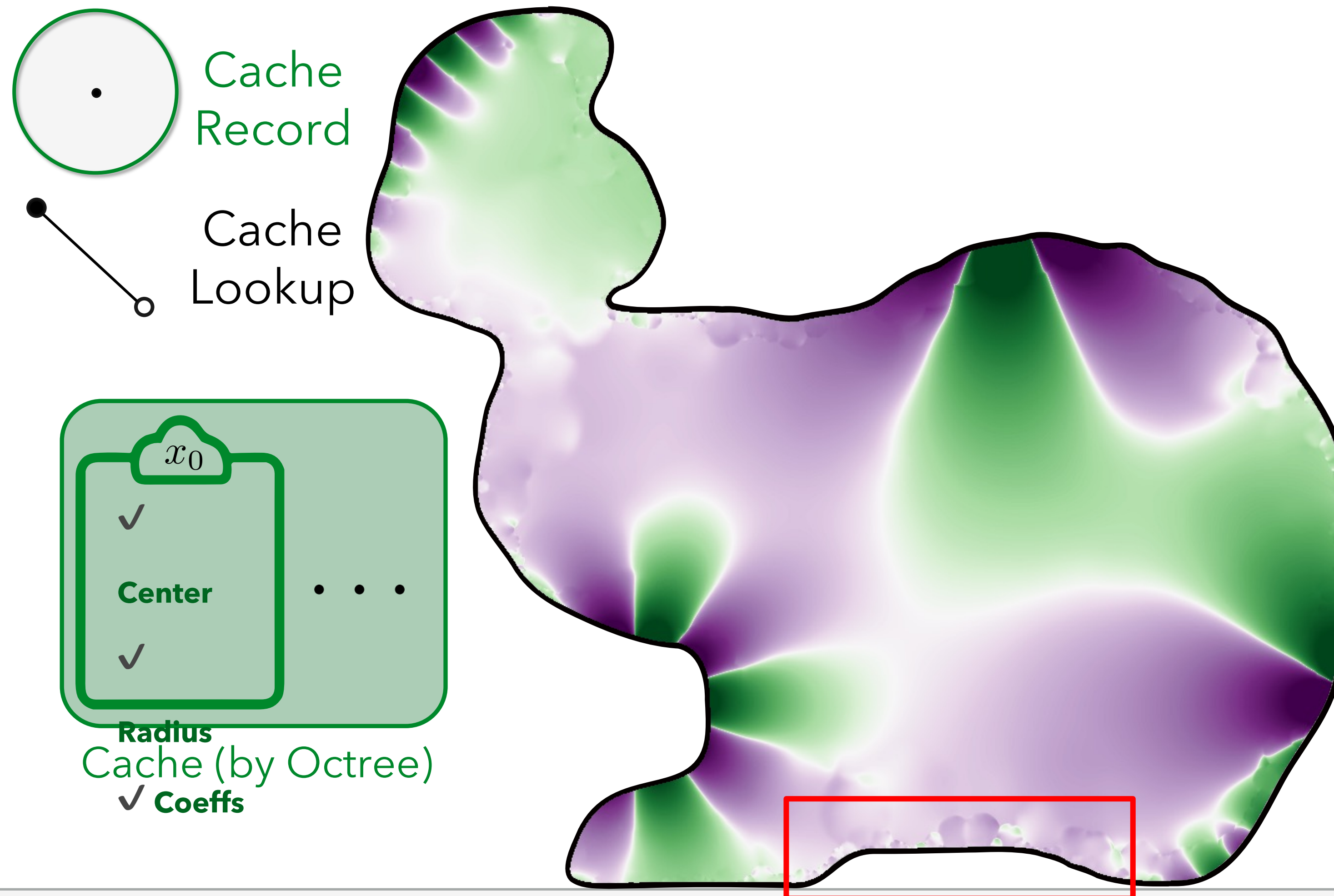


Cache Population Pass

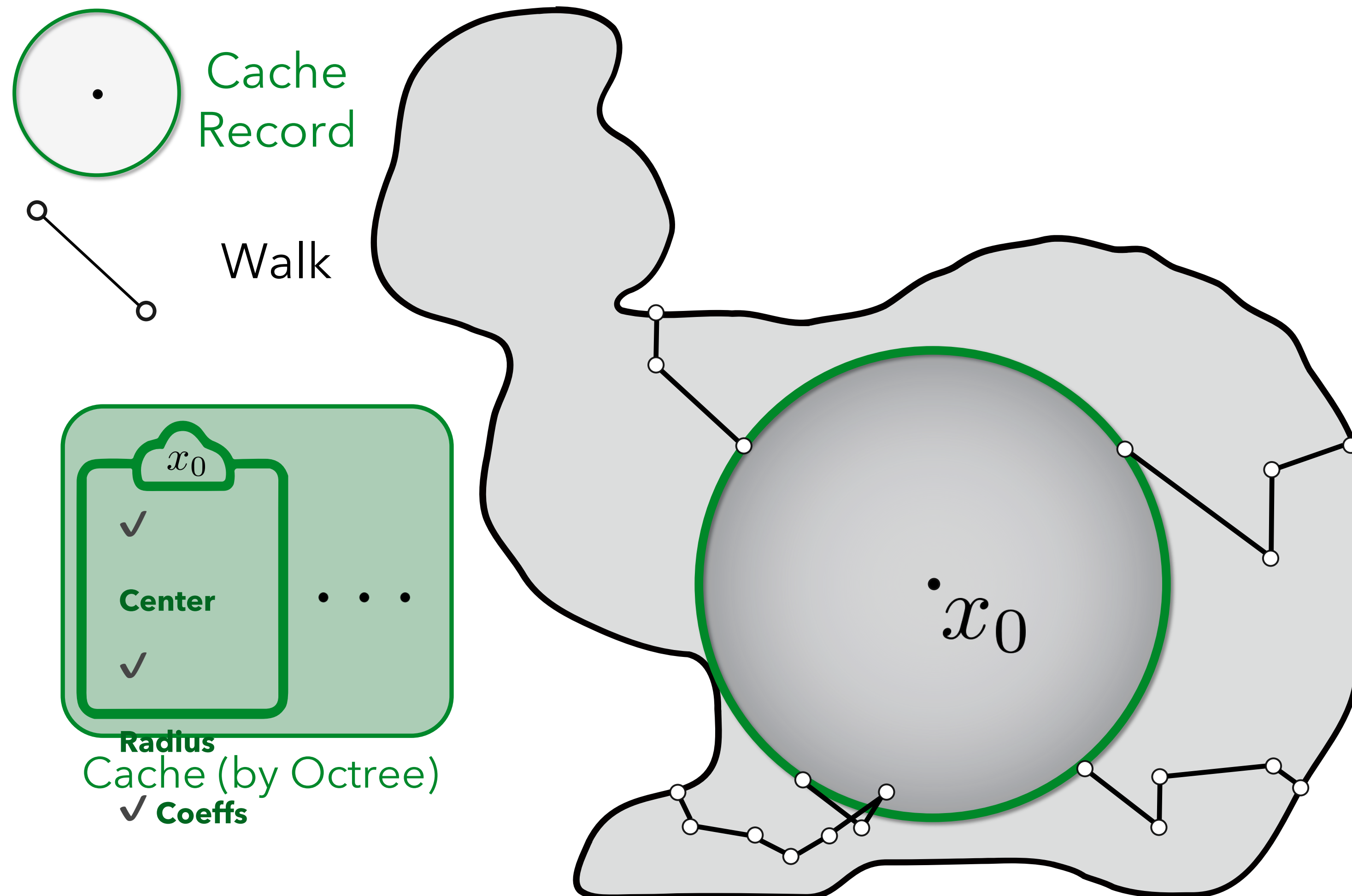
Reconstruction Pass

Reconstruction by series expansion

(Weighted) sum of series expansions from cache balls



Coefficients estimation can be noisy



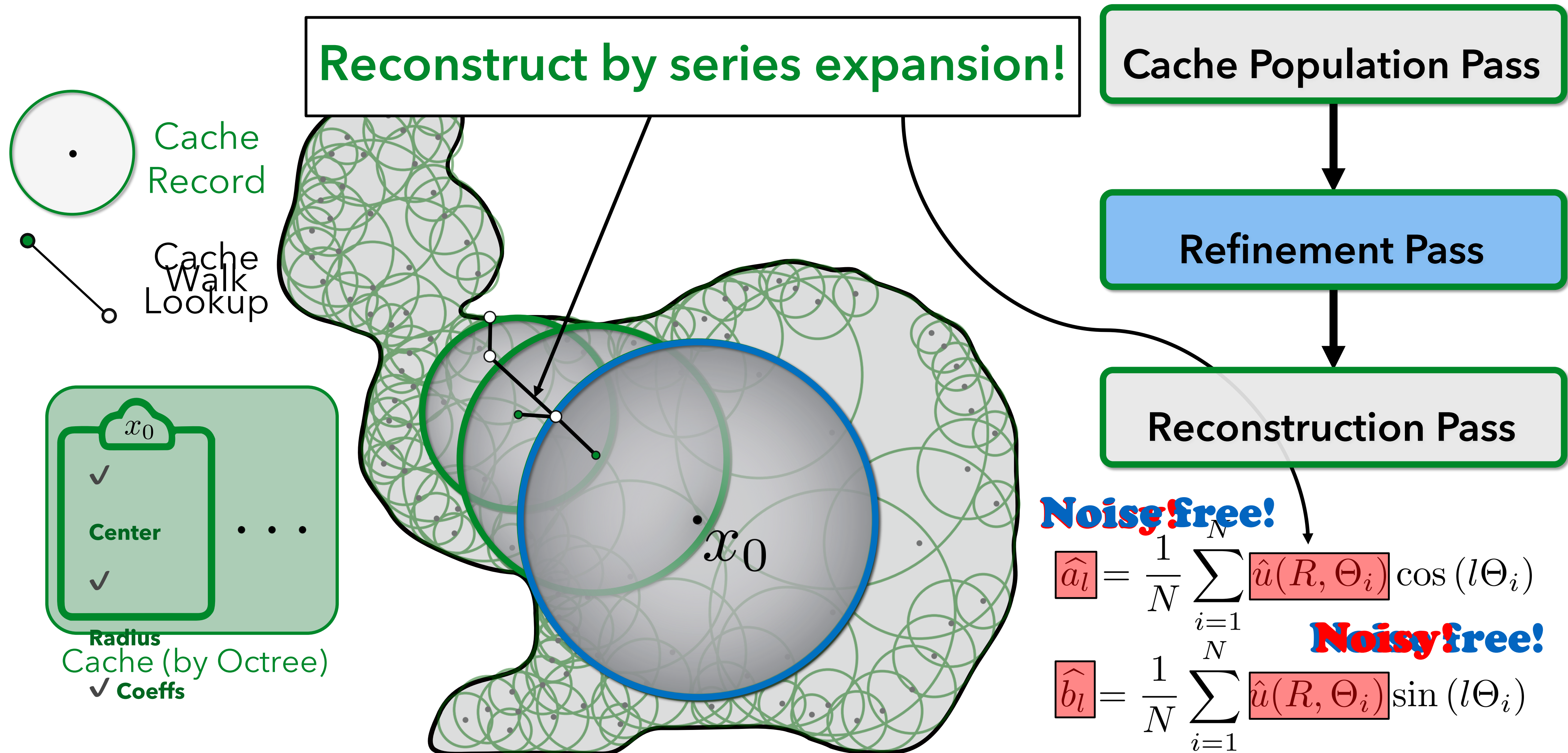
Noisy!

$$\hat{a}_l = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i) \cos(l\Theta_i)$$

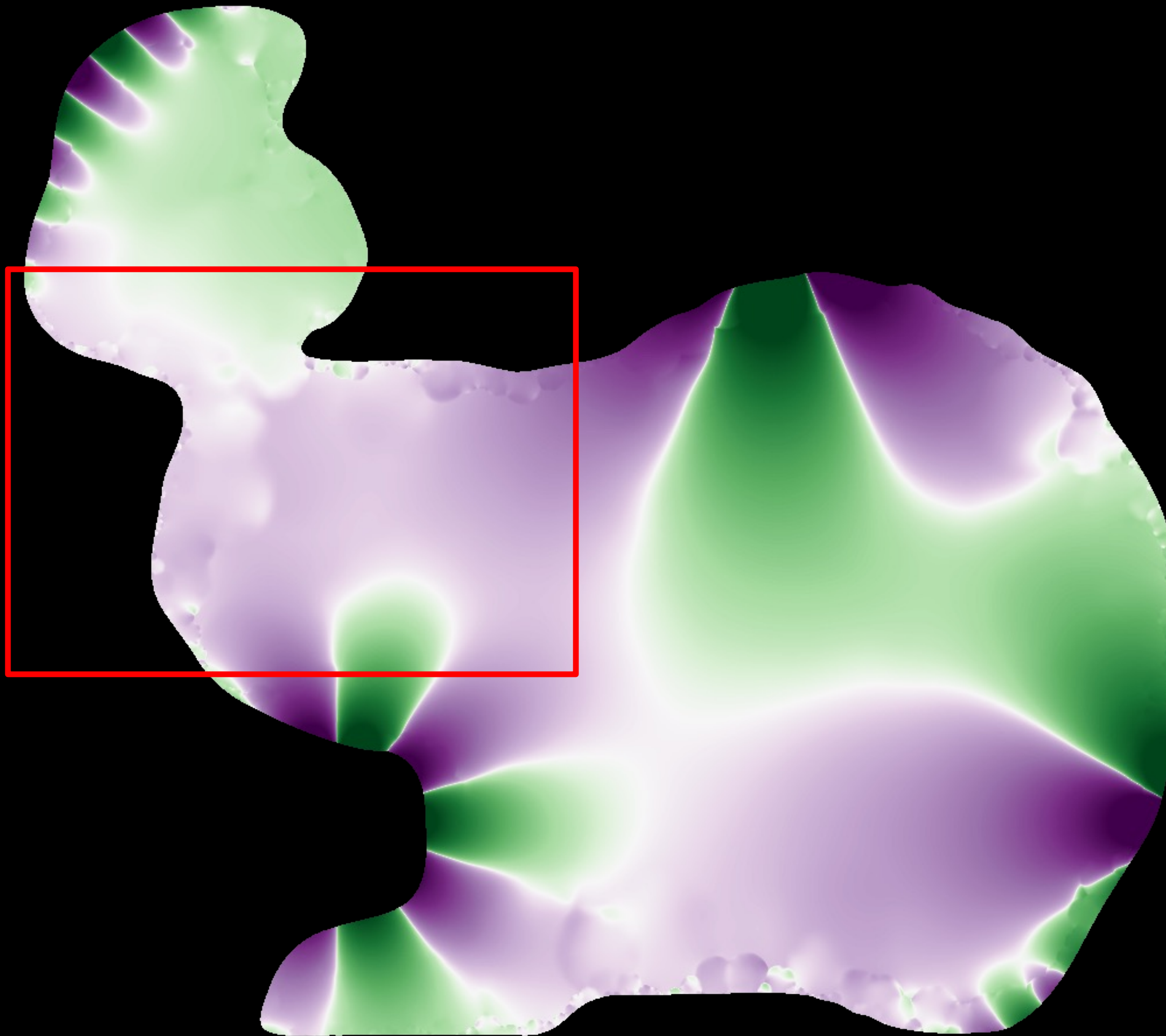
Noisy!

$$\hat{b}_l = \frac{1}{N} \sum_{i=1}^N \hat{u}(R, \Theta_i) \sin(l\Theta_i)$$

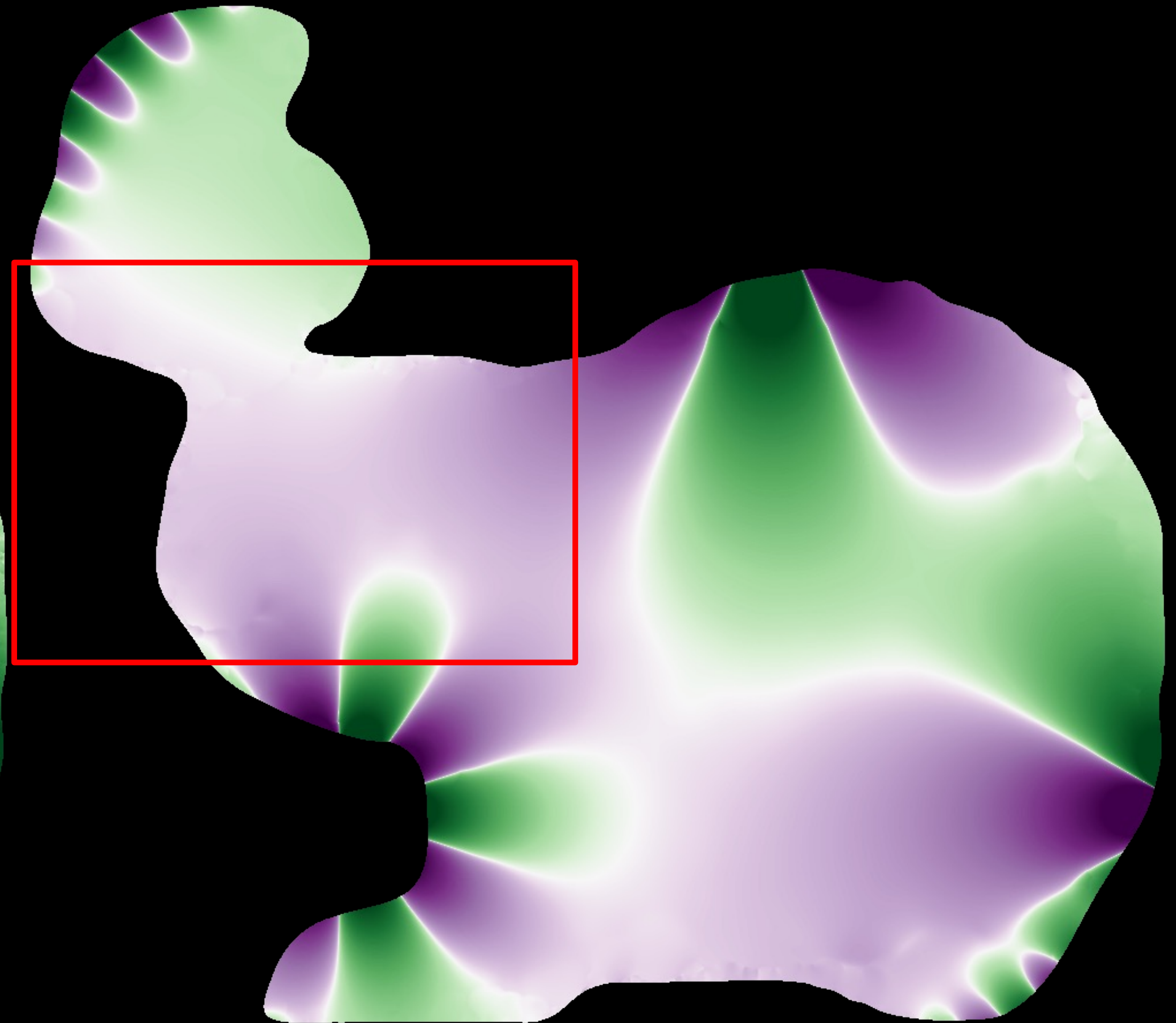
Additional refinement pass



Without refinement



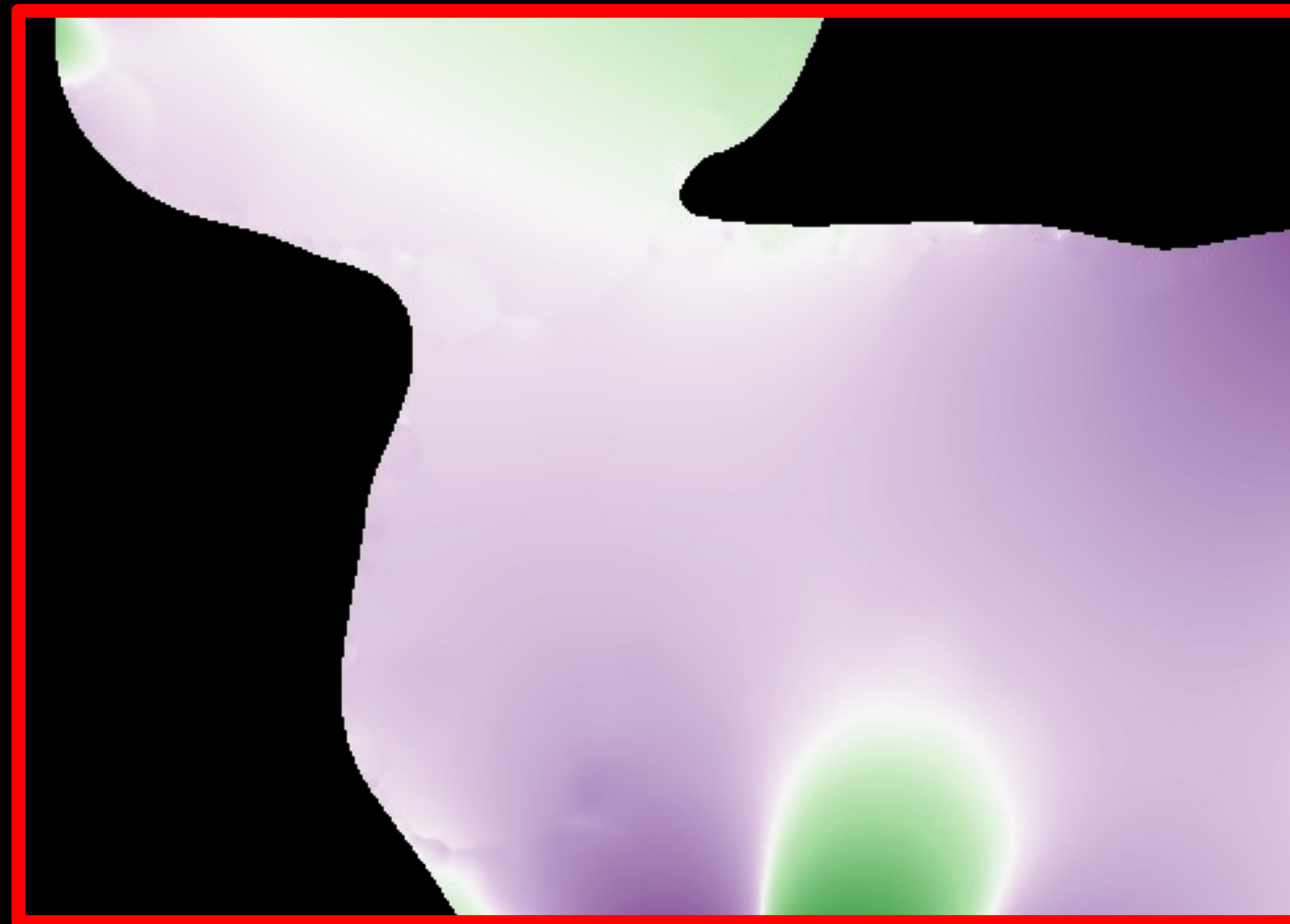
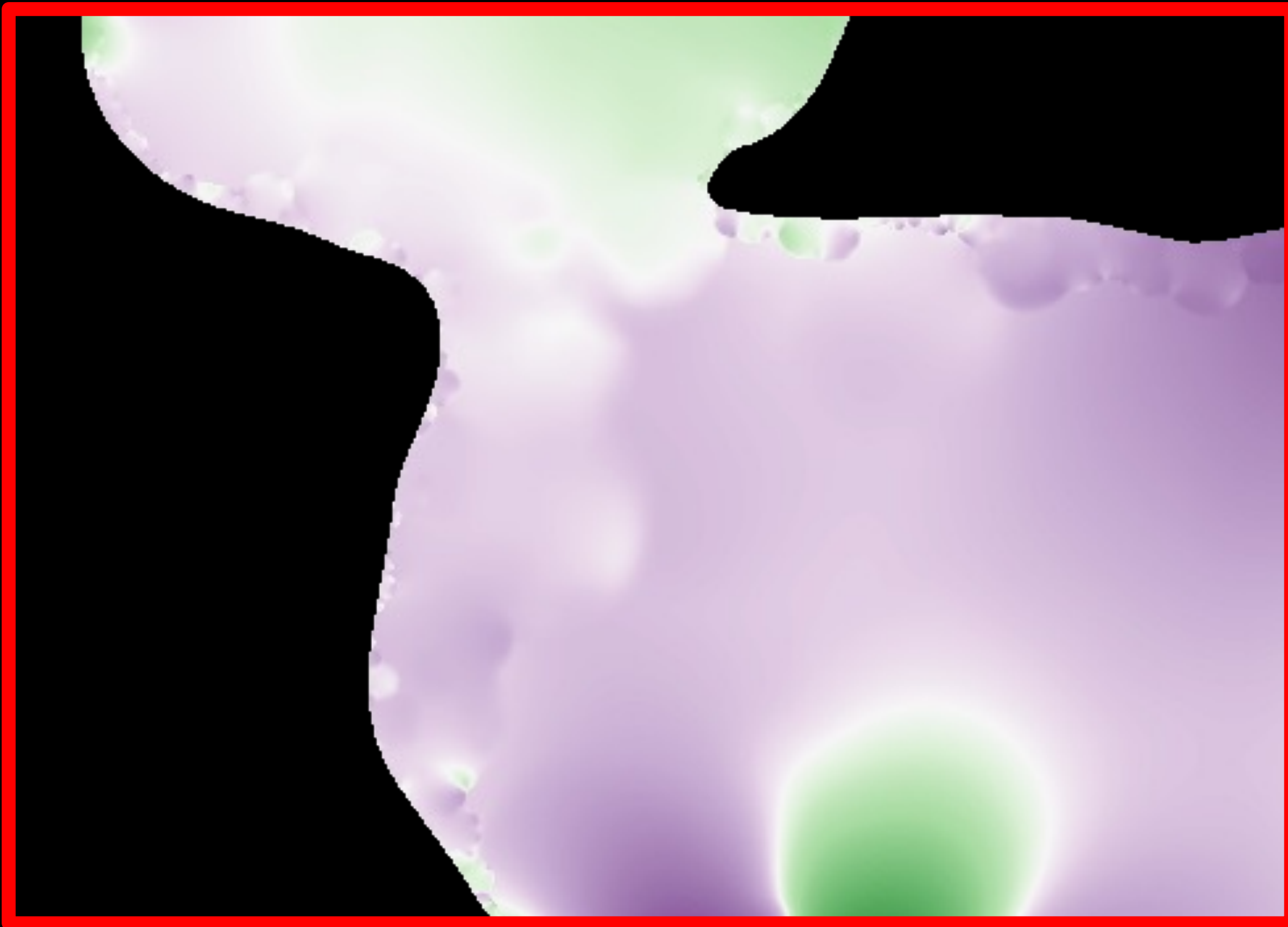
With refinement



(+10% More time)

Without refinement

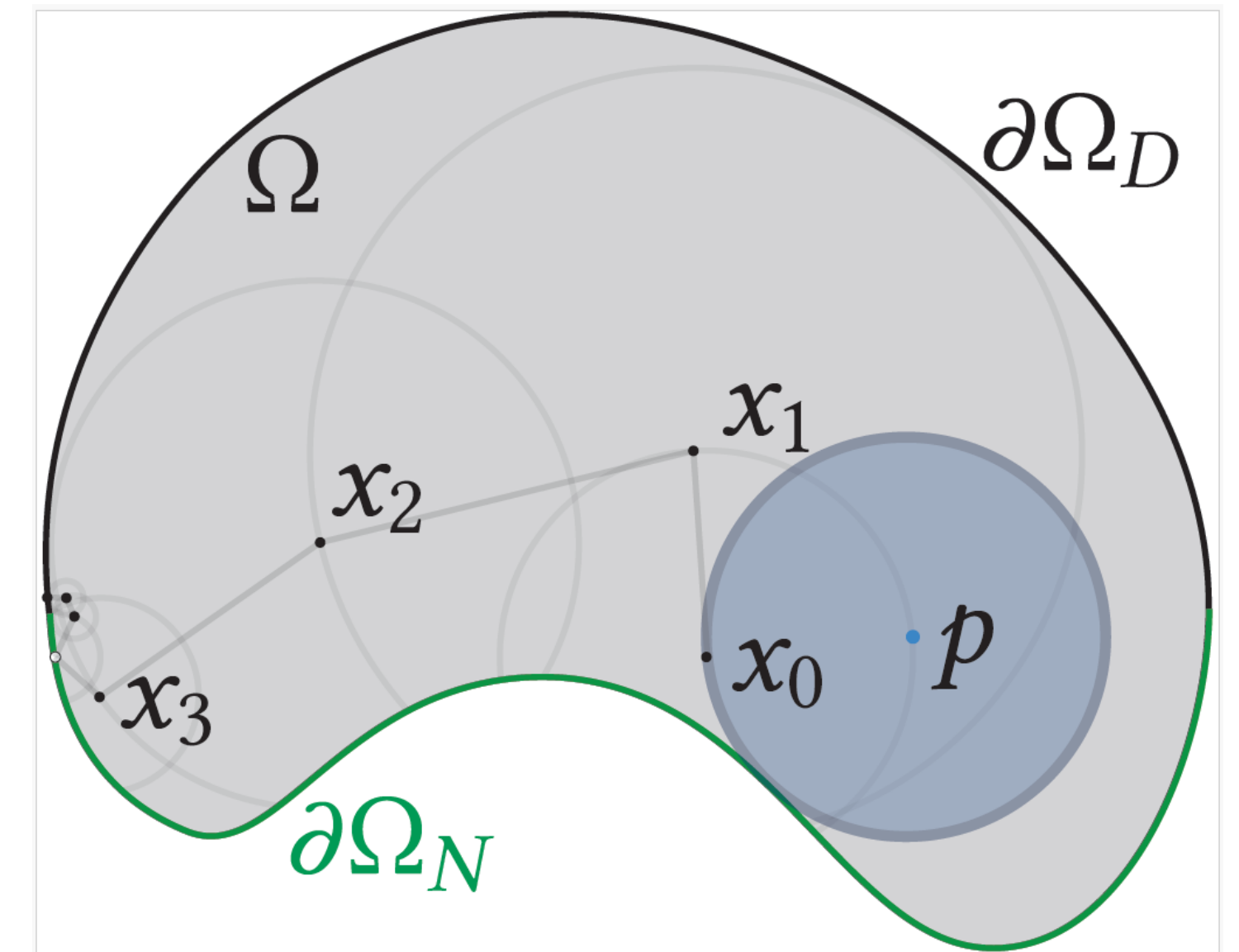
With refinement



(+10% More time)

Generalizations

- ✓ 2D, 3D
- ✓ Source term (Poisson equation)
- ✓ Screening term (screened Poisson eq)
- ✓ Reflecting and mixed BCs by WoSt [Miller et al. 2024]

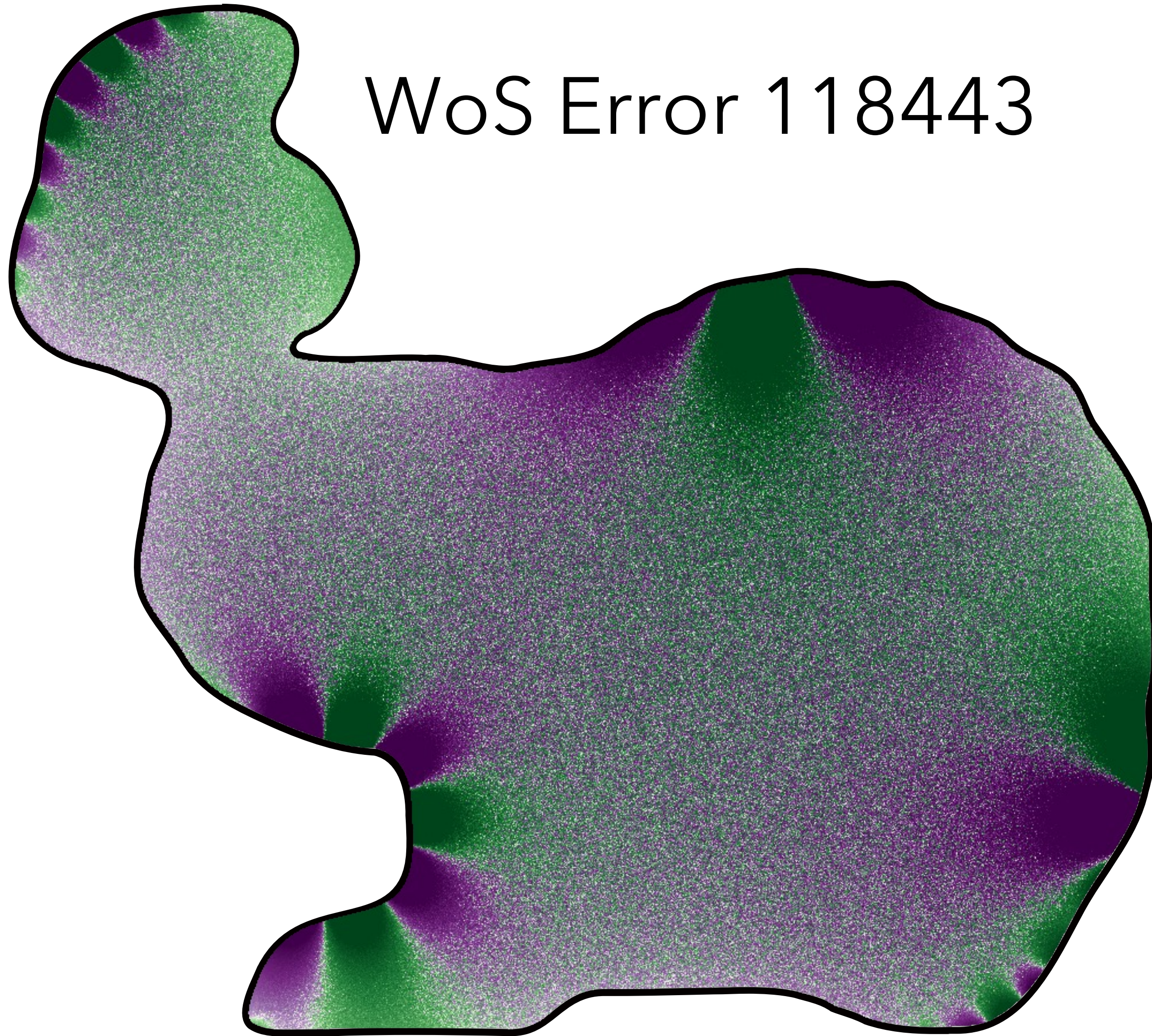




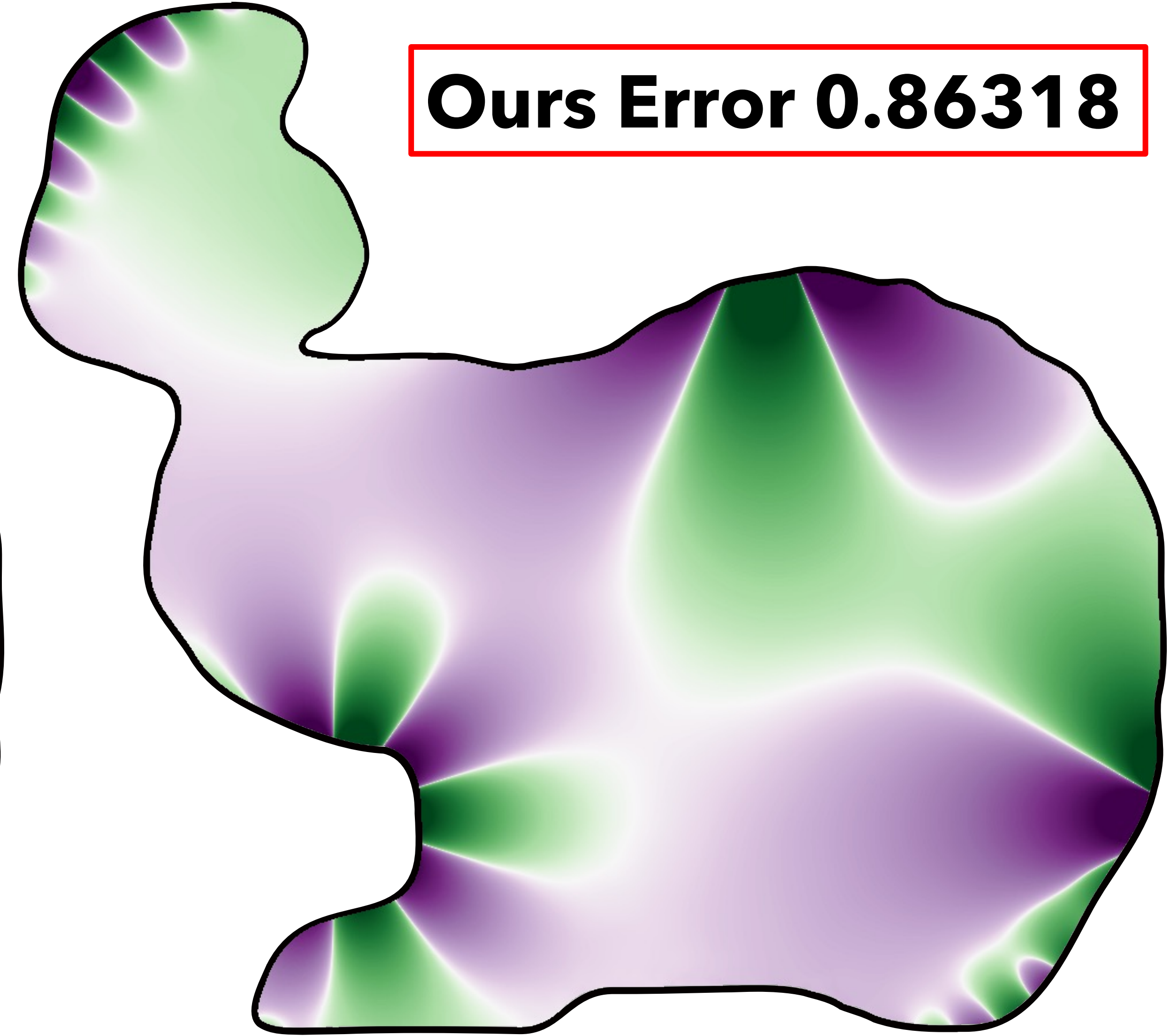
Evaluation

2D comparisons: WoS/Ours (Equal Time)

WoS Error 118443

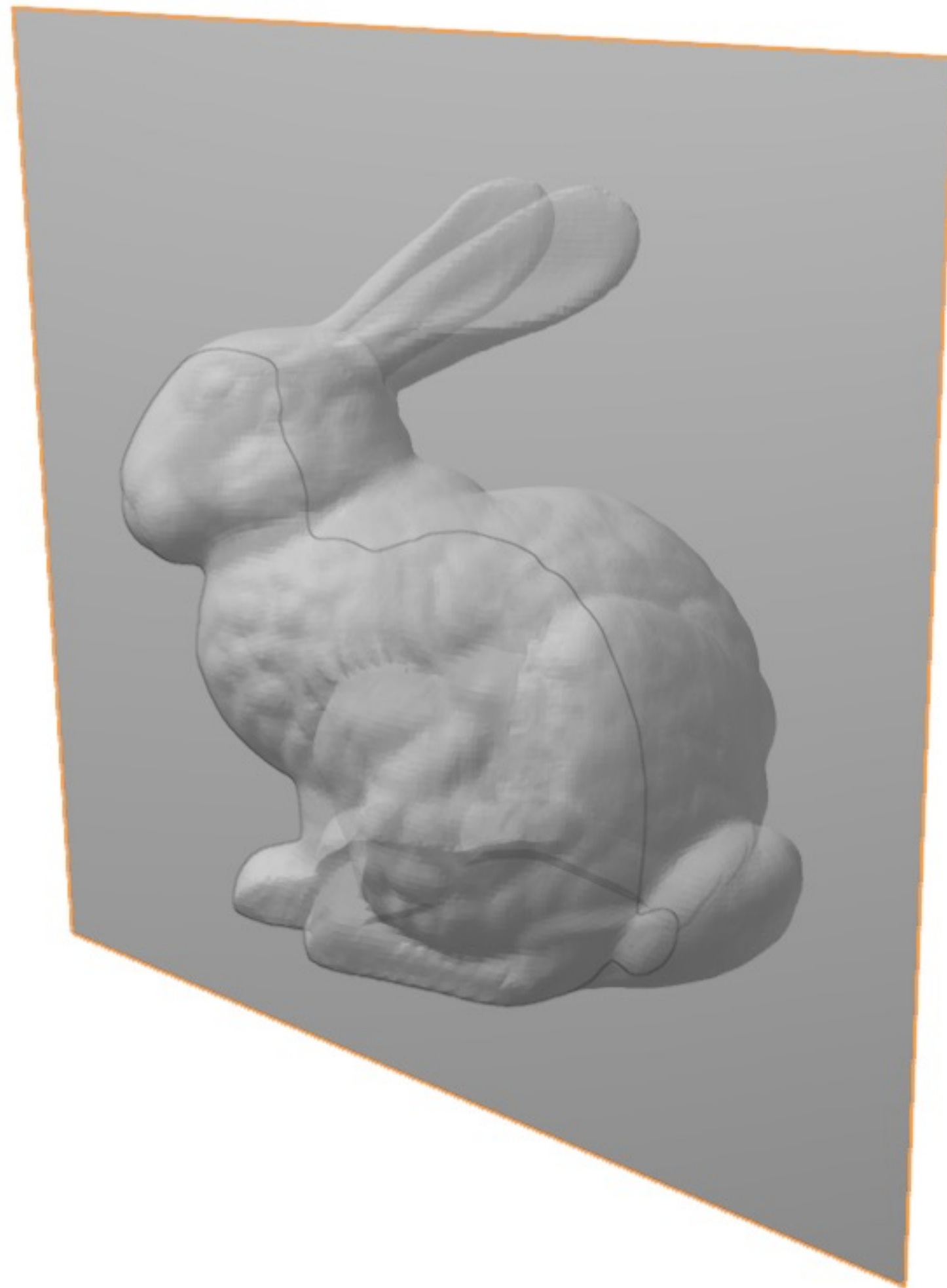


Ours Error 0.86318

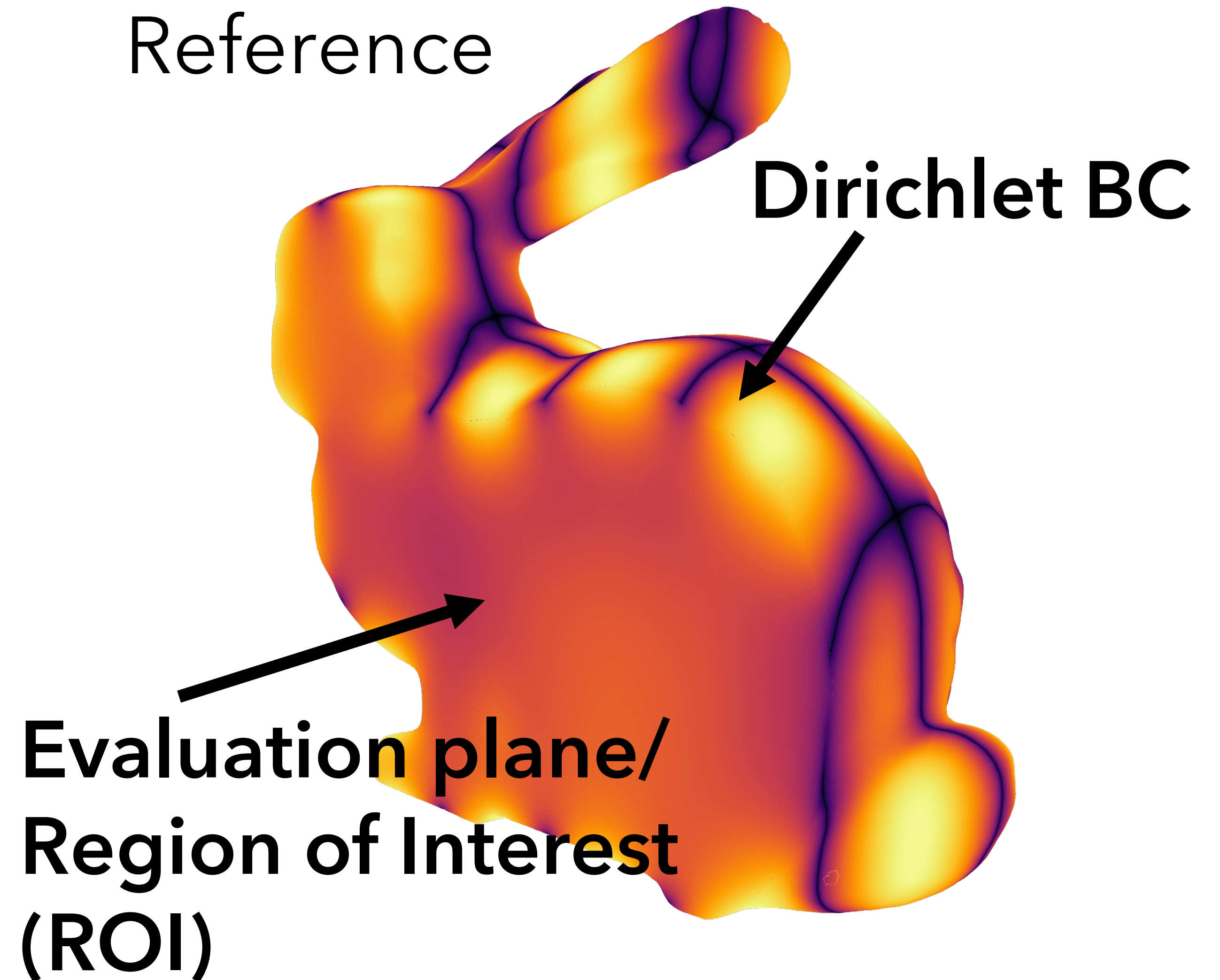


3D comparisons to related work

Setup

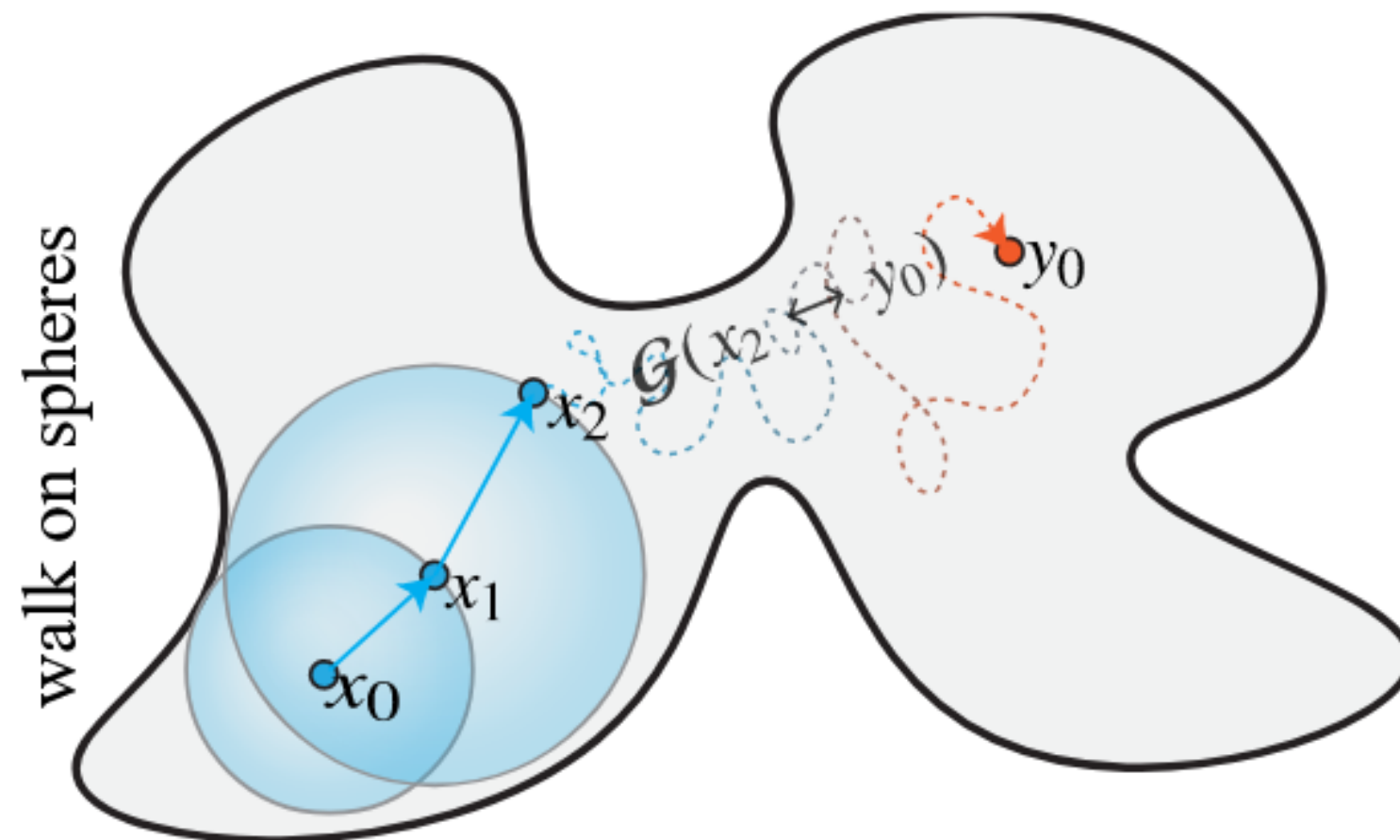


Reference

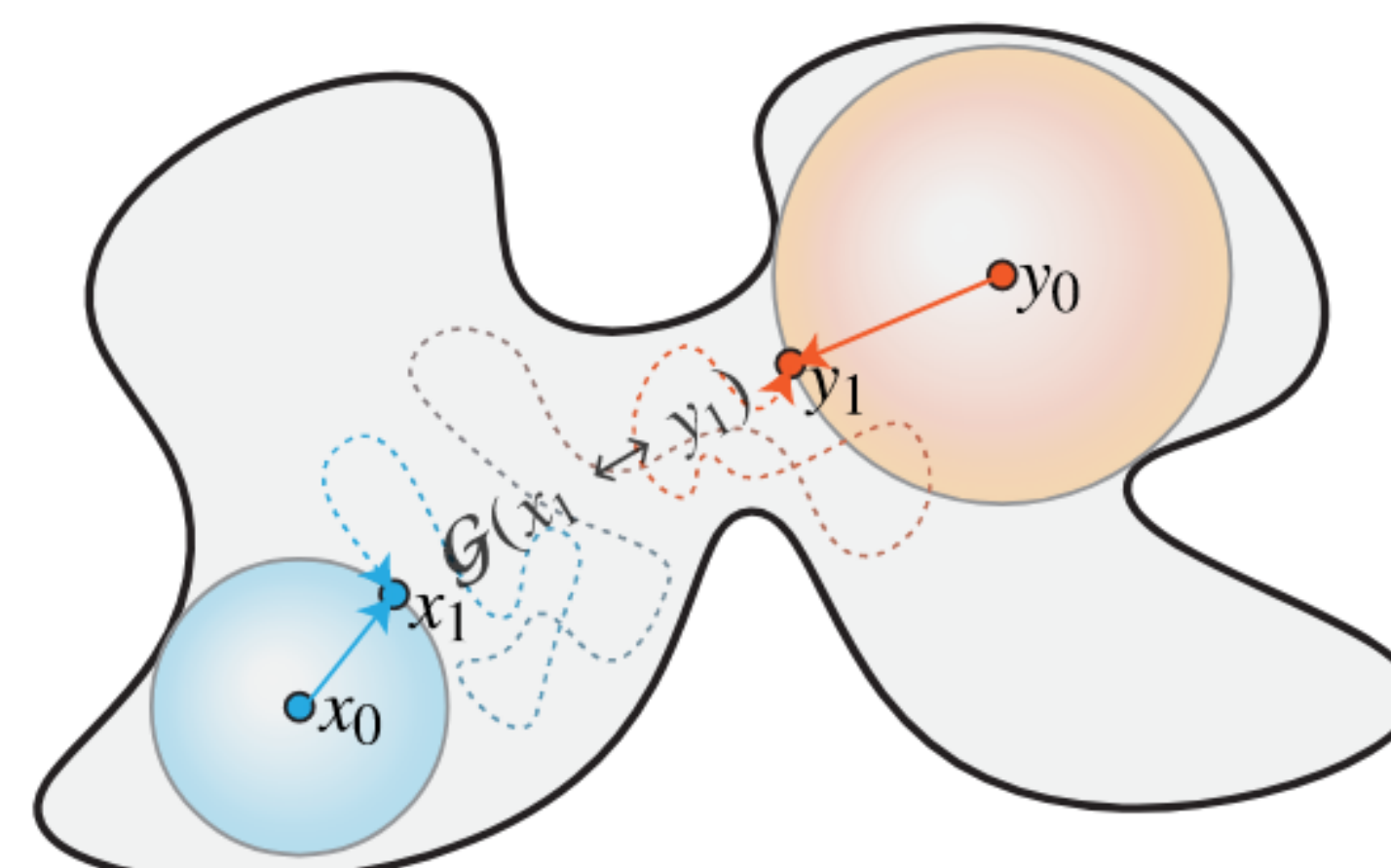


Reverse WoS [Qi et al. 2022]

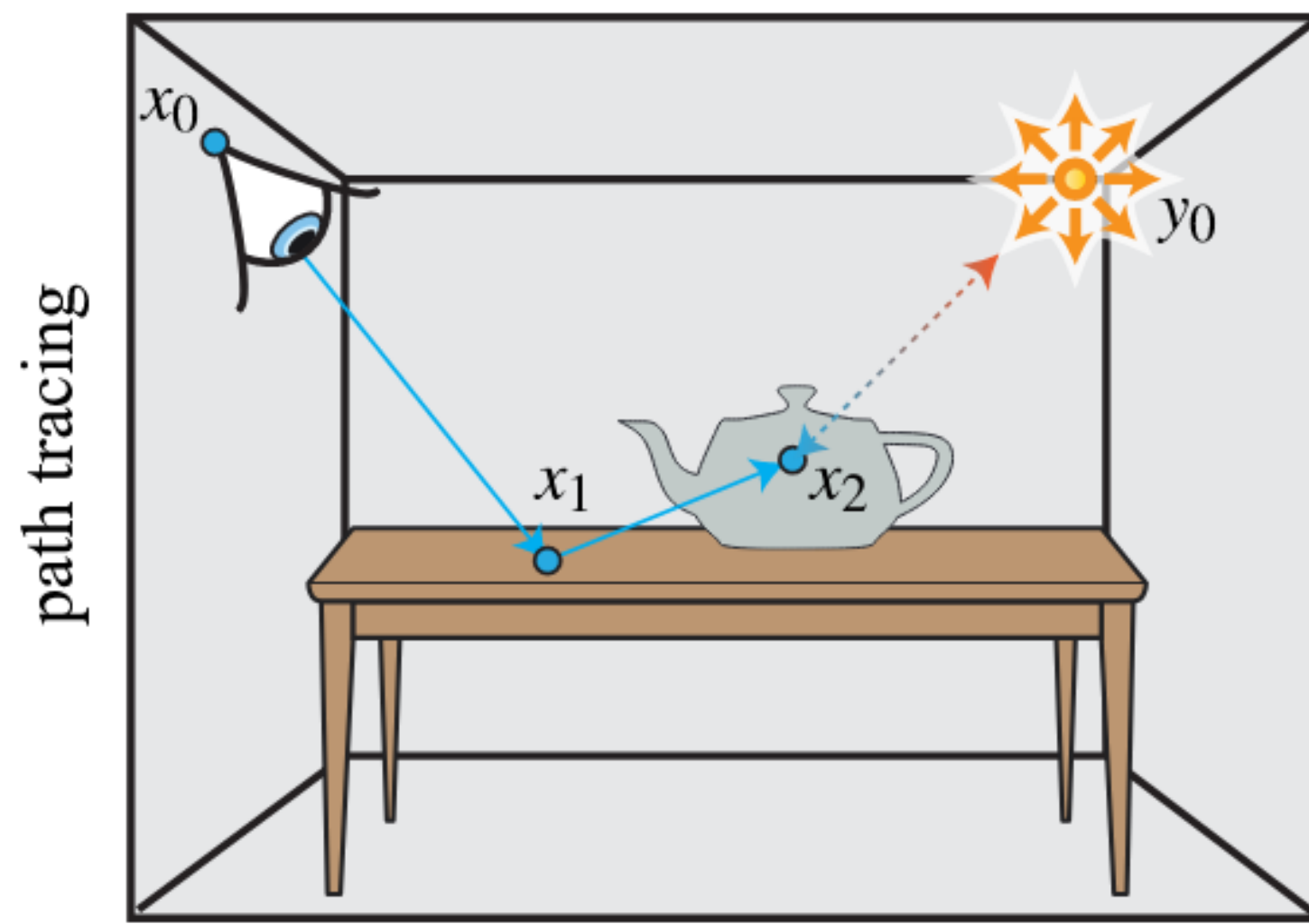
[Qi et al. 2022]



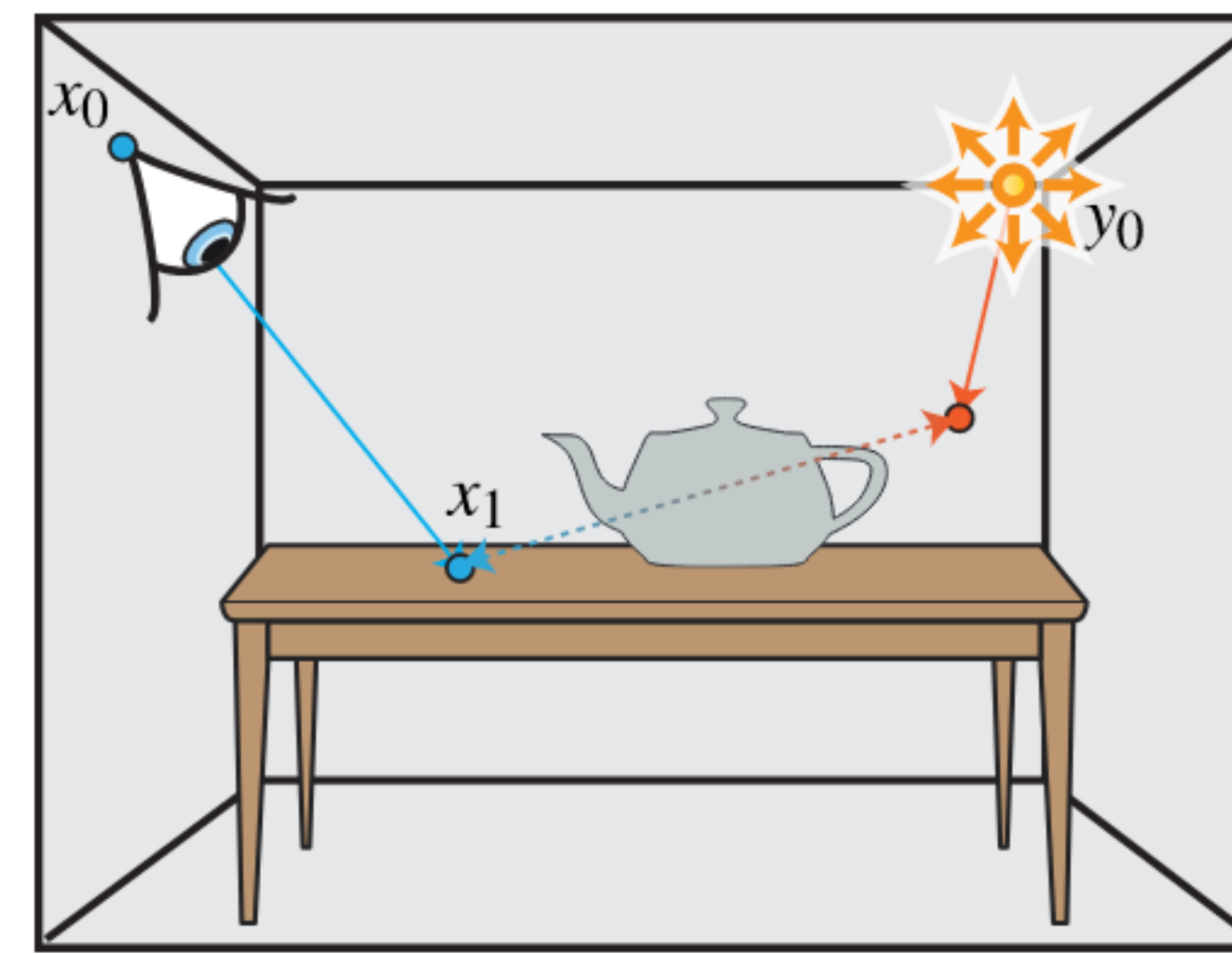
2 sensor segments, 0 source segments



1 sensor segment, 1 source segment



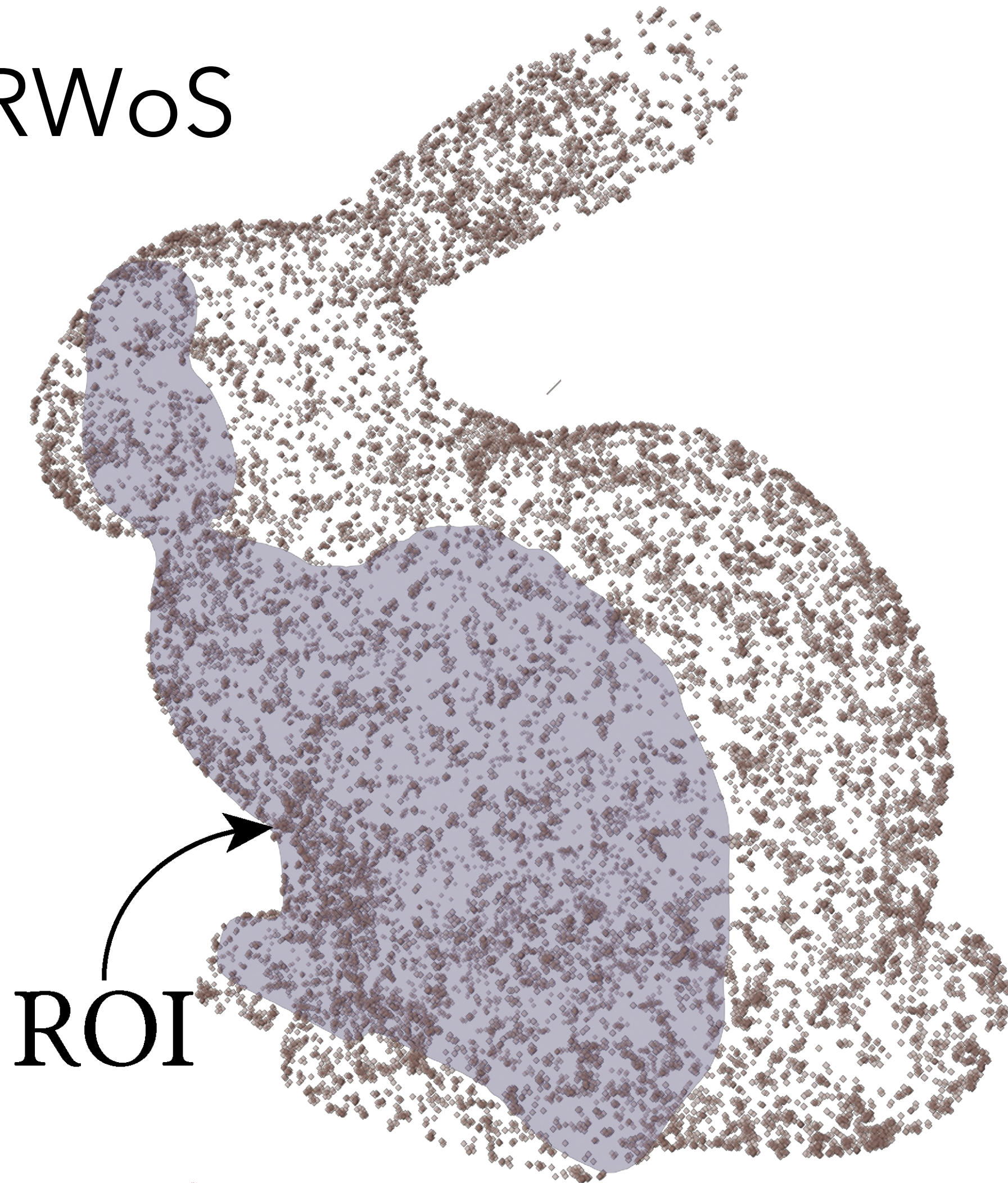
2 camera segments, 0 light segments



1 camera segment, 1 light segment

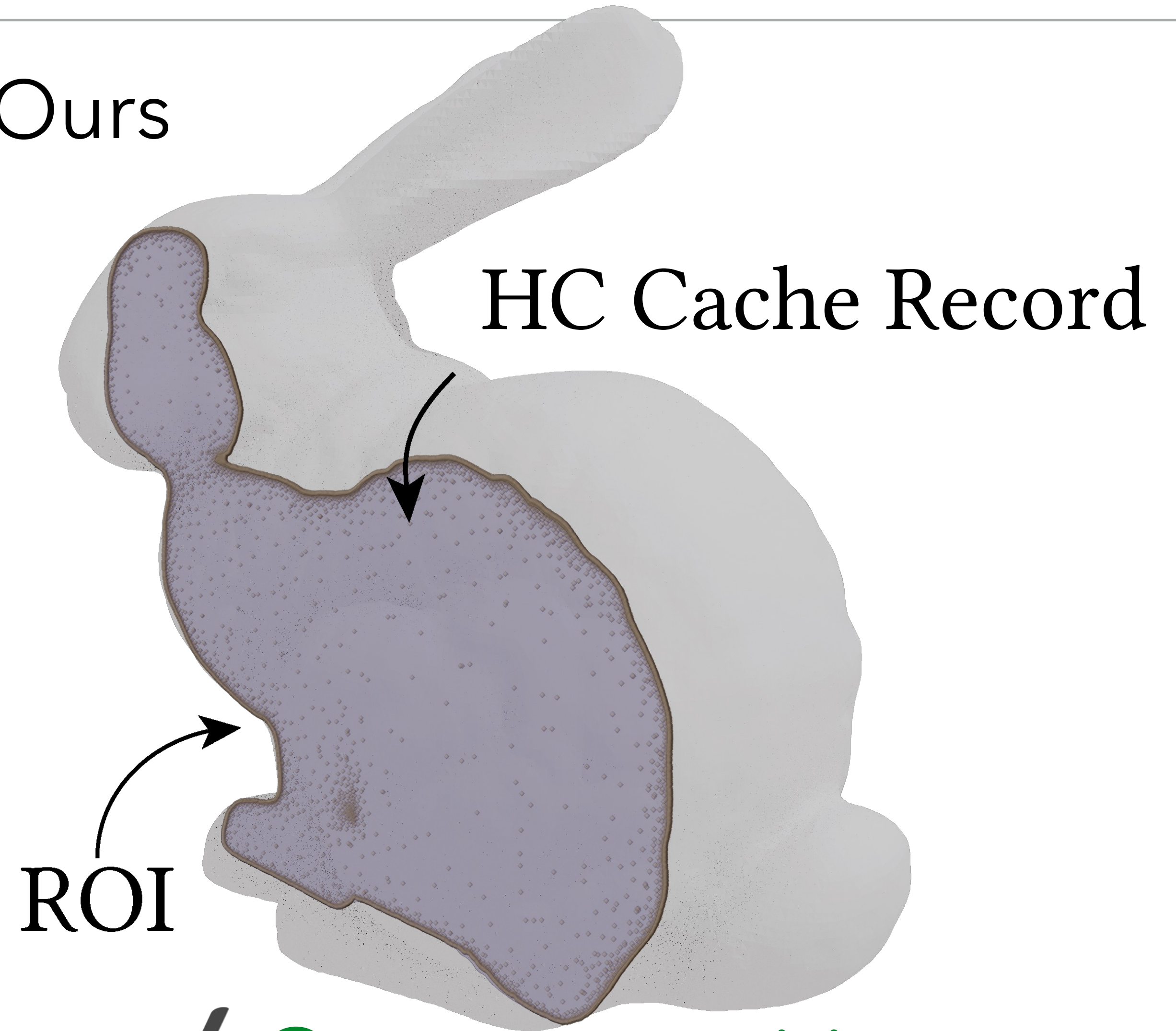
Reverse WoS vs Ours

RWoS



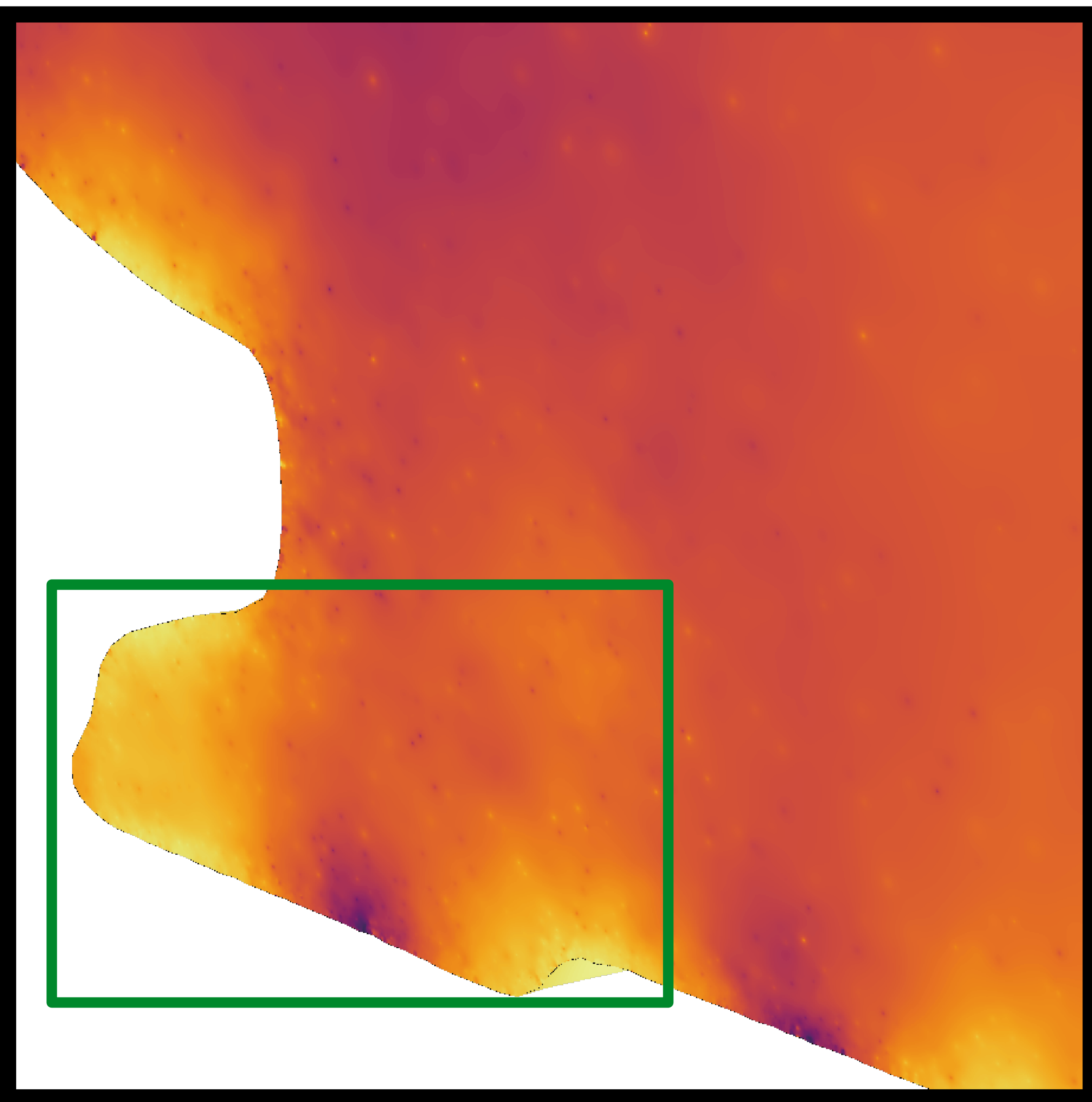
X Output sensitive

Ours

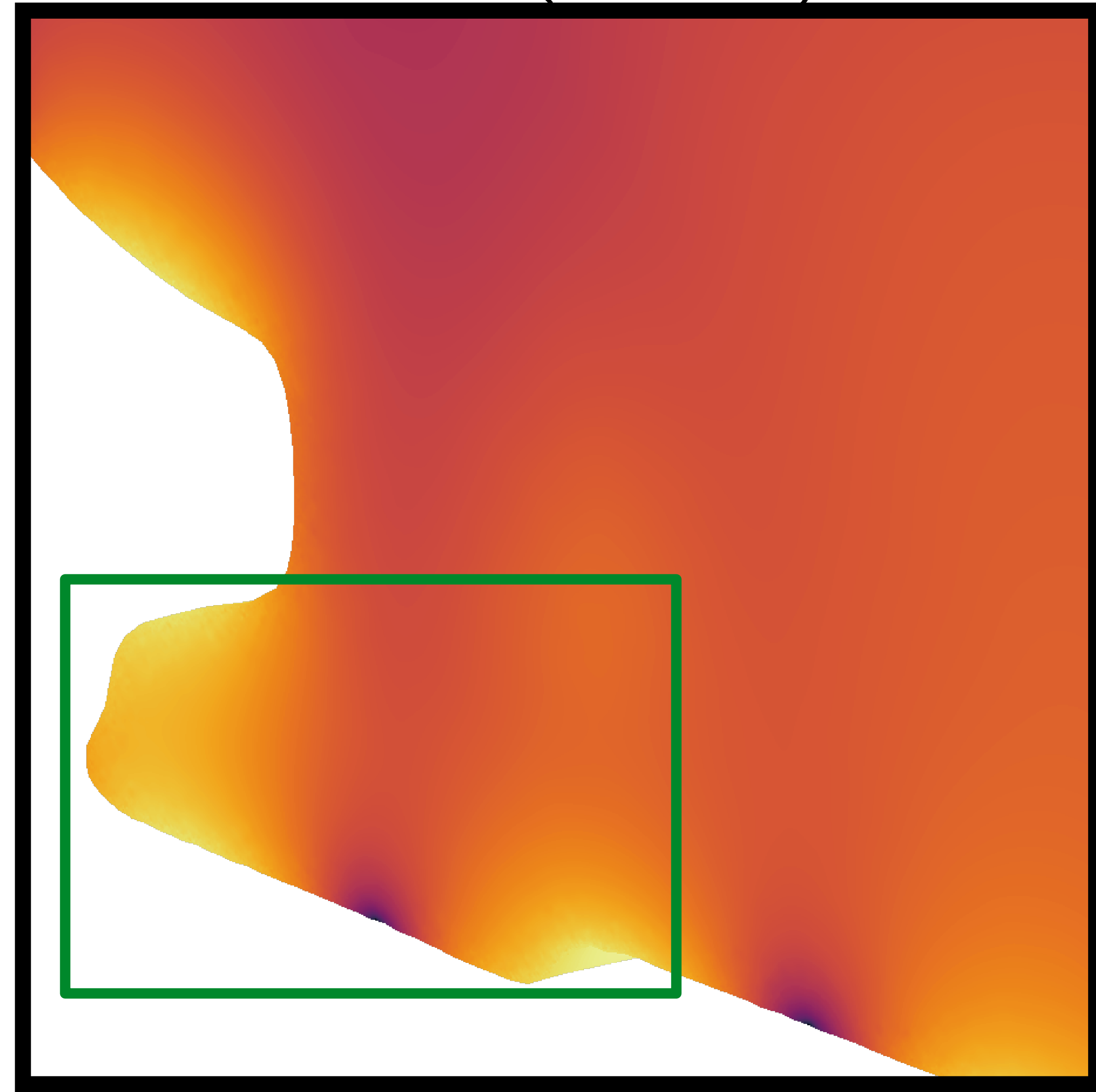


✓ Output sensitive

RWoS

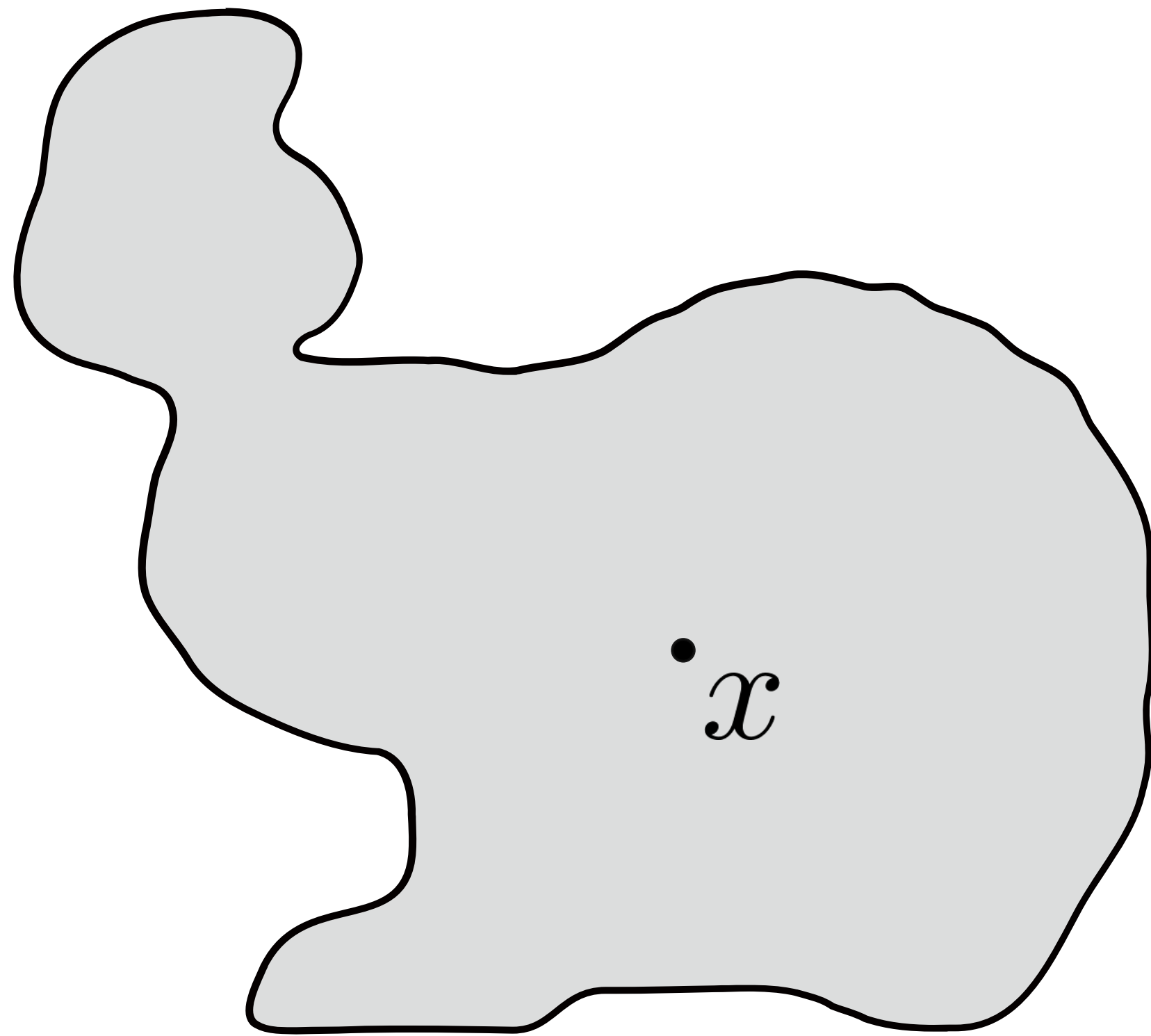


HC(Ours)

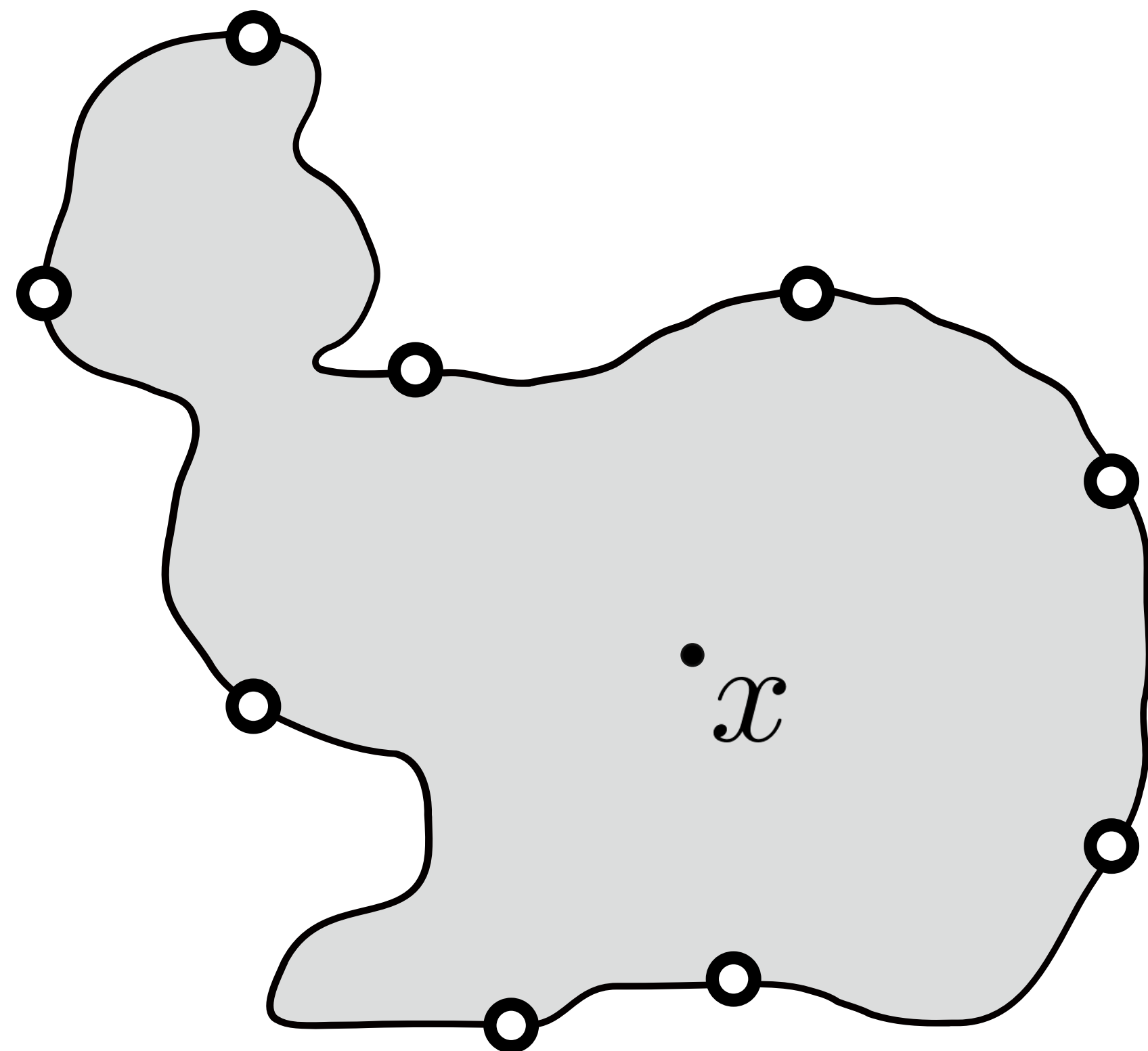


Boundary Value Caching [Miller et al. 2022]

Reference



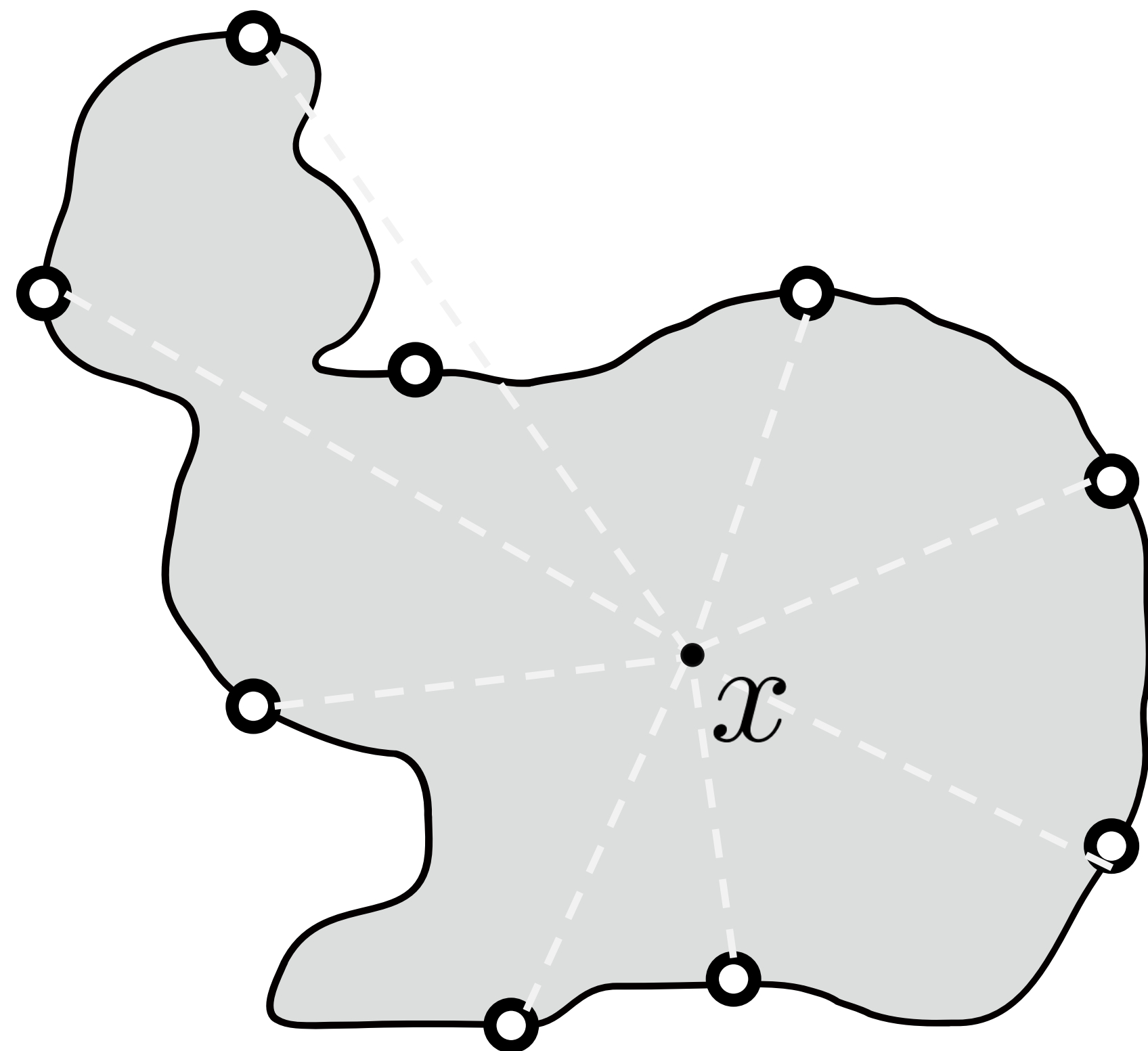
Boundary Value Caching [Miller et al. 2022]



Reference



Boundary Value Caching [Miller et al. 2022]



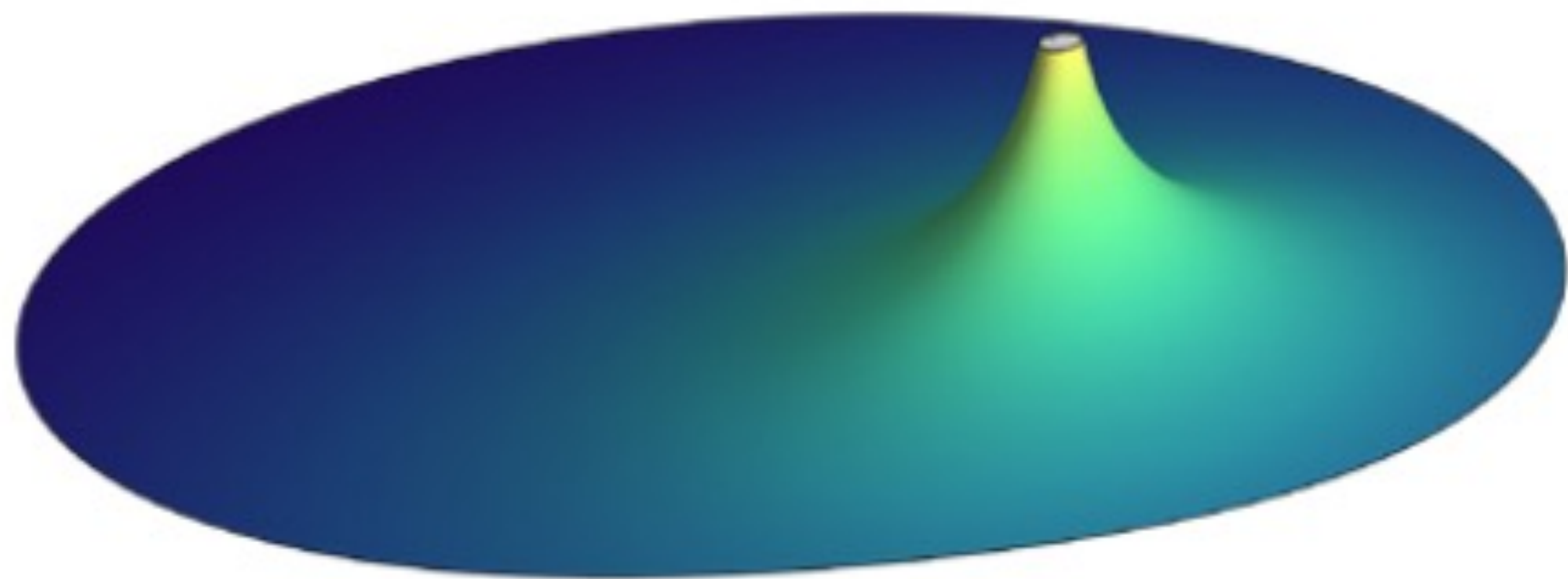
Reference



Boundary Value Caching [Miller et al. 2022]

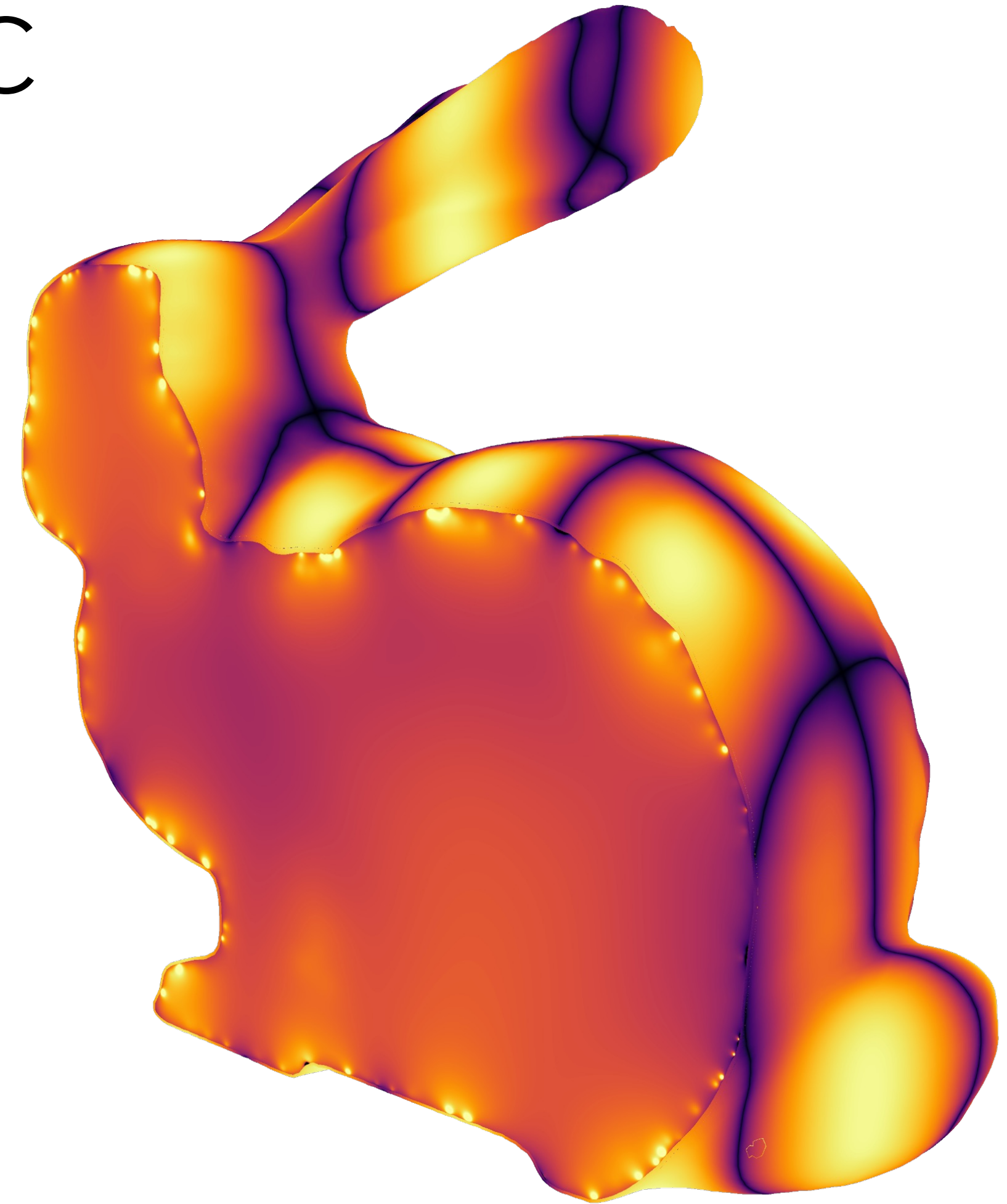
[Miller et al. 2022]

$G(x, y)$



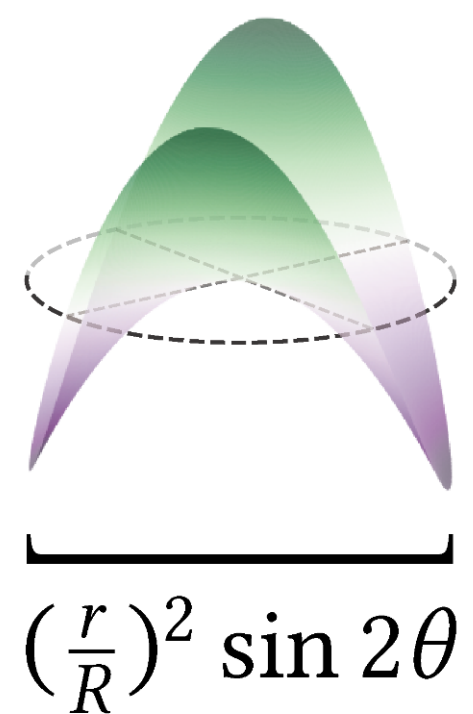
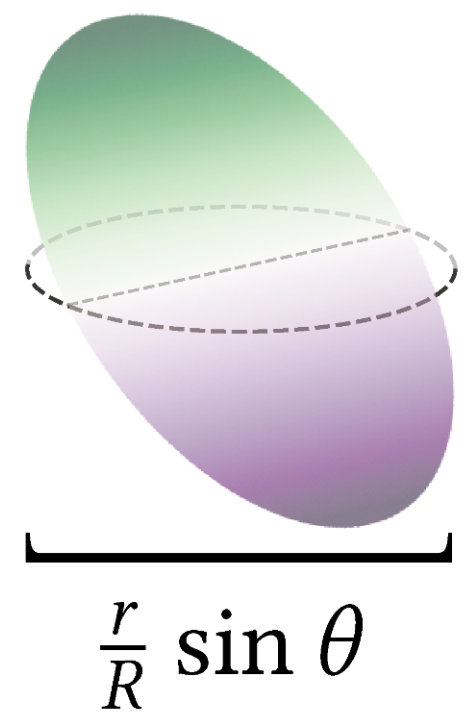
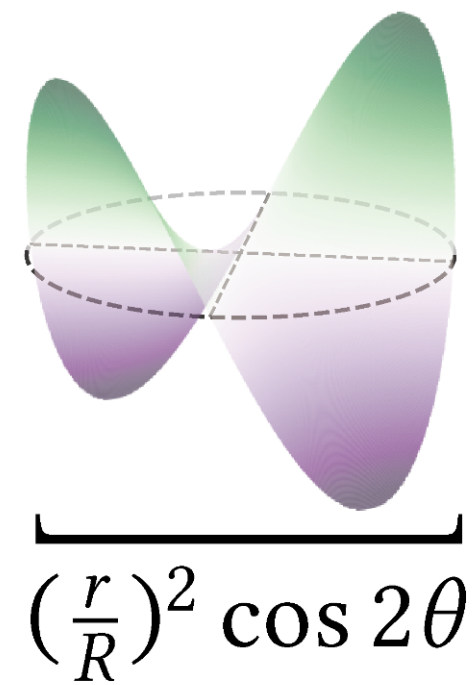
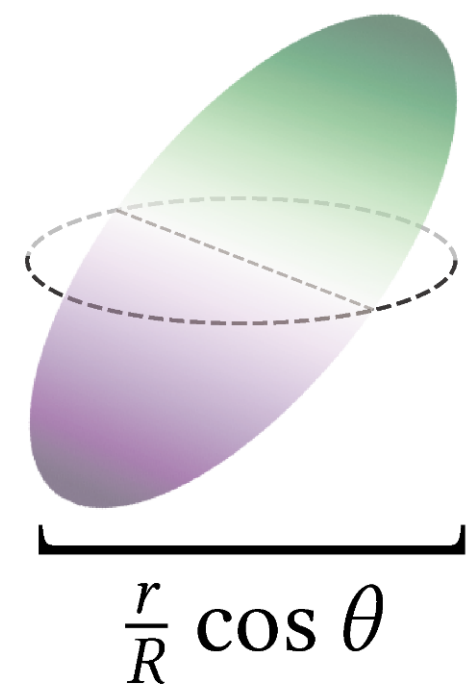
Singularity!

BVC



Harmonic Caching (Ours)

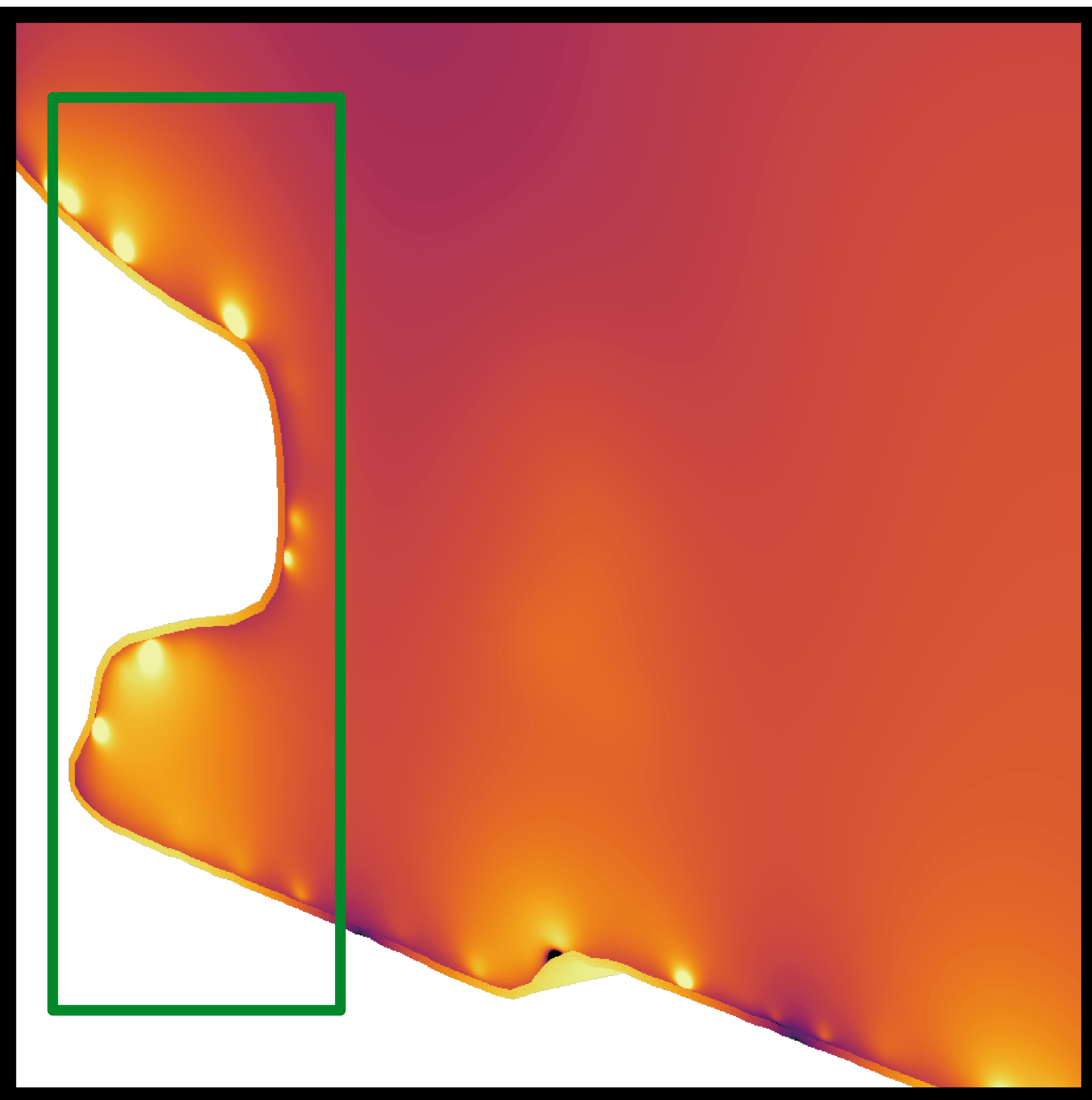
We use smooth basis functions: Ours



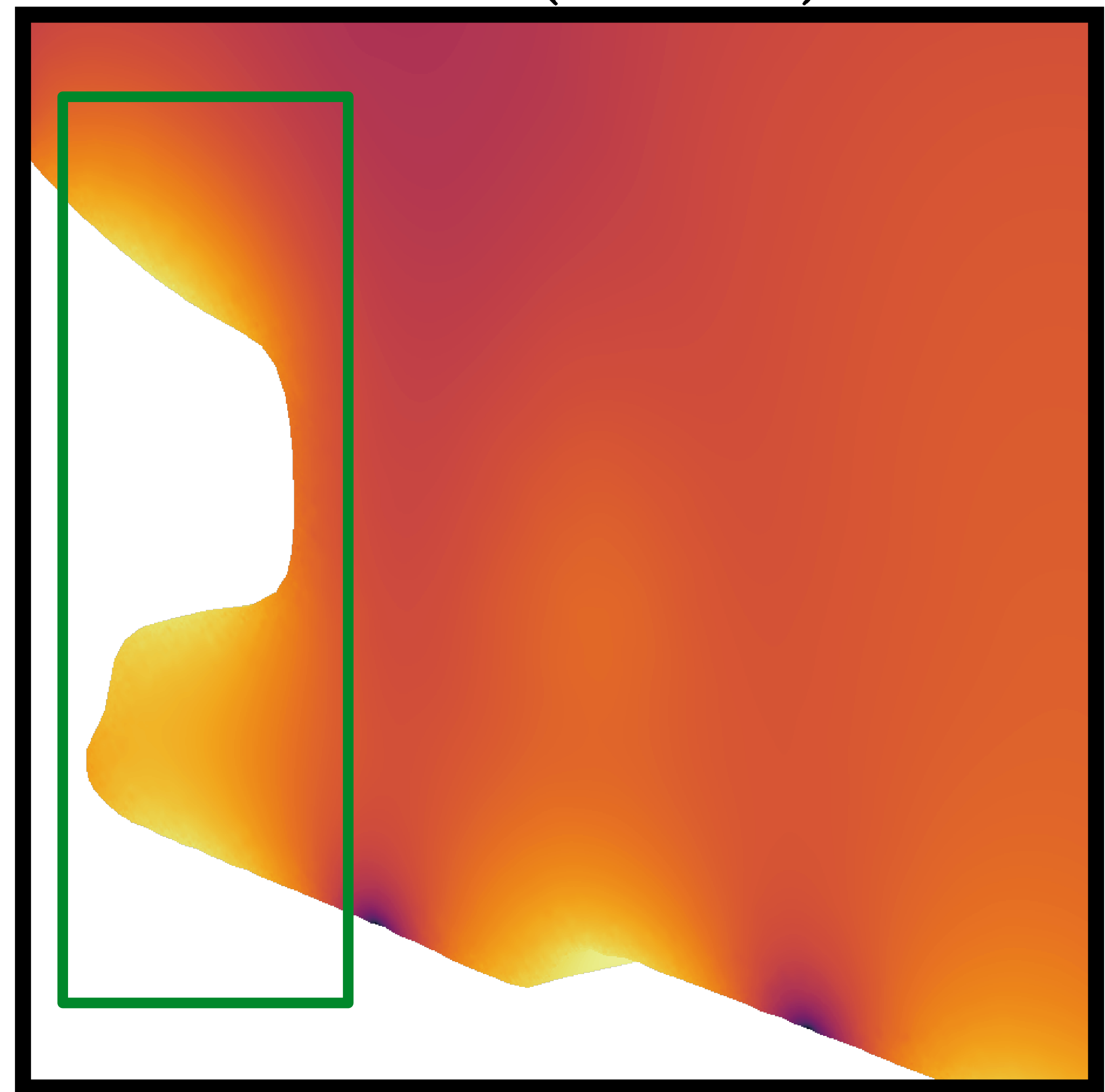
✓ Singularity free



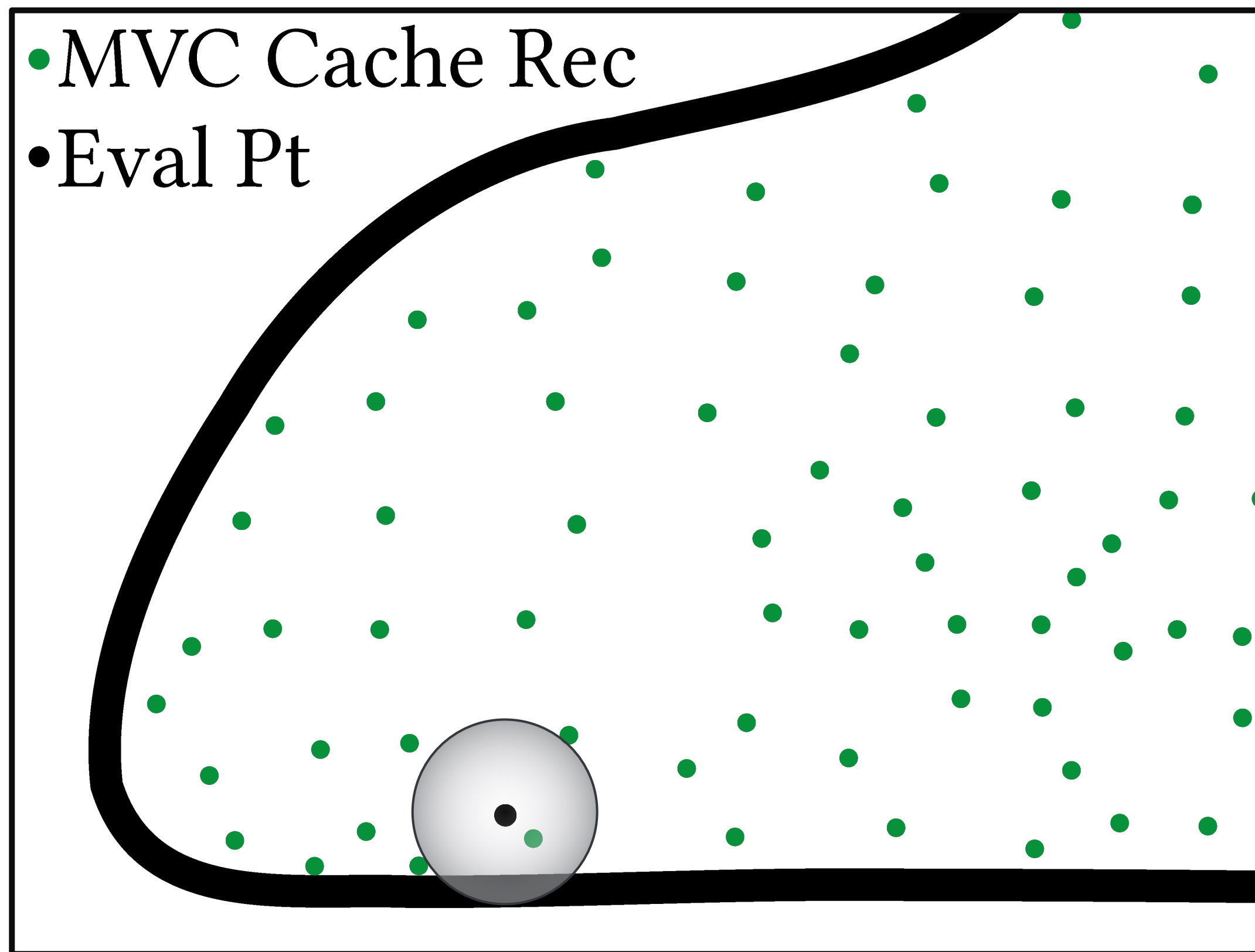
BVC



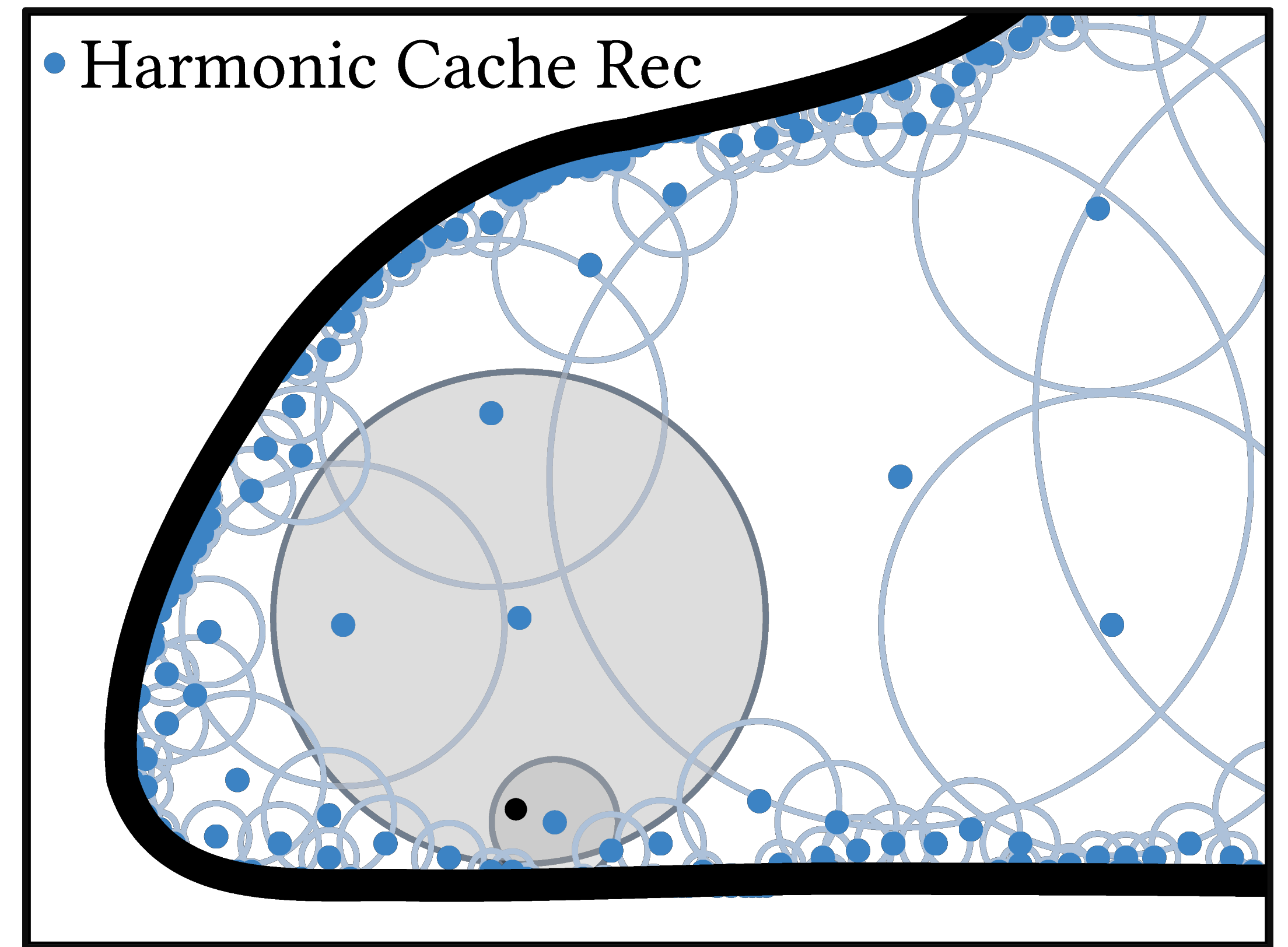
HC(Ours)



Mean Value Caching [Bakbouk et al. 2023]

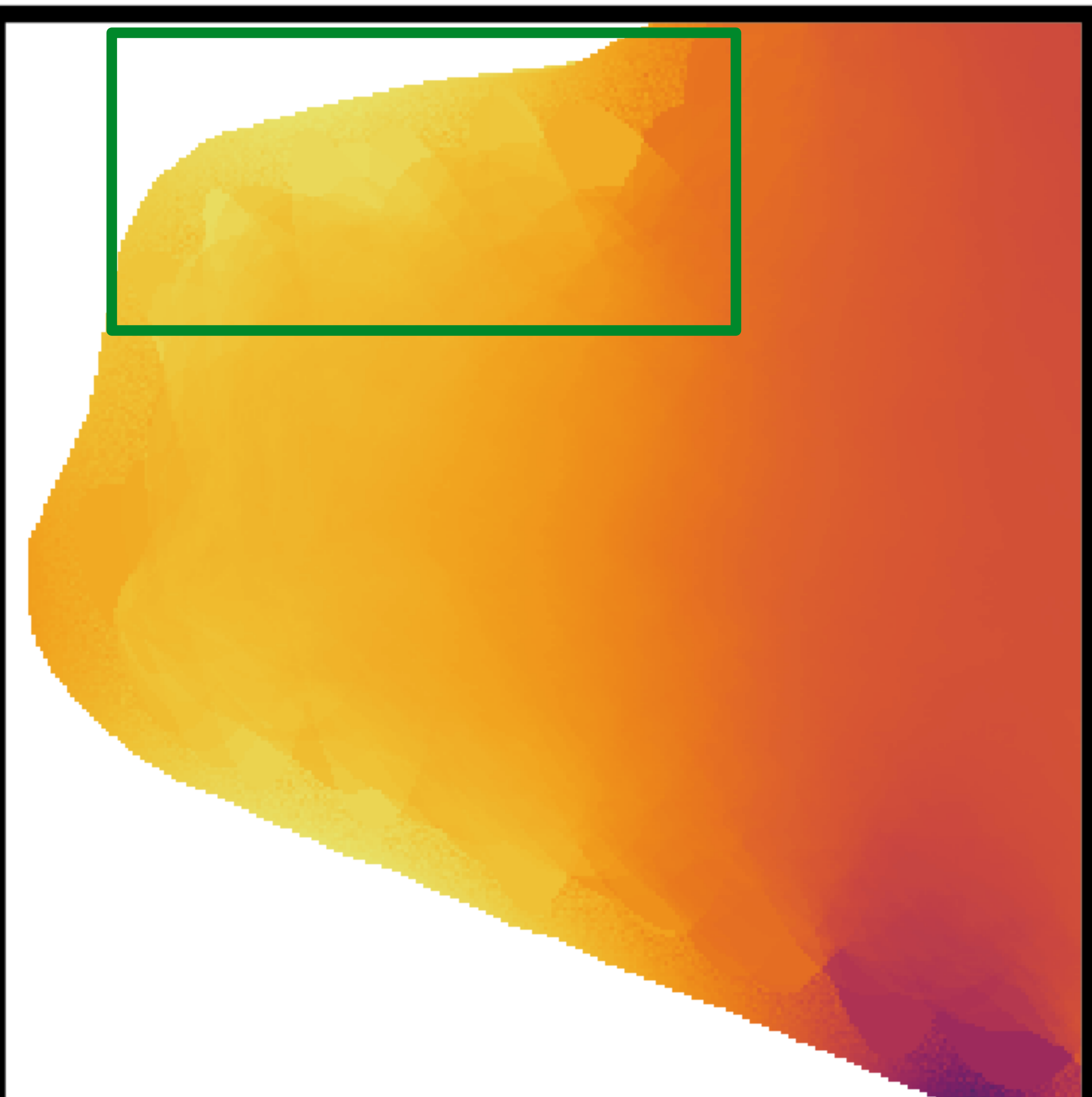


MVC expands the largest ball around the evaluation point: fewer records available.

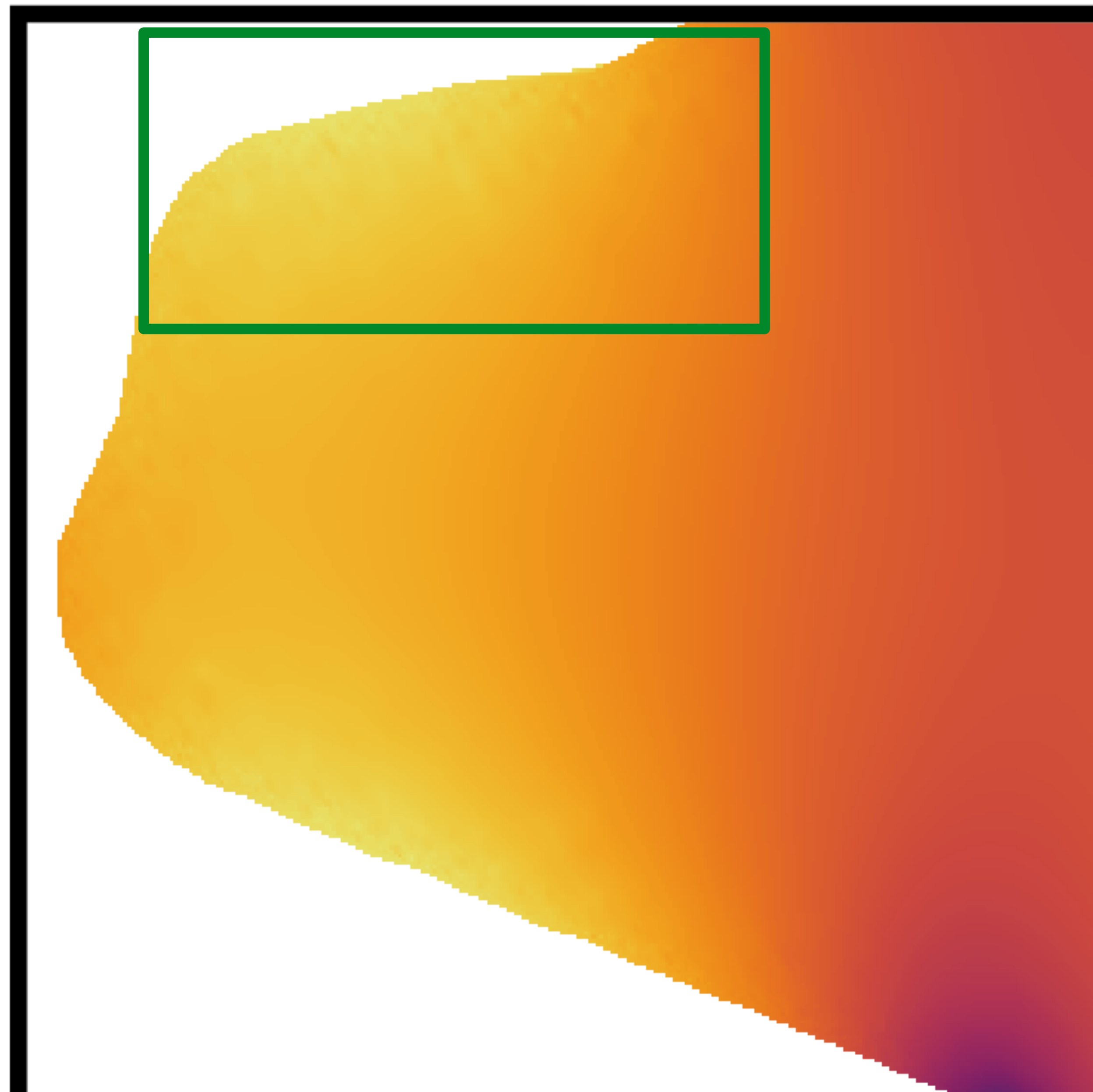


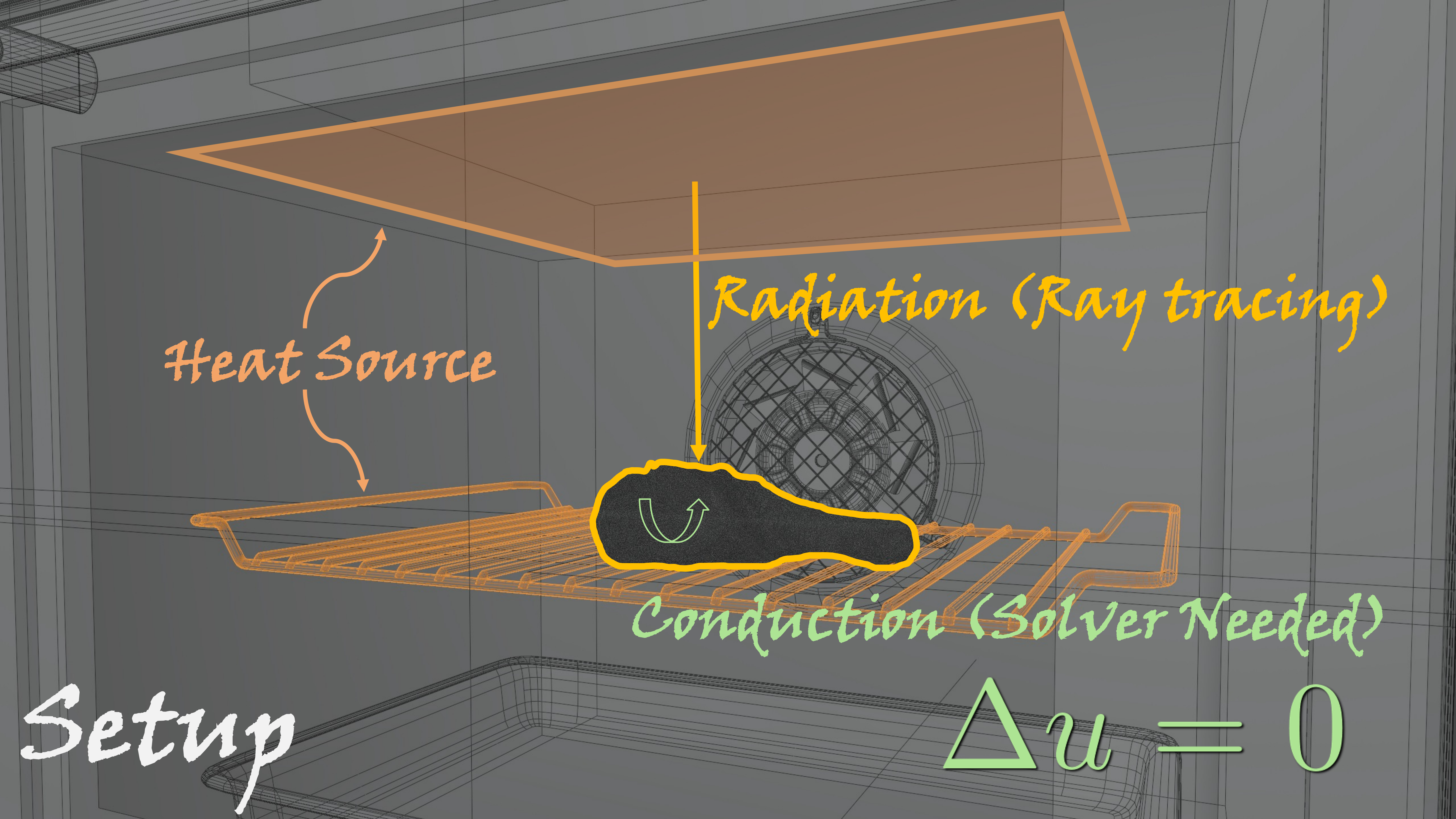
Ours gathers records from overlapping balls: more are available even near the boundary!

MVC



HC(Ours)





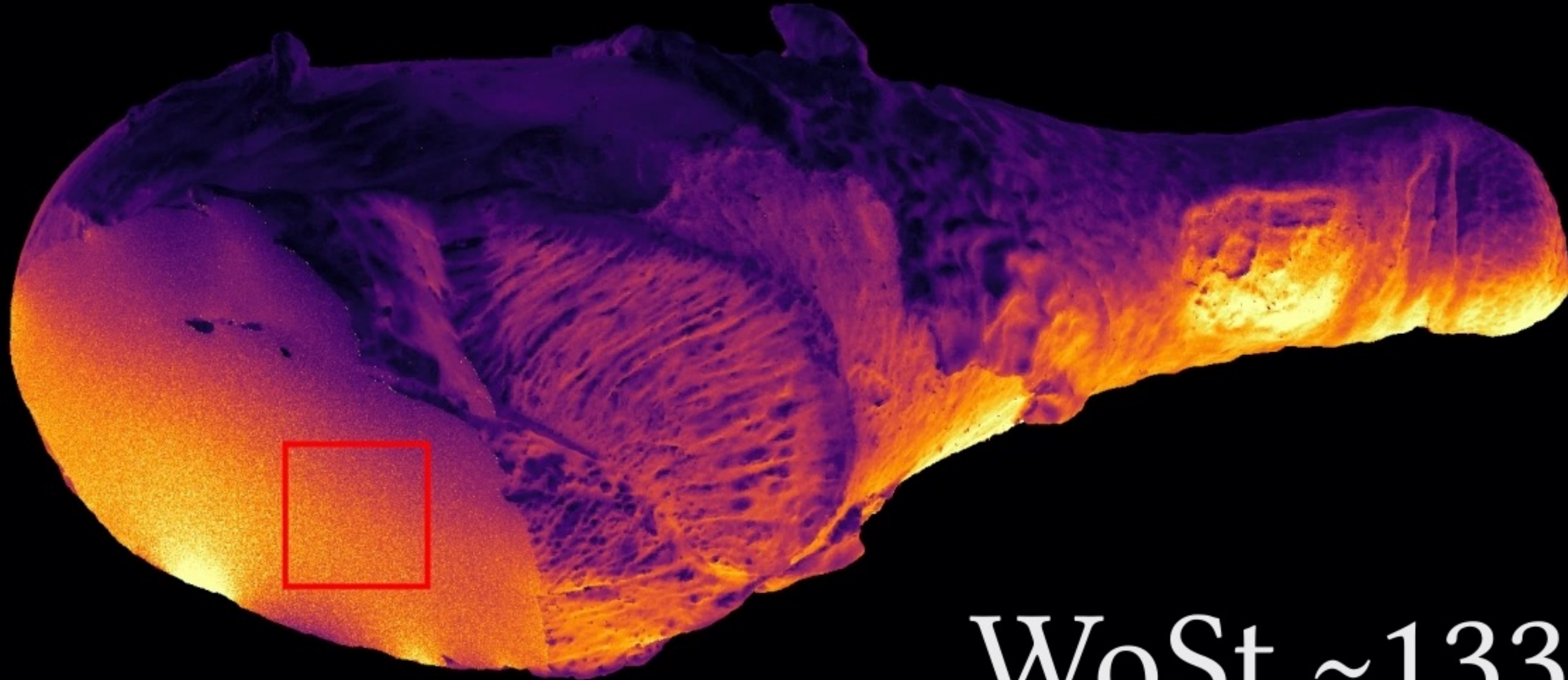
Heat Source

Radiation (Ray tracing)

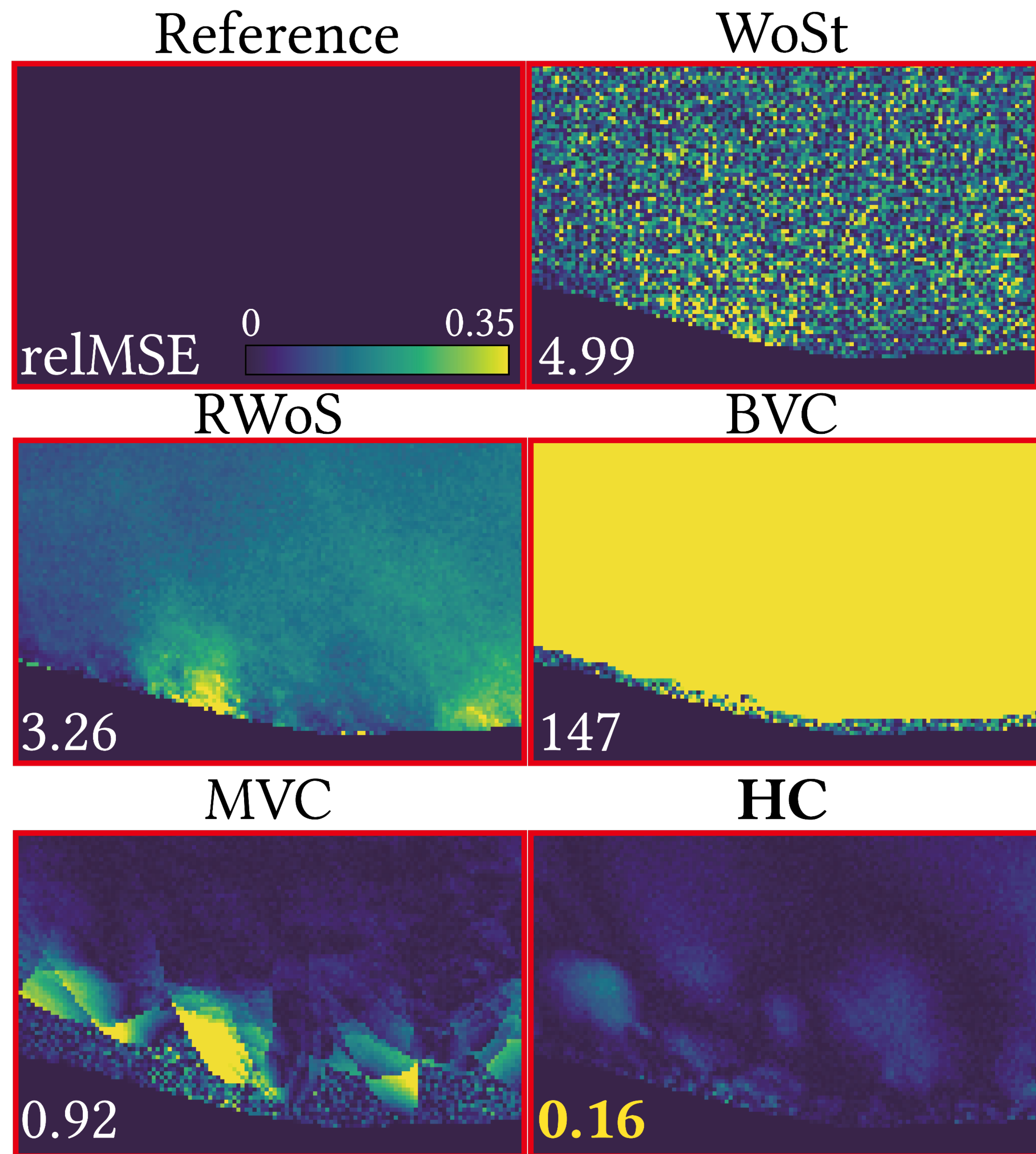
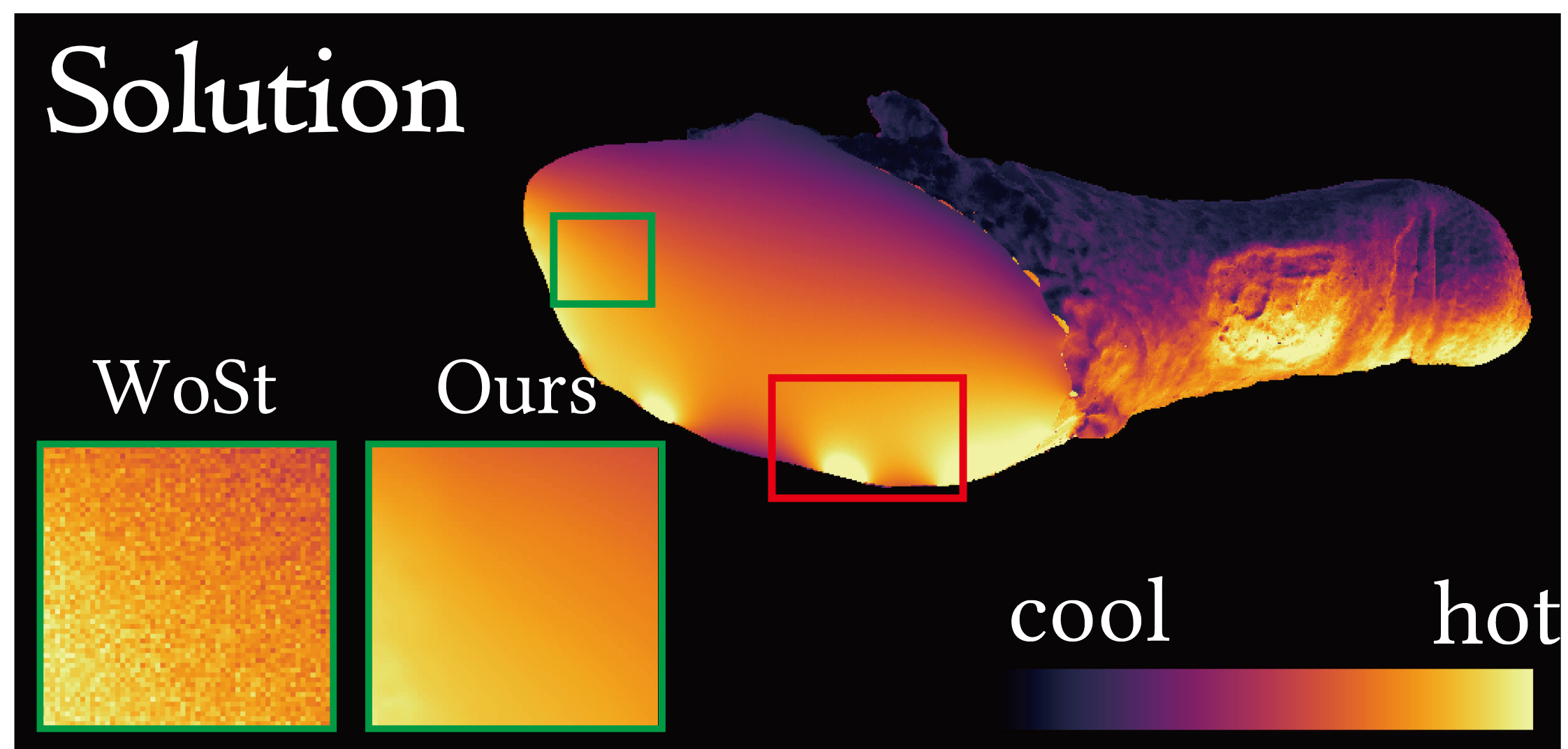
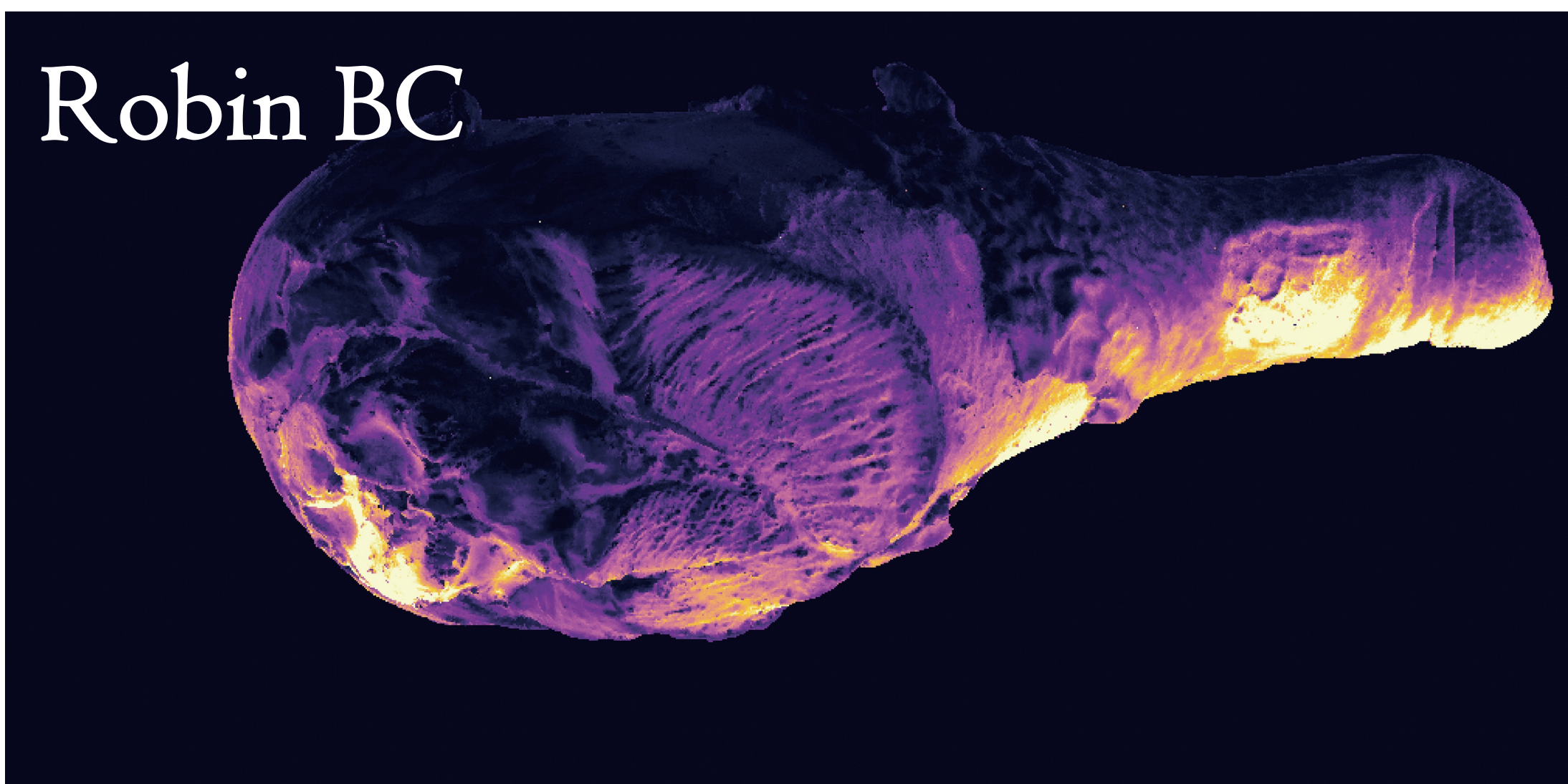
Conduction (Solver Needed)

Setup

$$\Delta u = 0$$



WoSt $\sim 133\text{s}/\text{frame}$





Conclusion

Conclusion

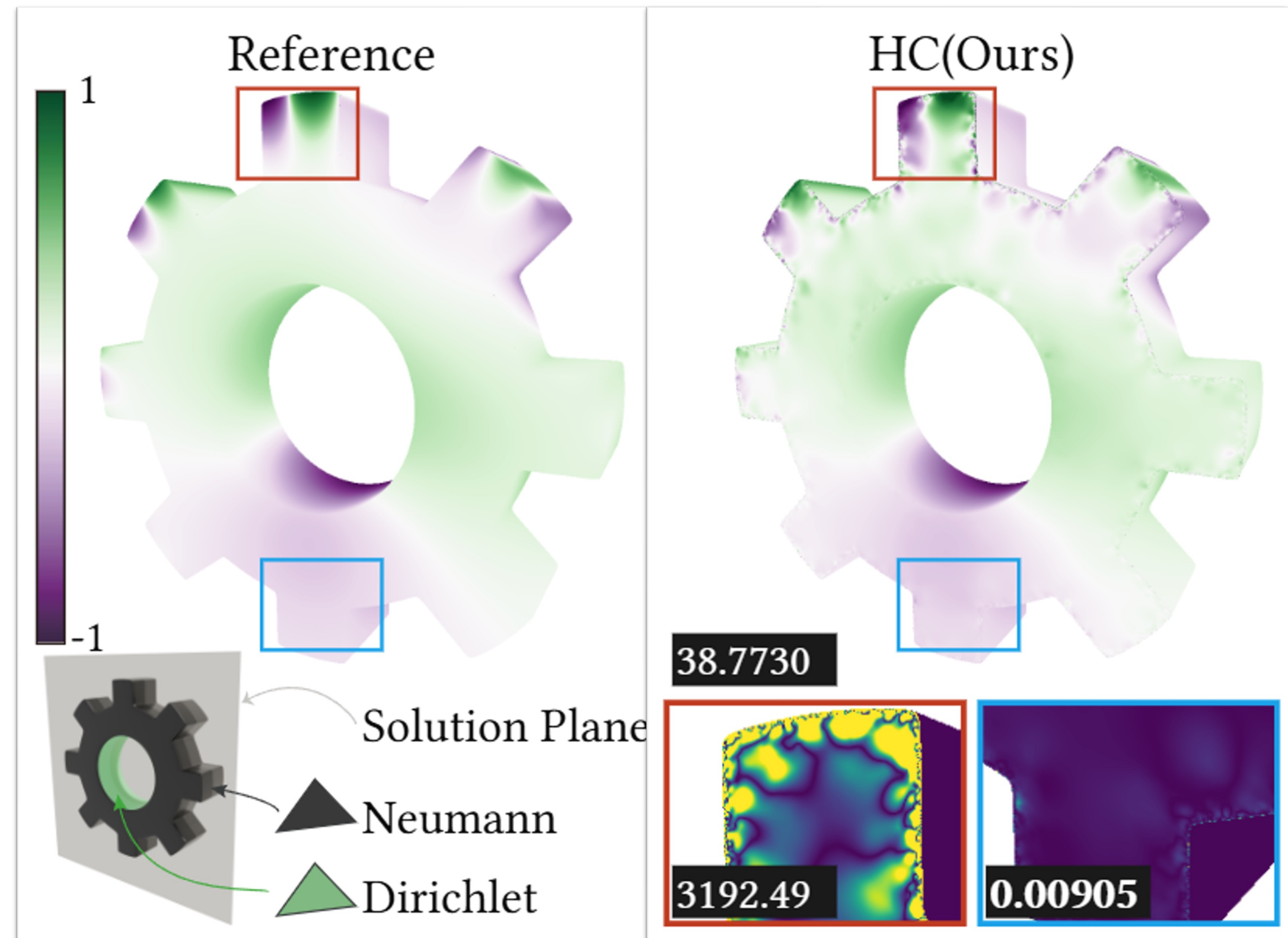
A novel caching algorithm for WoS:

- ✓ **Leverages harmonic structures in PDE solutions**
- ✓ **Reduces variance by orders of magnitude**
- ✓ **Easy to implement**
- ✓ **Supports various boundary conditions, sources and screening terms**

Future work

Other estimators for coefficients estimation?

Still more work to do for reducing artifacts!



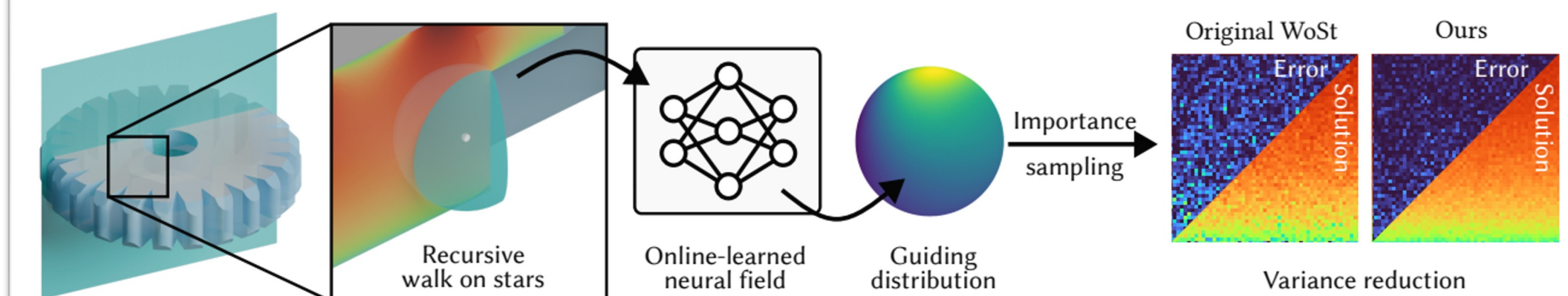
Guiding-Based Importance Sampling for Walk on Stars

TIANYU HUANG, School of Software and BNRist, Tsinghua University, China

JINGWANG LING, School of Software and BNRist, Tsinghua University, China

SHUANG ZHAO, University of California Irvine, United States of America

FENG XU*, School of Software and BNRist, Tsinghua University, China



[Huang et al. 2025]

Harmonic Caching for Walk on Spheres

Thank you!



Eugene d'Eon

NVIDIA



Rohan Sawhney

NVIDIA



Wojciech Jarosz

Dartmouth College